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Non-Isolated Resolving Number of Graph with Pendant Edges

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We consider V, E are respectively vertex and edge sets of a simple, nontrivial and connected graph G. For an ordered set $W = \{w_1, w_2, w_3, \ldots, w_k\}$ of vertices and a vertex $v \in G$, the ordered $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ of k-vector is representations of v with respect to W, where $d(v, w)$ is the distance between the vertices v and w. The set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. The metric dimension, denoted by $dim(G)$ is min of $|W|$. Furthermore, the resolving set W of graph G is called non-isolated resolving set if there is no $\forall v \in W$ induced by non-isolated vertex. While a non-isolated resolving number, denoted by $nr(G)$, is the minimum cardinality of non-isolated resolving set in graph. In this paper, we study the non isolated resolving number of graph with any pendant edges.

Keywords: Non isolated resolving number; non isolated resolving set; graph with pendant edges.

1. Introduction

In this paper, we consider that a graph $G = (V, E)$ is a connected graph, for more detail definition of graph in Refs. 1 and 2. The concept of metric dimension was independently introduced by Slater[3] and Harary and Melter.[4] In his paper, Slater called this concept as a locating set.

The metric dimension have been used to describe navigation in networks. In a network, each place represented as vertices and edges denote the connections between vertices. The

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vertices of a network where we place the machines (robots) are called landmarks. The minimum number of machines required to locate each and vertex of the network is termed as metric dimension and the set of all minimum possible number of landmarks constitute metric basis, for more detail this application in Ref. 5.

The distance $d(u, v)$ is the length of a shortest path between two vertices u and v in connected graph G. For an ordered set $W = \{w_1, w_2, ..., w_k\}$ subset of vertex set $V(G)$. The representation $r(v|W)$ of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is called resolving set of G if every vertices of G have distinct representation respect to W, let u and v be two any vertices in G if $r(u|W) = r(v|W)$ implies that $u = v$. Hence if W is a resolving set of cardinality k for a graph G, then the representation set $r(v|W), v \in V(G)$ consists of $|V(G)|$ distinct k-vector. A resolving set of minimum cardinality for a graph G is called a minimum resolving set for G and this cardinality is the metric dimension of G, denoted by $dim(G)$.

Saenpholphat and Zhang[8] introduced the concept of connected resolving set. A resolving set W of graph G is connected if each subgraph $\langle W \rangle$ induced by W has no isolated vertices in G. The minimum cardinality of a non-isolated set in a graph G is the non-isolated resolving number, denoted by $nr(G)$. For more detail notation of $nr(G)$ please see in Chitra and Arumugam.[6]

Until today, Chartrand *et al.*[7] determined the bounds of the metric dimensions for any connected graphs and determined the metric dimensions of some well known families of graphs such as tree, path, and complete graph. Saenpholphat *et al.* in Ref. 8 studied the connected partition dimension of graphs. On the other hand, Chitra and Arumugam[6] determined resolving Sets without Isolated vertices of some special graph and graph resulting of cartesian product. Furthermore, Baca *et al.*[9] showed the metric dimension of regular bipartite graphs, while Iswadi *et al.*[10] studied the metric dimension of corona product of graphs. Moreover Rodriguez-Velazquez *et al.*[11] showed the metric dimension of corona and strong product graph, Simanjuntak *et al.*[12] showed metric dimension of amalgamation of graphs, Saputro *et al.* in Ref. 13 described the metric dimension of a graph composition products with star. The last, Yero *et al.* in Ref. 14 obtained the metric dimension number of corona product graphs. Dafik *et al.*[15] studied non-isolated resolving number of special graphs and their operations and Alfarisi *et al.*[16,17] showed non-isolated resolving number of k-corona product of graphs and graphs with homogeneous pendant edges. We present some known results as follows.

Theorem 1.1 (Chitra and Arumugam).[6] *Let T be a tree which is not a path. Let s denote the number of vertices v of T and l_v is a leaf in T with $l_v > 1$. Then $nr(T) = dim(T) + s$.*

The results of non-isolated resolving number of well known graph are as follows.

