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ON THE LDS OF THE JOINT PRODUCT OF GRAPHS

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Abstract

Locating dominating set (LDS) is one of the topics of graph theory. A set $D \subseteq V$ of vertices in graph G is called a dominating set if every vertex $u \in V$ is either an element of D or adjacent to some element of D [3]. The domination number is the minimum cardinality of dominating set and denoted by $\gamma(G)$. Domination set D in graph $G = (V, E)$ is a locating dominating set if for each pair of distinct vertices u and v in $V(G) - D$ we have $N(u) \cap D \neq N(v) \cap D$, and

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$N(u) \cap D \neq \emptyset$, $N(v) \cap D \neq \emptyset$, where $N(u)$ is the neighboring set of u. Locating domination number which is denoted by $\gamma_L(G)$ is the minimum cardinality of locating dominating set [4]. This research aims to determine locating dominating set of some joint product of graphs and its relation with its basis graphs. Simple graphs used here are star graph, helm graph, triangular book graph and prism graph.

1. Introduction

Dominating set (DS) was studied mathematically for the first time in 1960s and rapidly increasing in 1970s. The set $D \subseteq V$ of vertices in graph $G = (V, E)$ is called a DS if each vertex $v \in V$ is either an element of D or adjacent to an element of S [3]. Domination number of G is the minimum cardinality of a DS in G, and denoted by $\gamma(G)$. One of the applications of DS that we can see in our daily life is at chess game. At the chess game, we determine minimum number of queen's movement to cover all the positions.

After DS was developed, other theories developed include independent dominating set (IDS), total dominating set (TDS), and then in 1987 locating dominating set (LDS) was introduced by Slater [4]. LDS is DS with additional condition. DS D in graph $G = (V, E)$ is an LDS if for each pair of distinct vertices u and v in $V(G) - D$, we have neighbor of u, v intersection with dominator D not having the same value and non-empty; denoted by $N(u) \cap D \neq N(v) \cap D$, $N(u) \cap D \neq \emptyset$ and $N(v) \cap D \neq \emptyset$, where $N(u)$ is the neighboring set of u. Locating domination number which is denoted by $\gamma_L(G)$ is the minimum cardinality of LDS [4]. Slater et al. [9-12] firstly studied the concept of LDS, an LDS of order $\gamma_L(G)$ is called a $\gamma_L(G)$-*set*. Some studies about domination number can be seen in [13-15].

In this paper, we determine the upper bound of locating domination number $\gamma_L(G)$ of the joint product of graphs. The definition of the joint product of graphs is taken from [5]. The joint product of graphs G and H which is denoted by $G = G + H$ is the graph with $V(G) = V(G) \cup V(H)$

and $E(G) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. The joint product of graphs is produced by connecting each vertex on G with each one of H.

2. Result

In this section, we have eight theorems about upper bounds of LDS of some joint product of graphs. The simple graphs used in this paper are triangular book graphs (BT_u), star graphs (S_u), helm graphs (H_u) and prism graph $(P_{k,2})$.

Theorem 1. *For* $u \geq 4$ *and* $k \geq 3$ *of* $H_u + Bt_k$, $\gamma_L(H_u + Bt_k) \leq u + k$.

Proof. The joint product graph $H_u + Bt_k$ is a connected graph with vertex set and edge set, respectively. $V(H_u + But_k) = \{a\} \cup \{x_s^r; r = 1, 2$ and $s = 1, ..., u\} \cup \{b_r; r = 1, 2\} \cup \{y_r; r = 1, ..., k\}$ and

$$E(H_u + Bt_k) = \{ax_r^1; r = 1, ..., u\} \cup \{x_r^1 x_r^2; r = 1, ..., u\}$$

$$\cup \{x_r^1 x_{r+1}^1; r = 1, ..., u-1\} \cup \{x_u^1 x_1^1\} \cup \{b_1 b_2\}$$

$$\cup \{b_r y_s; r = 1, 2 \text{ and } s = 1, ..., k\} \cup \{ab_r; r = 1, 2\}$$

$$\cup \{ay_r; r = 1, ..., k\} \cup \{b_r x_t^s; r = 1, 2, s = 1, 2 \text{ and } t = 1, ..., u\}$$

$$\cup \{x_s^r y_t; r = 1, 2, s = 1, ..., u \text{ and } t = 1, ..., k\}.$$

The cardinalities of $H_u + Bt_k$ are $|V(H_u + Bt_k)| = 2u + k + 3$ and $| E(H_u + Bt_k) = 2uk + 7u + 3k + 3|$.

