

# On the Local Irregularity Vertex Coloring of Related Grid Graph

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**Abstract:** All graph in this paper is connected and simple graph. Let graph  $d(u,v)$  be a distance between any vertex  $u$  and  $v$  in graph  $G = (V, E)$ . A function  $l: V(G) \rightarrow \{1, 2, \dots, k\}$  is called vertex irregular  $k$ -labelling and  $w: V(G) \rightarrow N$  where  $w(u) = \sum_{v \in N(u)} l(v)$ . If for every  $uv \in E(G), w(u) \neq w(v)$  and  $\max(l) = \min\{\max\{l_i\}; l_i, \text{ vertex irregular labelling}\}$  is called a local irregularity vertex coloring.  $\chi_{lis}(G)$  or chromatic number of local irregularity vertex coloring of graph  $(G)$  is the minimum cardinality of the largest label over all such local irregularity vertex coloring. In this paper, we will study about local irregularity vertex coloring of related grid graphs, and we have found the exact value of their chromatic number local irregularity, namely ladder graph, triangular ladder graph, and  $H$ -graph.

**Keywords :** local irregularity; vertex coloring; grid graph.

## 1. INTRODUCTION

Let  $G(V, E)$  be a connected and simple graphs with vertex set  $V$  and edge set  $E$ . In this paper, we combine the two concepts, namely combining the local antimagic vertex coloring and the distance irregular labelling. So, we study a new notion of coloring type of graph, namely a local irregularity vertex coloring. This concept firstly was introduced by Kristiana, et.al [3]. By the local irregularity vertex coloring, we recall a bijection  $l: V(G) \rightarrow \{1, 2, \dots, k\}$  is called vertex irregular  $k$ -labelling and  $w: V(G) \rightarrow N$  where  $w(u) = \sum_{v \in N(u)} l(v)$ ,  $l$  is called local irregularity vertex coloring. A condition for  $w$  to be a local irregularity vertex coloring, if  $\max(l) = \min\{\max\{l_i\}; l_i, \text{ vertex irregular labelling}\}$  and for every  $uv \in E(G), w(u) \neq w(v)$ . The chromatic number of local irregular denoted by  $\chi_{lis}(G)$ , is minimum of cardinality local irregularity vertex coloring.

For illustration local irregularity vertex coloring and chromatic number local irregularity is provided in Figure 1.

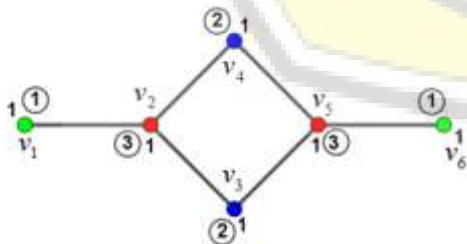


Fig. 1. Chromatic number local irregular,  $\chi_{lis}(G) = 4$

**Lemma.** Let  $G$  simple and connected graph,  $\chi_{lis}(G) \geq \chi(G)$ .

The last, Kristiana, et.al [3] have discovered the exact value of the chromatic number local irregular of some graph, namely path, cycle, complete, star, friendship, and related wheel graphs.

## 2. RESULT

**Theorem 2.1** Let  $G$  be a ladder graph  $(L_n)$ . For  $n \geq 2$ , the chromatic number local irregular is  $\chi_{lis}(L_n) = 4$ .

**Proof.** Let  $L_n$  for  $n \geq 2$  be a ladder. Then  $L_n$  has the vertex set  $V(L_n) = \{x_i, y_i: 1 \leq i \leq n\}$ , and the edge set  $E(L_n) = \{x_i x_{i+1}, y_i y_{i+1}: 1 \leq i \leq n-1\} \cup \{x_i y_i: 1 \leq i \leq n\}$ . If every vertex is labelled by 1, so we have  $w(x_i) = l(x_{i-1}) + l(x_{i+1}) + l(y_i) = 1 + 1 + 1 = 3$  and  $w(x_{i+1}) = l(x_i) + l(x_{i+2}) + l(y_{i+1}) = 1 + 1 + 1 = 3$ . It means a contradiction with definition. Since,  $x_i x_{i+1} \in E(L_n), w(x_i) \neq w(x_{i+1})$ . Hence  $\max(l) = 2$ .

