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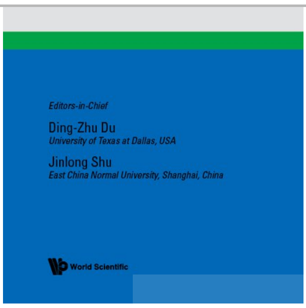

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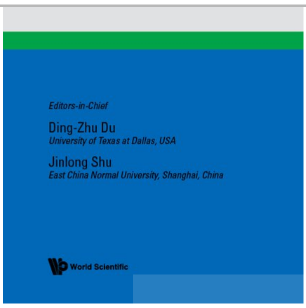

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r-Dynamic coloring of the corona product of graphs

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Let $G = (V, E)$ be a graph. A proper k-coloring of graph G is r-dynamic coloring if for every v, the neighbors of vertex v receive at least $\min\{r, d(v)\}$ different colors. The minimum k such that graph G has r-dynamic k coloring is called the r-dynamic chromatic number, denoted by $\chi_r(G)$. In this paper, we study the r-dynamic coloring of corona product of graph. The corona product of graph is obtained by taking a number of vertices $|V(G)|$ copy of H, and making the ith of $V(G)$ adjacent to every vertex of the ith copy of $V(H)$. We obtain the lower bound of r-dynamic chromatic number of corona product of graphs and some exact value.

Keywords: r-Dynamic chromatic number; corona product of graph.

Mathematics Subject Classification 2020: 05C78

1. Introduction

Let $G = (V, E)$ be a graph on the vertex set $V(G)$ and the set $E(G)$. The open neighborhood of the vertex v of G is denoted by $N(v)$. The maximum and minimum degrees of graph G, respectively, are denoted by $\Delta(G)$ and $\delta(G)$. An r-dynamic coloring of a graph G is defined to be a map from V to the set of colors such that (1) a proper coloring, and (2) for each vertex v, $|c(N(v))| \geq \min\{r, d(v)\}$. The minimum k for which G has an r-dynamic k coloring is called the r-dynamic chromatic number, $\chi_r(G)$. This concept was introduced by Lai and Montgomery [12]. They found lower bound of the r-dynamic chromatic number, $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$. The r-dynamic chromatic number has been studied by several authors, for instance, in [1–8, 13, 14]. In this paper, we use some observations as follows.

Observation 1.1 ([12]). Let $\Delta(G)$ be the maximum degree of graph G. It holds $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$.

Observation 1.2. Let $G \odot H$ be corona product of graph, then we have $\delta(G \odot H) = \delta(H) + 1$ and $\Delta(G \odot H) = \Delta(G) + |V(H)|$, where $\Delta(G \odot H)$ is the maximum degree of $G \odot H$ and $\delta(G \odot H)$ is the minimum degree of $G \odot H$.

Montgomery in [12] found for any tree, $\chi_r(G) = \min\{\Delta, r\} + 1$ and if $r \geq 2$ then $\chi_r(C_n) = 5$ for $n = 5$, $\chi_r(C_n) = 3$ if $n = 3k, k \geq 1$ and $\chi_r(C_n) = 4$ for n otherwise. Kang *et al.* [7] also found the r-dynamic chromatic number of grid graph. Akbari *et al.* [2] found 2-dynamic chromatic number of Cartesian of path graph and Cartesian product of cycle graph. Taherkhani *et al.* in [14], proved upper bound of regular graph.

Lai and Chang in [11] introduced the definition of corona product. Given graph G and H with n and m vertices, the corona product of G with respect to H is the graph $G \odot H$ with vertex set, $V(G \odot H) = V(G) \cup \{n$ distinct copies of $V(H)\}$ and edge set $E(G \odot H) = E(G) \cup \{n$ distinct copies of $E(H)\}$. Kristiana *et al.* in [9] studied the lower bound of the r-dynamic chromatic number of corona product product by wheel graphs and [10] found the corona product by complete graph. In this paper, we will continue to study of graph corona product.

2. Results

In this paper, we will study the lower bound of the r-dynamic chromatic number of corona product of graph and some new results of the exact value of some corona product of graph. We obtain $\chi_r(S_n \odot S_m), \chi_r(S_n \odot P_m), \chi_r(S_n \odot F_m), \chi_r(P_n \odot S_m)$ and $\chi_r(F_n \odot S_m)$ for $m, n \geq 3$.

Lemma 2.1. *Let $G \odot H$ be a corona product of graph, the lower bound of r-dynamic chromatic number is as follows*:

$$\chi_r(G \odot H) \geq \begin{cases} \delta(H) + 2, & 1 \leq r \leq \delta(H) + 1, \\ r + 1, & \delta(H) + 1 \leq r \leq \Delta(G) + |V(H)|, \\ \Delta(G) + |V(H)| + 1, & r \geq \Delta(G) + |V(H)| + 1. \end{cases}$$

Proof. Based on the definition of corona product of graph, the vertex set is $V(G \odot H) = V(S_n) \cup \bigcup_{i=1}^{n} V(H_i)$ and the maximum degree of $G \odot H$ is $\Delta(G \odot H) = \Delta(G) + |V(H)|$. For $1 \leq r \leq \delta(H) + 1$, based on Observation 1.1 and by choosing $r = \delta(H)+1$, we get $\chi_r(G \odot H) \geq \min\{ r, \Delta(G \odot H)\} + 1 = \min\{ \delta(H)+1, \Delta(G) + |V(H)|\} + 1 = (\delta(H) + 1) + 1 = \delta(H) + 2$.

For $\delta(H)+1 \leq r \leq \Delta(G)+|V(H)|$, respected to Observation 1.1 and by choosing $r_1 = \delta(H)+1$, we get $\chi_r(G \odot H) \geq \min\{ r, \Delta(G \odot H)\} + 1 = \min\{ \delta(H)+1, \Delta(G)+|V(H)|\}+1$. Since $\delta(H)$ is the minimum degree of graph H and $|V(H)|$ is the order of graph H, thus $\delta(H) \leq |V(H)|$ and $\chi_r(S_n \odot H) \geq \delta(H) + 1 + 1 = r_1 + 1$. If we choose $r_2 = \delta(H) + 2$, we get $\chi_r(G \odot H) \geq \min\{ \delta(H) + 2, \Delta(G) + |V(H)|\} + 1 = \delta(H) + 2 + 1 = r_2 + 1$ and so on. It gives the proof for $\delta(H) + 1 \leq r \leq n + |V(H)|, \chi_r(G \odot H) \geq r + 1$.

For $r \geq \Delta(G) + |V(H)| + 1$, based on Observation 1.1 and by choosing $r = \Delta(G) + |V(H)| + 1$, we get $\chi_r(G \odot H) \geq \min\{ r, \Delta(G \odot H)\} + 1 = \min\{\Delta(G) + |V(H)| + 1, \Delta(G) + |V(H)|\} + 1 = \Delta(G) + |V(H)| + 1$. It concludes the proof. □

Based on Lemma 2.1, we get some corollary of corona product of graph, as follows.

Corollary 2.2. *Let $S_n \odot H$ be a corona product of star graph and $H \neq K_m, C_m, W_m$, for $n \geq 3, m \geq 4$*

$$\chi_r(S_n \odot H) \geq \begin{cases} \delta(H) + 2, & 1 \leq r \leq \delta(H) + 1, \\ r + 1, & \delta(H) + 1 \leq r \leq n + |V(H)|, \\ n + |V(H)| + 1, & r \geq n + |V(H)| + 1. \end{cases}$$

Corollary 2.3. *Let $H \odot S_m$ be a corona product of star graph and $H \neq K_n, C_n, W_n$, for $n \geq 4, m \geq 3$*

$$\chi_r(H \odot S_m) \geq \begin{cases} 3, & r = 1, 2, \\ r + 1, & 3 \leq r \leq m + \Delta(H) + 1, \\ \Delta(H) + m + 2, & r \geq m + \Delta(H) + 2. \end{cases}$$

Corollary 2.4 ([10]). *Let $K_n \odot H$ be a corona product of complete graph and graph $H \neq C_m, W_m$, for $n \geq 4, m \geq 4$*

$$\chi_r(K_n \odot H) \geq \begin{cases} \chi_r(K_n), & 1 \leq r \leq n - 1, \\ r + 1, & n \leq r \leq n + |V(H)| - 1, \\ n + |V(H)|, & r \geq n + |V(H)|. \end{cases}$$

Corollary 2.5 ([10]). *Let $H \odot K_m$ be a corona product of $H \neq C_n, W_n$ and complete graph, for $n \geq 4, m \geq 4$*

$$\chi_r(H \odot K_m) \geq \begin{cases} \chi_r(K_m) + 1, & 1 \leq r \leq m, \\ r + 1, & m + 1 \leq r \leq m + \Delta(H), \\ \Delta(H) + m + 1, & r \geq m + \Delta(H) + 1. \end{cases}$$

Corollary 2.6 ([9]). *Let $H \odot W_m$ be a corona product of graph of H and wheel graph, for $m \geq 4$*

$$\chi_{r=1,2,3}(H \odot W_m) \geq \begin{cases} 4 & \text{for } m \text{ even,} \\ 5 & \text{for } m \text{ odd,} \end{cases}$$

$$\chi_{r=4,5}(H \odot W_m) \geq \begin{cases} r+1 & \text{for } m = 3k,\ k \geq 1, \\ 7 & \text{for } m = 5, \\ 6 & \text{for otherwise,} \end{cases}$$

$$\chi_r(H \odot W_m) \geq \begin{cases} r+1 & \text{for } 6 \leq r \leq \Delta(H) + m + 1, \\ \Delta(H) + m + 2 & \text{for } r \geq \Delta(H) + m + 2. \end{cases}$$

Corollary 2.7 ([9]). *Let $W_n \odot H$ be a corona product of wheel graph and $H \neq K_m, C_m$, for $n \geq 4, m \geq 4$*

$$\chi_{r=1,2}(W_n \odot H) \geq \begin{cases} 3 & \text{for } n \text{ even,} \\ 4 & \text{for } n \text{ odd,} \end{cases}$$

$$\chi_r(W_n \odot H) \geq \begin{cases} r+1 & \text{for } 3 \leq r \leq n + |V(H)| + 1, \\ n + |V(H)| + 2 & \text{for } r \geq n + |V(G)| + 2. \end{cases}$$

The following theorem is the exact value of r-dynamic chromatic number of the corona product of graph, namely, $W_n \odot S_m$, $S_n \odot F_m$, $W_n \odot F_m$, $F_n \odot S_m$ and $S_n \odot S_m$.