Proposition 1.2 (Chitra and Arumugam).[6] *Let G be a connected graph of order $n \geq 2$*

- *For the path P_n, $n \geq 2$, $nr(P_n) = 2$.*
- *For the complete graph K_n, $n \geq 3$, $nr(K_n) = n - 1$.*
- *For the complete bipartite graph $K_{m,n}$, $m, n \geq 2$, $nr(K_{m,n}) = m + n - 2$.*

- *For the friendship graph G with k-triangles, $k \geq 2$, $nr(G) = k + 1$.*
- *For the graph $P_n + K_1$, $n \geq 2$, $nr(P_n + K_1) = \lfloor n/2 \rfloor$.*

Let G be a connected graph of order n and H (not necessarily connected) be a graph of order m. A graph G corona product H, $G \odot H$, is defined as a graph obtained by taking one copy of G and $|V(G)|$ copies of graph $H_1, H_2, ..., H_n$ of H and connecting i-th vertex of G to the vertices of H_i, $1 \leq i \leq n$. By definition of corona product, we can say that

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$

$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} (E(H_i) \cup \{iu_i | u_i \in V(H_i)\}),$$

where $H_i \cong H$, for all $i \in V(G)$. If $H \cong K_1$, $G \odot H$ is equal to the graph produced by adding one pendant edge to every vertex of G. Generally, if $H \cong mK_1$ where mK_1 is union of trivial graph K_1, $G \odot H$ is equal to the graph produced by adding one m pendant edge to every vertex of G. We determine the metric dimension with non-isolated resolving set of graphs with pendant edges.

2. Main Results

In this section, we will determine the non-isolated resolving number of graph with pendant edges. Next, we will use the idea of Saenpholphat and Zhang in Ref. 8 have introduced the concept of distance similar in a graph G. The open neighbourhood $N(v)$ of a vertex v in a graph G is all of vertices in graph G which adjacent to v and the close neighbourhood $N[v]$ of a vertex v in a graph G is $N(v) \cup \{v\}$. For two any vertices u and v of a connected graph G are defined to be distance similar if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. They can be found some of their properties in the following observation.

Observation 2.1 (Saenpholphat and Zhang).[8] *Two vertices u and v of a connected graph G are distance similar if and only if*

(1) $uv \notin E(G)$ and $N(u) = N(v)$ or
(2) $uv \in E(G)$ and $N[u] = N[v]$

Observation 2.2 (Saenpholphat and Zhang).[8] *Distance similarity in a connected graph G is an equivalence relation on $V(G)$.*

Observation 2.3 (Saenpholphat and Zhang).[8] *If U is a distance similar equivalence class of a connected graph G, then U is either independent in G or in $\bar{G}$.*

Observation 2.4 (Saenpholphat and Zhang).[8] *If U is a distance similar equivalence class of a connected graph G with $|U| = p \geq 2$, then every resolving set of G contains at least $p - 1$ vertices from U.*

The following theorem determine the non-isolated resolving number of a graphs H, where H is a graph with pendant edges more general that of corona product $G \odot mK_1$. Every i-th vertex in G is joining with m_i number of pendant edges with $m_i \geq 2$, $1 \leq i \leq |V(G)|$.

Theorem 2.1. Let G be a connected graph, H be a graph with m_i pendant edges, $m_i \geq 2$ and $1 \leq i \leq |V(G)|$, then non-isolated resolving number of H is

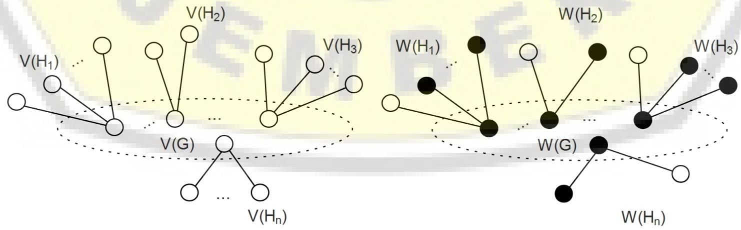
$$nr(H) \geq |V(G)| + \Sigma_{i=1}^{|V(G)|}(m_i - 1)$$