We will show that $\gamma_L(U_u + Bt_k) \leq u + k$ by choosing $D = \{x_r^1; r = 1, ..., u-1\} \cup \{y_s; s = 1, ..., k-1\} \cup \{a, b_1\}$ as the dominator set of $H_u + Bt_k$ for $u \geq 4$ and $k \geq 3$ so that $|D| = u + k$, and the non-dominator set of $H_u + Bt_k$ is $V - D = \{x_r^2; r = 1, ..., u-1\} \cup \{b_2, x_u^1, x_u^2, y_k\}$.

Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D as follows:

$$N(b_2) \cap D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{y_s;\ s = 1, \ldots, k-1\} \cup \{a, b_1\},$$

$$N(x_u^1) \cap D = \{y_s;\ s = 1, \ldots, k-1\} \cup \{a, b_1, x_1^1, x_{u-1}^1\},$$

$$N(x_r^2) \cap D = \{x_r^1\} \cup \{y_s;\ s = 1, \ldots, k-1\} \cup \{b_1\};\ r = 1, \ldots, u-1,$$

$$N(x_u^2) \cap D = \{y_s;\ s = 1, \ldots, k-1\} \cup \{b_1\},$$

$$N(y_k) \cap D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{a, b_1\}.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D are different and are also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $H_u + Bt_k$ and fulfill the condition of LDS. Thus, $\gamma_L(H_u + Bt_k) \leq u + k$ for $u \geq 4$ and $k \geq 3$. □

Since $\gamma_L(H_u) = u$ [8] and $\gamma_L(Bt_u) = u$ [7], the relation between LDS of $H_u + Bt_k$ with its basic graph is $\gamma_L(H_u + Bt_k) \leq \gamma_L(H_u) + \gamma_L(Bt_k)$ for $u \geq 4$ and $k \geq 3$.

Theorem 2. *For* $u \geq 4$ *and* $k \geq 3$ *of* $H_u + S_k$, $\gamma_L(H_u + S_k) \leq u + k + 1$.

Proof. The joint product graph $H_u + S_k$ is a connected graph with vertex set and edge set, respectively. $V(H_u + S_k) = \{a, b\} \cup \{x_s^r;\ r = 1, 2$ and $s = 1, \ldots, u\} \cup \{y_r;\ r = 1, \ldots, k\}$ and $E(H_u + S_k) = \{ax_r^1;\ r = 1, \ldots, u\} \cup \{x_r^1 x_r^2;\ r = 1, \ldots, u\} \cup \{x_r^1 x_{r+1}^1;\ r = 1, \ldots, u-1\} \cup \{x_u^1 x_1^1\} \cup \{by_r;\ r = 1, \ldots, k\} \cup \{ab\} \cup \{ay_r; r = 1, \ldots, k\} \cup \{bx_s^r;\ r = 1, 2$ and $s = 1, \ldots, u\} \cup \{x_s^r y_t;\ r = 1, 2,$ $s = 1, \ldots, u$ and $t = 1, \ldots, k\}$. The cardinalities of $H_u + Bt_k$ are $|V(H_u + S_k)| = 2u + k + 3$ and $|E(H_u + S_k)| = 2uk + 5u + 2k + 1$.

We will show that $\gamma_L(H_u + S_k) \leq u + k - 1$ by choosing $D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{y_s;\ s = 1, \ldots, k-1\} \cup \{b\}$ as the dominator set of $H_u + S_k$ for $u \geq 4$ and $k \geq 3$ so that $|D| = u + k - 1$, and the non-dominator set of $H_u + S_k$ is $V - D = \{a, x_u^1, x_u^2, y_k\} \cup \{x_r^2;\ r = 1, \ldots, u-1\}$. Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D as follows:

$$N(a) \cap D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{y_s;\ s = 1, \ldots, k-1\} \cup \{b\},$$