Theorem 2.8. *Let $W_n \odot S_m$ be a corona product of wheel graph and star graph*

$$\chi_r(W_n \odot S_m) = \begin{cases} 3, & r = 1, 2, n \text{ even,} \\ 4, & r = 1, 2, n \text{ odd,} \\ r+1, & 3 \leq r \leq n + m + 1, \\ m + n + 2, & r \geq n + m + 2. \end{cases}$$

Proof. $V(W_n \odot S_m) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq m\} \cup \{x_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{p\} \cup \{p_j; 1 \leq j \leq m\}$ and the order of graph is $|V(W_n \odot S_m)| = mn + m + 2n$. The edge set is $E(W_n \odot S_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{x x_i; 1 \leq i \leq n\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{x_i y_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{xp\} \cup \{x p_j; 1 \leq j \leq m\} \cup \{p p_j; 1 \leq j \leq m\}$ and the size of graph is $|E(W_n \odot S_m)| = nm + 3n + 2n + 1$. Thus, $\Delta(W_n \odot S_m) = m+n+1$ and $\delta(W_n \odot S_m) = 2$. We define three cases, namely, for $\chi_{r=1,2}(W_n \odot S_m), \chi_{3 \leq r \leq n+m+1}(W_n \odot S_m)$ and $\chi_{r \geq m+n++2}(W_n \odot S_m)$.

Case 1. For $r = 1, 2$

Based on Corollary 2.7, the lower bound for $r = 1, 2$ is $\chi_r(W_n \odot S_m) \geq \delta(S_m) + 2 = 1 + 2 = 3$. To show the upper bound, we divide two cases, namely, n even and n odd.

(1) n even, we define $c_1 : V(W_n \odot S_m) \to \{1, 2, 3\}$ where $n, m \geq 3$, as follows:

$$c_1(x) = 1,$$
$$c_1(y_i) = 1, \quad 1 \leq i \leq n,$$
$$c_1(p) = 2,$$
$$c_1(p_j) = 3, \quad 1 \leq j \leq m,$$
$$c_1(x_i) = \begin{cases} 2 & \text{for } i \text{ odd}, \ 1 \leq i \leq n, \\ 3 & \text{for } i \text{ even}, \ 1 \leq i \leq n, \end{cases}$$
$$c_1(y_{ij}) = \begin{cases} 2 & \text{for 1 odd}, \ 1 \leq i \leq n, \ 1 \leq j \leq m, \\ 3 & \text{for 1 even}, \ 1 \leq i \leq n, \ 1 \leq j \leq m. \end{cases}$$

Clearly, c_1 shows the upper bound for $\chi_{r=1,2}(W_n \odot S_m) \leq 3$. Hence, $\chi_{r=1,2}(W_n \odot S_m) = 3$ for n even.

(2) n odd, we define $c_1 : V(W_n \odot S_m) \to \{1, 2, 3, 4\}$, as follows:

$$c_1(x) = 1,$$
$$c_1(y_i) = 1, \quad 1 \leq i \leq n,$$
$$c_1(p) = 2,$$
$$c_1(p_j) = 3, \quad 1 \leq j \leq m,$$
$$c_1(x_i) = \begin{cases} 2 & \text{for } i \text{ odd}, \ 1 \leq i \leq n-1, \\ 3 & \text{for } i \text{ even}, \ 1 \leq i \leq n-1, \\ 4 & \text{for } i = n, \end{cases}$$
$$c_1(y_{ij}) = \begin{cases} 2 & \text{for } i \text{ odd}, \ 1 \leq i \leq n, \ 1 \leq j \leq m, \\ 3 & \text{for } i \text{ even}, \ 1 \leq i \leq n, \ 1 \leq j \leq m. \end{cases}$$

Clearly, c_1 shows the upper bound for $\chi_{r=1,2}(W_n \odot S_m) \leq 4$. Hence, $\chi_{r=1,2}(W_n \odot S_m) = 4$ for n odd.

Case 2. For $3 \leq r \leq n + m + 1$

Based on Corollary 2.7, we know $\delta(S_m) = 1$ and $|V(S_m)| = m + 1$ so $\chi_{3 \leq r \leq n+m+1} \geq r + 1$. To determine the upper bound, we construct the pattern as follows:

$$n = 3, \quad m = 3, \quad r = 3, \quad \chi_{\mathbf{3}}(W_3 \odot S_3) = \mathbf{4},$$
$$n = 3, \quad m = 3, \quad r = 3, \quad \chi_{\mathbf{3}}(W_3 \odot S_3) = \mathbf{4},$$

$$n = 3, \quad m = 3, \quad r = 4, \quad \chi_{\mathbf{4}}(W_3 \odot S_3) = \mathbf{5},$$
$$n = 3, \quad m = 4, \quad r = 4, \quad \chi_{\mathbf{4}}(W_3 \odot S_4) = \mathbf{5},$$
$$n = 3, \quad m = 4, \quad r = 5, \quad \chi_{\mathbf{5}}(W_3 \odot S_4) = \mathbf{6},$$
$$n = 4, \quad m = 4, \quad r = 5, \quad \chi_{\mathbf{5}}(W_4 \odot S_4) = \mathbf{6},$$
$$n = 4, \quad m = 4, \quad r = 6, \quad \chi_{\mathbf{6}}(W_4 \odot S_4) = \mathbf{7},$$
$$n = 4, \quad m = 4, \quad r = 7, \quad \chi_{\mathbf{7}}(W_4 \odot S_4) = \mathbf{8}.$$

Clearly, it shows that $\chi_{3 \le r \le n+m+1} \le r + 1$. Hence, $\chi_{3 \le r \le n+m+1} = r + 1$.

Case 3. For $r \ge n + m + 1$

Based on Corollary 2.7, the lower bound for $r \ge n + m + 2$ is $\chi_r(W_n \odot S_m) \ge n + |V(S_m)| + 1 = n + m + 2$. To find the upper bound, we define $c_2 : V(W_n \odot S_m) \to \{1, 2, \cdots, k\}$ where $n, m \ge 3$, as follows:

$$c_2(x) = 1,$$
$$c_2(x_i) = i + 1, \quad 1 \le i \le n,$$
$$c_2(y_i) = n + 2 + m,$$
$$c_2(y_{ij}) = n + 1 + j, \quad 1 \le i \le n, \ 1 \le j \le m,$$
$$c_2(p) = n + 2 + m,$$
$$c_2(p_j) = n + 1 + j.$$

Clearly, c_2 shows the upper bound for $\chi_{r \ge n+m+2}(W_n \odot S_m) \le n + 2 + m$. Finally, $\chi_{r \ge m+n+2}(W_m \odot S_n) = m + n + 2$. □

Theorem 2.9. *Let $S_n \odot F_m$ be a corona product of star graph and fan graph,*

$$\chi_r(S_n \odot F_m) = \begin{cases} 4, & r = 1, 2, 3, \\ r + 1, & 4 \le r \le (m + n + 1), \\ m + n + 2, & r \ge (m + n + 2). \end{cases}$$

Proof. The graph $S_n \odot F_m$ is a connected graph with vertex set, $V(S_n \odot F_m) = \{x\} \cup \{x_i; 1 \le i \le n\} \cup \{y_i; 1 \le i \le n+1\} \cup \{x_{ij}; 1 \le i \le n+1, 1 \le j \le m\}$ and the order of graph is $|V(S_n \odot F_m)| = mn + 2n + m + 2$. The edge set is $E(S_n \odot F_m) = \{xx_i; 1 \le i \le n\} \cup \{xx_{(n+1)j}; 1 \le j \le m\} \cup \{xy_{n+1}\} \cup \{y_{n+1}x_{(n+1)j}; 1 \le j \le m\} \cup \{x_{(n+1)j}x_{(n+1)(j+1)}; 1 \le j \le m-1\} \cup \{x_iy_i; 1 \le i \le n\} \cup \{x_{ij}x_i; 1 \le i \le n; 1 \le j \le m\} \cup \{x_{ij}x_{i(j+1)}; 1 \le i \le n, 1 \le j \le m-1\} \cup \{y_ix_{ij}; 1 \le i \le n, 1 \le j \le m\}$ and the size of graph is $|E(S_n \odot F_m)| = 3mn + n + 3m$. Thus, the minimum degree of graph G is $\delta(S_n \odot F_m) = 3$ and the maximum degree of graph G is $\Delta(S_n \odot F_m) = m + n + 1$.

We define three cases, namely, for $\chi_{1 \le r \le 3}(S_n \odot F_m)$, $\chi_{4 \le r \le n+m+1}(S_n \odot F_m)$ and $\chi_{r \ge m+n+2}(S_n \odot F_m)$.

Case 1. For $1 \leq r \leq 3$

Based on Corollary 2.2, for $1 \leq r \leq 3$, the lower bound is $\chi_r(S_n \odot F_m) \geq \delta(F_m)+2 = 2+2$. To find the upper bound for $\chi_r(S_n \odot F_m)$, we define $c_3 : V(S_n \odot F_m) \to \{1,2,3,4\}$ where $n, m \geq 3$, as follows:

$$c_3(x) = 1,$$

$$c_3(x_i) = 2, \quad 1 \leq i \leq n,$$

$$c_3(y_i) = \begin{cases} 1, & 1 \leq i \leq n, \\ 2, & i = n+1, \end{cases}$$

$$c_3(x_{ij}) = \begin{cases} 3 & \text{for } 1 \leq i \leq n+1,\ j \text{ odd},\ \ 1 \leq j \leq m, \\ 4 & \text{for } 1 \leq i \leq n+1,\ j \text{ even},\ 1 \leq j \leq m. \end{cases}$$

Clearly, c_3 shows the upper bound for $\chi_{1 \leq r \leq 3}(S_n \odot F_m) \leq 4$. Hence, $\chi_{1 \leq r \leq 3}(S_n \odot F_m) = 4$.

Case 2. For $4 \leq r \leq n+m+1$

The lower bound in Corollary 2.2, for $4 \leq r \leq n+m+1$, we know $\delta(F_m) = 2$ and $|V(F_m)| = m+1$ so $2+2 \leq r \leq n+|V(F_m)|$, the r-dynamic chromatic number, $\chi_{4 \leq r \leq n+m+1} \geq r+1$.

