Journal of Physics: Conference Series

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To cite this article: Arika Indah Kristiana *et al* 2019 *J. Phys.: Conf. Ser.* **1211** 012003

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IOP Conf. Series: Journal of Physics: Conf. Series **1211** (2019) 012003 doi:10.1088/1742-6596/1211/1/012003

# On the chromatic number local irregularity of related wheel graph

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**Abstract.** A function $f$ is called a local irregularity vertex coloring if (i) $l : V(G) \to \{1, 2, \cdots, k\}$ as vertex irregular $k$-labeling and $w : V(G) \to N$, for every $uv \in E(G), w(u) \neq w(v)$ where $w(u) = \Sigma_{v \in N(u)} l(v)$ and (ii) $max(l) = min\{max\{l_i\};$ $l_i$vertex irregular labeling$\}$. The chromatic number of local irregularity vertex coloring of $G$, denoted by $\chi_{lis}(G)$, is the minimum cardinality of the largest label over all such local irregularity vertex coloring. In this article, we study the local irregularity vertex coloring of related wheel graphs and we have found the exact value of their chromatic number local irregularity, namely web graph, helm graph, close helm graph, gear graph, fan graph, sun let graph, and double wheel graph.

## Introduction

Let $G = (V, E)$ be a graph of order $n$ and size $m$ having no isolated vertices. For $v \in V(G)$, $d(v)$ denoted the degree of $v$ in $G$. Amurugam [4] defined local antimagic labeling if for any two adjacent vertices $u$ and $v$, $w(u) \neq w(v)$, where $w(u) = \Sigma e \in E(u) f(e)$, and $E(u)$ is the set of edges incident to $u$. Futhermore Amurugam [4] written as $\chi_{la}(G)$ is the minimum number of colors taken over all colorings of $G$ induced by local antimagic labeling of $G$. Slamin [5] was presented the distance irregular labeling of graphs. The distance irregularity strength of $G$, denoted by $dis(G)$, is the minumum cardinality of the largest label $k$ over all such irregular assignments. Kristiana, et.al [3] have modified the two definitions above to be local irregularity vertex coloring.

**Definition 1** *[3] Suppose $l : V(G) \to \{1, 2, \cdots, k\}$ is called vertex irregular $k$-labeling and $w : V(G) \to N$ where $w(u) = \Sigma_{v \in N(u)} l(v)$, $l$ is called local irregularity vertex coloring, if*

*i.* $max(l) = min\{max\{l_i\};\ l_i,$ vertex irregular labeling$\}$

*ii. for every $uv \in E(G), w(u) \neq w(v)$.*

**Definition 2** *[3] The chromatic number local irregular denoted by $\chi_{lis}(G)$, is minimum of cardinality local irregularity vertex coloring.*

**Observation 1** *[3] Let graph $G$ with each two vertex adjacent have a different degree of the vertex, then the $max(l_i) \geq 2$*

**Lemma 1** *[3] Let $G$ simple and connected graph, $\chi_{lis}(G) \geq \chi(G)$*

**Proposition 1** *[3] Let $W_n$ be a wheel graph. For $n \geq 4$, the chromatic number local irregular is*

$$\chi_{lis}(W_n) = \begin{cases} 3, & \text{for } n \text{ even} \\ 4, & \text{for } n \text{ odd} \end{cases}$$

**The Results**

We will find the exact value the chromatic number local irregular, namely web graph, helm graph, close helm graph, gear graph, fan graph, sun let graph anda double wheel graph.