Proof: Let G be a graph of order $n \geq 2$ and vertex set $V(G) = \{u_i : 1 \leq i \leq n\}$ and $V((m_iK_1)_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m_i}\}$ with $1 \leq i \leq n$ $m_1 \neq m_2 \neq \cdots \neq m_n$ is copy of m_iK_1 by joining with u_i and $|V(G)|$ is the cardinality of vertices G. Let H be a connected graph with vertex set $V(H) = V(G) \cup \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ and the edge set $E(H) = E(G) \cup \{u_iv_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m_i\}$.

For every $i \in \{1, 2, \ldots, n\}$, every pair vertices $u, v \in (m_iK_1)_i$ holds $d(u, x) = d(v, x)$ where $x \in V(H) = \{u, v\}$. Thus, every subgraph $(m_iK_1)_i$ are a distance similar equivalence class of H. Based on Observation 2.4, we have resolving set W_i for every subgraph $(m_iK_1)_i$ at least $(m_i - 1)$, $1 \leq i \leq n$ where $W = \bigcup_{i \in V(G)}(W_i)$, $W_i \subset V((m_iK_1)_i)$. There is isolated vertex in W such that W is not resolving set with non-isolated vertex. If we will show that W is resolving set with non-isolated vertex, then we need to show that every vertices $v \in W_i \subset V((m_iK_1)_i)$ connected to each i-th vertex in graph G by edge set $\{u_iv_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ such that

$$\begin{aligned} |W| &\geq |\bigcup_{i \in V(G)}(W_i)| + |V(G)| \\ &\geq \Sigma_{i=1}^{|V(G)|}(m_i - 1) + |V(G)| \end{aligned}$$

It is easy to see that non-isolated resolving set W at least $\Sigma_{i=1}^{|V(G)|}(m_i - 1) + |V(G)|$ or the lower bound of non-isolated resolving number is $nr(H) \geq \Sigma_{i=1}^{|V(G)|}(m_i - 1) + |V(G)|$.



(a) (b)

Fig. 1. (a) Example of graph with pendant edges; (b) Example of non-isolated resolving set.

In the next theorem, we will determine the exact value of non isolated resolving number of $G \odot mK_1$ where G is a special family of graphs. The families are the path and cycle graphs, the complete graphs, and the star graph and wheel graphs.

Furthermore, non-isolated resolving number will be discussed with $G \cong P_n$, $G \odot K_1$ is isomorphic to a caterpillar $C_{n,1}$. For $n = 1$, then $P_1 \odot K_1$ isomorphic with P_2. Based on Proposition 1.2, we have $nr(P_2) = 1$ and for $n = 2$, then $P_2 \odot K_1$ isomorphic with P_3. Based on Proposition 1.2, we have $nr(P_3) = 1$. For $n \geq 3$, the non-isolated resolving number of $P_n \odot K_1$ in the following.

Theorem 2.2. Consider the path graph P_n for $n \geq 3$. Then

$$nr(P_n \odot K_1) = n$$

Proof: Let $P_n \odot K_1$ be a connected graph with vertex set $V(P_n \odot K_1) = \{u_i; 1 \leq i \leq n\} \cup \{v_j; 1 \leq j \leq n\}$ and edge set $E(P_n \odot K_1) = \{u_iu_{i+1}; 1 \leq i \leq n-1\} \cup \{u_iv_j; i = j, 1 \leq i \leq n\}$ where u_i is vertex in path P_n and v_j is a pendant vertex of u_i. The cardinality of vertices $|V(P_n \odot K_1)| = 2n$ and the cardinality of edges $|E(P_n \odot K_1)| = 2n - 1$.