To determine the upper bound, we construct the pattern as follows:

$$n = 3, \quad m = 3, \quad r = 4, \quad \text{so } \chi_4(S_3 \odot F_3) = 5,$$

$$n = 3, \quad m = 3, \quad r = 5, \quad \text{so } \chi_5(S_3 \odot F_3) = 6,$$

$$n = 3, \quad m = 4, \quad r = 4, \quad \text{so } \chi_4(S_3 \odot F_3) = 5,$$

$$n = 3, \quad m = 4, \quad r = 5, \quad \text{so } \chi_5(S_3 \odot F_3) = 6,$$

$$n = 4, \quad m = 4, \quad r = 6, \quad \text{so } \chi_6(S_4 \odot F_4) = 7,$$

$$n = 4, \quad m = 4, \quad r = 7, \quad \text{so } \chi_7(S_4 \odot F_4) = 8,$$

$$n = 4, \quad m = 4, \quad r = 8, \quad \text{so } \chi_8(S_4 \odot F_4) = 9.$$

It is easy that $\chi_{4 \leq r \leq n+m+1} \leq r+1$. Hence, $\chi_{4 \leq r \leq n+m+1} = r+1$.

Case 3. For $r \geq m+n+2$

Based on Corollary 2.2, $\chi_r(S_n \odot F_m) \geq n+|V(F_m)|+1 = n+(m+1)+1 = n+m+2$.

The upper bound for $\chi_r(S_n \odot F_m)$, we define $c_4 : V(S_n \odot F_m) \to \{1,2,\ldots,k\}$ where $n, m \geq 3$, as follows:

$$c_4(x) = 1,$$

$$c_4(y_i) = 2, \quad 1 \leq i \leq n,$$

$$c_4(x_{ij}) = j+2, \quad 1 \leq i \leq n,\ \ 1 \leq j \leq m,$$

$$c_4(x_i) = m+i+2, \quad 1 \leq i \leq n.$$

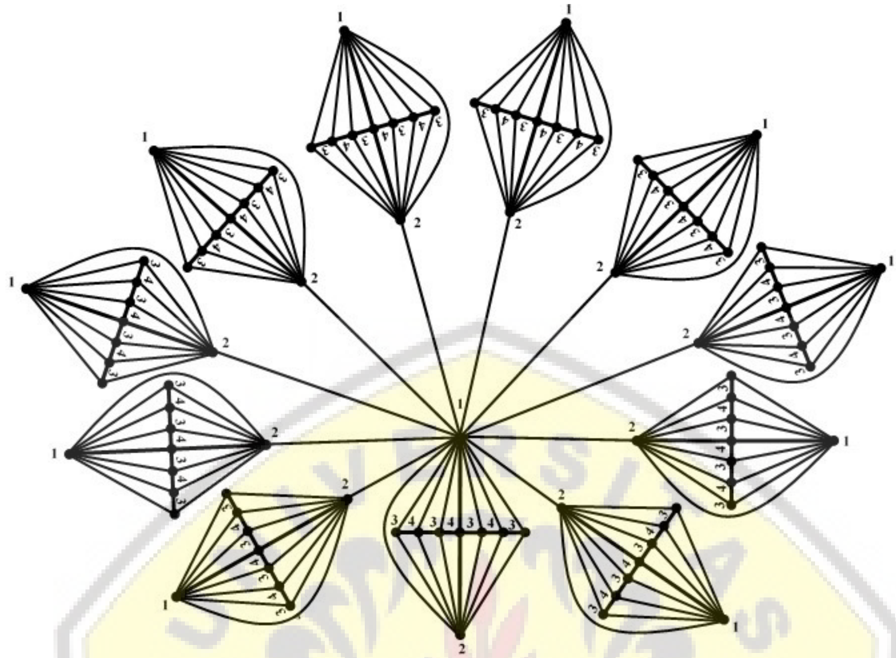

Fig. 1. Example of r-dynamic coloring of $S_{10} \odot F_8$.

Clearly, c_4 shows the upper bound for $\chi_{r \geq n+m+2}(S_n \odot P_m) \leq m+n+2$. Hence, $\chi_{r \geq m+n+2}(S_n \odot F_m) = m+n+2$. The Fig. 1 is the illustration of r-dynamic coloring of the corona product of star and fan graph. □

Theorem 2.10. *Let $W_n \odot F_m$ be a corona product of wheel graph and fan graph,*

$$\chi_r(W_n \odot F_m) = \begin{cases} 3, & r = 1,2,3,\ n \text{ even}, \\ 4, & r = 1,2,3,\ n \text{ odd}, \\ r+1, & 4 \leq r \leq m+n+1, \\ m+n+2, & r \geq m+n+2. \end{cases}$$

Proof. $V(W_n \odot F_m) = \{x, x_i, p, p_j, y_i, y_{ij};\ 1 \leq i \leq n; 1 \leq j \leq m\}$ and the order of graph, $|V(W_n \odot F_m)| = nm + 2n + m + 2$. The edge set is $E(W_n \odot F_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{x x_i; 1 \leq i \leq n\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{y_i y_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_{ij} y_{i(j+1)}; 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{xp\} \cup \{p_j p_{j+1}; 1 \leq j \leq m\} \cup \{pp_j; 1 \leq j \leq m\}$ and the size of graph is $|E(W_n \odot S_m)| = 2nm + n + 2m$. Thus, $\Delta(W_n \odot F_m) = m+n+1$ and $\delta(W_n \odot F_m) = 3$. We define three cases, namely, for $\chi_{r=1,2,3}(W_n \odot F_m)$, $\chi_{4 \leq r \leq n+m+1}(W_n \odot F_m)$ and $\chi_{r \geq m+n++2}(W_n \odot F_m)$.

Case 1. For $r = 1,2,3$

Based on Corollary 2.7, the lower bound for $r = 1, 2$ is $\chi_r(W_n \odot S_m) \geq \delta(S_m) + 2 = 1 + 2 = 3$. To show the upper bound, we define $c_5 : V(W_n \odot F_m) \to \{1, 2, 3, 4\}$ where $n, m \geq 3$, as follows:

$$c_5(x) = 1,$$
$$c_5(y_i) = 1, \quad 1 \leq i \leq n$$
$$c_5(p) = 2,$$
$$c_5(x_i) = c_5(p_j) = \begin{cases} 2 & \text{for } i \equiv 1 (\text{mod } 3) \;\; 1 \leq i \leq n, \\ 3 & \text{for } i \equiv 2 (\text{mod } 3) \;\; 1 \leq i \leq n, \\ 4 & \text{for } i \equiv 0 (\text{mod } 3) \;\; 1 \leq i \leq n, \end{cases}$$
$$c_5(y_{ij}) = \begin{cases} 2 & \text{for } i \equiv 2 (\text{mod } 3) \; j \text{ odd}, \;\; 1 \leq i \leq n, \; 1 \leq j \leq m \\ & \text{or } i \equiv 0 (\text{mod } 3) \; j \text{ odd}, \;\; 1 \leq i \leq n, \; 1 \leq j \leq m, \\ 3 & \text{for } i \equiv 1 (\text{mod } 3) \; j \text{ odd}, \;\; 1 \leq i \leq n, \; 1 \leq j \leq m \\ & \text{or } i \equiv 0 (\text{mod } 3) \; j \text{ even}, \;\; 1 \leq i \leq n, \; 1 \leq j \leq m, \\ 4 & \text{for } i \equiv 1 (\text{mod } 3) \; j \text{ even}, \;\; 1 \leq i \leq n, \; 1 \leq j \leq m \\ & \text{or } i \equiv 2 (\text{mod } 3) \; j \text{ even}, \;\; 1 \leq i \leq n, \; 1 \leq j \leq m. \end{cases}$$

Clearly, c_5 shows the upper bound for $\chi_{r=1,2,3}(W_n \odot F_m) \leq 3$. Hence, $\chi_{r=1,2,3}(W_n \odot F_m) = 4$ for n odd.

Case 2. For $4 \leq r \leq n + m + 1$

Based on Corollary 2.7, we know $\delta(F_m) = 2$ and $|V(F_m)| = m+1$ so $\chi_{4 \leq r \leq n+m+1} \geq r + 1$. To determine the upper bound, we construct the pattern, as follows:

$$\begin{aligned} &n = 3, && m = 3, && r = 4, && \chi_{\mathbf{4}}(W_3 \odot F_3) = \mathbf{5}, \\ &n = 3, && m = 3, && r = 5, && \chi_{\mathbf{5}}(W_3 \odot F_3) = \mathbf{6}, \\ &n = 3, && m = 3, && r = 4, && \chi_{\mathbf{4}}(W_3 \odot F_5) = \mathbf{5}, \\ &n = 3, && m = 5, && r = 5, && \chi_{\mathbf{5}}(W_3 \odot F_5) = \mathbf{6}, \\ &n = 4, && m = 5, && r = 5, && \chi_{\mathbf{5}}(W_4 \odot F_5) = \mathbf{5}, \\ &n = 4, && m = 5, && r = 6, && \chi_{\mathbf{6}}(W_4 \odot F_5) = \mathbf{7}, \\ &n = 4, && m = 6, && r = 7, && \chi_{\mathbf{7}}(W_4 \odot F_6) = \mathbf{8}, \\ &n = 4, && m = 6, && r = 8, && \chi_{\mathbf{8}}(W_4 \odot F_6) = \mathbf{9}. \end{aligned}$$

Clearly, it shows that $\chi_{4 \leq r \leq n+m+1} \leq r + 1$. Hence, $\chi_{4 \leq r \leq n+m+1} = r + 1$.