**Theorem 1** *Let $Wb_n$ be a web graph. For $n \geq 3$, the chromatic number local irregular is*

$$\chi_{lis}(Wb_n) = \begin{cases} 5, & \text{for } n = 4, 6 \\ 6, & \text{for } n = 5 \text{ or } n \geq 7,\ n \text{ even} \\ 7, & \text{for } n \geq 7,\ n \text{ odd} \end{cases}$$

**Proof**: The vertex set is $V(Wb_n) = \{x\} \cup \{x_{ji}; j = 1, 2, 3, 1 \leq i \leq n\}$ and the edge set is $E(Wb_n) = \{xx_{ii} : 1 \leq i \leq n\} \cup \{x_{1i}x_{2i}; 1 \leq i \leq n\} \cup \{x_{2i}x_{3i}\} \cup \{x_{1i}x_{1(i+1)}; 1 \leq i \leq n-1\} \cup \{x_{11}x_{1n}\} \cup \{x_{2i}y_{2(i+1)}; 1 \leq i \leq n-1\} \cup \{y_{21}y_{2n}\}$. The order and size of graph $Wb_n$, respectively are $|V(Wb_n)| = 3n + 1$ and $|E(Wb_n)| = 5n$. Thus, the minimum degree of graph $Wb_n$ is $\delta(Wb_n) = 1$ and the maximum degree of graph $Wb_n$ is $\Delta(Wb_n) = n$. Based on Obsevation 1 and definition of webs graph, $max(l_i(Wb)) = 2$. This proof can be divided into three cases in the following.

**Case 1**: For $n = 4, 6$

For the proof $\chi_{lis}(Wb_n) = 5;\ n = 4, 6$, based on Lemma 1, the lower bound is $\chi_{lis}(Wb_n) = 5 \geq \chi(Wb_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of web graph define $l : V(Wb_n) \to \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 2$$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j = 1,\ 1 \leq i \leq n; \text{ or } j = 2,\ 1 \leq i \leq n,\ i \text{ odd; or } j = 3,\ 1 \leq i \leq n \\ 2, & \text{for } j = 2,\ 1 \leq i \leq n,\ i \text{ odd} \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = n$$

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j = 3,\ 1 \leq i \leq n,\ i \text{ odd} \\ 2, & \text{for } j = 3,\ 1 \leq i \leq n,\ i \text{ even} \\ 4, & \text{for } j = 2,\ 1 \leq i \leq n,\ i \text{ even} \\ 5, & \text{for } j = 1,\ 1 \leq i \leq n,\ i \text{ odd} \\ 6, & \text{for } j = 1,\ 1 \leq i \leq n,\ i \text{ even; or } j = 2,\ 1 \leq i \leq n,\ i \text{ odd} \end{cases}$$

Clearly $|w(V(Wb_n))| = 5$, so the upper bound is $\chi_{lis}(Wb_n) \leq 5$. Hence, $\chi_{lis}(Wb_n) = 5$.

**Case 2**: For $n = 5,\ n \geq 7,\ n$ even

For the proof $\chi_{lis}(Wb_n) = 6;\ n = 5,\ n \geq 7,\ n$ even, based on Lemma 1, the lower bound is $\chi_{lis}(Wb_n) = 6 \geq \chi(Wb_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of web graph define $l : V(Wb_n) \to \{1, 2\}$ in following.

(i) for $n = 5$

The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 1$$

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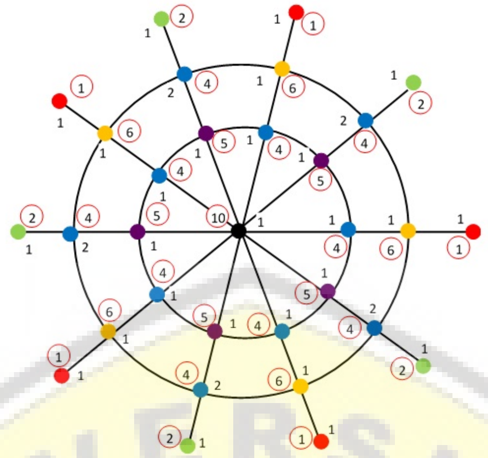

**Figure 1.** vertex coloring local irregularity of website graph, $\chi_{lis}(Wb_{10}) = 6$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j=1,\ i \text{ odd},\ 1 \le i \le 4 \text{ or } j=1, i=n \text{ or} j=2,\ 1 \le i \le 4 \text{ or } j=3,\ 2 \le i \le 4 \\ 2, & \text{for } j=1,\ i \text{ even},\ 1 \le i \le 4 \text{ or} j=2,\ i=5 \text{ or } j=3,\ i=1,\ 5 \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = 7$$

