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Research Papers

No Access

The strong domination problem in block graphs and proper interval graphs

Saikat Pal and D. Pradhan

1950063

https://doi.org/10.1142/S1793830919500630

Abstract | **PDF**

Preview Abstract

In a graph $G=(V,E)$, the *degree* of a vertex $v \in V$, denoted by $d_G(v)$, is defined as the number of edges incident on v. A set D of vertices of G is called a *strong dominating* set if for every $v \in V \setminus D$, there exists a vertex $u \in D$ such that $uv \in E$ and $d_G(u) \geq d_G(v)$. For a given graph G, Min-Strong-DS is the problem of finding a strong dominating set of minimum cardinality. The decision version of Min-Strong-DS is shown to be NP-complete for chordal graphs. In this paper, we present polynomial time algorithms for computing a strong dominating set in block graphs and proper interval graphs, two subclasses of chordal graphs. On the other hand, we show that for a graph G with n-vertices, Min-Strong-DS cannot be approximated within a factor of $(\frac{1}{2}-\varepsilon)\ln n$ for every $\varepsilon > 0$, unless NP $\subseteq$ DTIME($n^{O(\log\log n)}$). We also show that Min-Strong-DS is APX-complete for graphs with maximum degree 3. On the positive side, we show that Min-Strong-DS can be approximated within a factor of $O(\ln \Delta)$ for graphs with maximum degree Δ.

Research Papers

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$(1,0)$-Relaxed strong edge list coloring of planar graphs with girth 6

Kai Lin, Min Chen and Dong Chen

1950064

https://doi.org/10.1142/S1793830919500642

Abstract | **PDF**

Preview Abstract

Let G be a graph. An (s,t)-relaxed strong edge k-coloring is a mapping $\pi : E(G) \to \{1,2,\ldots,k\}$ such that for any edge e, there are at most s edges adjacent to e and t edges which are distance two apart from e assigned the same color as e. The (s,t)-relaxed strong chromatic index, denoted by $\chi'_{(s,t)}(G)$, is the minimum number k of an (s,t)-relaxed strong k-edge-coloring admitted by G. G is called (s,t)-relaxed strong edge L-colorable if for a given list assignment $L=\{L(e) \mid e \in E(G)\}$, there exists an (s,t)-relaxed strong edge coloring π of G such that $\pi(e) \in L(e)$ for all $e \in E(G)$. If G is (s,t)-relaxed strong edge L-colorable for any list assignment with $|L(e)| = k$ for all $e \in E(G)$, then G is said to be (s,t)-relaxed strong edge k-choosable. The (s,t)-relaxed strong list chromatic index, denoted by $\mathrm{ch}'_{(s,t)}(G)$, is defined to be the smallest integer k such that G is (s,t)-relaxed strong edge k-choosable.

In this paper, we prove that every planar graph G with girth 6 satisfies that $\mathrm{ch}'_{(1,0)}(G) \leq 3\Delta(G) - 1$. This strengthens a result which says that every planar graph G with girth 7 and $\Delta(G) \geq 4$ satisfies that $\chi'_{(1,0)}(G) \leq 3\Delta(G) - 1$.

Research Papers

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Some results for the two disjoint connected dominating sets problem

Xianliang Liu, Zishen Yang and Wei Wang

1950065

https://doi.org/10.1142/S1793830919500654

Abstract | **PDF**

Preview Abstract

As a variant of minimum connected dominating set problem, two disjoint connected dominating sets (DCDS) problem is to ask whether there are two DCDS V_1, V_2 in a connected graph $G = (V, E)$ with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, and if not, how to add an edge subset with minimum cardinality such that the new graph has a pair of DCDS. The two DCDS problem is so hard that it is NP-hard on trees. In this paper, if the vertex set V of a connected graph $G = (V, E)$ can be partitioned into two DCDS of G, then it is called a DCDS graph. First, a necessary but not sufficient condition is proposed for cubic (3-regular) graph to be a DCDS graph. To be exact, if a cubic graph is a DCDS graph, there are at most four disjoint triangles in it. Next, if a connected graph $G = (V, E)$ is a DCDS graph, a simple but nontrivial upper bound $6\log_2 \frac{2|V|+18}{9} + 2$ of the girth $g(G)$ is presented.

Research Papers

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Linear time algorithm for dominator chromatic number of trestled graphs

S. Arumugam and K. Raja Chandrasekar

1950066

https://doi.org/10.1142/S1793830919500666

Abstract | **PDF**

Preview Abstract

A *dominator coloring* (respectively, *total dominator coloring*) of a graph G is a proper coloring $\mathscr{C}$ of G such that each closed neighborhood (respectively, open neighborhood) of every vertex of G contains a color class of $\mathscr{C}$. The minimum number of colors required for a dominator coloring (respectively, total dominator coloring) of G is called the *dominator chromatic number* (respectively, *total dominator chromatic number*) of G and is denoted by $\chi_d(G)$ (respectively, $\chi_{td}(G)$). In this paper, we prove that the dominator coloring problem and the total dominator coloring problem are solvable in linear time for trestled graphs.

Research Papers

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Extremal trees with respect to the Steiner Wiener index

Jie Zhang, Guang-Jun Zhang, Hua Wang and Xiao-Dong Zhang

1950067

https://doi.org/10.1142/S1793830919500678

Abstract | **PDF**

Preview Abstract

The well-known Wiener index is defined as the sum of pairwise distances between vertices. Extremal problems with respect to it have been extensively studied for trees. A generalization of the Wiener index, called the Steiner Wiener index, takes the sum of minimum sizes of subgraphs that span k given vertices over all possible choices of the k vertices. We consider the extremal problems with respect to the

Steiner Wiener index among trees of a given degree sequence. First, it is pointed out minimizing the Steiner Wiener index in general may be a difficult problem, although the extremal structure may very likely be the same as that for the regular Wiener index. We then consider the upper bound of the general Steiner Wiener index among trees of a given degree sequence and study the corresponding extremal trees. With these findings, some further discussion and computational analysis are presented for chemical trees. We also propose a conjecture based on the computational results. In addition, we identify the extremal trees that maximize the Steiner Wiener index among trees with a given maximum degree or number of leaves.

Research Papers

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Graham's pebbling conjecture holds for the product of a graph and a sufficiently large complete bipartite graph

Nopparat Pleanmani

1950068

https://doi.org/10.1142/S179383091950068X

Abstract | **PDF**

Preview Abstract

A graph pebbling is a network optimization model for the transmission of consumable resources. A pebbling move on a connected graph G is the process of removing two pebbles from a vertex and placing one of them on an adjacent vertex after configuration of a fixed number of pebbles on the vertex set of G. The pebbling number of G, denoted by $\pi(G)$, is defined to be the least number of pebbles to guarantee that for any configuration of pebbles on G and arbitrary vertex v, there is a sequence of pebbling movement that places at least one pebble on v. For connected graphs G and H, Graham's conjecture asserted that $\pi(G \,\square\, H) \leq \pi(G)\pi(H)$. In this paper, we show that such conjecture holds when H is a complete bipartite graph with sufficiently large order in terms of $\pi(G)$ and the order of G.

Research Papers

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A short proof of a min–max relation for the bases packing of a matroid

Brahim Chaourar

1950069

https://doi.org/10.1142/S1793830919500691

Abstract | **PDF**

Preview Abstract

Let E be a finite set, and M be a matroid defined on E. Given $w \in \mathbb{R}_+^E$, we use the notations (w-maximum bases packing for the first one): $\lambda(w) = \text{Max}\{\sum_{B\text{basis}} \lambda_B$ such that $\sum_{B \ni e} \lambda_B \leq w(e)$ for any $e \in E$, and $\lambda_B \geq 0$ for any basis $B\}$, and $w_\ell = \text{Min}\{\frac{w(E)-w(U)}{r(E)-r(U)}$ such that $U \subset E$ and $r(U) \leq r(E) - 1\}$. In this paper, we give a short proof for the known min–max relation $\lambda(w) = w_\ell$. Moreover, we prove that the minimum w_ℓ can be restricted to single elements and semi locked subsets only. A subset $L \subset E$ is semi locked in M if $M^*|(E\backslash L)$ is closed and 2-connected, and $\min\{r(L), r^*(E\backslash L)\} \geq 2$. We deduce then a polynomial algorithm to compute w_ℓ in a large class of matroids by using a matroid oracle related to semi locked subsets.

Research Papers

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Some spectral properties of A_α-matrix

Shuang Zhang and Yan Zhu

1950070

https://doi.org/10.1142/S1793830919500708

Abstract | **PDF**

Preview Abstract

For a real number $\alpha \in [0, 1]$, the A_α-matrix of a graph G is defined to be $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and degree diagonal matrix of G, respectively. The A_α-spectral radius of G, denoted by $\rho_\alpha(G)$, is the largest eigenvalue of $A_\alpha(G)$. In this paper, we consider the upper bound of the A_α-spectral radius $\rho_\alpha(G)$, also we give some upper bounds for the second largest eigenvalue of A_α-matrix.

Research Papers

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Resolving domination number of graphs

Ridho Alfarisi, Dafik and Arika Indah Kristiana

1950071

https://doi.org/10.1142/S179383091950071X

Abstract | **PDF**

Preview Abstract

For a set $W = \{s_1, s_2, \ldots, s_k\}$ of vertices of a graph G, the representation multiset of a vertex v of G with respect to W is $r(v \mid W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$, where $d(v, s_i)$ is a distance between of the vertex v and the vertices in W together with their multiplicities. The set W is a resolving set of G if $r(v \mid W) \neq r(u \mid W)$ for every pair u, v of distinct vertices of G. The minimum resolving set W is a multiset basis of G. If G has a multiset basis, then its cardinality is called multiset dimension, denoted by $\mathrm{md}(G)$. A set W of vertices in G is a dominating set for G if every vertex of G that is not in W is adjacent to some vertex of W. The minimum cardinality of the dominating set is a domination number, denoted by $\gamma(G)$. A vertex set of some vertices in G that is both resolving and dominating set is a resolving dominating set. The minimum cardinality of resolving dominating set is called resolving domination number, denoted by $\gamma_r(G)$. In our paper, we investigate and establish sharp bounds of the resolving domination number of G and determine the exact value of some family graphs.

Research Papers

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The hull number of powers of cycle graphs under restricted conditions

Jameel Rwalah, Hasan Al-Ezeh and Manal Ghanem

1950072

https://doi.org/10.1142/S1793830919500721

Abstract | **PDF**

Preview Abstract

Let C_n be the cycle graph of order n on the vertices $v_0, v_1, \ldots, v_{n-1}$ and C_n^k be the kth power of C_n. In this paper, we find the hull number of C_n^k under restricted conditions on the vertices of the graph C_n^k namely the independent and connected hull numbers of C_n^k.