We will show that the lower bound of the non-isolated resolving number of $P_n \odot K_1$ is $nr(P_n \odot K_1) \geq n$. We assume that non-isolated resolving set W of $P_n \odot K_1$ with $|W| < n$. Without loss of generality, we choose $W = \{u_1, u_2, \ldots, u_{n-1}\}$ then there are at least two vertices v_{n-1} and $u_n \in V(P_n \odot K_1)$ such that have the same representation, namely $r(v_{n-1}|W) = \underbrace{(n-1, n-2, \ldots, 2, 1)}_{n-1\ times} = r(u_n|W)$, it is a contradiction. Hence, the lower bound of the non-isolated resolving number of $P_n \odot K_1$ is $nr(P_n \odot K_1) \geq n$.

Furthermore, we will show that the upper bound of the non-isolated resolving number of $P_n \odot K_1$ is $nr(P_n \odot K_1) \leq n$. We Choose $W \subset V(P_n \odot K_1)$ with $W = \{u_1, u_2, \ldots, u_n\}$ as a non-isolated resolving set of $P_n \odot K_1$. The cardinality of non-isolated resolving set is $|W| = |\{u_1, u_2, \ldots, u_n\}| = n$. Thus, the representation of vertices $v \in V(P_n \odot K_1) - W$ respect to W are as follows.

$r(v_j|W) = (t'_{j-1}, ..., t'_1, 1, t_1, ..., t_{n-j})$ for $t_k = k+1$, $1 \leq k \leq n-j$ and $t'_l = l+1$, $1 \leq l \leq j-1$, $1 \leq j \leq n$.

It is clear that every vertex $v \in V(P_n \odot K_1) - W$ has a distinct representation with respect to W. Furthermore, we need to shown that all vertices in W without isolated vertex. All vertices in vertex set $W = \{u_1, u_2, \ldots, u_n\}$ are connected with the edge set $\{u_iu_{i+1}; 1 \leq i \leq n-1\}$ which all vertices in W induces subgraph P_n. Hence, $\langle W \rangle$ has no isolated vertices. Thus, the upper bound of the non-isolated resolving number of $P_n \odot K_1$ is $nr(P_n \odot K_1) \leq n$.

Hence, we obtain that $n \leq nr(P_n \odot K_1) \leq n$, then it concludes that the non-isolated resolving number of $P_n \odot K_1$ is $nr(P_n \odot K_1) = n$

The star graph S_n is a tree with one central vertex and n leaves, for $n = 1$, then $S_1 \odot K_1$ isomorphic with P_4. Based on Proposition 1.2, we have $nr(P_4) = 1$ and for $n = 2$, then $S_2 \odot K_1$ isomorphic with $P_3 \odot K_1$. Based on Theorem 2.2, we have $nr(P_3 \odot K_1) = 3$ such that $nr(S_2 \odot K_1) = 3$. For $n \geq 3$, the non-isolated resolving number of $S_n \odot K_1$ in the following.

Theorem 2.3. Consider the star graph S_n for $n \geq 3$. Then

$$nr(S_n \odot K_1) = n + 1$$

Proof: Let $S_n \odot K_1$ be a connected graph with vertex set $V(S_n \odot K_1) = \{u, v, u_i, v_j; 1 \leq i \leq n\}$ and edge set $E(P_n \odot K_1) = \{uv, uu_i, u_iv_i; 1 \leq i \leq n\}$ where u is central vertex of star S_n and v, v_i is a pendant vertex of u, u_i with the cardinality of vertices $|V(S_n \odot K_1)| = 2n + 2$ and the cardinality of edges $|E(P_n \odot K_1)| = 2n + 1$.