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j=3,\ 1 \le i \le 4 \\ 2, & \text{for } j=3,\ i=5 \\ 4, & \text{for } j=1,\ i=2 \text{ or } j=2,\ i=3 \\ 5, & \text{for } j=1,\ i=1,4 \text{ or } j=2,\ i=2,5 \\ 6, & \text{for } j=1,\ i=3,5 \text{ or } j=2,\ i=1,4 \end{cases}$$

(ii) for $n \ge 7$, $n$ is even
The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 1$$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j=1, \text{ or } j=2,\ 1 \le i \le n, i \text{ odd or } j=3,\ 1 \le i \le n \\ 2, & \text{for } j=2\ 1 \le i \le n, i \text{ even} \end{cases}$$

$$w(x) = n$$

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j=3,\ 1 \le i \le n,\ i \text{ odd} \\ 2, & \text{for } j=3,\ 1 \le i \le n,\ i \text{ even} \\ 4, & \text{for } j=2,\ 1 \le i \le n,\ i \text{ even} \\ 5, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ odd} \\ 6, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ even; or } j=2,\ 1 \le i \le n,\ i \text{ odd} \end{cases}$$

Clearly $|w(V(Wb_n))| = 6$, so the upper bound is $\chi_{lis}(Wb_n) \le 6$. Hence, $\chi_{lis}(Wb_n) = 6$. **Case 3**: For $n \ge 7$, $n$ odd
For the proof $\chi_{lis}(Wb_n) = 5$; $n \ge 7$, $n$ odd, based on Lemma 1, the lower bound is $\chi_{lis}(Wb_n) = 7 \ge \chi(Wb_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of web graph define $l : V(Wb_n) \to \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 1$$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j=1,\ 1 \le i \le n; \text{ or } j=2,\ i=1,\ 2 \le i \le n,\ i \text{ even; or } j=3,\ 2 \le i \le n \\ 2, & \text{for } j=1,\ 1 \le i \le n; \text{ or } j=2,\ 2 \le i \le n,\ i \text{ odd; or } j=3,\ i=1,2 \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = \frac{3(n-1)}{2} + 1$$

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j=3,\ i=1,2,\ 3 \le i \le n,\ i \text{ even} \\ 2, & \text{for } j=3,\ 3 \le i \le n,\ i \text{ odd} \\ 4, & \text{for } j=1,\ 2 \le i \le n-1,\ i \text{ odd or } j=2,\ i=n, 2 \le i \le n-2,\ i \text{ odd} \\ 5, & \text{for } j=1,\ i=1 \\ 6, & \text{for } j=1,\ i=n; \text{ or } j=2,\ i=1,n-1 \\ 7, & \text{for } j=1,\ 2 \le i \le n-1,\ i \text{ even or } j=2,\ 2 \le i \le n-2,\ i \text{ even} \end{cases}$$

Clearly $|w(V(Wb_n))| = 7$, so the upper bound is $\chi_{lis}(Wb_n) \le 7$. Hence, $\chi_{lis}(Wb_n) = 7$. The proof is complete.

**Theorem 2** *Let $H_n$ be a helm graph. For $n \ge 3$, the chromatic number local irregular is*

$$\chi_{lis}(H_n) = \begin{cases} 5, & \textit{for } n \textit{ even} \\ 6, & \textit{for } n \textit{ odd} \end{cases}$$

**Proof**: The vertex set is $V(H_n) = \{x\} \cup \{x_{ji}; j = 1,2,1 \le i \le n\}$ and the edge set is $E(H_n) = \{xx_{1i} : 1 \le i \le n\} \cup \{x_{1i}x_{2i}; 1 \le i \le n\} \cup \{x_{1i}x_{1(i+1)}; 1 \le i \le n-1\} \cup \{x_{11}x_{1n}\}$. The order and size of graph $H_n$, respectively are $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. Thus, the minimum degree of graph $H_n$ is $\delta(H_n) = 1$ and the maximum degree of graph $H_n$ is $\Delta(H_n) = n$. Based on Obsevation 1 and definition of Helm graph, $max(l_i(H)) = 2$. This proof can be divided into two cases in the following.