Research Papers

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Adjacency and Laplacian spectra of variants of neighborhood corona of graphs constrained by vertex subsets

M. Gayathri and R. Rajkumar

1950073

https://doi.org/10.1142/S1793830919500733

Abstract | **PDF**

Preview Abstract

In this paper, we define some variants of corona of graphs namely, subdivision (respectively, R-graph, Q-graph, total) neighborhood corona, R-graph (respectively, Q-graph, total) semi-edge neighborhood corona, R-graph (respectively, total) semi-vertex neighborhood corona of graphs constrained by vertex subsets. These corona operations generalize some existing corona operations such as subdivision (R-graph, Q-graph, total) double neighborhood corona, subdivision vertex (respectively, edge) neighborhood corona, R-graph vertex (respectively, edge) neighborhood corona of graphs. First, we consider a matrix in specific form and determine its spectrum. Then by using this, we derive the characteristic polynomials of the adjacency and the Laplacian matrices of the new graphs when the base graph is regular. Also, we deduce the characteristic polynomials of the adjacency and Laplacian matrices of the above mentioned particular cases from our results.

Research Papers

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Counting dominating sets in generalized series-parallel graphs

Min-Sheng Lin

1950074

https://doi.org/10.1142/S1793830919500745

Abstract | **PDF**

Preview Abstract

Counting dominating sets in a graph is a #P-complete problem even in planar graphs. This paper studies this problem for generalized series-parallel graphs, which are a subclass of planar graphs. This work develops some linear-time algorithms for counting dominating sets and their two variants, independent dominating sets and connected dominating sets in generalized series-parallel graphs.

Research Papers

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Monotone submodular maximization over the bounded integer lattice with cardinality constraints

Lei Lai, Qiufen Ni, Changhong Lu, Chuanhe Huang and Weili Wu

1950075

https://doi.org/10.1142/S1793830919500757

Abstract | **PDF**

Preview Abstract

We consider the problem of maximizing monotone submodular function over the bounded integer lattice with a cardinality constraint. Function $f : \mathbb{Z}_+^E \to R_+$ is submodular over integer lattice if $f(\boldsymbol{x}) + f(\boldsymbol{y}) \geq f(\boldsymbol{x} \vee \boldsymbol{y}) + f(\boldsymbol{x} \wedge \boldsymbol{y})$, $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_+^E$, where ∨ and ∧ represent elementwise maximum and minimum, respectively. Let $\boldsymbol{B} \in \mathbb{Z}_+^E$, and $k \in \mathbb{Z}_+$, we study the problem of maximizing submodular function $f(\boldsymbol{x})$ with constraints $\boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{B}$ and $\boldsymbol{x}(\mathbb{E}) \leq k$. A random greedy $(1 - \frac{1}{e})$-approximation algorithm and a deterministic $\frac{1}{e}$-approximation algorithm are proposed in this paper. Both algorithms work in value oracle model. In the random greedy algorithm, we assume the monotone submodular function satisfies diminishing return property, which is not an equivalent definition of submodularity on integer lattice. Additionally, our random greedy algorithm makes $\mathscr{O}((|\mathbb{E}| + 1) \cdot k)$ value oracle queries and deterministic algorithm makes $\mathscr{O}(|\mathbb{E}| \cdot B \cdot k^3)$ value oracle queries.

Research Papers

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Succinct enumeration of distant vertex pairs

Ali Gholami Rudi

1950076

https://doi.org/10.1142/S1793830919500769

Abstract | **PDF**

Preview Abstract

The fastest known algorithms for finding the exact value of the diameter of general graphs are no faster than the algorithms that compute all-pairs shortest paths. An extension of the problem of computing graph diameter is enumerating pairs of vertices in a graph, ordered decreasingly by their distance. In this paper, we investigate this problem with the presence of memory constraints. We also show how our result can help the computation of graph Hyperbolicity, by lowering the memory complexity of computing the ordered list of far-apart vertex pairs.

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Resolving domination number of graphs

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For a set $W = \{s_1, s_2, \ldots, s_k\}$ of vertices of a graph G, the representation multiset of a vertex v of G with respect to W is $r(v\,|\,W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$, where $d(v, s_i)$ is a distance between of the vertex v and the vertices in W together with their multiplicities. The set W is a resolving set of G if $r(v\,|\,W) \neq r(u\,|\,W)$ for every pair u, v of distinct vertices of G. The minimum resolving set W is a multiset basis of G. If G has a multiset basis, then its cardinality is called multiset dimension, denoted by $\mathrm{md}(G)$. A set W of vertices in G is a dominating set for G if every vertex of G that is not in W is adjacent to some vertex of W. The minimum cardinality of the dominating set is a domination number, denoted by $\gamma(G)$. A vertex set of some vertices in G that is both resolving and dominating set is a resolving dominating set. The minimum cardinality of resolving dominating set is called resolving domination number, denoted by $\gamma_r(G)$. In our paper, we investigate and establish sharp bounds of the resolving domination number of G and determine the exact value of some family graphs.

Keywords: Resolving set; multiset dimension; dominating set; domination number; resolving dominating set; resolving domination number.

Mathematics Subject Classification 2010: 05C12

1. Introduction

In this paper, all graphs are nontrivial and connected graphs, for detailed definition of graph, see [1, 2, 4]. The concept of metric dimension was independently introduced by Slater [6], Harrary and Melter [3]. In his paper, Slater considered the minimum resolving set of a graph as the location of the placement of a minimum number of sonar/loran detecting devices in a network. So, the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Applications of metric dimension problem can also be found in network and verification, robot navigation, combinatorial optimization, pharmaceutical chemistry, and strategies for the mastermind game.

Simanjuntak *et al.* [7] started the definition of multiset dimension of G. Let G be a connected graph with vertex set $V(G)$. Suppose $W = \{s_1, s_2, \ldots, s_k\}$ is a subset of vertex set $V(G)$, the representation multiset of a vertex v of G with respect to W is $r(v \,|\, W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$, where $d(v, s_i)$ is a distance between v and the vertices in W together with their multiplicities. The resolving set W is a resolving set of G if $r(v \,|\, W) \neq r(u \,|\, W)$ for every pair of distances vertices u and v. The minimum resolving set W is a multiset basis of G. If G has a multiset basis, then its cardinality is called a multiset dimension, denoted by $\mathrm{md}(G)$.

A vertex v in a graph G is said to dominate itself as well as its neighbors. A set W of vertices in G is a dominating set for G if every vertex of G is dominated by some vertex of W. The minimum cardinality of a dominating set is domination number, denoted by $\gamma(G)$. In recent years, there exist additional properties for dominating set, for example independent dominating set requires a dominating set to be independent, the connected dominating set requires a dominating set to induce a connected graphs and total dominating sets are not defined for graphs having an isolated vertex. For more details about other conditional domination numbers see [5]. Some results of domination numbers of some special families graphs are as follows.

Proposition 1.1 ([5]). *Let P_n be a path graphs, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.*

The centipede graphs, denoted by Cp_n are the caterpillar graphs $C_{n,1}$.

Proposition 1.2 ([5]). *Let $C_{n,m}$ be a caterpillar graphs, $\gamma(C_{n,m}) = n$.*

We define the new notation that combines the concept multiset dimension and domination number of G, which is called the resolving domination number. We start the definition of resolving domination number as follows.

Definition 1.1. A vertex set W of some vertices in G that is both resolving and dominating set is a resolving dominating set. The minimum cardinality of resolving dominating set is called the resolving domination number, denoted by $\gamma_r(G)$.

We will illustrate these concepts in Fig. 1. In this case, we have the resolving set $W = \{v_1\}$ which is shown in Fig. 1(a) that $\mathrm{md}(G) = 1$ and the representations of $v \in V(G)$ with respect to W are distinct. On the other hand, the set $W = \{v_2, v_4\}$ is a dominating set of G and so we have $\gamma(G) = 2$ which is shown in Fig. 1(b). To determine the resolving domination number of G, (a) W is a resolving set but not a dominating set, (b) W is a dominating set but not a resolving set such that we observe the set $W = \{v_1, v_3, v_4\}$ in (c) with the given representation of the vertices

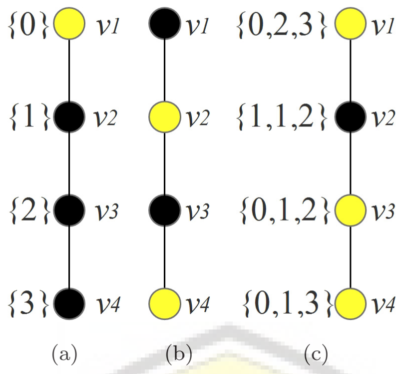

Fig. 1. (a) A graph with multiset dimension $\mathrm{md}(G) = 1$; (b) A graph with domination number $\gamma(G) = 2$; (c) A graph with resolving domination number $\gamma_r(G) = 3$.

of G with respect to W as follows:

$$r(v_1 \,|\, W) = \{0, 2, 3\}, \quad r(v_2 \,|\, W) = \{1, 1, 2\},$$
$$r(v_3 \,|\, W) = \{0, 1, 2\}, \quad r(v_4 \,|\, W) = \{0, 1, 3\},$$

and $v_2 \in V(G) - W$ adjacent to vertices in W, then W is a resolving set and a dominating set. Hence, $\gamma_r(G) = 3$.

Until now, there have been some results of multiset dimension in Simanjuntak *et al.* [7] as follows.

Theorem 1.3. *The multiset dimension of a graph G is one if and only if G is a path.*

Theorem 1.4. *Let G be a graph other than a path. Then* $\mathrm{md}(G) \geq 3$.

Theorem 1.5. *If G is a graph of diameter at most 2 other than a path, then* $\mathrm{md}(G) = \infty$.

Lemma 1.1. *If G contains a vertex which is adjacent to (at least) three pendant vertices, then* $\mathrm{md}(G) = \infty$.

2. Main Results

In this paper, we investigate and determine the exact values of a resolving domination number of some family of graph.

Proposition 2.1. *For every graph G,*

$$\max\{\gamma(G), \mathrm{md}(G)\} \leq \gamma_r(G).$$

Theorem 2.2. *Let G be a connected graph with $G \cong K_1, P_2$ if and only if the resolving domination number of G is $\gamma_r(G) = 1$.*

Proof. For this proof, we characterize for a graph $G \cong K_1, P_2$.

Case 1. Let K_1 be a trivial graph with order one (say $|V(K_1)| = 1$) such that we have $W = V(K_1) = \{u\}$ that is a resolving and dominating set, then $\gamma_M(K_1) = 1$. Now, we show that if $\gamma_M(K_1) = 1$, then G is trivial graph K_1. Let $W = \{u\}$ be a resolving dominating set of a graph G. Thus, $d(u, u) = 0$ with diameter 0, hence G is trivial graph K_1.

Case 2. Let P_2 be a path graph with order two. Then the set $W = \{u\}$ contains a pendant vertex of a path, which is resolving dominating set, thus $\gamma_M(P_2) = 1$. Now, we show that if $\gamma_M(P_2) = 1$, then G is path graph P_2. Let $W = \{u\}$ be a resolving dominating set of a graph G. Thus, $r(u \,|\, W) = \{d(u, u)\} = \{0\}$ and $r(v \,|\, W) = \{d(v, u)\} = \{1\}$, this implies that the diameter of G is 1, hence, G is complete graph K_2 isomorphic to path graph with order 2.