Case 3. For $r \geq n + m + 1$

Based on Corollary 2.7, the lower bound for $r \geq n + m + 2$ is $\chi_r(W_n \odot F_m) \geq n + |V(F_m)| + 1 = n + m + 2$. To find the upper bound, we define $c_6 : V(W_n \odot F_m) \to \{1, 2, \ldots, k\}$ where $n, m \geq 3$, as follows:

$$c_6(x) = 1,$$
$$c_6(x_i) = i + 1, \quad 1 \leq i \leq n,$$
$$c_6(y_i) = n + 2 + m, \quad 1 \leq i \leq n,$$

$$c_6(y_{ij}) = n+1+j, \quad 1 \le i \le n, \ 1 \le j \le m,$$
$$c_6(p) = n+2+m,$$
$$c_6(p_j) = n+1+j.$$

Clearly, c_5 shows the upper bound for $\chi_{r \ge n+m+2}(W_n \odot F_m) \le n+2+m$. Finally, $\chi_{r \ge m+n+2}(W_m \odot F_n) = m+n+2$. □

Theorem 2.11. *Let $F_n \odot S_m$ be a corona product of fan graph and star graph*

$$\chi_r(F_n \odot S_m) = \begin{cases} 3, & r = 1,2, \\ r+1, & 3 \le r \le n+m+1, \\ n+m+2, & \text{otherwise.} \end{cases}$$

Proof. The graph $F_n \odot S_m$ is a connected graph with vertex set $V(F_n \odot S_m) = \{x, z\} \cup \{x_i, y_i, y_{ij}, z_j; 1 \le i \le n; 1 \le j \le m\}$, edge set $E(F_n \odot S_m) = \{xx_i; 1 \le i \le n\} \cup \{x_iy_i; 1 \le i \le n\} \cup \{x_iy_{ij}; 1 \le i \le n; 1 \le j \le m\} \cup \{y_iy_{ij}; 1 \le i \le n; 1 \le j \le m\} \cup \{xz\} \cup \{zz_j; 1 \le j \le m\}$. The order and size of the graph $G = F_n \odot S_m$ are $|V(G_n \odot S_m)| = (n+1)(m+2)$, and $|E(F_n \odot S_m)| = 2mn + 2n + m + 1$, thus $\Delta(F_n \odot S_m) = n + m + 1$ and $\delta(F_n \odot S_m) = 2$. We define three cases, namely, for $\chi_{r=1,2}(F_n \odot S_m), \chi_{3 \le r \le n+m+1}(F_n \odot S_m)$, and $\chi_{r \ge m+n+2}(F_n \odot S_m)$.

Case 1. For $r = 1, 2$

Based on Corollary 2.3, for $r = 1, 2$, the lower bound is $\chi_r(F_n \odot S_m) \ge 3$. To find the upper bound $\chi_r(F_n \odot S_m)$, we define a map $c_7 : V(F_n \odot S_m) \to \{1, 2, \dots, k\}$ where $n, m \ge 3$, as follows:

$$c_7(x) = 1,$$
$$c_7(x_i) = \begin{cases} 2, & 1 \le i \le n, \ i = \text{odd}, \\ 3, & 1 \le i \le n, \ i = \text{even}, \end{cases}$$
$$c_7(y_i) = 1,$$
$$c_7(y_{ij}) = \begin{cases} 2, & 1 \le i \le n, \ i = \text{even}, \\ 3, & 1 \le i \le n, \ i = \text{odd}, \end{cases}$$
$$c_7(z) = 2,$$
$$c_7(z_j) = 3.$$

Clearly, c_7 shows the upper bound for $\chi_{r=1,2}(F_n \odot S_m) \le 3$. Hence, $\chi_{r=1,2}(F_n \odot S_m) = 3$.

Case 2. For $3 \leq r \leq n+m+1$

Based on Corollary 2.3, we know $\Delta(F_n) = n$ so $3 \leq r \leq m+n+1$, the lower bound is $\chi_{3\leq r\leq n+m+1} \geq r+1$.
The upper bound r-dynamic chromatic numbers are by explicit construction.

$$\begin{aligned}
n &= 3, \quad m = 3, \quad r = 3, \quad \text{so } \chi_{\mathbf{3}}(F_3 \odot S_3) = \mathbf{4},\\
n &= 3, \quad m = 3, \quad r = 4, \quad \text{so } \chi_{\mathbf{4}}(F_3 \odot S_3) = \mathbf{5},\\
n &= 3, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(F_3 \odot S_4) = \mathbf{6},\\
n &= 3, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(F_3 \odot S_4) = \mathbf{7},\\
n &= 4, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(F_4 \odot S_4) = \mathbf{6},\\
n &= 4, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(F_4 \odot S_4) = \mathbf{7},\\
n &= 4, \quad m = 4, \quad r = 7, \quad \text{so } \chi_{\mathbf{7}}(F_4 \odot S_4) = \mathbf{8}.
\end{aligned}$$

It is easy the $\chi_{3\leq r\leq n+m+1} \leq r+1$. Hence, $\chi_{3\leq r\leq n+m+1} = r+1$.

Case 3. For $r \geq m+n+2$

Based on Corollary 2.3, the lower bound is $\chi_r(F_n \odot S_m) \geq \Delta(F_n) + m + 2 = n+m+2$. To find the upper bound $\chi_r(F_n \odot S_m)$, we define a map $c_8 : V(F_n \odot S_m) \to \{1, 2, \ldots, k\}$ where $n, m \geq 3$, as follows:

$$\begin{aligned}
c_8(x) &= 1,\\
c_8(x_i) &= i+1, \quad 1 \leq i \leq n,\\
c_8(y_{ij}) &= n+j+1, \quad 1 \leq i \leq n, \ \ 1 \leq j \leq m,\\
c_8(y_i) &= n+m+2, \quad 1 \leq i \leq n,\\
c_8(z_j) &= n+j+1, \quad 1 \leq j \leq m,\\
c_8(z) &= n+m+2.
\end{aligned}$$

Clearly, c_8 shows the upper bound for $\chi_{r\geq m+n+2}(F_n \odot S_m) \leq m+n+2$. Hence, $\chi_{r\geq n+m+2}(F_n \odot S_m) = n+m+2$. The proof is complete. □

Theorem 2.12. *Let $S_n \odot S_m$ be a corona product of star graph and star graph,*

$$\chi_r(S_n \odot S_m) = \begin{cases} 3, & r = 1, 2,\\ r+1, & 3 \leq r \leq m+n+1,\\ m+n+2, & r \geq m+n+2. \end{cases}$$

Proof. The graph $S_n \odot S_m$ is a connected graph with vertex set, $V(S_n \odot S_m) = \{x, p\} \cup \{x_i, y_i, p_j; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\}$ and the order of graph $|V(S_n \odot S_m)| = nm + 2n + m + 2$. The edge set is $E(S_n \odot S_m) = \{xx_i; 1 \leq i \leq n\} \cup \{xp\} \cup \{xp_j; 1 \leq j \leq m\} \cup \{x_iy_i; 1 \leq i \leq n\} \cup \{x_iy_{ij}1 \leq$

$i \leq n, 1 \leq j \leq m\} \cup \{y_i y_{ij}; 1 \leq i \leq n, 1 \leq j\ leqm\}$ and the size of graph is $|E(S_n \odot S_m)| = 2nm + 2n + m + 1$. Thus, the minimum degree of graph G is $\delta(S_n \odot S_m) = 2$ and the maximum degree of graph G is $\Delta(S_n \odot S_m) = m + n + 1$.

We define three cases, namely, for $\chi_{r=1,2}(S_n \odot S_m), \chi_{3 \leq r \leq m+n+1}(S_n \odot S_m)$, and $\chi_{r \geq m+n+2}(S_n \odot S_m)$.

Case 1. For $1 \leq r \leq 2$

Based on Corollary 2.2, for $1 \leq r \leq 2$, the lower bound is $\chi_r(S_n \odot S_m) \geq 3$. To find the upper bound $\chi_r(S_n \odot S_m)$, we define $c_9 : V(S_n \odot S_m) \to \{1, 2, \ldots, k\}$ where $n, m \geq 3$, as follows:

$$\begin{aligned}
c_9(x) &= 1, \\
c_9(x_i) &= 2, \quad 1 \leq i \leq n, \\
c_9(y_i) &= 1, \quad 1 \leq i \leq n, \\
c_9(p) &= 2, \\
c_9(y_{ij}) &= 3, \quad 1 \leq i \leq n, \;\; 1 \leq j \leq m, \\
c_9(p_j) &= 3, \quad 1 \leq i \leq m.
\end{aligned}$$

Clearly, c_9 shows the upper bound for $\chi_{r=1,2}(S_n \odot S_m) \leq 3$. Hence, $\chi_{1 \leq r \leq 2}(S_n \odot S_m) = 3$.

Case 2. For $3 \leq r \leq n + m + 1$

Based on Corollary 2.2, we known $\Delta(S_n) = n$ so $3 \leq r \leq m + n + 1$, the lower bound $\chi_{3 \leq r \leq n+m+1} \geq r + 1$.

To determine the upper bound, we construct the pattern as follows.

$$\begin{aligned}
&n = 3, \quad m = 3, \quad r = 3, \quad \text{so } \chi_{\mathbf{3}}(S_3 \odot S_3) = \mathbf{4}, \\
&n = 3, \quad m = 3, \quad r = 4, \quad \text{so } \chi_{\mathbf{4}}(S_3 \odot S_3) = \mathbf{5}, \\
&n = 3, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(S_3 \odot S_4) = \mathbf{6}, \\
&n = 3, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(S_3 \odot S_4) = \mathbf{7}, \\
&n = 4, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(S_4 \odot S_4) = \mathbf{6}, \\
&n = 4, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(S_4 \odot S_4) = \mathbf{7}, \\
&n = 4, \quad m = 4, \quad r = 7, \quad \text{so } \chi_{\mathbf{7}}(S_4 \odot S_4) = \mathbf{8}.
\end{aligned}$$

It is easy to the $\chi_{3 \leq r \leq n+m+1} \leq r + 1$. Hence, $\chi_{3 \leq r \leq n+m+1} = r + 1$.

Case 3. For $r \geq m + n + 2$

Based on Corollary 2.2, the lower bound is $\chi_r(S_n \odot S_m) \geq n + |V(S_m)| + 1 = n + (m + 1) + 1 = n + m + 2$.

To find the upper bound $\chi_r(S_n \odot S_m)$, we define $c_{10} : V(S_n \odot S_m) \to \{1, 2, \dots, k\}$ where $n, m \leq 3$, as follows:

$$c_{10}(x) = 1,$$
$$c_{10}(x_i) = 1 + i, \quad 1 \leq i \leq n,$$
$$c_{10}(y_{ij}) = 1 + n + j, \quad 1 \leq i \leq n, \ 1 \leq j \leq m,$$
$$c_{10}(p_j) = 1 + n + j, \quad 1 \leq j \leq m,$$
$$c_{10}(y_i) = 2 + n + m, \quad 1 \leq i \leq n,$$
$$c_{10}(p) = 2 + n + m.$$

Clearly, c_{10} shows the upper bound for $\chi_{r \geq m+n+2}(S_n \odot S_m) \leq m + n + 2$. Hence, $\chi_{r \geq m+n+2}(S_n \odot S_m) = m + n + 2$. □

3. Conclusion

We conclude the lower bound of the r-dynamic chromatic number of corona product of graph and some the exact value of its. Moreover, we are interested to characterize the lower bound of $\chi_r(G \odot H)$ for any connected graphs G and H.