**Case 1**: For $n$ even

For the proof $\chi_{lis}(H_n) = 5$; $n$ even, based on Lemma 1, the lower bound is $\chi_{lis}(H_n) = 5 \ge \chi(H_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of helm graph define $l : V(H_n) \to \{1,2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 2$$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j=1,\ 1 \le i \le n, i \text{ odd; or } j=2,\ 1 \le i \le n \\ 2, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ even} \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = \frac{3n}{2}$$

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j=2,\ 1 \le i \le n,\ i \text{ odd} \\ 2, & \text{for } j=2,\ 1 \le i \le n,\ i \text{ even} \\ 5, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ even} \\ 7, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ odd} \end{cases}$$

Clearly $|w(V(H_n))| = 5$, so the upper bound is $\chi_{lis}(H_n) \le 5$. Hence, $\chi_{lis}(H_n) = 5$.

**Case 2**: For $n$ odd

For the proof $\chi_{lis}(H_n) = 6$; $n$ even, based on Lemma 1, the lower bound is $\chi_{lis}(H_n) = 6 \ge \chi(H_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of helm graph define $l : V(H_n) \to \{1,2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 2$$

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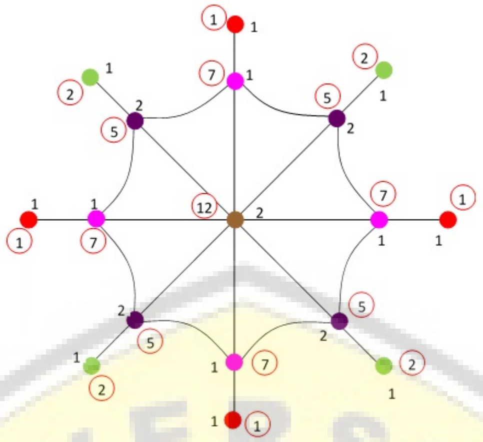

**Figure 2.** vertex coloring local irregularity of helm graph, $\chi_{lis}(H_8) = 6$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j = 1,\ i = 1 \text{ or } j = 1,\ 2 \le i \le n, i \text{ even; or } j = 2,\ i \neg 2 \\ 2, & \text{for } j = 1,\ i \text{ odd},\ 2 \le i \le n \text{ or} j = 2,\ i = 2 \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = \tfrac{3(n-1)}{2} + 1$$

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j = 2,\ i = 1,\ 2 \le i \le n,\ i \text{ even} \\ 2, & \text{for } j = 2,\ 2 \le i \le n,\ i \text{ odd} \\ 5, & \text{for } j = 1,\ 2 \le i \le n,\ i \text{ odd} \\ 6, & \text{for } j = 1,\ i = 1 \\ 7, & \text{for } j = 1,\ 2 \le i \le n,\ i \text{ even} \end{cases}$$

Clearly $|w(V(H_n))| = 6$, so the upper bound is $\chi_{lis}(H_n) \le 6$. Hence, $\chi_{lis}(H_n) = 6$. The proof is complete.

**Theorem 3** *Let $CH_n$ be a close helm graph. For $n \ge 4$, the chromatic number local irregular is*

$$\chi_{lis}(CH_n) = \begin{cases} 4, & \text{for } n = 4 \\ 5, & \text{for } n \ge 5 \text{ even} \\ 6, & \text{for } n \ge 5 \text{ odd} \end{cases}$$