Based on Lemma, the lower bound for the chromatic number  $\chi_{lis}(L_n) \geq \chi(L_n) = 2$ , so that  $\chi_{lis}(L_n) \geq 2$ .

Furthermore, the upper bound for the chromatic number local irregular, we define  $l: V(L_n) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows :

$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ odd} \\ 2, & \text{for } i \text{ even} \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ even} \\ 2, & \text{for } i \text{ odd} \end{cases}$$

Hence,  $\max(l) = 2$  and the labelling provides vertex-weight as follows:

for  $n$  odd

$$w(x_i) = \begin{cases} 3, & \text{for } i = \text{even}, 2 \leq i \leq n-1 \\ 6, & \text{for } i = \text{odd}, 2 \leq i \leq n-1 \\ 4, & \text{for } i = 1 \text{ and } i = n \end{cases}$$

$$w(y_i) = \begin{cases} 3, & \text{for } i = \text{odd}, 2 \leq i \leq n-1 \\ 6, & \text{for } i = \text{even}, 2 \leq i \leq n-1 \\ 2, & \text{for } i = 1 \text{ and } i = n \end{cases}$$

for  $n$  even

$$w(x_i) = \begin{cases} 4, & \text{for } i = 1 \\ 3, & \text{for } i = \text{even}, 2 \leq i \leq n-1 \\ 6, & \text{for } i = \text{odd}, 2 \leq i \leq n-1 \\ 2, & \text{for } i = n \end{cases}$$

$$w(y_i) = \begin{cases} 2, & \text{for } i = 1 \\ 3, & \text{for } i = \text{even}, 2 \leq i \leq n-1 \\ 6, & \text{for } i = \text{odd}, 2 \leq i \leq n-1 \\ 4, & \text{for } i = n \end{cases}$$

For every  $uv \in E(L_n)$ , take any  $u = x_i$  and  $v = x_{i+1}$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(x_i) \neq w(x_{i+1})$ , then  $u = y_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(y_i) \neq w(y_{i+1})$  and  $u = x_i$  and  $v = y_i$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(x_i) \neq w(y_i)$ . Based on that, we have the set of vertices weight of the ladder graph  $L_n$  is  $W(L_n) = \{2, 3, 4, 6\}$ , so  $|w(V(L_n))| = 4$ . Hence,  $\chi_{lis}(L_n) = 4$ .

**Theorem 2.2** Let  $G$  be a triangular ladder graph  $(TL_n)$ . For  $n \geq 5$ , the chromatic number local irregular is

$$\chi_{lis}(TL_n) = \begin{cases} 5, & \text{for } n = 5 \\ 6, & \text{for } n \geq 6 \end{cases}$$

**Proof.** Let  $TL_n$  for  $n \geq 2$  be a triangular ladder. Then  $TL_n$  has the vertex set  $V(TL_n) = \{x_i, y_i : 1 \leq i \leq n\}$ , and the edge set  $E(TL_n) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\} \cup \{x_i y_{i+1} : 1 \leq i \leq n\}$ . If every  $x \in V(TL_n)$  is labelled by 1 and  $y \in V(TL_n)$  is labelled by 2, for  $1 \leq i \leq n$ , so we have  $w(x_i) = w(x_{i+1})$ . It means a contradiction with definition. Since,  $x_i x_{i+1} \in E(TL_n)$ ,  $w(x_i) \neq w(x_{i+1})$ . Hence  $\max(l) = 3$ . Based on Lemma, the lower bound for the chromatic number  $\chi_{lis}(TL_n) \geq \chi(TL_n) = 3$ , so that  $\chi_{lis}(TL_n) \geq 3$ .