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r-Dynamic coloring of the corona product of graphs

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Let $G = (V, E)$ be a graph. A proper k-coloring of graph G is r-dynamic coloring if for every v, the neighbors of vertex v receive at least $\min\{r, d(v)\}$ different colors. The minimum k such that graph G has r-dynamic k coloring is called the r-dynamic chromatic number, denoted by $\chi_r(G)$. In this paper, we study the r-dynamic coloring of corona product of graph. The corona product of graph is obtained by taking a number of vertices $|V(G)|$ copy of H, and making the ith of $V(G)$ adjacent to every vertex of the ith copy of $V(H)$. We obtain the lower bound of r-dynamic chromatic number of corona product of graphs and some exact value.

Keywords: r-Dynamic chromatic number; corona product of graph.

Mathematics Subject Classification 2020: 05C78

1. Introduction

Let $G = (V, E)$ be a graph on the vertex set $V(G)$ and the set $E(G)$. The open neighborhood of the vertex v of G is denoted by $N(v)$. The maximum and minimum degrees of graph G, respectively, are denoted by $\Delta(G)$ and $\delta(G)$. An r-dynamic coloring of a graph G is defined to be a map from V to the set of colors such that (1) a proper coloring, and (2) for each vertex v, $|c(N(v))| \geq \min\{r, d(v)\}$. The minimum k for which G has an r-dynamic k coloring is called the r-dynamic chromatic number, $\chi_r(G)$. This concept was introduced by Lai and Montgomery [12]. They found lower bound of the r-dynamic chromatic number, $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$. The r-dynamic chromatic number has been studied by several authors, for instance, in [1–8, 13, 14]. In this paper, we use some observations as follows.

Observation 1.1 (**[12]**). Let $\Delta(G)$ be the maximum degree of graph G. It holds $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$.

Observation 1.2. Let $G \odot H$ be corona product of graph, then we have $\delta(G \odot H) = \delta(H) + 1$ and $\Delta(G \odot H) = \Delta(G) + |V(H)|$, where $\Delta(G \odot H)$ is the maximum degree of $G \odot H$ and $\delta(G \odot H)$ is the minimum degree of $G \odot H$.

Montgomery in [12] found for any tree, $\chi_r(G) = \min\{\Delta, r\} + 1$ and if $r \geq 2$ then $\chi_r(C_n) = 5$ for $n = 5$, $\chi_r(C_n) = 3$ if $n = 3k, k \geq 1$ and $\chi_r(C_n) = 4$ for n otherwise. Kang *et al.* [7] also found the r-dynamic chromatic number of grid graph. Akbari *et al.* [2] found 2-dynamic chromatic number of Cartesian of path graph and Cartesian product of cycle graph. Taherkhani *et al.* in [14], proved upper bound of regular graph.

Lai and Chang in [11] introduced the definition of corona product. Given graph G and H with n and m vertices, the corona product of G with respect to H is the graph $G \odot H$ with vertex set, $V(G \odot H) = V(G) \cup \{n$ distinct copies of $V(H)\}$ and edge set $E(G \odot H) = E(G) \cup \{n$ distinct copies of $E(H)\}$. Kristiana *et al.* in [9] studied the lower bound of the r-dynamic chromatic number of corona product product by wheel graphs and [10] found the corona product by complete graph. In this paper, we will continue to study of graph corona product.

2. Results

In this paper, we will study the lower bound of the r-dynamic chromatic number of corona product of graph and some new results of the exact value of some corona product of graph. We obtain $\chi_r(S_n \odot S_m), \chi_r(S_n \odot P_m), \chi_r(S_n \odot F_m), \chi_r(P_n \odot S_m)$ and $\chi_r(F_n \odot S_m)$ for $m, n \geq 3$.

Lemma 2.1. *Let $G \odot H$ be a corona product of graph, the lower bound of r-dynamic chromatic number is as follows:*

$$\chi_r(G \odot H) \geq \begin{cases} \delta(H) + 2, & 1 \leq r \leq \delta(H) + 1, \\ r + 1, & \delta(H) + 1 \leq r \leq \Delta(G) + |V(H)|, \\ \Delta(G) + |V(H)| + 1, & r \geq \Delta(G) + |V(H)| + 1. \end{cases}$$

Proof. Based on the definition of corona product of graph, the vertex set is $V(G \odot H) = V(S_n) \cup \bigcup_{i=1}^{n} V(H_i)$ and the maximum degree of $G \odot H$ is $\Delta(G \odot H) = \Delta(G) + |V(H)|$. For $1 \le r \le \delta(H) + 1$, based on Observation 1.1 and by choosing $r = \delta(H)+1$, we get $\chi_r(G \odot H) \ge \min\{ r, \Delta(G \odot H)\}+1 = \min\{ \delta(H)+1, \Delta(G) + |V(H)|\} + 1 = (\delta(H) + 1) + 1 = \delta(H) + 2$.

For $\delta(H)+1 \le r \le \Delta(G)+|V(H)|$, respected to Observation 1.1 and by choosing $r_1 = \delta(H)+1$, we get $\chi_r(G \odot H) \ge \min\{ r, \Delta(G \odot H)\}+1 = \min\{ \delta(H)+1, \Delta(G)+|V(H)|\}+1$. Since $\delta(H)$ is the minimum degree of graph H and $|V(H)|$ is the order of graph H, thus $\delta(H) \le |V(H)|$ and $\chi_r(S_n \odot H) \ge \delta(H) + 1 + 1 = r_1 + 1$. If we choose $r_2 = \delta(H) + 2$, we get $\chi_r(G \odot H) \ge \min\{ \delta(H) + 2, \Delta(G) + |V(H)|\} + 1 = \delta(H) + 2 + 1 = r_2 + 1$ and so on. It gives the proof for $\delta(H) + 1 \le r \le n + |V(H)|$, $\chi_r(G \odot H) \ge r + 1$.

For $r \ge \Delta(G) + |V(H)| + 1$, based on Observation 1.1 and by choosing $r = \Delta(G) + |V(H)| + 1$, we get $\chi_r(G \odot H) \ge \min\{ r, \Delta(G \odot H)\} + 1 = \min\{\Delta(G) + |V(H)| + 1, \Delta(G) + |V(H)|\} + 1 = \Delta(G) + |V(H)| + 1$. It concludes the proof. □

Based on Lemma 2.1 we get some corollary of corona product of graph, as follows.

Corollary 2.2. *Let $S_n \odot H$ be a corona product of star graph and $H \ne K_m, C_m, W_m$, for $n \ge 3, m \ge 4$*

$$\chi_r(S_n \odot H) \ge \begin{cases} \delta(H) + 2, & 1 \le r \le \delta(H) + 1, \\ r + 1, & \delta(H) + 1 \le r \le n + |V(H)|, \\ n + |V(H)| + 1, & r \ge n + |V(H)| + 1. \end{cases}$$

Corollary 2.3. *Let $H \odot S_m$ be a corona product of star graph and $H \ne K_n, C_n, W_n$, for $n \ge 4, m \ge 3$*

$$\chi_r(H \odot S_m) \ge \begin{cases} 3, & r = 1, 2, \\ r + 1, & 3 \le r \le m + \Delta(H) + 1, \\ \Delta(H) + m + 2, & r \ge m + \Delta(H) + 2. \end{cases}$$

Corollary 2.4 ([10]). *Let $K_n \odot H$ be a corona product of complete graph and graph $H \ne C_m, W_m$, for $n \ge 4, m \ge 4$*

$$\chi_r(K_n \odot H) \ge \begin{cases} \chi_r(K_n), & 1 \le r \le n - 1, \\ r + 1, & n \le r \le n + |V(H)| - 1, \\ n + |V(H)|, & r \ge n + |V(H)|. \end{cases}$$

Corollary 2.5 ([10]). *Let $H \odot K_m$ be a corona product of $H \ne C_n, W_n$ and complete graph, for $n \ge 4, m \ge 4$*

$$\chi_r(H \odot K_m) \ge \begin{cases} \chi_r(K_m) + 1, & 1 \le r \le m, \\ r + 1, & m + 1 \le r \le m + \Delta(H), \\ \Delta(H) + m + 1, & r \ge m + \Delta(H) + 1. \end{cases}$$

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Corollary 2.6 ([9]). *Let $H \odot W_m$ be a corona product of graph of H and wheel graph, for $m \geq 4$*

$$\chi_{r=1,2,3}(H \odot W_m) \geq \begin{cases} 4 & \text{for } m \text{ even}, \\ 5 & \text{for } m \text{ odd}, \end{cases}$$

$$\chi_{r=4,5}(H \odot W_m) \geq \begin{cases} r+1 & \text{for } m = 3k,\ k \geq 1, \\ 7 & \text{for } m = 5, \\ 6 & \text{for otherwise}, \end{cases}$$

$$\chi_r(H \odot W_m) \geq \begin{cases} r+1 & \text{for } 6 \leq r \leq \Delta(H)+m+1, \\ \Delta(H)+m+2 & \text{for } r \geq \Delta(H)+m+2. \end{cases}$$

Corollary 2.7 ([9]). *Let $W_n \odot H$ be a corona product of wheel graph and $H \neq K_m, C_m$, for $n \geq 4, m \geq 4$*

$$\chi_{r=1,2}(W_n \odot H) \geq \begin{cases} 3 & \text{for } n \text{ even}, \\ 4 & \text{for } n \text{ odd}, \end{cases}$$

$$\chi_r(W_n \odot H) \geq \begin{cases} r+1 & \text{for } 3 \leq r \leq n+|V(H)|+1, \\ n+|V(H)|+2 & \text{for } r \geq n+|V(G)|+2. \end{cases}$$

The following theorem is the exact value of r-dynamic chromatic number of the corona product of graph, namely, $W_n \odot S_m$, $S_n \odot F_m$, $W_n \odot F_m$, $F_n \odot S_m$ and $S_n \odot S_m$.