**Proof**: The vertex set is $V(CH_n) = \{x\} \cup \{x_{ji}; j = 1, 2, 1 \le i \le n\}$ and the edge set is $E(CH_n) = \{xx_{1i} : 1 \le i \le n\} \cup \{x_{1i}x_{2i}; 1 \le i \le n\} \cup \{x_{1i}x_{1(i+1)}; 1 \le i \le n-1\} \cup \{x_{11}x_{1n}\} \cup \{x_{2i}x_{2(i+1)}; 1 \le i \le n-1\} \cup \{x_{21}x_{2n}\}$. The order and size of graph $CH_n$, respectively are $|V(CH_n)| = 2n+1$ and $|E(CH_n)| = 4n$. Thus, the minimum degree of graph $CH_n$ is $\delta(CH_n) = 3$ and the maximum degree of graph $CH_n$ is $\Delta(CH_n) = n$. Based on Obsevation 1 and definition of Helm graph, $max(l_i(CH)) = 2$. This proof can be divided into two cases in the following.

**Case 1**: For $n = 4$

For the proof $\chi_{lis}(CH_n) = 4$; $n = 4$, based on Lemma 1, the lower bound is $\chi_{lis}(CH_n) = 4 \ge \chi(CH_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of close helm graph define $l : V(CH_n) \to \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 2$$

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$$l(x_{ji}) = \begin{cases} 1, & \text{for } j=1,\ 1 \le i \le n, i \text{ odd; or } j=2,\ 1 \le i \le n,\ i \text{ even} \\ 2, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ even; or } j=2,\ 1 \le i \le n,\ i \text{ odd} \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = 6$$

$$w(x_{ji}) = \begin{cases} 3, & \text{for } j=2,\ 1 \le i \le n,\ i \text{ odd} \\ 4, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ even} \\ 6, & \text{for } j=2,\ 1 \le i \le n,\ i \text{ even} \\ 7, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ odd} \end{cases}$$

Clearly $|w(V(CH_n))| = 4$, so the upper bound is $\chi_{lis}(CH_n) \le 4$. Hence, $\chi_{lis}(CH_n) = 4$.
**Case 2**: For $n \ge 5,\ n$ even
For the proof $\chi_{lis}(CH_n) = 5;\ n$ even, based on Lemma 1, the lower bound is $\chi_{lis}(CH_n) = 5 \ge \chi(CH_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of close helm graph define $l : V(CH_n) \to \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 1$$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j=1,\ 1 \le i \le n, i \text{ odd or } j=2,\ 1 \le i \le n, i \text{ even} \\ 2, & \text{for } j=1,\ 1 \le i \le n, i \text{ even or } j=2,\ 1 \le i \le n, i \text{ odd} \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = \tfrac{3n}{2}$$

$$w(x_{ji}) = \begin{cases} 3, & \text{for } j=2,\ 1 \le i \le n,\ i \text{ odd} \\ 4, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ even} \\ 6, & \text{for } j=2,\ 1 \le i \le n,\ i \text{ even} \\ 7, & \text{for } j=1,\ 1 \le i \le n,\ i \text{ odd} \end{cases}$$

Clearly $|w(V(CH_n))| = 5$, so the upper bound is $\chi_{lis}(CH_n) \le 5$. Hence, $\chi_{lis}(CH_n) = 5$.
**Case 3**: For $n \ge 5,\ n$ odd
For the proof $\chi_{lis}(CH_n) = 6;\ n$ odd, based on Lemma 1, the lower bound is $\chi_{lis}(CH_n) = 6 \ge \chi(CH_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of close helm graph define $l : V(CH_n) \to \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 1$$

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j=1,\ 1 \le i \le n-1, i \text{ odd or } j=2,\ 1 \le i \le n, i \text{ even} \\ 2, & \text{for } j=1,\ i=n,\ 1 \le i \le n-1, i \text{ even or } j=2,\ 1 \le i \le n, i \text{ odd} \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = \tfrac{3(n-1}{2} + 2$$

$$w(x_{ji}) = \begin{cases} 3, & \text{for } j=2,\ 2 \le i \le n-2,\ i \text{ odd} \\ 4, & \text{for } j=1,\ 1 \le i \le n-3,\ i \text{ even; or } j=2,\ i=1 \\ 5, & \text{for } j=1,\ i=n-2; \text{ or } j=2,\ i=n \\ 6, & \text{for } j=1,\ i=n-1; \text{ or } j=2,\ 2 \le i \le n-2,\ i \text{ even} \\ 7, & \text{for } j=1,\ 1 \le i \le n-3,\ i \text{ odd} \end{cases}$$

Clearly $|w(V(CH_n))| = 6$, so the upper bound is $\chi_{lis}(CH_n) \le 6$. Hence, $\chi_{lis}(CH_n) = 6$. The proof is complete.