From both cases, for $G \cong K_1, P_2$, if and only if the resolving domination number $\gamma_r(G) = 1$. □

Theorem 2.3. *Let G be a connected graph with diameter one except P_2, then the resolving domination number of G is $\gamma_r(G) = \infty$.*

Proof. If G has a diameter at most one expect K_1 and P_2, then every vertex is adjacent to other vertices. We choose the vertices in W as $w_1, w_2, w_3, \ldots, w_k$, where $i \in [1, k]$ such that we have $r(w_i \,|\, W) = \{0, 1^{k-1}\}$ that is same representation and $w_i \in W$ is also dominator for vertices in G. For $r(u \,|\, W) = \{1^k\}$ for $u \in V(G) - W$ has same representation. Therefore, W is not resolving dominating set of G. □

Lemma 2.1. *No graphs G has resolving domination number 2.*

Proof. Let G be a connected graph with order at least 2. Assume that $\gamma_r(G) = 2$ for any graphs. We choose resolving dominating set $W = \{u, v\}$, then we have $r(u \,|\, W) = \{0, d(u, v)\} = \{d(v, u), 0\} = r(v \,|\, W)$, where $d(u, v) = d(v, u)$, it is a contradiction. Hence, all graphs do not have the resolving domination number 2. □

From Lemma 2.1, Theorems 2.2 and 2.3, we have lemma as follows.

Lemma 2.2. *Let G be a connected graph with diameter at least two, then the resolving domination number of G is $\gamma_r(G) \geq 3$.*

Proof. Based on Theorem 2.2 that $G \cong K_1, P_2 \leftrightarrow \gamma_r(G) = 1$ and Theorem 2.3 and Lemma 2.1 that no graph has multiset dominating number two. Hence, $\gamma_r(G) \geq 3$ for diameter at least 2. □

Lemma 2.3. *If G contains a vertex which is adjacent to (at least) three pendant vertices, then the resolving domination number is $\gamma_r(G) = \infty$.*

Proof. Let W be a resolving dominating set of vertex set in G. We have u_1, u_2, u_3 that is, three pendant vertices for some vertices in G. Therefore, there exist at least two vertices of pendant vertices (u_1 and u_2) are in W, or at least two vertices of pendant vertices (u_1 and u_2) aren't in W. We know that the distance v_1, v_2 to other vertex v of vertex set in G (say $d(v_1, v) = d(v_2, v)$), then in both cases these vertices cannot be resolved or dominated. □

The following theorem is a corollary of Theorem 2.3.

Corollary 2.1. *Let K_m be a complete graph with order $m \geq 3$, then resolving domination number of K_m is $\gamma_M(K_m) = \infty$.*

The following theorem is a corollary of Lemma 2.3.

Corollary 2.2. *Let S_m be a star graph with order $m \geq 2$, then the resolving domination number of S_m is $\gamma_M(S_m) = \infty$.*

Corollary 2.3. *Let $\mathrm{Br}_{n,m}$ be a broom graph with order $n, m \geq 3$, then the resolving domination number of $\mathrm{Br}_{n,m}$ is $\gamma_M(S_m) = \infty$.*

Corollary 2.4. *Let $\mathrm{DS}_{n,m}$ be a double star with order $n, m \geq 3$, then the resolving domination number of $\mathrm{DS}_{n,m}$ is $\gamma_M(S_m) = \infty$.*

For any two graphs G and H, a corona product of G and H, denoted by $G \odot H$, is a connected graph which is formed by taking n copies of graphs H_i, $1 \leq i \leq n$ of H and connecting ith vertex of G to the vertices of H_i.

Theorem 2.4. *Let $G \odot mK_1$ be a corona product of G order n and mK_1 is trivial graph with $m \geq 3$, then the resolving domination number of $G \odot mK_1$ is $\gamma_M(G \odot mK_1) = \infty$.*

Furthermore, we determine the exact value of some families graphs for the resolving domination number, namely path, centipede graphs and tadpole $T_{4,n}$. The results of $\gamma_r(G)$ as follows.

Theorem 2.5. *Let P_n be a path with order $n \geq 2$, then the resolving domination number of P_n is*

$$\gamma_{M(P_n)} = \begin{cases} 1, & \text{if } n = 2 \\ \infty, & \text{if } n = 3 \\ 3, & \text{if } n \in \{4, 5, 6\} \\ \left\lceil \frac{n}{3} \right\rceil, & \text{if } n \geq 7, n \neq 0 \pmod 3 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n \geq 7, n = 0 \pmod 3. \end{cases}$$

Proof. Path graph, denoted by P_n, is a tree graph with n vertices. Vertex set and edge set of P_n, respectively, are $V(P_n) = \{x_i : 1 \leq i \leq n\}$ and $E(P_n) = \{x_{i-1}x_i : 1 \leq i \leq n-1\}$. For this proof, we divide the proof into two cases as follows.

Case 1. For $n = 2$.

Based on Theorem 2.2 that $\gamma_M(P_2) = 1$.

Case 2. For $n = 3$.

Based on Lemma 2.1 that $\gamma_M(P_3) \geq 3$. Furthermore, we prove that $\gamma_M(P_3) \leq 3$? we can construct the resolving dominating set of P_3, namely $W = V(P_3) = \{x_1, x_2, x_3\}$. The representation of vertex in P_3 is as follows:

$$r(x_1 \,|\, W) = \{0, 1, 2\} \quad r(x_2 \,|\, W) = \{0, 1, 1\} \quad r(x_3 \,|\, W) = \{0, 1, 2\}.$$

There are same representations, namely $r(x_1 \,|\, W) = r(x_3 \,|\, W)$. We know that W is not a resolving set such that W is not a resolving dominating set. Thus, we obtain that $\gamma_M(P_3) \neq 3$. It concludes that $\gamma_M(P_3) = \infty$.

Case 3. For $n = 4, 5, 6$.

Based on Lemma 2.1 that $\gamma_M(P_n) \geq 3$. Furthermore, we prove that $\gamma_M(P_n) \leq 3$, we can construct the resolving dominating set of P_n. The representation of vertex in P_n is as follows:

	P_4 with $W = \{x_1, x_2, x_4\}$	P_5 with $W = \{x_1, x_2, x_5\}$	P_6 with $W = \{x_2, x_5, x_6\}$
$r(x_1 \,|\, W)$	$\{0, 1, 3\}$	$\{0, 1, 4\}$	$\{1, 4, 5\}$
$r(x_2 \,|\, W)$	$\{0, 1, 2\}$	$\{0, 1, 3\}$	$\{0, 3, 4\}$
$r(x_3 \,|\, W)$	$\{1, 1, 3\}$	$\{1, 2, 2\}$	$\{1, 2, 3\}$
$r(x_4 \,|\, W)$	$\{0, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 2\}$
$r(x_5 \,|\, W)$		$\{0, 3, 4\}$	$\{0, 1, 3\}$
$r(x_6 \,|\, W)$			$\{0, 1, 4\}$

All vertices in P_n have distinct representations. We know that W is resolving set and dominating set such that W is resolving dominating set. Thus, we obtain that $\gamma_M(P_n) \leq 3$. It concludes that $\gamma_M(P_n) = 3$.

Case 4. For $n \geq 7$ and $n \equiv 1 \pmod 3$.

Based on Proposition 2.1 that $\gamma_M(P_n) \geq \max\{\gamma(P_n), \mathrm{md}(P_n)\} = \{\lceil \frac{n}{3} \rceil, 1\} = \lceil \frac{n}{3} \rceil$. Furthermore, we prove that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$, we can construct the resolving dominating set of P_n, namely $W = \{x_i, x_{n-1}; i \equiv 2 \pmod 3\}$. The vertex $x_i; i \neq 2 \pmod 3$ is dominated by vertices in W. We have the properties to show that all vertices have distinct representation as follows:

(i) We know that $d(x_l, x_{n-1}) \neq d(x_k, x_{n-1})$, for $1 \leq l, k \leq n-1$ and $x_l, x_k \notin W$.

(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i \,|\, W) = \{d(x_i, x_s) : x_s \in W - \{x_{n-1}\}\} \cup \{d(x_i, x_{n-1})\}$.

(iii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l \,|\, W - \{y_n\}) = \{d(x_l, x_s) : x_s \in W - \{x_{n-1}\}\} = \{d(x_k, x_s) : x_s \in W - \{x_{n-1}\}\} = r(x_k \,|\, W - \{x_{n-1}\})$ for $l + k = n + 1$ and $1 \leq l, k \leq n-1$.

(iv) Based on (i)–(iii) that $r(x_l \mid W) \neq r(x_k \mid W)$ for $1 \leq l, k \leq n-1$.
(v) We know that $r(x_i \mid W) \neq r(x_n \mid W)$ for $1 \leq i \leq n-1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil$. Thus, we obtain that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$. It concludes that $\gamma_M(P_n) = \lceil \frac{n}{3} \rceil$.

Case 5. For $n \geq 7$ and $n \equiv 2 \pmod 3$.

Based on Proposition 2.1 that $\gamma_M(P_n) \geq \max\{\gamma(P_n), \mathrm{md}(P_n)\} = \{\lceil \frac{n}{3} \rceil, 1\} = \lceil \frac{n}{3} \rceil$. Furthermore, we prove that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$, we can construct the resolving dominating set of P_n, namely $W = \{x_i, x_{n-1}; i \equiv 2 \pmod 3\}$. The vertex $x_i; i \not\equiv 2 \pmod 3$ is dominated by vertices in W. We have the properties to show that all vertices have distinct representation as follows:

(i) We know that $d(x_l, x_{n-1}) \neq d(x_k, x_{n-1})$, for $1 \leq l, k \leq n-1$ and $x_l, x_k \notin W$.
(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i \mid W) = \{d(x_i, x_s) : x_s \in W - \{x_{n-1}\}\} \cup \{d(x_i, x_{n-1})\}$.
(iii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l \mid W - \{x_{n-1}\}) = \{d(x_l, x_s) : x_s \in W - \{x_{n-1}\}\} = \{d(x_k, x_s) : x_s \in W - \{x_{n-1}\}\} = r(x_k \mid W - \{x_{n-1}\})$ for $l + k = n-1$ and $1 \leq l, k \leq n-1$.
(iv) Based on (i)–(iii) that $r(x_l \mid W) \neq r(x_k \mid W)$ for $1 \leq l, k \leq n-1$.
(v) We know that $r(x_i \mid W) \neq r(x_n \mid W)$ for $1 \leq i \leq n-1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil$. Thus, we obtain that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$. It concludes that $\gamma_M(P_n) = \lceil \frac{n}{3} \rceil$.

Case 6. For $n \geq 7$ and $n \equiv 0 \pmod 3$.

Based on Proposition 2.1 that $\gamma_M(P_n) \geq \max\{\gamma(P_n), \mathrm{md}(P_n)\} = \{\lceil \frac{n}{3} \rceil, 1\} = \lceil \frac{n}{3} \rceil$. Assume that $|W| = \lceil \frac{n}{3} \rceil$, namely $W = \{x_i; i \equiv 2 \pmod 3\}$. There are at least two vertices which have same representation. We can construct the representation as follows:

(i) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i \mid W) = \{d(x_i, x_s) : x_s \in W, i \equiv 2 \pmod 3\}$.
(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l \mid W) = \{d(x_l, x_s) : x_s \in W\} = \{d(x_k, x_s) : x_s \in W\} = r(x_k \mid W)$ for $l + k = n+1$ and $1 \leq l, k \leq n-1$.
(iii) Based on (i)–(ii) that $r(x_l \mid W) = r(x_k \mid W)$ for $1 \leq l, k \leq n-1$.