Theorem 2.8. *Let $W_n \odot S_m$ be a corona product of wheel graph and star graph*

$$\chi_r(W_n \odot S_m) = \begin{cases} 3, & r = 1,2, n \text{ even}, \\ 4, & r = 1,2, n \text{ odd}, \\ r+1, & 3 \leq r \leq n+m+1, \\ m+n+2, & r \geq n+m+2. \end{cases}$$

Proof. $V(W_n \odot S_m) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq m\} \cup \{x_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{p\} \cup \{p_j; 1 \leq j \leq m\}$ and the order of graph is $|V(W_n \odot S_m)| = mn + m + 2n$. The edge set is $E(W_n \odot S_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{x x_i; 1 \leq i \leq n\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{x_i y_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{xp\} \cup \{x p_j; 1 \leq j \leq m\} \cup \{p p_j; 1 \leq j \leq m\}$ and the size of graph is $|E(W_n \odot S_m)| = nm + 3n + 2n + 1$. Thus, $\Delta(W_n \odot S_m) = m+n+1$ and $\delta(W_n \odot S_m) = 2$. We define three cases, namely, for $\chi_{r=1,2}(W_n \odot S_m), \chi_{3 \leq r \leq n+m+1}(W_n \odot S_m)$ and $\chi_{r \geq m+n++2}(W_n \odot S_m)$.

Case 1. For $r = 1, 2$

Based on Corollary 2.7, the lower bound for $r = 1, 2$ is $\chi_r(W_n \odot S_m) \geq \delta(S_m) + 2 = 1 + 2 = 3$. To show the upper bound, we divide two cases, namely, n even and n odd.

(1) n even, we define $c_1 : V(W_n \odot S_m) \to \{1, 2, 3\}$ where $n, m \geq 3$, as follows:

$$c_1(x) = 1,$$
$$c_1(y_i) = 1, \quad 1 \leq i \leq n,$$
$$c_1(p) = 2,$$
$$c_1(p_j) = 3, \quad 1 \leq j \leq m,$$
$$c_1(x_i) = \begin{cases} 2 & \text{for } i \text{ odd}, \quad 1 \leq i \leq n, \\ 3 & \text{for } i \text{ even}, \ 1 \leq i \leq n, \end{cases}$$
$$c_1(y_{ij}) = \begin{cases} 2 & \text{for 1 odd}, \quad 1 \leq i \leq n, \ 1 \leq j \leq m, \\ 3 & \text{for 1 even}, \ 1 \leq i \leq n, \ 1 \leq j \leq m. \end{cases}$$

Clearly, c_1 shows the upper bound for $\chi_{r=1,2}(W_n \odot S_m) \leq 3$. Hence, $\chi_{r=1,2}(W_n \odot S_m) = 3$ for n even.

(2) n odd, we define $c_1 : V(W_n \odot S_m) \to \{1, 2, 3, 4\}$, as follows:

$$c_1(x) = 1,$$
$$c_1(y_i) = 1, \quad 1 \leq i \leq n,$$
$$c_1(p) = 2,$$
$$c_1(p_j) = 3, \quad 1 \leq j \leq m,$$
$$c_1(x_i) = \begin{cases} 2 & \text{for } i \text{ odd}, \quad 1 \leq i \leq n-1, \\ 3 & \text{for } i \text{ even}, \ 1 \leq i \leq n-1, \\ 4 & \text{for } i = n, \end{cases}$$
$$c_1(y_{ij}) = \begin{cases} 2 & \text{for } i \text{ odd}, \quad 1 \leq i \leq n, \ 1 \leq j \leq m, \\ 3 & \text{for } i \text{ even}, \ 1 \leq i \leq n, \ 1 \leq j \leq m. \end{cases}$$

Clearly, c_1 shows the upper bound for $\chi_{r=1,2}(W_n \odot S_m) \leq 4$. Hence, $\chi_{r=1,2}(W_n \odot S_m) = 4$ for n odd.

Case 2. For $3 \leq r \leq n + m + 1$

Based on Corollary 2.7, we know $\delta(S_m) = 1$ and $|V(S_m)| = m+1$ so $\chi_{3 \leq r \leq n+m+1} \geq r + 1$. To determine the upper bound, we construct the pattern as follows:

$$n = 3, \quad m = 3, \quad r = 3, \quad \chi_{\mathbf{3}}(W_3 \odot S_3) = \mathbf{4},$$
$$n = 3, \quad m = 3, \quad r = 3, \quad \chi_{\mathbf{3}}(W_3 \odot S_3) = \mathbf{4},$$

$$n = 3,\quad m = 3,\quad r = 4,\quad \chi_{\mathbf{4}}(W_3 \odot S_3) = \mathbf{5},$$
$$n = 3,\quad m = 4,\quad r = 4,\quad \chi_{\mathbf{4}}(W_3 \odot S_4) = \mathbf{5},$$
$$n = 3,\quad m = 4,\quad r = 5,\quad \chi_{\mathbf{5}}(W_3 \odot S_4) = \mathbf{6},$$
$$n = 4,\quad m = 4,\quad r = 5,\quad \chi_{\mathbf{5}}(W_4 \odot S_4) = \mathbf{6},$$
$$n = 4,\quad m = 4,\quad r = 6,\quad \chi_{\mathbf{6}}(W_4 \odot S_4) = \mathbf{7},$$
$$n = 4,\quad m = 4,\quad r = 7,\quad \chi_{\mathbf{7}}(W_4 \odot S_4) = \mathbf{8}.$$

Clearly, it shows that $\chi_{3 \le r \le n+m+1} \le r + 1$. Hence, $\chi_{3 \le r \le n+m+1} = r + 1$.

Case 3. For $r \ge n + m + 1$

Based on Corollary 2.7, the lower bound for $r \ge n + m + 2$ is $\chi_r(W_n \odot S_m) \ge n + |V(S_m)| + 1 = n + m + 2$. To find the upper bound, we define $c_2 : V(W_n \odot S_m) \to \{1, 2, \cdots, k\}$ where $n, m \ge 3$, as follows:

$$c_2(x) = 1,$$
$$c_2(x_i) = i + 1,\quad 1 \le i \le n,$$
$$c_2(y_i) = n + 2 + m,$$
$$c_2(y_{ij}) = n + 1 + j,\quad 1 \le i \le n,\ 1 \le j \le m,$$
$$c_2(p) = n + 2 + m,$$
$$c_2(p_j) = n + 1 + j.$$

Clearly, c_2 shows the upper bound for $\chi_{r \ge n+m+2}(W_n \odot S_m) \le n + 2 + m$. Finally, $\chi_{r \ge m+n+2}(W_m \odot S_n) = m + n + 2$. □

Theorem 2.9. *Let $S_n \odot F_m$ be a corona product of star graph and fan graph,*

$$\chi_r(S_n \odot F_m) = \begin{cases} 4, & r = 1, 2, 3, \\ r + 1, & 4 \le r \le (m + n + 1), \\ m + n + 2, & r \ge (m + n + 2). \end{cases}$$

Proof. The graph $S_n \odot F_m$ is a connected graph with vertex set, $V(S_n \odot F_m) = \{x\} \cup \{x_i; 1 \le i \le n\} \cup \{y_i; 1 \le i \le n+1\} \cup \{x_{ij}; 1 \le i \le n+1, 1 \le j \le m\}$ and the order of graph is $|V(S_n \odot F_m)| = mn + 2n + m + 2$. The edge set is $E(S_n \odot F_m) = \{xx_i; 1 \le i \le n\} \cup \{xx_{(n+1)j}; 1 \le j \le m\} \cup \{xy_{n+1}\} \cup \{y_{n+1}x_{(n+1)j}; 1 \le j \le m\} \cup \{x_{(n+1)j}x_{(n+1)(j+1)}; 1 \le j \le m-1\} \cup \{x_iy_i; 1 \le i \le n\} \cup \{x_{ij}x_i; 1 \le i \le n; 1 \le j \le m\} \cup \{x_{ij}x_{i(j+1)}; 1 \le i \le n, 1 \le j \le m-1\} \cup \{y_ix_{ij}; 1 \le i \le n, 1 \le j \le m\}$ and the size of graph is $|E(S_n \odot F_m)| = 3mn + n + 3m$. Thus, the minimum degree of graph G is $\delta(S_n \odot F_m) = 3$ and the maximum degree of graph G is $\Delta(S_n \odot F_m) = m+n+1$.

We define three cases, namely, for $\chi_{1 \le r \le 3}(S_n \odot F_m)$, $\chi_{4 \le r \le n+m+1}(S_n \odot F_m)$ and $\chi_{r \ge m+n+2}(S_n \odot F_m)$.

Case 1. For $1 \leq r \leq 3$

Based on Corollary 2.2, for $1 \leq r \leq 3$, the lower bound is $\chi_r(S_n \odot F_m) \geq \delta(F_m)+2 = 2+2$. To find the upper bound for $\chi_r(S_n \odot F_m)$, we define $c_3 : V(S_n \odot F_m) \to \{1,2,3,4\}$ where $n, m \geq 3$, as follows:

$$c_3(x) = 1,$$

$$c_3(x_i) = 2, \quad 1 \leq i \leq n,$$

$$c_3(y_i) = \begin{cases} 1, & 1 \leq i \leq n, \\ 2, & i = n+1, \end{cases}$$

$$c_3(x_{ij}) = \begin{cases} 3 & \text{for } 1 \leq i \leq n+1,\ j \text{ odd},\ 1 \leq j \leq m, \\ 4 & \text{for } 1 \leq i \leq n+1,\ j \text{ even},\ 1 \leq j \leq m. \end{cases}$$

Clearly, c_3 shows the upper bound for $\chi_{1\leq r\leq 3}(S_n \odot F_m) \leq 4$. Hence, $\chi_{1\leq r\leq 3}(S_n \odot F_m) = 4$.

Case 2. For $4 \leq r \leq n+m+1$

The lower bound in Corollary 2.2, for $4 \leq r \leq n+m+1$, we know $\delta(F_m) = 2$ and $|V(F_m)| = m+1$ so $2+2 \leq r \leq n+|V(F_m)|$, the r-dynamic chromatic number, $\chi_{4\leq r\leq n+m+1} \geq r+1$.