Case 1: for  $n = 5$

Furthermore, the upper bound for the chromatic number local irregular, we define  $l: V(TL_n) \rightarrow \{1, 2, 3\}$  with vertex irregular 3-labelling as follows :

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{3} \\ 2, & \text{for } i \equiv 2 \pmod{3} \\ 3, & \text{for } i \equiv 0 \pmod{3} \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i \equiv 2 \pmod{3} \\ 2, & \text{for } i \equiv 0 \pmod{3} \\ 3, & \text{for } i \equiv 1 \pmod{3} \end{cases}$$

Hence,  $\max(l) = 3$  and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 2, & \text{for } i = n \\ 6, & \text{for } i = 1 \\ 7, & \text{for } i = 2 \\ 8, & \text{for } i = 3 \\ 9, & \text{for } i = 4 \end{cases}$$

$$w(y_i) = \begin{cases} 2, & \text{for } i = 1 \\ 6, & \text{for } i = n \\ 7, & \text{for } i = 4 \\ 8, & \text{for } i = 2 \\ 9, & \text{for } i = 3 \end{cases}$$

For every  $uv \in E(TL_n)$ , take any  $u = x_i$  and  $v = x_{i+1}$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(x_i) \neq w(x_{i+1})$ , then  $u = y_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(y_i) \neq w(y_{i+1})$  and  $u = x_i$  and  $v = y_i$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(x_i) \neq w(y_i)$ , and the last, take  $u = x_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(x_i) \neq w(y_{i+1})$ . Based on that, we have the set of vertices weight of the triangular ladder graph  $TL_n$  is  $W(TL_n) = \{2, 6, 7, 8, 9\}$ , so  $|w(V(TL_n))| = 5$ . Hence,  $\chi_{lis}(TL_n) = 5$ .

Case 2: for  $n \geq 6, n \equiv 1 \pmod{3}$

Furthermore, the upper bound for the chromatic number local irregular, we define  $l: V(TL_n) \rightarrow \{1, 2, 3\}$  with vertex irregular 3-labelling as follows :

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{3} \\ 2, & \text{for } i \equiv 2 \pmod{3} \\ 3, & \text{for } i \equiv 0 \pmod{3} \end{cases}$$

$$l(y_i) = 2, \text{ for } 1 \leq i \leq n$$

Hence,  $\max(l) = 3$  and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 5, & \text{for } i = n \\ 6, & \text{for } i = 1 \\ 7, & \text{for } i \equiv 0 \pmod{3}, n \geq 3 \\ 8, & \text{for } i \equiv 2 \pmod{3}, n \geq 2 \\ 9, & \text{for } i \equiv 1 \pmod{3}, n \geq 4 \end{cases}$$

$$w(y_i) = \begin{cases} 3, & \text{for } i = 1 \\ 6, & \text{for } i = n \\ 7, & \text{for } i \equiv 2 \pmod{3}, n \geq 2 \\ 8, & \text{for } i \equiv 1 \pmod{3}, n \geq 4 \\ 9, & \text{for } i \equiv 0 \pmod{3}, n \geq 3 \end{cases}$$

For every  $uv \in E(TL_n)$ , take any  $u = x_i$  and  $v = x_{i+1}$ ,  $i = 1, 2, 3, \dots, n-1$ ,  $w(x_i) \neq w(x_{i+1})$ , then  $u = y_i$  and

$v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(y_i) \neq w(y_{i+1})$  and  $u = x_i$  and  $v = y_i$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(y_i)$ , and the last, take  $u = x_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(y_{i+1})$ . Based on that, we have the set of vertices weight of the triangular ladder graph  $TL_n$  is  $W(TL_n) = \{3, 5, 6, 7, 8, 9\}$ , so  $|w(V(TL_n))| = 6$  Hence,  $\chi_{lis}(TL_n) = 6$ .