Based on the assumption above, there are same representations, which is a contradiction. Thus, $\gamma_M(P_n) \geq \lceil \frac{n}{3} \rceil + 1$. Furthermore, we prove that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil + 1$, we can construct the resolving dominating set of P_n, namely $W = \{x_i, x_n; i \equiv 2 \pmod 3\}$. The vertex $x_i; i \not\equiv 2 \pmod 3$ is dominated by vertices in W. We have the properties to show that all vertices have distinct representation as follows:

(i) We know that $d(x_l, x_n) \neq d(x_k, x_n)$, for $1 \leq l, k \leq n-1$ and $x_l, x_k \notin W$.

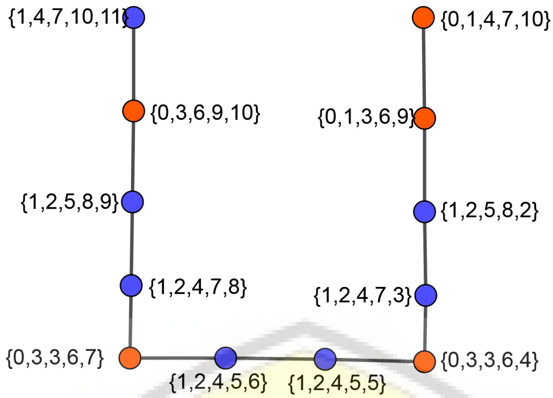

Fig. 2. A graph with resolving domination number $\gamma_M(P_{12}) = 5$.

(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i \,|\, W) = \{d(x_i, x_s) : x_s \in W - \{x_n\}\} \cup \{d(x_i, x_n)\}$.

(iii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l \,|\, W - \{x_n\}) = \{d(x_l, x_s) : x_s \in W - \{x_n\}\} = \{d(x_k, x_s) : x_s \in W - \{x_n\}\} = r(x_k \,|\, W - \{x_n\})$ for $l + k = n + 1$ and $1 \le l, k \le n - 1$.

(iv) Based on (i)–(iii) that $r(x_l \,|\, W) \ne r(x_k \,|\, W)$ for $1 \le l, k \le n - 1$.

(v) We know that $r(x_i \,|\, W) \ne r(x_n \,|\, W)$ for $1 \le i \le n - 1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil + 1$. Thus, we obtain that $\gamma_M(P_n) \le \lceil \frac{n}{3} \rceil + 1$. It concludes that $\gamma_M(P_n) = \lceil \frac{n}{3} \rceil + 1$. □

Theorem 2.6. *Let* Cp_n *be a centipede with order* $n \ge 2$*, then the resolving domination number of* Cp_n *is*

$$\gamma_M(\mathrm{Cp}_n) = \begin{cases} 3, & if\ n = 2, \\ n, & if\ n \ge 3. \end{cases}$$

Proof. Centipede graph, denoted by Cp_n, is a tree graph with $2n$ vertices. Vertex set and edge set of Cp_n, respectively, are $V(\mathrm{Cp}_n) = \{x_i, y_j : 1 \le i \le n\}$ and $E(\mathrm{Cp}_n) = \{x_{i-1}x_i : 1 \le i \le n-1\} \cup \{x_i y_i ; 1 \le i \le n\}$. The vertice x_i is a backbone and the vertice y_i is a pendant vertex. For this proof, we divide the proof into two cases as follows.

Case 1. For $n = 2$.

Centipede graph Cp_2 has four vertices (two vertices as backbone and two vertices in pendant vertex), based on the definition that Centipede graph Cp_2 isomorphic

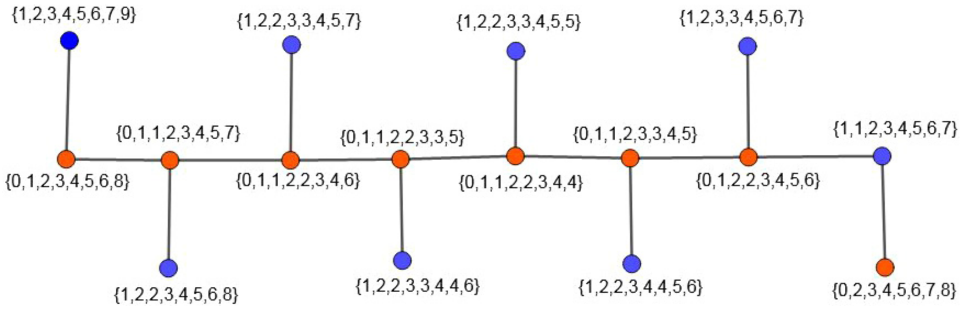

Fig. 3. A graph with resolving domination number $\gamma_M(\mathrm{Cp}_8) = 8$.

to path with four vertices. It is based on Lemma 2.2 that $\gamma_M(\mathrm{Cp}_2) \geq 3$. Furthermore, we prove that $\gamma_M(\mathrm{Cp}_2) \leq 3$, we can construct the resolving dominating set of Cp_2, namely $W = \{x_1, y_1, y_2\}$. The vertex x_2 is dominated by y_2 or x_1. The representation of vertex in Cp_n is as follows:

$$r(x_1 \mid W) = \{0, 1, 2\} \quad r(x_2 \mid W) = \{1, 1, 2\}$$
$$r(y_1 \mid W) = \{0, 1, 3\} \quad r(y_2 \mid W) = \{0, 2, 3\}.$$

From the representation, all vertices are distinct. We know that W is resolving set and dominating set such that W is a resolving dominating set with $|W| = 3$. Thus, we obtain that $\gamma_M(\mathrm{Cp}_2) \leq 3$. It concludes that $\gamma_M(\mathrm{Cp}_2) = 3$.

Case 2. For $n \geq 3$.

Centipede graph Cp_n has $2n$ vertices (n vertices as backbone and n vertices in pendant vertex), based on Proposition 2.1 that $\gamma_M(\mathrm{Cp}_2) \geq \max\{\gamma(\mathrm{Cp}_n), \mathrm{md}(\mathrm{Cp}_n)\} = \{n, n\} = n$. Furthermore, we prove that $\gamma_M(\mathrm{Cp}_n) \leq n$, we can construction the resolving dominating set of Cp_n, namely $W = \{x_1, \ldots, x_{n-1}, y_n\}$. The vertex x_n is dominated by y_n or x_{n-1} and the vertex y_i, $1 \leq i \leq n-1$ dominated by x_i, $1 \leq i \leq n-1$. The representation of vertex in Cp_n is shown in Table 1.

From Table 1, we have the properties that all vertices have distinct representation as follows:

(i) We know that $d(y_l, y_n) \neq d(y_k, y_n) \neq d(x_n, y_n)$, for $1 \leq l, k \leq n-1$.
(ii) We have the representation of y_i in Cp_n, namely $r(y_i \mid W) = \{d(y_1, x_s) : x_s \in W - \{y_n\}\} \cup \{d(y_i, y_n)\}$.
(iii) We have the representation of y_i in Cp_n, namely $r(y_l \mid W - \{y_n\}) = \{d(y_l, x_s) : x_s \in W - \{y_n\}\} = \{d(y_k, x_s) : x_s \in W - \{y_n\}\} = r(y_k \mid W - \{y_n\})$ for $l + k = n$ and $1 \leq l, k \leq n-1$.

Table 1. The representation of Cp_n.

	y_1	y_2	y_3	y_4	y_5	$\cdots$	y_{n-2}	y_{n-1}	x_n
x_1	1	2	3	4	5	$\cdots$	$n-2$	$n-1$	$n-1$
x_2	2	1	2	3	4	$\cdots$	$n-3$	$n-2$	$n-2$
x_3	3	2	1	2	3	$\cdots$	$n-4$	$n-3$	$n-3$
x_4	4	3	2	1	2	$\cdots$	$n-5$	$n-4$	$n-4$
x_5	5	4	3	2	1	$\cdots$	$n-6$	$n-5$	$n-5$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
x_{n-2}	$n-2$	$n-3$	$n-4$	$n-5$	$n-6$	$\cdots$	1	2	2
x_{n-1}	$n-1$	$n-2$	$n-3$	$n-4$	$n-5$	$\cdots$	2	1	1
y_n	$n+1$	n	$n-1$	$n-2$	$n-3$	$\cdots$	4	3	1

(iv) Based on (i)–(iii) that $r(y_l \,|\, W) \neq r(y_k \,|\, W)$ for $1 \leq l, k \leq n-1$.

(v) We know that $r(y_i \,|\, W) \neq r(x_n \,|\, W)$ for $1 \leq i \leq n-1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = n$. Thus, we obtain that $\gamma_M(\mathrm{Cp}_n) \leq n$. It concludes that $\gamma_M(\mathrm{Cp}_n) = n$. □

Theorem 2.7. *Let $T_{4,n}$ be a tadpole graph with order $n \in N$, then resolving domination number of $T_{4,n}$ is*

$$\gamma_M(T_{4,n}) = \begin{cases} 4, & \text{if } n = 3, \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{if } n \neq 3. \end{cases}$$

Proof. Tadpole graph, denoted by $T_{4,n}$, is a unicyclic graph which is obtained by joining a cycle C_4 and path P_n with a bridge. Vertex set and edge set of $T_{4,n}$, respectively, are $V(T_{4,n}) = \{x_i, y_j : 1 \leq i \leq 4, 1 \leq j \leq n\}$ and $E(T_{4,n}) = \{y_{j-1}y_j : 1 \leq j \leq n-1\} \cup \{x_1y_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$. The edge x_1y_1 is a bridge in tadpole graphs. For this proof, we divide the proof into two cases as follows:

Case 1. For $n = 3$.

Based on Lemma 2.1 that $\gamma_M(T_{4,n}) \geq 3$. Assume that $|W| = 3$, such that we have the same representation as follows:

(i) If we choose the set $W = \{x_3, y_1, y_3\}$, then we know that $d(x_2, x_1) = d(x_4, x_1)$ and $d(x_2, x_3) = d(x_4, x_3) = 1$. Thus, $r(x_2 \,|\, W) = r(x_4 \,|\, W) = \{1, 2, 4\}$.

(ii) If we choose the set $W = \{x_3, x_4, y_2\}$, then we know that $d(x_2, x_4) = d(y_1, x_4)$ and $d(x_2, y_2) = d(y_1, x_3)$. Thus, $r(x_2 \,|\, W) = r(y_1 \,|\, W) = \{1, 2, 3\}$.