To determine the upper bound, we construct the pattern as follows:

$$n = 3, \quad m = 3, \quad r = 4, \quad \text{so } \chi_4(S_3 \odot F_3) = 5,$$
$$n = 3, \quad m = 3, \quad r = 5, \quad \text{so } \chi_5(S_3 \odot F_3) = 6,$$
$$n = 3, \quad m = 4, \quad r = 4, \quad \text{so } \chi_4(S_3 \odot F_3) = 5,$$
$$n = 3, \quad m = 4, \quad r = 5, \quad \text{so } \chi_5(S_3 \odot F_3) = 6,$$
$$n = 4, \quad m = 4, \quad r = 6, \quad \text{so } \chi_6(S_4 \odot F_4) = 7,$$
$$n = 4, \quad m = 4, \quad r = 7, \quad \text{so } \chi_7(S_4 \odot F_4) = 8,$$
$$n = 4, \quad m = 4, \quad r = 8, \quad \text{so } \chi_8(S_4 \odot F_4) = 9.$$

It is easy that $\chi_{4\leq r\leq n+m+1} \leq r+1$. Hence, $\chi_{4\leq r\leq n+m+1} = r+1$.

Case 3. For $r \geq m+n+2$

Based on Corollary 2.2, $\chi_r(S_n \odot F_m) \geq n+|V(F_m)|+1 = n+(m+1)+1 = n+m+2$.

The upper bound for $\chi_r(S_n \odot F_m)$, we define $c_4 : V(S_n \odot F_m) \to \{1,2,\ldots,k\}$ where $n, m \geq 3$, as follows:

$$c_4(x) = 1,$$

$$c_4(y_i) = 2, \quad 1 \leq i \leq n,$$

$$c_4(x_{ij}) = j+2, \quad 1 \leq i \leq n, \ 1 \leq j \leq m,$$

$$c_4(x_i) = m+i+2, \quad 1 \leq i \leq n.$$

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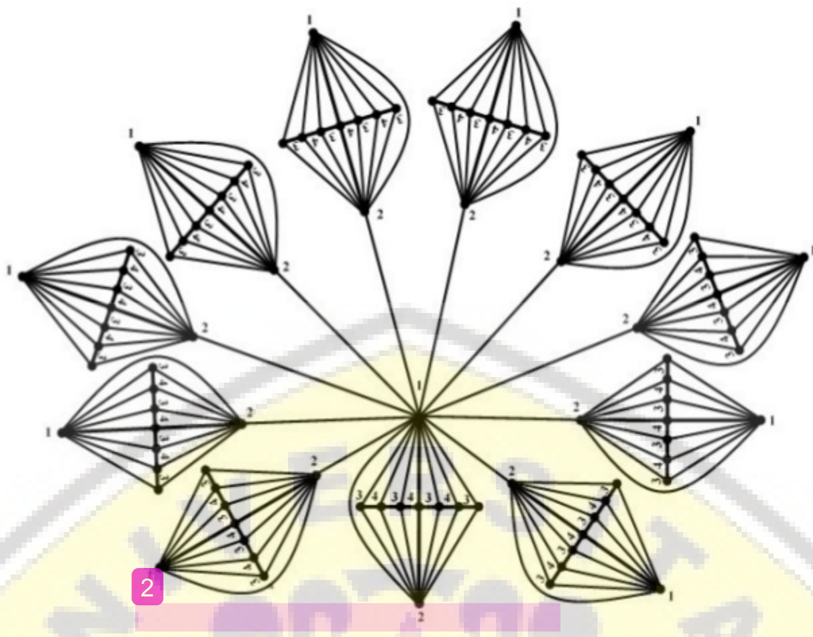

Fig. 1. Example of r-dynamic coloring of $S_{10} \odot F_8$.

Clearly, c_4 shows the upper bound for $\chi_{r \geq n+m+2}(S_n \odot P_m) \leq m + n + 2$. Hence, $\chi_{r \geq m+n+2}(S_n \odot F_m) = m+n+2$. The Fig. 1 is the illustration of r-dynamic coloring of the corona product of star and fan graph. □

Theorem 2.10. *Let $W_n \odot F_m$ be a corona product of wheel graph and fan graph,*

$$\chi_r(W_n \odot F_m) = \begin{cases} 3, & r = 1,2,3,\ n \text{ even}, \\ 4, & r = 1,2,3,\ n \text{ odd}, \\ r+1, & 4 \leq r \leq m+n+1, \\ m+n+2, & r \geq m+n+2. \end{cases}$$

Proof. $V(W_n \odot F_m) = \{x, x_i, p, p_j, y_i, y_{ij};\ 1 \leq i \leq n; 1 \leq j \leq m\}$ and the order of graph, $|V(W_n \odot F_m)| = nm + 2n + m + 2$. The edge set is $E(W_n \odot F_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{x x_i; 1 \leq i \leq n\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{y_i y_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_{ij} y_{i(j+1)}; 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{xp\} \cup \{p_j p_{j+1}; 1 \leq j \leq m\} \cup \{pp_j; 1 \leq j \leq m\}$ and the size of graph is $|E(W_n \odot S_m)| = 2nm + n + 2m$. Thus, $\Delta(W_n \odot F_m) = m+n+1$ and $\delta(W_n \odot F_m) = 3$. We define three cases, namely, for $\chi_{r=1,2,3}(W_n \odot F_m)$, $\chi_{4 \leq r \leq n+m+1}(W_n \odot F_m)$ and $\chi_{r \geq m+n++2}(W_n \odot F_m)$.

Case 1. For $r = 1,2,3$

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Based on Corollary 2.7, the lower bound for $r = 1, 2$ is $\chi_r(W_n \odot S_m) \geq \delta(S_m) + 2 = 1 + 2 = 3$. To show the upper bound, we define $c_5 : V(W_n \odot F_m) \to \{1, 2, 3, 4\}$ where $n, m \geq 3$, as follows:

$$c_5(x) = 1,$$
$$c_5(y_i) = 1, \quad 1 \leq i \leq n$$
$$c_5(p) = 2,$$
$$c_5(x_i) = c_5(p_j) = \begin{cases} 2 & \text{for } i \equiv 1 (\text{mod } 3) \ \ 1 \leq i \leq n, \\ 3 & \text{for } i \equiv 2 (\text{mod } 3) \ \ 1 \leq i \leq n, \\ 4 & \text{for } i \equiv 0 (\text{mod } 3) \ \ 1 \leq i \leq n, \end{cases}$$
$$c_5(y_{ij}) = \begin{cases} 2 & \text{for } i \equiv 2 (\text{mod } 3) \ j \text{ odd}, \ \ 1 \leq i \leq n, \ 1 \leq j \leq m \\ & \text{or } i \equiv 0 (\text{mod } 3) \ j \text{ odd}, \ \ 1 \leq i \leq n, \ 1 \leq j \leq m, \\ 3 & \text{for } i \equiv 1 (\text{mod } 3) \ j \text{ odd}, \ \ 1 \leq i \leq n, \ 1 \leq j \leq m \\ & \text{or } i \equiv 0 (\text{mod } 3) \ j \text{ even}, \ \ 1 \leq i \leq n, \ 1 \leq j \leq m, \\ 4 & \text{for } i \equiv 1 (\text{mod } 3) \ j \text{ even}, \ \ 1 \leq i \leq n, \ 1 \leq j \leq m \\ & \text{or } i \equiv 2 (\text{mod } 3) \ j \text{ even}, \ \ 1 \leq i \leq n, \ 1 \leq j \leq m. \end{cases}$$

Clearly, c_5 shows the upper bound for $\chi_{r=1,2,3}(W_n \odot F_m) \leq 3$. Hence, $\chi_{r=1,2,3}(W_n \odot F_m) = 4$ for n odd.

Case 2. For $4 \leq r \leq n + m + 1$

Based on Corollary 2.7, we know $\delta(F_m) = 2$ and $|V(F_m)| = m+1$ so $\chi_{4 \leq r \leq n+m+1} \geq r + 1$. To determine the upper bound, we construct the pattern, as follows:

$$n = 3, \quad m = 3, \quad r = 4, \quad \chi_4(W_3 \odot F_3) = \mathbf{5},$$
$$n = 3, \quad m = 3, \quad r = 5, \quad \chi_5(W_3 \odot F_3) = \mathbf{6},$$
$$n = 3, \quad m = 3, \quad r = 4, \quad \chi_4(W_3 \odot F_5) = \mathbf{5},$$
$$n = 3, \quad m = 5, \quad r = 5, \quad \chi_5(W_3 \odot F_5) = \mathbf{6},$$
$$n = 4, \quad m = 5, \quad r = 5, \quad \chi_5(W_4 \odot F_5) = \mathbf{5},$$
$$n = 4, \quad m = 5, \quad r = 6, \quad \chi_6(W_4 \odot F_5) = \mathbf{7},$$
$$n = 4, \quad m = 6, \quad r = 7, \quad \chi_7(W_4 \odot F_6) = \mathbf{8},$$
$$n = 4, \quad m = 6, \quad r = 8, \quad \chi_8(W_4 \odot F_6) = \mathbf{9}.$$

Clearly, it shows that $\chi_{4 \leq r \leq n+m+1} \leq r + 1$. Hence, $\chi_{4 \leq r \leq n+m+1} = r + 1$.

Case 3. For $r \geq n + m + 1$

Based on Corollary 2.7, the lower bound for $r \geq n + m + 2$ is $\chi_r(W_n \odot F_m) \geq n + |V(F_m)| + 1 = n + m + 2$. To find the upper bound, we define $c_6 : V(W_n \odot F_m) \to \{1, 2, \ldots, k\}$ where $n, m \geq 3$, as follows:

$$c_6(x) = 1,$$
$$c_6(x_i) = i + 1, \quad 1 \leq i \leq n,$$
$$c_6(y_i) = n + 2 + m, \quad 1 \leq i \leq n,$$

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$$c_6(y_{ij}) = n+1+j, \quad 1 \le i \le n, \ 1 \le j \le m,$$
$$c_6(p) = n+2+m,$$
$$c_6(p_j) = n+1+j.$$

Clearly, c_5 shows the upper bound for $\chi_{r\ge n+m+2}(W_n \odot F_m) \le n+2+m$. Finally, $\chi_{r\ge m+n+2}(W_m \odot F_n) = m+n+2$. □

Theorem 2.11. *Let $F_n \odot S_m$ be a corona product of fan graph and star graph*

$$\chi_r(F_n \odot S_m) = \begin{cases} 3, & r = 1,2, \\ r+1, & 3 \le r \le n+m+1, \\ n+m+2, & \text{otherwise.} \end{cases}$$

Proof. The graph $F_n \odot S_m$ is a connected graph with vertex set $V(F_n \odot S_m) = \{x, z\} \cup \{x_i, y_i, y_{ij}, z_j; 1 \le i \le n; 1 \le j \le m\}$, edge set $E(F_n \odot S_m) = \{xx_i; 1 \le i \le n\} \cup \{x_iy_i; 1 \le i \le n\} \cup \{x_iy_{ij}; 1 \le i \le n; 1 \le j \le m\} \cup \{y_iy_{ij}; 1 \le i \le n; 1 \le j \le m\} \cup \{xz\} \cup \{zz_j; 1 \le j \le m\}$. The order and size of the graph $G = F_n \odot S_m$ are $|V(G_n \odot S_m)| = (n+1)(m+2)$, and $|E(F_n \odot S_m)| = 2mn + 2n + m + 1$, thus $\Delta(F_n \odot S_m) = n+m+1$ and $\delta(F_n \odot S_m) = 2$. We define three cases, namely, for $\chi_{r=1,2}(F_n \odot S_m)$, $\chi_{3\le r\le n+m+1}(F_n \odot S_m)$, and $\chi_{r\ge m+n+2}(F_n \odot S_m)$.