$$N(x_u^1) \cap D = \{y_s;\ s = 1, \ldots, k-1\} \cup \{x_1^1, x_{u-1}^1, b\},$$

$$N(x_r^2) \cap D = \{x_r^1\} \cup \{y_s;\ s = 1, \ldots, k-1\} \cup \{b\};\ r = 1, \ldots, u-1,$$

$$N(x_u^2) \cap D = \{y_s;\ s = 1, \ldots, k-1\} \cup \{b\},$$

$$N(y_k) \cap D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{b\}.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and dominator set D are different and also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $H_u + S_k$ and fulfills the condition of LDS. Thus, $\gamma_L(H_u + S_k) \leq u + k - 1$ for $u \geq 4$ and $k \geq 3$. □

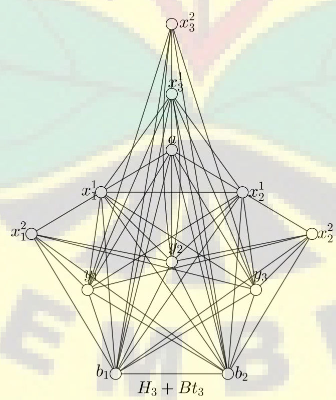

Figure 1. Join product of H_3 and Bt_3.

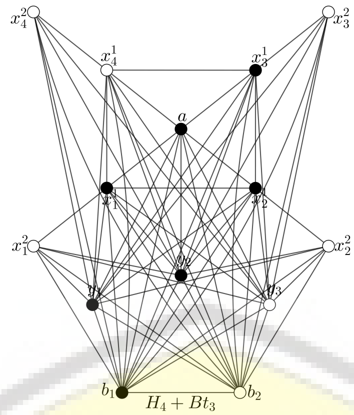

Figure 2. LDS of $H_4 + Bt_3$.

Since $\gamma_L(H_u) = u$ [8] and $\gamma_L(S_u) = u$ [6], the relation between LDS of $H_u + S_k$ with its basic graph is $\gamma_L(H_u + S_k) \leq \gamma_L(H_u) + \gamma_L(S_k) - 1$ for $u \geq 4$ and $k \geq 3$.

Theorem 3. *For* $u \geq 2$ *and* $k \geq 3$ *of* $Bt_u + S_k$, $\gamma_L(Bt_u + S_k) \leq u + k$.

Proof. The joint product graph $Bt_u + S_k$ is a connected graph with vertex set and edge set, respectively. $V(Bt_u + S_k) = \{a_r;\ r = 1, 2\} \cup \{x_r;\ r = 1, \ldots, u\} \cup \{b\} \cup \{y_r;\ r = 1, \ldots, k\}$ and $E(Bt_u + S_k) = \{a_1 a_2\} \cup \{a_r x_s;\ r = 1, 2$ and $s = 1, \ldots, u\} \cup \{by_r;\ i = 1, \ldots, k\} \cup \{a_r b;\ r = 1, 2\} \cup \{a_r y_s;\ r = 1, 2$ and $s = 1, \ldots, k\} \cup \{bx_r;\ r = 1, \ldots, u\} \cup \{x_r y_s;\ r = 1, \ldots, u$ and $s = 1, \ldots, k\}$. The cardinalities of $Bt_u + S_k$ are $|V(Bt_u + S_k)| = u + k + 3$ and $|E(Bt_u + S_k)| = uk + 3(u + k + 1)$.

We will show that $\gamma_L(Bt_u + S_k) \leq u + k$ by choosing $D = \{a_1, b\} \cup \{x_r;\ r = 1, \ldots, u - 1\} \cup \{y_s;\ s = 1, \ldots, u - 1\}$ as the dominator set of $Bt_u + S_k$ for $u \geq 2$ and $k \geq 3$ so that $|D| = u + k$, and the non-dominator set of $Bt_u + S_k$ is $V - D = \{a_2, x_u, y_k\}$. Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and dominator set D as follows:

$$N(a_2) \cap D = \{a_1, b\} \cup \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_s;\ s = 1, \ldots, k-1\},$$

$$N(x_u) \cap D = \{a_1, b\} \cup \{y_s;\ s = 1, \ldots, k-1\},$$

$$N(y_k) \cap D = \{a_1, b\} \cup \{x_r;\ r = 1, \ldots, u-1\}.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and dominator set D are different and also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $H_u + S_k$ and fulfills the condition of LDS. Thus, $\gamma_L(Bt_u + S_k) \le u + k$ for $u \ge 2$ and $k \ge 3$. □