There are same representations such that $\gamma_M(T_{4,n}) \geq 4$. Furthermore, we prove that $\gamma_M(T_{4,n}) \leq 4$, we can construct the resolving dominating set of $T_{4,n}$. The representation of vertex in $T_{4,n}$ is as follows:

	$T_{4,n}$ with $W = \{x_3, x_4, y_1, y_3\}$
$r(x_1 \mid W)$	$\{1, 1, 2, 3\}$
$r(x_2 \mid W)$	$\{1, 2, 2, 4\}$
$r(x_3 \mid W)$	$\{0, 1, 3, 5\}$
$r(x_4 \mid W)$	$\{0, 1, 2, 4\}$
$r(y_1 \mid W)$	$\{0, 2, 2, 3\}$
$r(y_2 \mid W)$	$\{1, 1, 3, 4\}$
$r(y_3 \mid W)$	$\{0, 2, 4, 5\}$

All vertices in $T_{4,n}$ have distinct representations. We know that W is a resolving set and dominating set such that W is a resolving dominating set. Thus, we obtain that $\gamma_M(T_{4,n}) \leq 4$. It concludes that $\gamma_M(T_{4,n}) = 4$.

Case 2. For $n \equiv 0, 2 \pmod 3$.

We prove that $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Assume that $|W| = \lceil \frac{n}{3} \rceil + 1$, namely $W = \{y_j; j \equiv 2 \pmod 3\} \cup \{x_3\}$. There are at least two vertices which have same representation. We can construct the representation as follows:

(i) We have $d(x_2, x_3) = d(x_4, x_3)$ and $d(x_2, x_1) = d(x_4, x_1)$.
(ii) We know that $d(x_2, y_s) = d(x_2, x_1) + d(x_1, y_s) = d(x_4, x_1) + d(x_1, y_s) = d(x_4, y_s)$ for $y_s \in W$.
(iii) We know that $r(x_2 \mid W) = \{d(x_2, y_s) : y_s \in W\} = \{d(x_4, y_s) : y_s \in W\} = r(x_4 \mid W)$.

Based on the assumption above, there are same representations, which is a contradiction. Thus, $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Furthermore, we prove that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$, we can construct the resolving dominating set of $T_{4,n}$, namely $W = \{y_j, x_3, x_4; j \equiv 2 \pmod 3\}$. The vertex $y_j; j \not\equiv 2 \pmod 3$ dominated by vertices in W. We have the properties this show that all vertices have distinct representations as follows:

(i) We know that $d(y_l, x_4) \neq d(y_k, x_4)$, for $1 \leq l, k \leq n$ and $y_l, y_k \notin W$.
(ii) We have the representation of $y_j \in V(T_{4,n}) - W$, namely $r(y_j \mid W) = \{d(y_j, y_s) : y_s \in W - \{x_3, x_4\}\} \cup \{d(y_j, x_4)\}$.
(iii) We have the representation of $y_j \in V(T_{4,n})$, namely $r(y_l \mid W - \{x_3, x_4\}) = \{d(y_l, y_s) : y_s \in W - \{x_3, x_4\}\} = \{d(y_k, y_s) : y_s \in W - \{x_3, x_4\}\} = r(y_k \mid W - \{x_3, x_4\})$ for $l + k = n + 2$ and $1 \leq l, k \leq n$.
(iv) Based on (i)–(iii) that $r(y_l \mid W) \neq r(y_k \mid W)$ for $1 \leq l, k \leq n$.
(v) We know that $r(y_j \mid W) \neq r(x_1 \mid W) \neq r(x_2 \mid W)$ for $1 \leq j \leq n$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil + 2$. Thus, we obtain that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$. It concludes that $\gamma_M(T_{4,n}) = \lceil \frac{n}{3} \rceil + 2$.

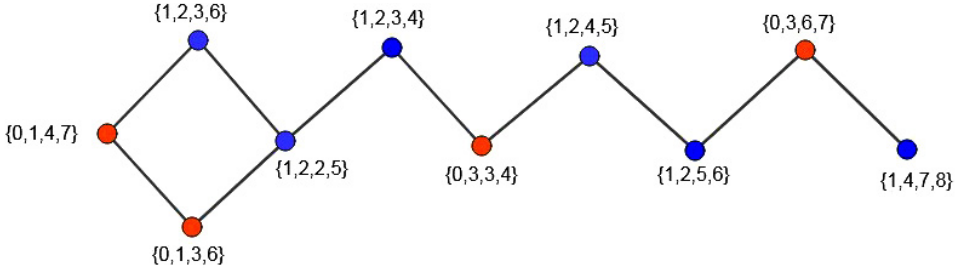

Fig. 4. A graph with resolving domination number $\gamma_M(V(T_{4,6})) = 4$.

Case 3. For $n \equiv 1 \pmod 3$.

We prove that $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Assume that $|W| = \lceil \frac{n}{3} \rceil + 1$, namely $W = \{y_j, y_n; j \equiv 2 \pmod 3\} \cup \{x_3\}$. There are at least two vertices which have same representation. We can construction the representation as follows:

(i) We have $d(x_2, x_3) = d(x_4, x_3)$ and $d(x_2, x_1) = d(x_4, x_1)$.

(ii) We know that $d(x_2, y_s) = d(x_2, x_1) + d(x_1, y_s) = d(x_4, x_1) + d(x_1, y_s) = d(x_4, y_s)$ for $y_s \in W$.

(iii) We know that $r(x_2 \,|\, W) = \{d(x_2, y_s) : y_s \in W\} = \{d(x_4, y_s) : y_s \in W\} = r(x_4 \,|\, W)$.

Based on the assumption above, there are same representations, which is a contradiction. Thus, $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Furthermore, we prove that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$, we can construct the resolving dominating set of $T_{4,n}$, namely $W = \{y_j, x_3, x_4, y_n; j \equiv 2 \pmod 3\}$. The vertex $y_j; j \not\equiv 2 \pmod 3$ is dominated by vertices in W. We have the properties this show that all vertices have distinct representations as follows:

(i) We know that $d(y_l, x_4) \neq d(y_k, x_4)$, for $1 \leq l, k \leq n$ and $y_l, y_k \notin W$.

(ii) We have the representation of $y_j \in V(T_{4,n}) - W$, namely $r(y_j \,|\, W) = \{d(y_j, y_s) : y_s \in W - \{x_3, x_4, y_n\}\} \cup \{d(y_j, x_4)\}$.

(iii) We have the representation of $y_j \in V(T_{4,n})$, namely $r(y_l \,|\, W - \{x_3, x_4, y_n\}) = \{d(y_l, y_s) : y_s \in W - \{x_3, x_4, y_n\}\} = \{d(y_k, y_s) : y_s \in W - \{x_3, x_4, y_n\}\} = r(y_k \,|\, W - \{x_3, x_4, y_n\})$ for $l + k = n$ and $1 \leq l, k \leq n$.

(iv) Based on (i)–(iii) that $r(y_l \,|\, W) \neq r(y_k \,|\, W)$ for $1 \leq l, k \leq n$.

(v) We know that $r(y_j \,|\, W) \neq r(x_1 \,|\, W) \neq r(x_2 \,|\, W)$ for $1 \leq j \leq n$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil + 2$. Thus, we obtain that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$. It concludes that $\gamma_M(T_{4,n}) = \lceil \frac{n}{3} \rceil + 2$. □

3. Conclusion

In this paper, we have given results on the lower bound of a resolving domination number and determine the exact values of some special graphs. Hence, the following problems arise naturally.

Open Problem 3.1. Determine the resolving domination number of family graph namely family tree, unicyclic, regular graphs, and others.

Open Problem 3.2. Determine the resolving domination number of operation graph namely corona product, cartesian product, joint, comb product, and others.

Open Problem 3.3. Characterize the resolving domination number $\gamma_r(G) = n - 1$.

Acknowledgment

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Resolving domination number of graphs

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For a set $W = \{s_1, s_2, \ldots, s_k\}$ of vertices of a graph G, the representation multiset of a vertex v of G with respect to W is $r(v \mid W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$, where $d(v, s_i)$ is a distance between of the vertex v and the vertices in W together with their multiplicities. The set W is a resolving set of G if $r(v \mid W) \neq r(u \mid W)$ for every pair u, v of distinct vertices of G. The minimum resolving set W is a multiset basis of G. If G has a multiset basis, then its cardinality is called multiset dimension, denoted by $\mathrm{md}(G)$. A set W of vertices in G is a dominating set for G if every vertex of G that is not in W is adjacent to some vertex of W. The minimum cardinality of the dominating set is a domination number, denoted by $\gamma(G)$. A vertex set of some vertices in G that is both resolving and dominating set is a resolving dominating set. The minimum cardinality of resolving dominating set is called resolving domination number, denoted by $\gamma_r(G)$. In our paper, we investigate and establish sharp bounds of the resolving domination number of G and determine the exact value of some family graphs.

Keywords: Resolving set; multiset dimension; dominating set; domination number; resolving dominating set; resolving domination number.

Mathematics Subject Classification 2010: 05C12

1. Introduction

In this paper, all graphs are nontrivial and connected graphs, for detailed definition of graph, see [1, 2, 4]. The concept of metric dimension was independently introduced by Slater [6], Harrary and Melter [3]. In his paper, Slater considered the minimum resolving set of a graph as the location of the placement of a minimum number of sonar/loran detecting devices in a network. So, the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Applications of metric dimension problem can also be found in network and verification, robot navigation, combinatorial optimization, pharmaceutical chemistry, and strategies for the mastermind game.

Simanjuntak *et al.* [7] started the definition of multiset dimension of G. Let G be a connected graph with vertex set $V(G)$. Suppose $W = \{s_1, s_2, \ldots, s_k\}$ is a subset of vertex set $V(G)$, the representation multiset of a vertex v of G with respect to W is $r(v \mid W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$, where $d(v, s_i)$ is a distance between v and the vertices in W together with their multiplicities. The resolving set W is a resolving set of G if $r(v \mid W) \neq r(u \mid W)$ for every pair of distances vertices u and v. The minimum resolving set W is a multiset basis of G. If G has a multiset basis, then its cardinality is called a multiset dimension, denoted by $\mathrm{md}(G)$.

A vertex v in a graph G is said to dominate itself as well as its neighbors. A set W of vertices in G is a dominating set for G if every vertex of G is dominated by some vertex of W. The minimum cardinality of a dominating set is domination number, denoted by $\gamma(G)$. In recent years, there exist additional properties for dominating set, for example independent dominating set requires a dominating set to be independent, the connected dominating set requires a dominating set to induce a connected graphs and total dominating sets are not defined for graphs having an isolated vertex. For more details about other conditional domination numbers see [5]. Some results of domination numbers of some special families graphs are as follows.

Proposition 1.1 ([5]). *Let P_n be a path graphs,* $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.

The centipede graphs, denoted by Cp_n are the caterpillar graphs $C_{n,1}$.

Proposition 1.2 ([5]). *Let $C_{n,m}$ be a caterpillar graphs,* $\gamma(C_{n,m}) = n$.

We define the new notation that combines the concept multiset dimension and domination number of G, which is called the resolving domination number. We start the definition of resolving domination number as follows.

Definition 1.1. A vertex set W of some vertices in G that is both resolving and dominating set is a resolving dominating set. The minimum cardinality of resolving dominating set is called the resolving domination number, denoted by $\gamma_r(G)$.