We will show that the lower bound of the non-isolated resolving number of $S_n \odot K_1$ is $nr(S_n \odot K_1) \geq n + 1$. We assume that non-isolated resolving set W of $S_n \odot K_1$ with $|W| < n + 1$. Without loss of generality, we can describe the reasons as follows:

(a) If we choose $W = \{u_1, u_2, \ldots, u_n\}$, then all vertices $u' \in V(S_n \odot K_1) - W$ has the distinct representation. Based on edge set uu_i for $1 \leq i \leq n$ such that there is isolated vertex in W. Hence, $\langle W \rangle$ has isolated vertices, it is a contradiction.

(b) If we choose $W = \{u_1, u_2, \ldots, u_{n-1}\} \cup \{u\}$, then based on edge set uu_i for $1 \leq i \leq n$ such that every vertices in W are connected. Hence, $\langle W \rangle$ has no isolated vertices but there are at least two vertices u_n and $v \in V(S_n \odot K_1)$ which have the same representation, namely $r(u_n|W) = (\underbrace{2, ..., 2}_{n-1\ times}) = r(v|W)$, it is a contradiction.

Furthermore, we will show that the upper bound of the non-isolated resolving number of $S_n \odot K_1$ is $nr(S_n \odot K_1) \leq n + 1$. We choose $W \subset V(S_n \odot K_1)$ with $W = \{u_1, u_2, \ldots, u_n\} \cup \{u\}$ is a non-isolated resolving set of $S_n \odot K_1$ and the cardinality of non-isolated resolving set is $|W| = |\{u_1, u_2, \ldots, u_n\}| + |\{u\}| = n + 1$. Thus, the representation of vertices $v \in V(S_n \odot K_1) - W$ respect to W are as follows.

$$r(v|W) = (\underbrace{2, ..., 2}_{n\ times}, 1)$$

$$r(v_i|W) = (\underbrace{3, ..., 3}_{i-1\ times}, 1, \underbrace{3, ..., 3}_{n-i\ times}, 2) : 1 \leq i \leq n$$

It can be seen that all representation of every vertices $v \in V(S_n \odot K_1) - W$ respect to W are distinct. Furthermore, we need to shown that all vertices in W without isolated vertex. All vertices in vertex set $W = \{u_1, u_2, \ldots, u_n\} \cup \{u\}$ are connected by the edge set $\{uu_i; 1 \leq i \leq n\}$ which all vertices in W induces subgraph S_n. Hence, $\langle W \rangle$ has no isolated vertices. So, the upper bound non-isolated resolving number of $S_n \odot K_1$ is $nr(S_n \odot K_1) \leq n + 1$.

Thus, the lower bound and upper bound of the non-isolated resolving number of $S_n \odot K_1$ are $n + 1 \leq nr(S_n \odot K_1) \leq n + 1$. It concludes that the non-isolated resolving number of $S_n \odot K_1$ is $nr(S_n \odot K_1) = n + 1$.

Let $G \cong K_n$ be a complete graph, for $n = 1$, then $K_1 \odot K_1$ isomorphic with P_2. Based on Proposition 1.2, we have $nr(P_2) = 1$ and for $n = 2$, then $K_2 \odot K_1$ isomorphic with

$P_2 \odot K_1$. Based on Proposition 1.2, we have $nr(P_4) = 1$. For $n \geq 3$, the non-isolated resolving number of $K_n \odot K_1$ in the following.

Theorem 2.4. Consider the complete graph K_n for $n \geq 3$. Then

$$nr(K_n \odot K_1) = n - 1$$

Proof: Let $K_n \odot K_1$ be a connected graph with vertex set $V(K_n \odot K_1) = \{u_i, v_i; 1 \leq i \leq n\}$ and edge set $E(K_n \odot K_1) = \{u_i u_{i+k}; 1 \leq i \leq n, 1 \leq k \leq n-i\} \cup \{u_i v_i; 1 \leq i \leq n\}$ where u_i is a vertex of complete graph K_n and v_i is a pendant vertex of u_i with the cardinality of vertices $|V(K_n \odot K_1)| = 2n$ and the cardinality of edges $|E(K_n \odot K_1)| = \frac{n(n-1)}{2} + n$.