Since $\gamma_L(Bt_u) = n$ [7] and $\gamma_L(S_n) = n$ [6], the relation between LDS of $Bt_u + S_k$ with its basic graph is $\gamma_L(Bt_u + S_k) \le \gamma_L(Bt_u) + \gamma_L(S_k)$ for $u \ge 2$ and $k \ge 3$.

Theorem 4. *For* $u, k \ge 4$ *of* $H_u + P_{k,2}$, $\gamma_L(H_u + P_{k,2}) \le u + k + 2$.

Proof. The joint product graph $H_u + P_{k,2}$ is a connected graph with vertex set and edge set, respectively. $V(H_u + P_{k,2}) = \{a\} \cup \{x_s^r;\ r = 1, 2$ and $s = 1, \ldots, u\} \cup \{y_s^r;\ r = 1, 2$ and $s = 1, \ldots, k\}$ and $E(H_u + P_{k,2}) = \{ax_r^1;\ r = 1, \ldots, u\} \cup \{x_r^1 x_r^2;\ r = 1, \ldots, u\} \cup \{x_r^1 x_{r+1}^1;\ r = 1, \ldots, u-1\} \cup \{x_1^1 x_u^1\} \cup \{y_r^1 y_{r+1}^1; r = 1, \ldots, k-1\} \cup \{y_1^1 y_k^1\} \cup \{y_r^2 y_{r+1}^2;\ r = 1, \ldots, k-1\} \cup \{y_1^2 y_k^2\} \cup \{y_r^1 y_r^2;\ r = 1, \ldots, k\} \cup \{ay_s^r;\ r = 1, 2$ and $s = 1, \ldots, u\} \cup \{x_s^r y_l^t;\ r = 1, 2,\ s = 1, \ldots, u,\ t = 1, 2$ and $l = 1, \ldots, k\}$. The cardinalities of $H_u + P_{k,2}$ are $|V(H_u + P_{k,2})| = 2u + k + 1$ and $|E(H_u + P_{k,2})| = 4uk + 3u + 5k$.

We will show that $\gamma_L(H_u + P_{k,2})| \le u + k - 2$ by choosing $D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\}$ as the dominator set of $H_u + P_{k,2}$ for $u, k \ge 4$ so that $|D| = u + k - 2$, and the non-dominator set of $H_u + P_{k,2}$ is $V - D = \{a, x_u^1, x_u^2, y_k^1, y_k^2\} \cup \{x_r^2;\ r = 1, \ldots, u-1\}$

$\bigcup \{y_s^2;\ s = 1, \dots, k-1\}$. Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D as follows:

$$N(a) \cap D = \{x_r^1;\ r = 1, \dots, u-1\} \cup \{y_s^1;\ s = 1, \dots, k-1\},$$

$$N(x_u^1) \cap D = \{y_s^1;\ s = 1, \dots, k-1\} \cup \{x_1^1, x_{u-1}^1\},$$

$$N(x_r^2) \cap D = \{x_r^1\} \cup \{y_s^1;\ s = 1, \dots, k-1\};\ r = 1, \dots, u-1,$$

$$N(x_u^2) \cap D = \{y_s^1;\ s = 1, \dots, k-1\},$$

$$N(y_k^1) \cap D = \{x_r^1;\ r = 1, \dots, u-1\} \cup \{y_1^1, y_{k-1}^1\},$$

$$N(y_s^2) \cap D = \{x_r^1;\ r = 1, \dots, u-1\} \cup \{y_s^1\};\ s = 1, \dots, k-1,$$

$$N(y_k^2) \cap D = \{x_r^1;\ r = 1, \dots, u-1\}.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D are different and also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $H_u + P_{k,2}$ and fulfills the condition of LDS. Thus, $\gamma_L(H_u + P_{k,2}) \leq u + k - 2$ for $u, k \geq 4$. □

Since $\gamma_L(H_u) = n$ [8] and $\gamma_L(P_{u,2}) = u$ [6], the relation between LDS of $H_u + P_{k,2}$ with its basic graph is $\gamma_L(H_u + P_{k,2}) \leq \gamma_L(H_u) + \gamma_L(P_{k,2}) - 2$ for $u, k \geq 4$.