We will illustrate these concepts in Fig. 1. In this case, we have the resolving set $W = \{v_1\}$ which is shown in Fig. 1(a) that $\mathrm{md}(G) = 1$ and the representations of $v \in V(G)$ with respect to W are distinct. On the other hand, the set $W = \{v_2, v_4\}$ is a dominating set of G and so we have $\gamma(G) = 2$ which is shown in Fig. 1(b). To determine the resolving domination number of G, (a) W is a resolving set but not a dominating set, (b) W is a dominating set but not a resolving set such that we observe the set $W = \{v_1, v_3, v_4\}$ in (c) with the given representation of the vertices

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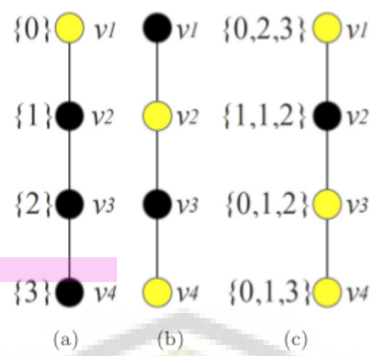

Fig. 1. (a) A graph with multiset dimension $\mathrm{md}(G) = 1$; (b) A graph with domination number $\gamma(G) = 2$; (c) A graph with resolving domination number $\gamma_r(G) = 3$.

of G with respect to W as follows:

$$r(v_1 \,|\, W) = \{0, 2, 3\}, \quad r(v_2 \,|\, W) = \{1, 1, 2\},$$
$$r(v_3 \,|\, W) = \{0, 1, 2\}, \quad r(v_4 \,|\, W) = \{0, 1, 3\},$$

and $v_2 \in V(G) - W$ adjacent to vertices in W, then W is a resolving set and a dominating set. Hence, $\gamma_r(G) = 3$.

Until now, there have been some results of multiset dimension in Simanjuntak *et al.* [7] as follows.

Theorem 1.3. *The multiset dimension of a graph G is one if and only if G is a path.*

Theorem 1.4. *Let G be a graph other than a path. Then* $\mathrm{md}(G) \geq 3$.

Theorem 1.5. *If G is a graph of diameter at most 2 other than a path, then* $\mathrm{md}(G) = \infty$.

Lemma 1.1. *If G contains a vertex which is adjacent to (at least) three pendant vertices, then* $\mathrm{md}(G) = \infty$.

2. Main Results

In this paper, we investigate and determine the exact values of a resolving domination number of some family of graph.

Proposition 2.1. *For every graph G,*

$$\max\{\gamma(G), \mathrm{md}(G)\} \leq \gamma_r(G).$$

Theorem 2.2. *Let G be a connected graph with $G \cong K_1, P_2$ if and only if the resolving domination number of G is $\gamma_r(G) = 1$.*

Proof. For this proof, we characterize for a graph $G \cong K_1, P_2$.

Case 1. Let K_1 be a trivial graph with order one (say $|V(K_1)| = 1$) such that we have $W = V(K_1) = \{u\}$ that is a resolving and dominating set, then $\gamma_M(K_1) = 1$. Now, we show that if $\gamma_M(K_1) = 1$, then G is trivial graph K_1. Let $W = \{u\}$ be a resolving dominating set of a graph G. Thus, $d(u, u) = 0$ with diameter 0, hence G is trivial graph K_1.

Case 2. Let P_2 be a path graph with order two. Then the set $W = \{u\}$ contains a pendant vertex of a path, which is resolving dominating set, thus $\gamma_M(P_2) = 1$. Now, we show that if $\gamma_M(P_2) = 1$, then G is path graph P_2. Let $W = \{u\}$ be a resolving dominating set of a graph G. Thus, $r(u\,|\,W) = \{d(u, u)\} = \{0\}$ and $r(v\,|\,W) = \{d(v, u)\} = \{1\}$, this implies that the diameter of G is 1, hence, G is complete graph K_2 isomorphic to path graph with order 2.

From both cases, for $G \cong K_1, P_2$, if and only if the resolving domination number $\gamma_r(G) = 1$. □

Theorem 2.3. *Let G be a connected graph with diameter one except P_2, then the resolving domination number of G is $\gamma_r(G) = \infty$.*

Proof. If G has a diameter at most one expect K_1 and P_2, then every vertex is adjacent to other vertices. We choose the vertices in W as $w_1, w_2, w_3, \ldots, w_k$, where $i \in [1, k]$ such that we have $r(w_i\,|\,W) = \{0, 1^{k-1}\}$ that is same representation and $w_i \in W$ is also dominator for vertices in G. For $r(u\,|\,W) = \{1^k\}$ for $u \in V(G) - W$ has same representation. Therefore, W is not resolving dominating set of G. □

Lemma 2.1. *No graphs G has resolving domination number 2.*

Proof. Let G be a connected graph with order at least 2. Assume that $\gamma_r(G) = 2$ for any graphs. We choose resolving dominating set $W = \{u, v\}$, then we have $r(u\,|\,W) = \{0, d(u, v)\} = \{d(v, u), 0\} = r(v\,|\,W)$, where $d(u, v) = d(v, u)$, it is a contradiction. Hence, all graphs do not have the resolving domination number 2. □

From Lemma 2.1, Theorems 2.2 and 2.3, we have lemma as follows.

Lemma 2.2. *Let G be a connected graph with diameter at least two, then the resolving domination number of G is $\gamma_r(G) \geq 3$.*

Proof. Based on Theorem 2.2 that $G \cong K_1, P_2 \leftrightarrow \gamma_r(G) = 1$ and Theorem 2.3 and Lemma 2.1 that no graph has multiset dominating number two. Hence, $\gamma_r(G) \geq 3$ for diameter at least 2. □

Lemma 2.3. *If G contains a vertex which is adjacent to (at least) three pendant vertices, then the resolving domination number is $\gamma_r(G) = \infty$.*

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Proof. Let W be a resolving dominating set of vertex set in G. We have u_1, u_2, u_3 that is, three pendant vertices for some vertices in G. Therefore, there exist at least two vertices of pendant vertices (u_1 and u_2) are in W, or at least two vertices of pendant vertices (u_1 and u_2) aren't in W. We know that the distance v_1, v_2 to other vertex v of vertex set in G (say $d(v_1, v) = d(v_2, v)$), then in both cases these vertices cannot be resolved or dominated. □

The following theorem is a corollary of Theorem 2.3.

Corollary 2.1. *Let K_m be a complete graph with order $m \geq 3$, then resolving domination number of K_m is $\gamma_M(K_m) = \infty$.*

The following theorem is a corollary of Lemma 2.3.

Corollary 2.2. *Let S_m be a star graph with order $m \geq 2$, then the resolving domination number of S_m is $\gamma_M(S_m) = \infty$.*

Corollary 2.3. *Let $\mathrm{Br}_{n,m}$ be a broom graph with order $n, m \geq 3$, then the resolving domination number of $\mathrm{Br}_{n,m}$ is $\gamma_M(S_m) = \infty$.*

Corollary 2.4. *Let $\mathrm{DS}_{n,m}$ be a double star with order $n, m \geq 3$, then the resolving domination number of $\mathrm{DS}_{n,m}$ is $\gamma_M(S_m) = \infty$.*

For any two graphs G and H, a corona product of G and H, denoted by $G \odot H$, is a connected graph which is formed by taking n copies of graphs H_i, $1 \leq i \leq n$ of H and connecting ith vertex of G to the vertices of H_i.

Theorem 2.4. *Let $G \odot mK_1$ be a corona product of G order n and mK_1 is trivial graph with $m \geq 3$, then the resolving domination number of $G \odot mK_1$ is $\gamma_M(G \odot mK_1) = \infty$.*

Furthermore, we determine the exact value of some families graphs for the resolving domination number, namely path, centipede graphs and tadpole $T_{4,n}$. The results of $\gamma_r(G)$ as follows.

Theorem 2.5. *Let P_n be a path with order $n \geq 2$, then the resolving domination number of P_n is*

$$\gamma_{M(P_n)} = \begin{cases} 1, & \text{if } n = 2 \\ \infty, & \text{if } n = 3 \\ 3, & \text{if } n \in \{4, 5, 6\} \\ \left\lceil \frac{n}{3} \right\rceil, & \text{if } n \geq 7, n \neq 0 \pmod 3 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n \geq 7, n = 0 \pmod 3. \end{cases}$$

Proof. Path graph, denoted by P_n, is a tree graph with n vertices. Vertex set and edge set of P_n, respectively, are $V(P_n) = \{x_i : 1 \leq i \leq n\}$ and $E(P_n) = \{x_{i-1}x_i : 1 \leq i \leq n-1\}$. For this proof, we divide the proof into two cases as follows.

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Case 1. For $n = 2$.

Based on Theorem 2.2 that $\gamma_M(P_2) = 1$.

Case 2. For $n = 3$.

Based on Lemma 2.1 that $\gamma_M(P_3) \geq 3$. Furthermore, we prove that $\gamma_M(P_3) \leq 3$? we can construct the resolving dominating set of P_3, namely $W = V(P_3) = \{x_1, x_2, x_3\}$. The representation of vertex in P_3 is as follows:

$$r(x_1 \,|\, W) = \{0, 1, 2\} \quad r(x_2 \,|\, W) = \{0, 1, 1\} \quad r(x_3 \,|\, W) = \{0, 1, 2\}.$$

There are same representations, namely $r(x_1 \,|\, W) = r(x_3 \,|\, W)$. We know that W is not a resolving set such that W is not a resolving dominating set. Thus, we obtain that $\gamma_M(P_3) \neq 3$. It concludes that $\gamma_M(P_3) = \infty$.

Case 3. For $n = 4, 5, 6$.

Based on Lemma 2.1 that $\gamma_M(P_n) \geq 3$. Furthermore, we prove that $\gamma_M(P_n) \leq 3$, we can construct the resolving dominating set of P_n. The representation of vertex in P_n is as follows:

	P_4 with $W = \{x_1, x_2, x_4\}$	P_5 with $W = \{x_1, x_2, x_5\}$	P_6 with $W = \{x_2, x_5, x_6\}$
$r(x_1 \,\|\, W)$	$\{0, 1, 3\}$	$\{0, 1, 4\}$	$\{1, 4, 5\}$
$r(x_2 \,\|\, W)$	$\{0, 1, 2\}$	$\{0, 1, 3\}$	$\{0, 3, 4\}$
$r(x_3 \,\|\, W)$	$\{1, 1, 3\}$	$\{1, 2, 2\}$	$\{1, 2, 3\}$
$r(x_4 \,\|\, W)$	$\{0, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 2\}$
$r(x_5 \,\|\, W)$		$\{0, 3, 4\}$	$\{0, 1, 3\}$
$r(x_6 \,\|\, W)$			$\{0, 1, 4\}$

All vertices in P_n have distinct representations. We know that W is resolving set and dominating set such that W is resolving dominating set. Thus, we obtain that $\gamma_M(P_n) \leq 3$. It concludes that $\gamma_M(P_n) = 3$.

Case 4. For $n \geq 7$ and $n \equiv 1 \pmod 3$.

Based on Proposition 2.1 that $\gamma_M(P_n) \geq \max\{\gamma(P_n), \mathrm{md}(P_n)\} = \{\lceil \frac{n}{3} \rceil, 1\} = \lceil \frac{n}{3} \rceil$. Furthermore, we prove that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$, we can construct the resolving dominating set of P_n, namely $W = \{x_i, x_{n-1}; i \equiv 2 \pmod 3\}$. The vertex $x_i; i \neq 2 \pmod 3$ is dominated by vertices in W. We have the properties to show that all vertices have distinct representation as follows:

(i) We know that $d(x_l, x_{n-1}) \neq d(x_k, x_{n-1})$, for $1 \leq l, k \leq n-1$ and $x_l, x_k \notin W$.