Case 3: for  $n \geq 6, n \equiv 0 \pmod{3}$

Furthermore, the upper bound for the chromatic number local irregular, we define  $l: V(TL_n) \rightarrow \{1, 2, 3\}$  with vertex irregular 3-labelling as follows :

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{3} \\ 2, & \text{for } i \equiv 2 \pmod{3} \\ 3, & \text{for } i \equiv 0 \pmod{3} \end{cases}$$

$$l(y_i) = 3, \text{ for } 1 \leq i \leq n$$

Hence,  $\max(l) = 3$  and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 5, & \text{for } i = n \\ 8, & \text{for } i = 1 \\ 9, & \text{for } i \equiv 0 \pmod{3}, n \geq 3 \\ 10, & \text{for } i \equiv 2 \pmod{3}, n \geq 2 \\ 11, & \text{for } i \equiv 1 \pmod{3}, n \geq 4 \end{cases}$$

$$w(y_i) = \begin{cases} 4, & \text{for } i = 1 \\ 8, & \text{for } i = n \\ 9, & \text{for } i \equiv 2 \pmod{3}, n \geq 2 \\ 10, & \text{for } i \equiv 1 \pmod{3}, n \geq 4 \\ 11, & \text{for } i \equiv 0 \pmod{3}, n \geq 3 \end{cases}$$

For every  $uv \in E(TL_n)$ , take any  $u = x_i$  and  $v = x_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(x_{i+1})$ , then  $u = y_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(y_i) \neq w(y_{i+1})$  and  $u = x_i$  and  $v = y_i$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(y_i)$ , and the last, take  $u = x_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(y_{i+1})$ . Based on that, we have the set of vertices weight of the triangular ladder graph  $TL_n$  is  $W(TL_n) = \{4, 5, 8, 9, 10, 11\}$ , so  $|w(V(TL_n))| = 6$ . Hence,  $\chi_{lis}(TL_n) = 6$ .

Case 4: for  $n \geq 6, n \equiv 2 \pmod{3}$

Furthermore, the upper bound for the chromatic number local irregular, we define  $l: V(TL_n) \rightarrow \{1, 2, 3\}$  with vertex irregular 3-labelling as follows :

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{3} \\ 2, & \text{for } i \equiv 2 \pmod{3} \\ 3, & \text{for } i \equiv 0 \pmod{3} \end{cases}$$

$$l(y_i) = 3, \text{ for } 1 \leq i \leq n$$

Hence,  $\max(l) = 3$  and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 4, & \text{for } i = n \\ 8, & \text{for } i = 1 \\ 9, & \text{for } i \equiv 0 \pmod{3}, n \geq 3 \\ 10, & \text{for } i \equiv 2 \pmod{3}, n \geq 2 \\ 11, & \text{for } i \equiv 1 \pmod{3}, n \geq 4 \end{cases}$$

$$w(y_i) = \begin{cases} 4, & \text{for } i = 1 \\ 6, & \text{for } i = n \\ 9, & \text{for } i \equiv 2 \pmod{3}, n \geq 2 \\ 10, & \text{for } i \equiv 1 \pmod{3}, n \geq 4 \\ 11, & \text{for } i \equiv 0 \pmod{3}, n \geq 3 \end{cases}$$

For every  $uv \in E(TL_n)$ , take any  $u = x_i$  and  $v = x_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(x_{i+1})$ , then  $u = y_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(y_i) \neq w(y_{i+1})$  and  $u = x_i$  and  $v = y_i$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(y_i)$ , and the last, take  $u = x_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, n - 1$ ,  $w(x_i) \neq w(y_{i+1})$ . Based on that, we have the set of vertices weight of the triangular ladder graph  $TL_n$  is  $W(TL_n) = \{4, 6, 8, 9, 10, 11\}$ , so  $|w(V(TL_n))| = 6$ . Hence,  $\chi_{lis}(TL_n) = 6$ .

**Theorem 2.3** Let  $G$  be a  $H$  graph  $(H_n)$ . For  $n \geq 2$ , the chromatic number local irregular is  $\chi_{lis}(H_n) = 4$ .