Theorem 5. *For $u \geq 2$ and $k \geq 4$ of $Bt_u + P_{k,2}$, $\gamma_L(Bt_u + P_{k,2}) \leq u + k - 1$.*

Proof. The joint product graph $Bt_u + P_{k,2}$ is a connected graph with vertex set and edge set, respectively. $V(Bt_u + P_{k,2}) = \{a_r;\ r = 1, 2\} \cup \{x_r;\ r = 1, \dots, u\} \cup \{y_s^r;\ s = 1, 2 \text{ and } r = 1, \dots, k\}$ and $E(Bt_u + P_{k,2}) = \{a_1a_2\}$

$\cup \{a_r x_s;\ r = 1, 2$ and $s = 1, \ldots, u\} \cup \{y_s^r y_{s+1}^r;\ r = 1, 2$ and $s = 1, \ldots, k-1\} \cup \{y_1^r y_k^r;\ r = 1, 2\} \cup \{y_r^1 y_r^2;\ r = 1, \ldots, k\} \cup \{a_r y_t^s;\ r = 1, 2, s = 1, 2$ and $t = 1, \ldots, k\} \cup \{x_r y_t^s;\ r = 1, \ldots, u, s = 1, 2$ and $t = 1, \ldots, k\}$. The cardinalities of $Bt_u + P_{k,2}$ are $|V(Bt_u + P_{k,2})| = u + 2k + 2$ and $|E(Bt_u + P_{k,2})| = 2uk + 2u + 7k + 1$.

We will show that $\gamma_L(Bt_u + P_{k,2}) \le u + k - 1$ by choosing $D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\} \cup \{a_1\}$ as the dominator set of $Bt_u + P_{k,2}$ for $u \ge 2$ and $k \ge 4$ so that $|D| = u + k - 1$, and the non-dominator set of $Bt_u + P_{k,2}$ is $V - D = \{a_2, x_u, y_k^1, y_k^2\} \cup \{y_s^2, s = 1, \ldots, k-1\}$. Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D as follows:

$$N(a_2) \cap D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\} \cup \{a_1\},$$

$$N(x_u) \cap D = \{y_s^1;\ s = 1, \ldots, k-1\} \cup \{a_1\},$$

$$N(y_k^1) \cap D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{a_1, y_1^1, y_{k-1}^1\},$$

$$N(y_s^2) \cap D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_s^1\} \cup \{a_1\};\ s = 1, \ldots, k-1,$$

$$N(y_k^2) \cap D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{a_1\}.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D are different and also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $Bt_u + P_{k,2}$ and fulfills the condition of LDS. Thus, $\gamma_L(Bt_u + P_{k,2}) \le u + k - 1$ for $u \ge 2$ and $k \ge 4$. □

Since $\gamma_L(Bt_u) = u$ [7] and $\gamma_L(P_{u,2}) = u$ [6], the relation between

LDS of $Bt_u + P_{k,2}$ with its basic graph is $\gamma_L(BT_u + P_{k,2}) \le \gamma_L(Bt_u) + \gamma_L(P_{k,2}) - 1$ for $u \ge 2$ and $k \ge 4$.

Theorem 6. *For* $u, k \ge 3$ *of* $S_u + S_k$, $\gamma_L(S_u + S_k) \le u + k - 1$.

Proof. The joint product graph $S_u + S_k$ is a connected graph with vertex set and edge set, respectively. $V(S_u + S_k) = \{a, b\} \cup \{x_r; r - 1, ..., u\} \cup \{y_r; r = 1, ..., k\}$ and $E(S_u + S_k) = \{ax_r; r = 1, ..., u\} \cup \{by_r; r = 1, ..., k\} \cup \{ab\} \cup \{ay_r; r = 1, ..., k\} \cup \{bx_r; r = 1, ..., u\} \cup \{x_r y_s; i = 1, ..., u$ and $s = 1, ..., k\}$. The cardinalities of $S_u + S_k$ are $|V(S_u + S_k)| = u + k + 2$ and $|E(S_u + S_k)| = uk + 2(u + k) + 1$.