(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i \,|\, W) = \{d(x_i, x_s) : x_s \in W - \{x_{n-1}\}\} \cup \{d(x_i, x_{n-1})\}$.

(iii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l \,|\, W - \{y_n\}) = \{d(x_l, x_s) : x_s \in W - \{x_{n-1}\}\} = \{d(x_k, x_s) : x_s \in W - \{x_{n-1}\}\} = r(x_k \,|\, W - \{x_{n-1}\})$ for $l + k = n + 1$ and $1 \leq l, k \leq n-1$.

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(iv) Based on (i)–(iii) that $r(x_l\,|\,W) \neq r(x_k\,|\,W)$ for $1 \leq l, k \leq n-1$.
(v) We know that $r(x_i\,|\,W) \neq r(x_n\,|\,W)$ for $1 \leq i \leq n-1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil$. Thus, we obtain that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$. It concludes that $\gamma_M(P_n) = \lceil \frac{n}{3} \rceil$.

Case 5. For $n \geq 7$ and $n \equiv 2 \pmod 3$.

Based on Proposition 2.1 that $\gamma_M(P_n) \geq \max\{\gamma(P_n), \mathrm{md}(P_n)\} = \{\lceil \frac{n}{3} \rceil, 1\} = \lceil \frac{n}{3} \rceil$. Furthermore, we prove that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$, we can construct the resolving dominating set of P_n, namely $W = \{x_i, x_{n-1}; i \equiv 2 \pmod 3\}$. The vertex $x_i; i \neq 2 \pmod 3$ is dominated by vertices in W. We have the properties to show that all vertices have distinct representation as follows:

(i) We know that $d(x_l, x_{n-1}) \neq d(x_k, x_{n-1})$, for $1 \leq l, k \leq n-1$ and $x_l, x_k \notin W$.
(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i\,|\,W) = \{d(x_i, x_s) : x_s \in W - \{x_{n-1}\}\} \cup \{d(x_i, x_{n-1})\}$.
(iii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l\,|\,W - \{x_{n-1}\}) = \{d(x_l, x_s) : x_s \in W - \{x_{n-1}\}\} = \{d(x_k, x_s) : x_s \in W - \{x_{n-1}\}\} = r(x_k\,|\,W - \{x_{n-1}\})$ for $l + k = n-1$ and $1 \leq l, k \leq n-1$.
(iv) Based on (i)–(iii) that $r(x_l\,|\,W) \neq r(x_k\,|\,W)$ for $1 \leq l, k \leq n-1$.
(v) We know that $r(x_i\,|\,W) \neq r(x_n\,|\,W)$ for $1 \leq i \leq n-1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil$. Thus, we obtain that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil$. It concludes that $\gamma_M(P_n) = \lceil \frac{n}{3} \rceil$.

Case 6. For $n \geq 7$ and $n \equiv 0 \pmod 3$.

Based on Proposition 2.1 that $\gamma_M(P_n) \geq \max\{\gamma(P_n), \mathrm{md}(P_n)\} = \{\lceil \frac{n}{3} \rceil, 1\} = \lceil \frac{n}{3} \rceil$. Assume that $|W| = \lceil \frac{n}{3} \rceil$, namely $W = \{x_i; i \equiv 2 \pmod 3\}$. There are at least two vertices which have same representation. We can construct the representation as follows:

(i) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i\,|\,W) = \{d(x_i, x_s) : x_s \in W, i \equiv 2 \pmod 3\}$.
(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l\,|\,W) = \{d(x_l, x_s) : x_s \in W\} = \{d(x_k, x_s) : x_s \in W\} = r(x_k\,|\,W)$ for $l + k = n+1$ and $1 \leq l, k \leq n-1$.
(iii) Based on (i)–(ii) that $r(x_l\,|\,W) = r(x_k\,|\,W)$ for $1 \leq l, k \leq n-1$.

Based on the assumption above, there are same representations, which is a contradiction. Thus, $\gamma_M(P_n) \geq \lceil \frac{n}{3} \rceil + 1$. Furthermore, we prove that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil + 1$, we can construct the resolving dominating set of P_n, namely $W = \{x_i, x_n; i \equiv 2 \pmod 3\}$. The vertex $x_i; i \neq 2 \pmod 3$ is dominated by vertices in W. We have the properties to show that all vertices have distinct representation as follows:

(i) We know that $d(x_l, x_n) \neq d(x_k, x_n)$, for $1 \leq l, k \leq n-1$ and $x_l, x_k \notin W$.

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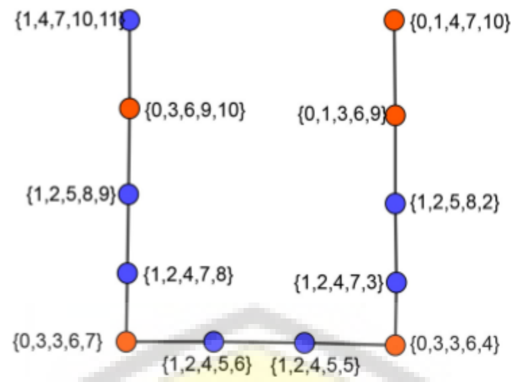

Fig. 2. A graph with resolving domination number $\gamma_M(P_{12}) = 5$.

(ii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_i \mid W) = \{d(x_i, x_s) : x_s \in W - \{x_n\}\} \cup \{d(x_i, x_n)\}$.

(iii) We have the representation of $x_i \in V(P_n) - W$, namely $r(x_l \mid W - \{x_n\}) = \{d(x_l, x_s) : x_s \in W - \{x_n\}\} = \{d(x_k, x_s) : x_s \in W - \{x_n\}\} = r(x_k \mid W - \{x_n\})$ for $l + k = n + 1$ and $1 \leq l, k \leq n - 1$.

(iv) Based on (i)–(iii) that $r(x_l \mid W) \neq r(x_k \mid W)$ for $1 \leq l, k \leq n - 1$.

(v) We know that $r(x_i \mid W) \neq r(x_n \mid W)$ for $1 \leq i \leq n - 1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil + 1$. Thus, we obtain that $\gamma_M(P_n) \leq \lceil \frac{n}{3} \rceil + 1$. It concludes that $\gamma_M(P_n) = \lceil \frac{n}{3} \rceil + 1$. □

Theorem 2.6. *Let* Cp_n *be a centipede with order* $n \geq 2$, *then the resolving domination number of* Cp_n *is*

$$\gamma_M(\mathrm{Cp}_n) = \begin{cases} 3, & if\ n = 2, \\ n, & if\ n \geq 3. \end{cases}$$

Proof. Centipede graph, denoted by Cp_n, is a tree graph with $2n$ vertices. Vertex set and edge set of Cp_n, respectively, are $V(\mathrm{Cp}_n) = \{x_i, y_j : 1 \leq i \leq n\}$ and $E(\mathrm{Cp}_n) = \{x_{i-1}x_i : 1 \leq i \leq n-1\} \cup \{x_i y_i; 1 \leq i \leq n\}$. The vertice x_i is a backbone and the vertice y_i is a pendant vertex. For this proof, we divide the proof into two cases as follows.

Case 1. For $n = 2$.

Centipede graph Cp_2 has four vertices (two vertices as backbone and two vertices in pendant vertex), based on the definition that Centipede graph Cp_2 isomorphic

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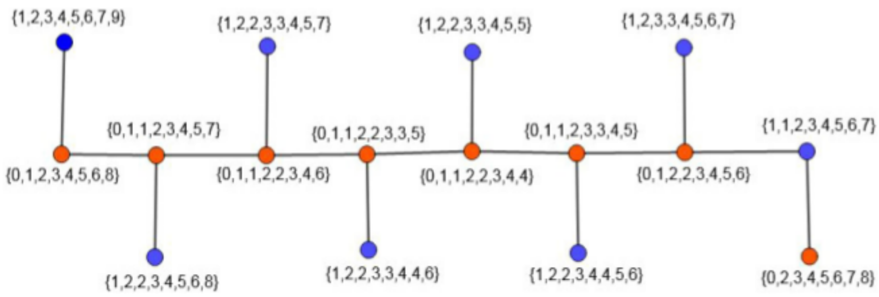

Fig. 3. A graph with resolving domination number $\gamma_M(\mathrm{Cp}_8) = 8$.

to path with four vertices. It is based on Lemma 2.2 that $\gamma_M(\mathrm{Cp}_2) \geq 3$. Furthermore, we prove that $\gamma_M(\mathrm{Cp}_2) \leq 3$, we can construct the resolving dominating set of Cp_2, namely $W = \{x_1, y_1, y_2\}$. The vertex x_2 is dominated by y_2 or x_1. The representation of vertex in Cp_n is as follows:

$$r(x_1 \,|\, W) = \{0, 1, 2\} \quad r(x_2 \,|\, W) = \{1, 1, 2\}$$

$$r(y_1 \,|\, W) = \{0, 1, 3\} \quad r(y_2 \,|\, W) = \{0, 2, 3\}.$$

From the representation, all vertices are distinct. We know that W is resolving set and dominating set such that W is a resolving dominating set with $|W| = 3$. Thus, we obtain that $\gamma_M(\mathrm{Cp}_2) \leq 3$. It concludes that $\gamma_M(\mathrm{Cp}_2) = 3$.

Case 2. For $n \geq 3$.

Centipede graph Cp_n has $2n$ vertices (n vertices as backbone and n vertices in pendant vertex), based on Proposition 2.1 that $\gamma_M(\mathrm{Cp}_2) \geq \max\{\gamma(\mathrm{Cp}_n), \mathrm{md}(\mathrm{Cp}_n)\} = \{n, n\} = n$. Furthermore, we prove that $\gamma_M(\mathrm{Cp}_n) \leq n$, we can construction the resolving dominating set of Cp_n, namely $W = \{x_1, \ldots, x_{n-1}, y_n\}$. The vertex x_n is dominated by y_n or x_{n-1} and the vertex y_i, $1 \leq i \leq n-1$ dominated by x_i, $1 \leq i \leq n-1$. The representation of vertex in Cp_n is shown in Table 1.

From Table 1, we have the properties that all vertices have distinct representation as follows:

(i) We know that $d(y_l, y_n) \neq d(y_k, y_n) \neq d(x_n, y_n)$, for $1 \leq l, k \leq n-1$.
(ii) We have the representation of y_i in Cp_n, namely $r(y_i \,|\, W) = \{d(y_1, x_s) : x_s \in W - \{y_n\}\} \cup \{d(y_i, y_n)\}$.
(iii) We have the representation of y_i in Cp_n, namely $r(y_l \,|\, W - \{y_n\}) = \{d(y_l, x_s) : x_s \in W - \{y_n\}\} = \{d(y_k, x_s) : x_s \in W - \{y_n\}\} = r(y_k \,|\, W - \{y_n\})$ for $l + k = n$ and $1 \leq l, k \leq n-1$.