**Proof.**  $V(H_n) = \{x_i, y_i : 1 \leq i \leq 2n\} \cup \{z_j : 1 \leq j \leq 2n\}$ , and the edge set  $E(H_n) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq 2n - 1\} \cup \{z_j z_{j+1} : 1 \leq j \leq n\} \cup \{x_i z_j : 1 \leq i = j \leq 2n\} \cup \{y_i z_j : 1 \leq i = j \leq 2n\}$ . If every  $x \in V(H_n), y \in V(H_n), z \in V(H_n)$  is labelled by 1, so we have  $w(x_i) = l(x_{i-1}) + l(x_{i+1}) + l(z_j) = 1 + 1 + 1 = 3$  and  $w(x_{i+1}) = l(x_i) + l(x_{i+2}) + l(z_{j+1}) = 1 + 1 + 1 = 3$ . It means a contradiction with definition. Since,  $x_i x_{i+1} \in E(H_n), w(x_i) \neq w(x_{i+1})$ . Hence  $\max(l) = 2$ .

Based on Lemma, the lower bound for the chromatic number  $\chi_{lis}(H_n) \geq \chi(H_n) = 2$ , so that  $\chi_{lis}(H_n) \geq 2$ .

Furthermore, the upper bound for the chromatic number local irregular, we define  $l: V(H_n) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows :

$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ odd} \\ 2, & \text{for } i \text{ even} \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ odd} \\ 2, & \text{for } i \text{ even} \end{cases}$$

$$l(z_j) = \begin{cases} 1, & \text{for } j \text{ even} \\ 2, & \text{for } j \text{ odd} \end{cases}$$

Hence,  $\max(l) = 2$  and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 4, & \text{for } i = 1 \\ 3, & \text{for } i = \text{even}, 2 \leq i \leq 2n - 1 \\ 6, & \text{for } i = \text{odd}, 2 \leq i \leq 2n - 1 \\ 2, & \text{for } i = n \end{cases}$$

$$w(y_i) = \begin{cases} 4, & \text{for } i = 1 \\ 3, & \text{for } i = \text{even}, 2 \leq i \leq n - 1 \\ 6, & \text{for } i = \text{odd}, 2 \leq i \leq n - 1 \\ 2, & \text{for } i = n \end{cases}$$

$$w(z_j) = \begin{cases} 3, & \text{for } j \text{ odd}, j = 1, 2, 3, \dots, 2n \\ 6, & \text{for } j \text{ even}, j = 1, 2, 3, \dots, 2n \end{cases}$$

For every  $uv \in E(H_n)$ , take any  $u = x_i$  and  $v = x_{i+1}$ ,  $i = 1, 2, 3, \dots, 2n$ ,  $w(x_i) \neq w(x_{i+1})$ , then  $u = y_i$  and  $v = y_{i+1}$ ,  $i = 1, 2, 3, \dots, 2n$ ,  $w(y_i) \neq w(y_{i+1})$  also take  $u = z_j$  and  $v = z_{j+1}$ ,  $i = 1, 2, 3, \dots, 2n$ ,  $w(z_j) \neq w(z_{j+1})$ . Then  $u = x_i$  and  $v = z_j$ ,  $1 \leq i = j \leq 2n$ ,  $w(x_i) \neq w(z_j)$ , and the last, take  $u = y_i$  and  $v = z_j$ ,  $1 \leq i = j \leq 2n$ ,  $w(y_i) \neq w(z_j)$ . Based on that, we have the set of vertices weight of the  $H$  graph is  $W(H_n) = \{2, 3, 4, 6\}$ , so  $|w(V(H_n))| = 4$ . Hence,  $\chi_{lis}(H_n) = 4$ .

### 3. CONCLUSION

In this paper, we have studied local irregularity vertex coloring of ladder graph, triangular ladder graph, and  $H$  graph. We have concluded the exact value of the chromatic number local irregular of ladder graph, namely  $\chi_{lis}(L_n) = 4$ , the chromatic number local irregular of triangular ladder graph, namely for  $n = 5$  is  $\chi_{lis}(TL_n) = 5$  and for  $n \geq 6$  is  $\chi_{lis}(TL_n) = 6$  and the chromatic number local irregular of  $H$  graph, namely  $\chi_{lis}(H_n) = 4$ . Hence the following problem arises naturally.

### 4. ACKNOWLEDGEMENT

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