We will show that $\gamma_L(S_u + S_k) \le u + k - 1$ by choosing $D = \{x_r; r = 1, .., u - 1\} \cup \{y_s; s = 1, .., k - 1\} \cup \{b\}$ as the dominator set of $S_u + S_k$ for $u, k \ge 3$ so that $|D| = u + k - 1$, and the non-dominator set of $S_u + S_k$ is $V - D = \{a, x_u, y_k\}$. Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D as follows:

$$N(a) \cap D = \{x_r; r = 1, ..., u - 1\} \cup \{y_s; s = 1, ..., k - 1\} \cup \{b\},$$

$$N(x_u) \cap D = \{y_s; s = 1, ..., k - 1\} \cup \{b\},$$

$$N(y_k) \cap D = \{x_r; r = 1, ..., u - 1\} \cup \{b\}.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D are different and also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $S_u + S_k$ and fulfills the condition of LDS. Thus, $\gamma_L(S_u + S_k) \le u + k - 1$ for $u, k \ge 3$. □

Since $\gamma_L(S_u) = u$ [6], the relation between LDS of $S_u + S_k$ with its basic graph is $\gamma_L(S_u + S_k) \le \gamma_L(S_u) + \gamma_L(S_k) - 1$ for $u, k \ge 3$.

Theorem 7. *For* $u \geq 3$ *and* $k \geq 4$ *of* $S_u + P_{k,2}$, $\gamma_L(S_u + P_{k,2}) \leq u + k - 2$.

Proof. The joint product graph $S_u + P_{k,2}$ is a connected graph with vertex set and edge set, respectively. $V(S_u + P_{k,2}) = \{a\} \cup \{x_r;\ r = 1, \ldots, u\} \cup \{y_s^r;\ r = 1, 2 \text{ and } s = 1, \ldots, k\}$ and $E(S_u + P_{k,2}) = \{ax_r;\ r = 1, \ldots, u\} \cup \{y_s^r y_{s+1}^r;\ r = 1, 2 \text{ and } s = 1, \ldots, k-1\} \cup \{y_1^r y_k^r;\ r = 1, 2\} \cup \{y_r^1 y_r^2;\ r = 1, \ldots, k\} \cup \{ay_s^r;\ r = 1, 2 \text{ and } s = 1, \ldots, k\} \cup \{x_r y_t^s;\ r = 1, \ldots, u,\ s = 1, 2 \text{ and } t = 1, \ldots, k\}$. The cardinalities of $S_u + P_{k,2}$ are $|V(S_u + P_{k,2})| = u + 2k + 1$ and $|E(S_u + P_{k,2})| = 2uk + u + 5k$.

We will show that $\gamma_L(S_u + P_{k,2}) \leq u + k - 2$ by choosing $D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\}$ as the dominator set of $S_u + P_{k,2}$ for $u \geq 3$ and $k \geq 4$ so that $|D| = u + k - 2$, and the non-dominator set of $S_u + P_{k,2}$ is $V - D = \{a, x_u, y_k^1, y_k^2\} \cup \{y_s^2;\ s = 1, \ldots, k-1\}$. Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D as follows:

$$N(a) \cap D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\},$$

$$N(x_u) \cap D = \{y_s^1;\ s = 1, \ldots, k-1\},$$

$$N(y_k^1) \cap D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_1^1, y_{k-1}^1\},$$

$$N(y_s^2) \cap D = \{x_r;\ r = 1, \ldots, u-1\} \cup \{y_s^1\};\ s = 1, \ldots, k-1,$$

$$N(y_k^2) \cap D = \{x_r;\ r = 1, \ldots, u-1\}.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D are different and also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $S_u + P_{k,2}$ and fulfills the condition of LDS. Thus, $\gamma_L(S_u + P_{k,2}) \leq u + k - 2$ for $u \geq 3$ and $k \geq 4$. □

Since $\gamma_L(S_u) = u$ and $\gamma_L(P_{u,2}) = u$ [6], the relation between LDS of $S_u + P_{k,2}$ with its basic graph is $\gamma_L(S_u + P_{k,2}) \leq \gamma_L(S_u) + \gamma_L(P_{k,2}) - 2$ for $u \geq 2$ and $k \geq 4$.