Case 1. For $r = 1, 2$

Based on Corollary 2.3, for $r = 1, 2$, the lower bound is $\chi_r(F_n \odot S_m) \ge 3$. To find the upper bound $\chi_r(F_n \odot S_m)$, we define a map $c_7 : V(F_n \odot S_m) \to \{1, 2, \ldots, k\}$ where $n, m \ge 3$, as follows:

$$c_7(x) = 1,$$
$$c_7(x_i) = \begin{cases} 2, & 1 \le i \le n, \ i = \text{odd}, \\ 3, & 1 \le i \le n, \ i = \text{even}, \end{cases}$$
$$c_7(y_i) = 1,$$
$$c_7(y_{ij}) = \begin{cases} 2, & 1 \le i \le n, \ i = \text{even}, \\ 3, & 1 \le i \le n, \ i = \text{odd}, \end{cases}$$
$$c_7(z) = 2,$$
$$c_7(z_j) = 3.$$

Clearly, c_7 shows the upper bound for $\chi_{r=1,2}(F_n \odot S_m) \le 3$. Hence, $\chi_{r=1,2}(F_n \odot S_m) = 3$.

Case 2. For $3 \leq r \leq n+m+1$

Based on Corollary 2.3, we know $\Delta(F_n) = n$ so $3 \leq r \leq m+n+1$, the lower bound is $\chi_{3 \leq r \leq n+m+1} \geq r+1$.
The upper bound r-dynamic chromatic numbers are by explicit construction.

$$\begin{aligned}
n &= 3, \quad m = 3, \quad r = 3, \quad \text{so } \chi_{\mathbf{3}}(F_3 \odot S_3) = \mathbf{4},\\
n &= 3, \quad m = 3, \quad r = 4, \quad \text{so } \chi_{\mathbf{4}}(F_3 \odot S_3) = \mathbf{5},\\
n &= 3, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(F_3 \odot S_4) = \mathbf{6},\\
n &= 3, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(F_3 \odot S_4) = \mathbf{7},\\
n &= 4, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(F_4 \odot S_4) = \mathbf{6},\\
n &= 4, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(F_4 \odot S_4) = \mathbf{7},\\
n &= 4, \quad m = 4, \quad r = 7, \quad \text{so } \chi_{\mathbf{7}}(F_4 \odot S_4) = \mathbf{8}.
\end{aligned}$$

It is easy the $\chi_{3 \leq r \leq n+m+1} \leq r+1$. Hence, $\chi_{3 \leq r \leq n+m+1} = r+1$.

Case 3. For $r \geq m+n+2$

Based on Corollary 2.3, the lower bound is $\chi_r(F_n \odot S_m) \geq \Delta(F_n) + m + 2 = n+m+2$. To find the upper bound $\chi_r(F_n \odot S_m)$, we define a map $c_8 : V(F_n \odot S_m) \rightarrow \{1, 2, \ldots, k\}$ where $n, m \geq 3$, as follows:

$$\begin{aligned}
c_8(x) &= 1,\\
c_8(x_i) &= i+1, \quad 1 \leq i \leq n,\\
c_8(y_{ij}) &= n+j+1, \quad 1 \leq i \leq n, \ 1 \leq j \leq m,\\
c_8(y_i) &= n+m+2, \quad 1 \leq i \leq n,\\
c_8(z_j) &= n+j+1, \quad 1 \leq j \leq m,\\
c_8(z) &= n+m+2.
\end{aligned}$$

Clearly, c_8 shows the upper bound for $\chi_{r \geq m+n+2}(F_n \odot S_m) \leq m+n+2$. Hence, $\chi_{r \geq n+m+2}(F_n \odot S_m) = n+m+2$. The proof is complete. □

Theorem 2.12. *Let $S_n \odot S_m$ be a corona product of star graph and star graph,*

$$\chi_r(S_n \odot S_m) = \begin{cases} 3, & r = 1, 2,\\ r+1, & 3 \leq r \leq m+n+1,\\ m+n+2, & r \geq m+n+2. \end{cases}$$

Proof. The graph $S_n \odot S_m$ is a connected graph with vertex set, $V(S_n \odot S_m) = \{x, p\} \cup \{x_i, y_i, p_j; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\}$ and the order of graph $|V(S_n \odot S_m)| = nm + 2n + m + 2$. The edge set is $E(S_n \odot S_m) = \{xx_i; 1 \leq i \leq n\} \cup \{xp\} \cup \{xp_j; 1 \leq j \leq m\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{x_i y_{ij} 1 \leq$

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$i \leq n, 1 \leq j \leq m\} \cup \{y_i y_{ij}; 1 \leq i \leq n, 1 \leq j\ leqm\}$ and the size of graph is $|E(S_n \odot S_m)| = 2nm + 2n + m + 1$. Thus, the minimum degree of graph G is $\delta(S_n \odot S_m) = 2$ and the maximum degree of graph G is $\Delta(S_n \odot S_m) = m + n + 1$.

We define three cases, namely, for $\chi_{r=1,2}(S_n \odot S_m), \chi_{3 \leq r \leq m+n+1}(S_n \odot S_m)$, and $\chi_{r \geq m+n+2}(S_n \odot S_m)$.

Case 1. For $1 \leq r \leq 2$

Based on Corollary 2.2, for $1 \leq r \leq 2$, the lower bound is $\chi_r(S_n \odot S_m) \geq 3$. To find the upper bound $\chi_r(S_n \odot S_m)$, we define $c_9 : V(S_n \odot S_m) \to \{1, 2, \ldots, k\}$ where $n, m \geq 3$, as follows:

$$
\begin{aligned}
c_9(x) &= 1, \\
c_9(x_i) &= 2, \quad 1 \leq i \leq n, \\
c_9(y_i) &= 1, \quad 1 \leq i \leq n, \\
c_9(p) &= 2, \\
c_9(y_{ij}) &= 3, \quad 1 \leq i \leq n, \ 1 \leq j \leq m, \\
c_9(p_j) &= 3, \quad 1 \leq i \leq m.
\end{aligned}
$$

Clearly, c_9 shows the upper bound for $\chi_{r=1,2}(S_n \odot S_m) \leq 3$. Hence, $\chi_{1 \leq r \leq 2}(S_n \odot S_m) = 3$.

Case 2. For $3 \leq r \leq n + m + 1$

Based on Corollary 2.2 we known $\Delta(S_n) = n$ so $3 \leq r \leq m + n + 1$, the lower bound $\chi_{3 \leq r \leq n+m+1} \geq r + 1$.

To determine the upper bound, we construct the pattern as follows.

$$
\begin{aligned}
&n = 3, \quad m = 3, \quad r = 3, \quad \text{so } \chi_{\mathbf{3}}(S_3 \odot S_3) = \mathbf{4}, \\
&n = 3, \quad m = 3, \quad r = 4, \quad \text{so } \chi_{\mathbf{4}}(S_3 \odot S_3) = \mathbf{5}, \\
&n = 3, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(S_3 \odot S_4) = \mathbf{6}, \\
&n = 3, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(S_3 \odot S_4) = \mathbf{7}, \\
&n = 4, \quad m = 4, \quad r = 5, \quad \text{so } \chi_{\mathbf{5}}(S_4 \odot S_4) = \mathbf{6}, \\
&n = 4, \quad m = 4, \quad r = 6, \quad \text{so } \chi_{\mathbf{6}}(S_4 \odot S_4) = \mathbf{7}, \\
&n = 4, \quad m = 4, \quad r = 7, \quad \text{so } \chi_{\mathbf{7}}(S_4 \odot S_4) = \mathbf{8}.
\end{aligned}
$$

It is easy to the $\chi_{3 \leq r \leq n+m+1} \leq r + 1$. Hence, $\chi_{3 \leq r \leq n+m+1} = r + 1$.

Case 3. For $r \geq m + n + 2$

Based on Corollary 2.2 the lower bound is $\chi_r(S_n \odot S_m) \geq n + |V(S_m)| + 1 = n + (m + 1) + 1 = n + m + 2$.

To find the upper bound $\chi_r(S_n \odot S_m)$, we define $c_{10} : V(S_n \odot S_m) \to \{1, 2, \ldots, k\}$ where $n, m \leq 3$, as follows:

$$c_{10}(x) = 1,$$

$$c_{10}(x_i) = 1 + i, \quad 1 \leq i \leq n,$$

$$c_{10}(y_{ij}) = 1 + n + j, \quad 1 \leq i \leq n, \; 1 \leq j \leq m,$$

$$c_{10}(p_j) = 1 + n + j, \quad 1 \leq j \leq m,$$

$$c_{10}(y_i) = 2 + n + m, \quad 1 \leq i \leq n,$$

$$c_{10}(p) = 2 + n + m.$$

Clearly, c_{10} shows the upper bound for $\chi_{r \geq m+n+2}(S_n \odot S_m) \leq m + n + 2$. Hence, $\chi_{r \geq m+n+2}(S_n \odot S_m) = m + n + 2$. □

3. Conclusion

We conclude the lower bound of the r-dynamic chromatic number of corona product of graph and some the exact value of its. Moreover, we are interested to characterize the lower bound of $\chi_r(G \odot H)$ for any connected graphs G and H.

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