**Theorem 4** *Let $G_n$ be a gear graph. For $n \ge 4$, the chromatic number local irregular is $\chi_{lis}(G_n) = 3$*

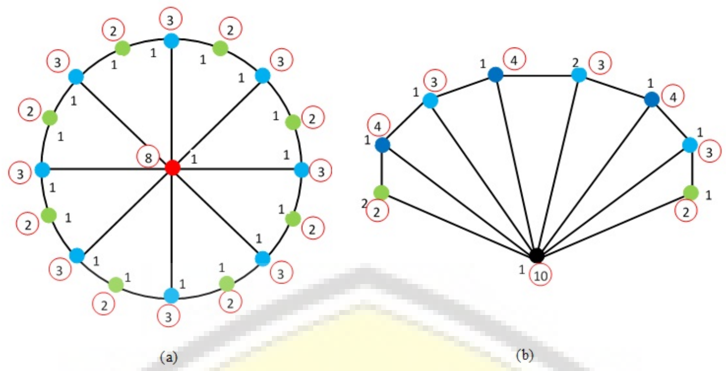


**Figure 3.** vertex coloring local irregularity: (a) gear graph, $\chi_{lis}(G_8) = 3$; (b) fan graph, $\chi_{lis}(F_8) = 4$

**Proof**: The vertex set is $V(G_n) = \{x\} \cup \{x_i; 1 \leq i \leq 2n\}$ and the edge set is $E(G_n) = \{xx_i : 1 \leq i \leq 2n, i\text{odd}\} \cup \{\mathrm{x_i x_{i+1}}; 1 \leq \mathrm{i} \leq 2\mathrm{n} - 1\} \cup \{\mathrm{x_1 x_2 n}\}$. The order and size of graph $G_n$, respectively are $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$. Thus, the minimum degree of graph $G_n$ is $\delta(G_n) = 2$ and the maximum degree of graph $G_n$ is $\Delta(G_n) = n$. Based on Obsevation 1 and definition of gear graph, $max(l_i(G)) = 1$. The degrees of the vertex gear graph, respectively are $d(x) = 1$, $d(x_i) = 3$, $i$ odd and $d(x_i) = 2$, $i$ even. This results in the different color in two an adjacent vertex. Hence $b(V(G_n)) = 3|$, so the proof $\chi_{lis}(G_n) = 3$ is complete.

**Theorem 5** *Let $F_n$ be a fan graph. For $n \geq 4$, the chromatic number local irregular is $\chi_{lis}(F_n) = 4$*

**Proof**: The vertex set is $V(F_n) = \{x\} \cup \{x_i; 1 \leq i \leq n\}$ and the edge set is $E(F_n) = \{xx_i : 1 \leq i \leq n\} \cup \{x_i x_{(i+1)}; 1 \leq i \leq n - 1\}$. The order and size of graph $F_n$, respectively are $|V(F_n)| = n+1$ and $|E(F_n)| = 2n-1$. Thus, the minimum degree of graph $F_n$ is $\delta(F_n) = 2$ and the maximum degree of graph $F_n$ is $\Delta(F_n) = n$. Based on Obsevation 1 and definition of fan graph, $max(l_i(H)) = 2$. This proof can be divided into two cases in the following.
**Case 1**: For $n$ even
For the proof $\chi_{lis}(F_n) = 4$; $n$ even, based on Lemma 1, the lower bound is $\chi_{lis}(F_n) = 4 \geq \chi(F_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of fan graph define $l : V(F_n) \rightarrow \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 1$$