We will show that the lower bound of the non-isolated resolving number of $K_n \odot K_1$ is $nr(K_n \odot K_1) \geq n-1$. We assume that non-isolated resolving set W of $K_n \odot K_1$ with $|W| < n-1$. Without loss of generality, If we choose $W = \{u_1, u_2, \ldots, u_{n-2}\}$ then based on edge set $u_i u_{i+k}$ for $1 \leq i \leq n$ and $1 \leq k \leq n-i$ then every vertices in W are connected. Hence, $\langle W \rangle$ has non-isolated vertex but there are at least two vertices u_n and $u_{n-1} \in V(K_n \odot K_1)$ which have the same representation, namely $r(u_n|W) = (\underbrace{1, ..., 1}_{n-2\ times}) = r(u_{n-1}|W)$ or two vertices in pendant edges namely v_n and $v_{n-1} \in V(K_n \odot K_1)$ have the same representation, namely $r(v_n|W) = (\underbrace{2, ..., 2}_{n-2\ times}) = r(v_{n-1}|W)$, it is a contradiction.

Furthermore, we will show that the upper bound of the non-isolated resolving number of $K_n \odot K_1$ is $nr(K_n \odot K_1) \leq n-1$. We choose $W \subset V(K_n \odot K_1)$ with $W = \{u_1, u_2, \ldots, u_{n-1}\}$ is a non-isolated resolving set of $K_n \odot K_1$ and the cardinality of non-isolated resolving set is $|W| = |\{u_1, u_2, \ldots, u_{n-1}\} = n - 1$. Thus, the representation of vertices $v \in V(K_n \odot K_1) - W$ respect to W are as follows.

$$r(v_i|W) = (\underbrace{2, ..., 2}_{i-1\ times}, 1, \underbrace{2, ..., 2}_{n-i-1\ times}); 1 \leq i \leq n-1$$

$$r(u_n|W) = (\underbrace{1, ..., 1}_{n-1\ times})$$

$$r(v_n|W) = (\underbrace{2, ..., 2}_{n-1\ times})$$

It is clear that all representation of every vertex $v \in V(K_n \odot K_1) - W$ respect to W are distinct. Furthermore, we need to shown that all vertices in W without isolated vertex. All vertices in vertex set $W = \{u_1, u_2, \ldots, u_{n-1}\}$ are connected by the edge set $\{u_i u_{i+k}; 1 \leq i \leq n, 1 \leq k \leq n-i\}$ which all vertices in W induces subgraph K_n. Hence, $\langle W \rangle$ has no isolated vertices. So, the upper bound non-isolated resolving number of $K_n \odot K_1$ is $nr(K_n \odot K_1) \leq n-1$.

Thus, the lower bound and upper bound of the non-isolated resolving number of $K_n \odot K_1$ are $n-1 \leq nr(K_n \odot K_1) \leq n-1$. It concludes that the non-isolated resolving number of $K_n \odot K_1$ is $nr(K_n \odot K_1) = n-1$.

Let $G \cong C_n$ be a cycle graph, for $n = 1$, then $K_1 \odot K_1$ isomorphic with P_2. Based on Proposition 1.2, we have $nr(P_2) = 1$, for $n = 2$, C_2 is not simple graph so we do not consider the $n = 2$ case computing $nr(C_n \odot K_1)$. For $n = 3$, then $C_3 \odot K_1$ isomorphic with $K_3 \odot K_1$. Based on Theorem 2.4, we have $nr(K_3 \odot K_1) = 2$ such that $nr(C_3 \odot K_1) = 2$. For $n \geq 4$, the non-isolated resolving number of $K_n \odot K_1$ in the following.