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Table 1. The representation of Cp_n.

	y_1	y_2	y_3	y_4	y_5	$\cdots$	y_{n-2}	y_{n-1}	x_n
x_1	1	2	3	4	5	$\cdots$	$n-2$	$n-1$	$n-1$
x_2	2	1	2	3	4	$\cdots$	$n-3$	$n-2$	$n-2$
x_3	3	2	1	2	3	$\cdots$	$n-4$	$n-3$	$n-3$
x_4	4	3	2	1	2	$\cdots$	$n-5$	$n-4$	$n-4$
x_5	5	4	3	2	1	$\cdots$	$n-6$	$n-5$	$n-5$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
x_{n-2}	$n-2$	$n-3$	$n-4$	$n-5$	$n-6$	$\cdots$	1	2	2
x_{n-1}	$n-1$	$n-2$	$n-3$	$n-4$	$n-5$	$\cdots$	2	1	1
y_n	$n+1$	n	$n-1$	$n-2$	$n-3$	$\cdots$	4	3	1

(iv) Based on (iii) that $r(y_l \,|\, W) \neq r(y_k \,|\, W)$ for $1 \leq l, k \leq n-1$.
(v) We know that $r(y_i \,|\, W) \neq r(x_n \,|\, W)$ for $1 \leq i \leq n-1$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = n$. Thus, we obtain that $\gamma_M(\mathrm{Cp}_n) \leq n$. It concludes that $\gamma_M(\mathrm{Cp}_n) = n$. □

Theorem 2.7. *Let $T_{4,n}$ be a tadpole graph with order $n \in N$, then resolving domination number of $T_{4,n}$ is*

$$\gamma_M(T_{4,n}) = \begin{cases} 4, & \text{if } n = 3, \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{if } n \neq 3. \end{cases}$$

Proof. Tadpole graph, denoted by $T_{4,n}$, is a unicyclic graph which is obtained by joining a cycle C_4 and path P_n with a bridge. Vertex set and edge set of $T_{4,n}$, respectively, are $V(T_{4,n}) = \{x_i, y_j : 1 \leq i \leq 4, 1 \leq j \leq n\}$ and $E(T_{4,n}) = \{y_{j-1}y_j : 1 \leq j \leq n-1\} \cup \{x_1y_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$. The edge x_1y_1 is a bridge in tadpole graphs. For this proof, we divide the proof into two cases as follows:

Case 1. For $n = 3$.

Based on Lemma 2.1 that $\gamma_M(T_{4,n}) \geq 3$. Assume that $|W| = 3$, such that we have the same representation as follows:

(i) If we choose the set $W = \{x_3, y_1, y_3\}$, then we know that $d(x_2, x_1) = d(x_4, x_1)$ and $d(x_2, x_3) = d(x_4, x_3) = 1$. Thus, $r(x_2 \,|\, W) = r(x_4 \,|\, W) = \{1, 2, 4\}$.
(ii) If we choose the set $W = \{x_3, x_4, y_2\}$, then we know that $d(x_2, x_4) = d(y_1, x_4)$ and $d(x_2, y_2) = d(y_1, x_3)$. Thus, $r(x_2 \,|\, W) = r(y_1 \,|\, W) = \{1, 2, 3\}$.

There are same representations such that $\gamma_M(T_{4,n}) \geq 4$. Furthermore, we prove that $\gamma_M(T_{4,n}) \leq 4$, we can construct the resolving dominating set of $T_{4,n}$. The representation of vertex in $T_{4,n}$ is as follows:

	$T_{4,n}$ with $W = \{x_3, x_4, y_1, y_3\}$
$r(x_1 \mid W)$	$\{1, 1, 2, 3\}$
$r(x_2 \mid W)$	$\{1, 2, 2, 4\}$
$r(x_3 \mid W)$	$\{0, 1, 3, 5\}$
$r(x_4 \mid W)$	$\{0, 1, 2, 4\}$
$r(y_1 \mid W)$	$\{0, 2, 2, 3\}$
$r(y_2 \mid W)$	$\{1, 1, 3, 4\}$
$r(y_3 \mid W)$	$\{0, 2, 4, 5\}$

All vertices in $T_{4,n}$ have distinct representations. We know that W is a resolving set and dominating set such that W is a resolving dominating set. Thus, we obtain that $\gamma_M(T_{4,n}) \leq 4$. It concludes that $\gamma_M(T_{4,n}) = 4$.

Case 2. For $n \equiv 0, 2 \pmod 3$.

We prove that $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Assume that $|W| = \lceil \frac{n}{3} \rceil + 1$, namely $W = \{y_j; j \equiv 2 \pmod 3\} \cup \{x_3\}$. There are at least two vertices which have same representation. We can construct the representation as follows:

(i) We have $d(x_2, x_3) = d(x_4, x_3)$ and $d(x_2, x_1) = d(x_4, x_1)$.
(ii) We know that $d(x_2, y_s) = d(x_2, x_1) + d(x_1, y_s) = d(x_4, x_1) + d(x_1, y_s) = d(x_4, y_s)$ for $y_s \in W$.
(iii) We know that $r(x_2 \mid W) = \{d(x_2, y_s) : y_s \in W\} = \{d(x_4, y_s) : y_s \in W\} = r(x_4 \mid W)$.

Based on the assumption above, there are same representations, which is a contradiction. Thus, $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Furthermore, we prove that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$, we can construct the resolving dominating set of $T_{4,n}$, namely $W = \{y_j, x_3, x_4; j \equiv 2 \pmod 3\}$. The vertex $y_j; j \not\equiv 2 \pmod 3$ dominated by vertices in W. We have the properties this show that all vertices have distinct representations as follows:

(i) We know that $d(y_l, x_4) \neq d(y_k, x_4)$, for $1 \leq l, k \leq n$ and $y_l, y_k \notin W$.
(ii) We have the representation of $y_j \in V(T_{4,n}) - W$, namely $r(y_j \mid W) = \{d(y_j, y_s) : y_s \in W - \{x_3, x_4\}\} \cup \{d(y_j, x_4)\}$.
(iii) We have the representation of $y_j \in V(T_{4,n})$, namely $r(y_l \mid W - \{x_3, x_4\}) = \{d(y_l, y_s) : y_s \in W - \{x_3, x_4\}\} = \{d(y_k, y_s) : y_s \in W - \{x_3, x_4\}\} = r(y_k \mid W - \{x_3, x_4\})$ for $l + k = n + 2$ and $1 \leq l, k \leq n$.
(iv) Based on (i)–(iii) that $r(y_l \mid W) \neq r(y_k \mid W)$ for $1 \leq l, k \leq n$.
(v) We know that $r(y_j \mid W) \neq r(x_1 \mid W) \neq r(x_2 \mid W)$ for $1 \leq j \leq n$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil + 2$. Thus, we obtain that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$. It concludes that $\gamma_M(T_{4,n}) = \lceil \frac{n}{3} \rceil + 2$.

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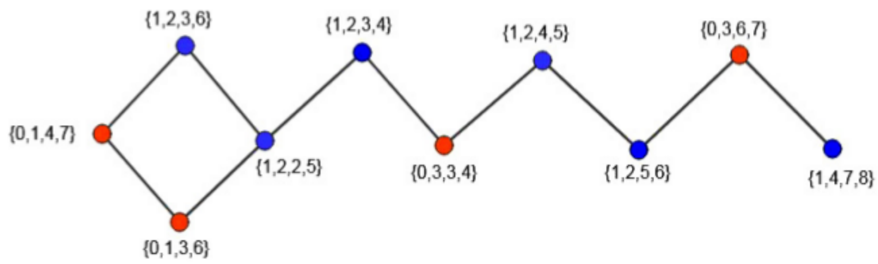

Fig. 4. A graph with resolving domination number $\gamma_M(V(T_{4,6})) = 4$.

Case 3. For $n \equiv 1 \pmod 3$.

We prove that $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Assume that $|W| = \lceil \frac{n}{3} \rceil + 1$, namely $W = \{y_j, y_n; j \equiv 2 \pmod 3\} \cup \{x_3\}$. There are at least two vertices which have same representation. We can construction the representation as follows:

(i) We have $d(x_2, x_3) = d(x_4, x_3)$ and $d(x_2, x_1) = d(x_4, x_1)$.
(ii) We know that $d(x_2, y_s) = d(x_2, x_1) + d(x_1, y_s) = d(x_4, x_1) + d(x_1, y_s) = d(x_4, y_s)$ for $y_s \in W$.
(iii) We know that $r(x_2 \,|\, W) = \{d(x_2, y_s) : y_s \in W\} = \{d(x_4, y_s) : y_s \in W\} = r(x_4 \,|\, W)$.

Based on the assumption above, there are same representations, which is a contradiction. Thus, $\gamma_M(T_{4,n}) \geq \lceil \frac{n}{3} \rceil + 2$. Furthermore, we prove that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$, we can construct the resolving dominating set of $T_{4,n}$, namely $W = \{y_j, x_3, x_4, y_n; j \equiv 2 \pmod 3\}$. The vertex $y_j; j \not\equiv 2 \pmod 3$ is dominated by vertices in W. We have the properties this show that all vertices have distinct representations as follows:

(i) We know that $d(y_l, x_4) \neq d(y_k, x_4)$, for $1 \leq l, k \leq n$ and $y_l, y_k \notin W$.
(ii) We have the representation of $y_j \in V(T_{4,n}) - W$, namely $r(y_j \,|\, W) = \{d(y_j, y_s) : y_s \in W - \{x_3, x_4, y_n\}\} \cup \{d(y_j, x_4)\}$.
(iii) We have the representation of $y_j \in V(T_{4,n})$, namely $r(y_l \,|\, W - \{x_3, x_4, y_n\}) = \{d(y_l, y_s) : y_s \in W - \{x_3, x_4, y_n\}\} = \{d(y_k, y_s) : y_s \in W - \{x_3, x_4, y_n\}\} = r(y_k \,|\, W - \{x_3, x_4, y_n\})$ for $l + k = n$ and $1 \leq l, k \leq n$.
(iv) Based on (i)–(iii) that $r(y_l \,|\, W) \neq r(y_k \,|\, W)$ for $1 \leq l, k \leq n$.
(v) We know that $r(y_j \,|\, W) \neq r(x_1 \,|\, W) \neq r(x_2 \,|\, W)$ for $1 \leq j \leq n$.

From the representation, all vertices are distinct. We know that W is a resolving set and dominating set such that W is a resolving dominating set with $|W| = \lceil \frac{n}{3} \rceil + 2$. Thus, we obtain that $\gamma_M(T_{4,n}) \leq \lceil \frac{n}{3} \rceil + 2$. It concludes that $\gamma_M(T_{4,n}) = \lceil \frac{n}{3} \rceil + 2$. □

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3. Conclusion

In this paper, we have given results on the lower bound of a resolving domination number and determine the exact values of some special graphs. Hence, the following problems arise naturally.

Open Problem 3.1. Determine the resolving domination number of family graph namely family tree, unicyclic, regular graphs, and others.

Open Problem 3.2. Determine the resolving domination number of operation graph namely corona product, cartesian product, joint, comb product, and others.

Open Problem 3.3. Characterize the resolving domination number $\gamma_r(G) = n - 1$.

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