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1, 2, 3 (\text{mod } 4) \\ 2, & \text{for } i \equiv 0 (\text{mod } 4) \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = \begin{cases} \frac{5(n-2)}{4} + 1, & \text{for } n \equiv 2(\text{mod } 4) \\ \frac{5n}{4}, & \text{for } n \equiv 0(\text{mod } 4) \end{cases}$$

$$w(x_{ji}) = \begin{cases} 2, & \text{for } i = 1,\ n \\ 3, & \text{for } 2 \leq i \leq n-1,\ i \text{ even} \\ 4, & \text{for } 2 \leq i \leq n-1,\ i \text{ odd} \end{cases}$$

Clearly $|w(V(F_n))| = 4$, so the upper bound is $\chi_{lis}(F_n) \leq 4$. Hence, $\chi_{lis}(F_n) = 4$.
**Case 2**: For $n$ odd

For the proof $\chi_{lis}(F_n) = 4$; $n$ odd, based on Lemma 1, the lower bound is $\chi_{lis}(F_n) = 4 \geq \chi(F_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of fan graph define $l : V(F_n) \to \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x) = 1$$

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 0, 2, 3 (\text{mod } 4) \\ 2, & \text{for } i = 1, i \equiv 1 (\text{mod } 4) \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x) = \begin{cases} \frac{5(n-1)}{4} + 2, & \text{for } n \equiv 1 (\text{mod } 4) \\ \frac{5(n-3}{4} + 4, & \text{for } n \equiv 3 (\text{mod } 4) \end{cases}$$

$$w(x_{ji}) = \begin{cases} 2, & \text{for } i = 1,\ n \\ 3, & \text{for } 2 \leq i \leq n-1,\ i \text{ odd} \\ 4, & \text{for } 2 \leq i \leq n-1,\ i \text{ even} \end{cases}$$

Clearly $|w(V(F_n))| = 4$, so the upper bound is $\chi_{lis}(F_n) \leq 4$. Hence, $\chi_{lis}(F_n) = 4$. The proof is complete.

**Theorem 6** *Let $SL_n$ be a sun let graph. For $n \geq 4$, the chromatic number local irregular is*

$$\chi_{lis}(H_n) = \begin{cases} 3, & \textit{for } n \textit{ even} \\ 5, & \textit{for } n \textit{ odd} \end{cases}$$

**Proof**: The vertex set is $V(SL_n) = \{x_{ji}; j = 1, 2, 1 \leq i \leq n\}$ and the edge set is $E(SL_n) = \{x_{1i}x_{2i}; 1 \leq i \leq n\} \cup \{x_{1i}x_{1(i+1)}; 1 \leq i \leq n-1\} \cup \{x_{11}x_{1n}\}$. The order and size of graph $SL_n$, respectively are $|V(SL_n)| = 2n$ and $|E(SL_n)| = 2n$. Thus, the minimum degree of graph $SL_n$ is $\delta(SL_n) = 1$ and the maximum degree of graph $SL_n$ is $\Delta(SL_n) = 3$. Based on Obsevation 1 and definition of sun let graph, $max(l_i(SL)) = 2$. This proof can be divided into two cases in the following.

**Case 1**: For $n$ even

For the proof $\chi_{lis}(SL_n) = 3$; $n$ even, based on Lemma 1, the lower bound is $\chi_{lis}(SL_n) = 3 \geq \chi(SL_n) = 3$. In order to prove the upper bound of the chromatic number local irregular of sun let graph define $l : V(SL_n) \to \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j = 1,\ 1 \leq i \leq n; \text{ or } j = 2,\ 1 \leq i \leq n,\ i \text{ odd} \\ 2, & \text{for } j = 2,\ 1 \leq i \leq n,\ i \text{ even} \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j = 2,\ 1 \leq i \leq n \\ 3, & \text{for } j = 1,\ 1 \leq i \leq n,\ i \text{ odd} \\ 4, & \text{for } j = 1,\ 1 \leq i \leq n,\ i \text{ even} \end{cases}$$

Clearly $|w(V(SL_n))| = 4$, so the upper bound is $\chi_{lis}(SL_n) \leq 4$. Hence, $\chi_{lis}(SL_n) = 4$.