Theorem 8. *For* $u \geq 3$ *and* $k \geq 4$ *of* $P_{u,2} + P_{k,2}$, $\gamma_L(P_{u,2} + P_{k,2}) \leq u + k - 2$.

Proof. The joint product graph $P_{u,2} + P_{k,2}$ is a connected graph with vertex set and edge set, respectively. $V(P_{u,2} + P_{k,2}) = \{x_s^r;\ r = 1, 2$ and $s = 1, \ldots, u\} \cup \{y_s^r;\ r = 1, 2$ and $s = 1, \ldots, k\}$ and $E(P_{u,2} + P_{k,2}) = \{x_s^r x_{s+1}^r;$ $r = 1, 2$ and $s = 1, \ldots, u-1\} \cup \{x_1^r x_u^r;\ r = 1, 2\} \cup \{x_r^1 x_r^2;\ r = 1, \ldots, u\} \cup \{y_s^r y_{s+1}^r;$ $r = 1, 2$ and $s = 1, \ldots, k-1\} \cup \{y_1^r y_m^r;\ i = 1, 2\} \cup \{y_i^1 y_i^2;\ i = 1, \ldots, m\} \cup \{x_j^i y_l^k;$ $i = 1, 2,\ j = 1, \ldots, n,\ k = 1, 2$ and $l = 1, \ldots, k\}$. The cardinalities of $P_{u,2} + P_{k,2}$ are $|V(P_{u,2} + P_{k,2})| = 2(u + k)$ and $|E(P_{u,2} + P_{k,2})| = 4uk + 3u + 3k$.

We will show that $\gamma_L(P_{u,2} + P_{k,2}) \leq u + k - 2$ by choosing $D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\}$ as the dominator set of $P_{u,2} + P_{k,2}$ for $u \geq 3$ and $k \geq 4$ so that $|D| = u + k - 2$, and the non-dominator set of $P_{u,2} + P_{k,2}$ is $V - D = \{x_u^1, x_u^2, y_k^1, y_k^2\} \cup \{x_r^2;\ r = 1, \ldots, u-1\}$ $\cup \{y_r^2;\ r = 1, \ldots, k-1\}$. Furthermore, we can determine the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D as follows:

$$N(x_u^1) \cap D = \{x_1^1, x_{u-1}^1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\},$$

$$N(y_k^1) \cap D = \{y_1^1, y_{k-1}^1\} \cup \{x_r^1;\ r = 1, \ldots, u-1\},$$

$$N(x_u^2) \cap D = \{y_s^1;\ s = 1, \ldots, m-1\},$$

$$N(y_k^2) \cap D = \{x_r^1;\ r = 1, \ldots, u-1\},$$

$$N(x_r^2) \cap D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{y_s^1;\ s = 1, \ldots, k-1\},$$

$$N(y_s^2) \cap D = \{x_r^1;\ r = 1, \ldots, u-1\} \cup \{y_j^1\};\ s = 1, \ldots, k-1.$$

Based on the explanation above, the intersection between the neighborhood $N(v)$ with $v \in V(G) - D$ and the dominator set D are different and also not an empty set, so that we can say that the dominator set D *dominates* all the vertices on $P_{u,2} + P_{k,2}$ and fulfills the condition of LDS. Thus, $\gamma_L(P_{u,2} + P_{k,2}) \le u + k - 2$ for $u \ge 3$ and $k \ge 4$. □

Since $\gamma_L(P_{u,2}) = u$ [6], the relation between LDS of $P_{u,2} + P_{k,2}$ with its basic graph is $\gamma_L(P_{u,2} + P_{k,2}) \le \gamma_L(P_{u,2}) + \gamma_L(P_{k,2}) - 2$ for $u \ge 3$ and $k \ge 4$.

3. Concluding Remarks

In this paper, we have determined upper bound of locating domination number of some joint product of graphs $G + H$. But, it still gives the following open problem:

Open problem 1. For a connected graph G, determine $\gamma_L(G)$ in any of the operation graphs.

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