**Case 2**: For $n$ odd

For the proof $\chi_{lis}(SL_n) = 5$; $n$ odd, based on Lemma 1, the lower bound is $\chi_{lis}(SL_n) = 5 \geq \chi(SL_n) = 3$. However, we can not reach the sharpest lower bound. In order to prove the upper bound of the chromatic number local irregular of sun let graph define $l : V(SL_n) \rightarrow \{1, 2\}$ in following. The vertex irregular 2-labeling uses the formula as follows:

$$l(x_{ji}) = \begin{cases} 1, & \text{for } j = 1,\ 1 \leq i \leq n-1; \text{ or } j = 2,\ 3 \leq i \leq n-1 \\ 2, & \text{for } j = 1,\ i = n \text{ or } j = 2,\ i = 1, 2, n \end{cases}$$

Hence, $max(l) = 2$ and the labeling gives vertex-weight as follows:

$$w(x_{ji}) = \begin{cases} 1, & \text{for } j = 2,\ 1 \leq i \leq n-1 \\ 2, & \text{for } j = 2,\ i = n \\ 3, & \text{for } j = 1,\ 2 \leq i \leq n,\ i \text{ odd} \\ 4, & \text{for } j = 1,\ 2 \leq i \leq n-1,\ i \text{ even} \\ 6, & \text{for } j = 1,\ i = 1,\ n-1 \end{cases}$$

Clearly $|w(V(SL_n))| = 5$, so the upper bound is $\chi_{lis}(SL_n) \leq 5$. Hence, $\chi_{lis}(SL_n) = 5$. The proof is complete.

**Theorem 7** *Let $DW_n$ be a double wheel graph. For $n \geq 4$, the chromatic number local irregular is*

$$\chi_{lis}(DW_n) = \begin{cases} 3, & \textit{for } n \textit{ even} \\ 4, & \textit{for } n \textit{ odd} \end{cases}$$

**Proof**: The vertex set is $V(DW_n) = \{x, x_i, y_j; 1 \leq i \leq 2n, 1 \leq j \leq n\}$ and the edge set is $E(DW_n) = \{xx_i; 1 \leq i \leq 2n\} \cup \{x_iy_j; 1 \leq i \leq 2n, \imath\text{even}, 1 \leq \text{j} \leq \text{n}\} \cup \{\text{x}_\text{i}\text{x}_{(\text{i}+1)}; 1 \leq \text{i} \leq 2\text{n} - 1\} \cup \{\text{x}_1\text{x}_{2\text{n}}\} \cup \{\text{y}_\text{j}\text{y}_{(\text{j}+1)}; 1 \leq \text{i} \leq \text{n} - 1\} \cup \{\text{y}_1\text{y}_\text{n}\}$. The order and size of graph $DW_n$, respectively are $|V(DW_n)| = 3n + 1$ and $|E(DW_n)| = 6n$. Thus, the minimum degree of graph $DW_n$ is $\delta(DW_n) = 2$ and the maximum degree of graph $DW_n$ is $\Delta(DW_n) = 2n$. a double wheel graph is a graph consisting of two cycles of size n where both cycles are connected to one center. This shows the double wheel graph is identical with the wheel graph so that the chromatic number local irregular of the double wheel graph is as same as the wheel graph.

## Conclusion

In this paper we have studied local irregularity vertex coloring of related wheel graph. We have concluded the exact value of the chromatic number local irregular of realted wheel graph, namely $\chi_{lis}(Wb_n)$, $\chi_{lis}(H_n)$, $\chi_{lis}(CH_n)$, $\chi_{lis}(G_n)$, $\chi_{lis}(F_n)$, $\chi_{lis}(SL_n)$, and $\chi_{lis}(DW_n)$. Hence the following problem arises naturally.

**Open Problem 1** *Determine lower and upper bound of the vertex coloring local irregularity of graph operation including cartesian, comb product, corona and others?*

## Acknowledgement

We gratefully acknowledge the support from CGANT - University of Jember of year 2018.

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