Theorem 2.5. Consider the cycle graph C_n for $n \geq 4$. Then

$$nr(C_n \odot K_1) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ odd} \\ \frac{n+2}{2} & \text{if } n \text{ even} \end{cases}$$

Proof: Let $C_n \odot K_1$ be a connected graph with vertex set $V(C_n \odot K_1) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\}$ and edge set $E(C_n \odot K_1) = \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i v_i; 1 \leq i \leq n\} \cup \{u_1 u_n\}$ where u_i is vertex in cycle C_n and v_i is a pendant vertex of u_i with the cardinality of vertices $|V(C_n \odot K_1)| = 2n$ and the cardinality of edges $|E(C_n \odot K_1)| = 2n$.

Case 1. For n odd, we will show that the lower bound of the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) \geq \frac{n+1}{2}$. We assume that non-isolated resolving set W of $C_n \odot K_1$ with $|W| < \frac{n+1}{2}$. Without loss of generality, we choose $W = \{u_1, u_2, \ldots, u_{\frac{n+1}{2}-1}\}$ then there are at least two vertices $v_{\frac{n+1}{2}-1}$ and $u_{\frac{n+1}{2}} \in V(C_n \odot K_1)$ which have the same representation, namely $r(v_{\frac{n+1}{2}-1}|W) = (t'_{\frac{n-3}{2}}, \ldots, t'_1, 1) = r(u_{\frac{n+1}{2}}|W)$ with $t'_l = l + 1$, $1 \leq l \leq \frac{n-3}{2}$, it is a contradiction. Hence, the lower bound of the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) \geq \frac{n+1}{2}$.

Furthermore, we will show that the upper bound of the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) \leq \frac{n+1}{2}$. We choose $W \subset V(C_n \odot K_1)$ with $W = \{u_1, u_2, \ldots, u_{\frac{n+1}{2}}\}$ is a non-isolated resolving set of $C_n \odot K_1$ and the cardinality of non-isolated resolving set is $|W| = |\{u_1, u_2, \ldots, u_{\frac{n+1}{2}}\}| = \frac{n+1}{2}$. Thus, the representation of vertices $v \in V(C_n \odot K_1) - W$ respect to W are as follows.

$r(v_i|W) = (t'_{i-1}, \ldots, t'_1, 1, t_1, \ldots, t_{\frac{n+1}{2}-i})$ for $t_k = k + 1$, $1 \leq k \leq \frac{n+1}{2} - i$ and $t'_l = l + 1$, $1 \leq l \leq i - 1$, $1 \leq i \leq \frac{n+1}{2}$,

$r(u_i|W) = (t'_{i-\frac{n+3}{2}}, \ldots, t'_1, \frac{n-1}{2}, \frac{n-1}{2}, t_1, \ldots, t_{n-i})$ for $t_k = \frac{n-1}{2} - k$, $1 \leq k \leq n - i$ and $t'_l = \frac{n-1}{2} - l$, $1 \leq l \leq i - \frac{n+3}{2}$, $\frac{n+3}{2} \leq i \leq n$,

$r(v_i|W) = (t'_{i-\frac{n+3}{2}}, \ldots, t'_1, \frac{n+1}{2}, \frac{n+1}{2}, t_1, \ldots, t_{n-i})$ for $t_k = \frac{n-1}{2} - k + 1$, $1 \leq k \leq n - i$ and $t'_l = \frac{n-1}{2} - l + 1$, $1 \leq l \leq i - \frac{n+3}{2}$, $\frac{n+3}{2} \leq i \leq n$.

It is clear that every vertex $v \in V(C_n \odot K_1) - W$ has a distinct representation with respect to W. Furthermore, we need to shown that all vertices in W without isolated vertex. All vertices in vertex set $W = \{u_1, u_2, \ldots, u_{\frac{n+1}{2}}\}$ are connected by the edge set $\{u_i u_{i+1}; 1 \leq i \leq \frac{n+1}{2} - 1\}$ which all vertices in W induces subgraph $P_{\frac{n+1}{2}}$. Hence, $\langle W \rangle$ has no isolated vertices. So, the upper bound non-isolated resolving number of $C_n \odot K_1$

is $nr(C_n \odot K_1) \leq \frac{n+1}{2}$ for n odd. It concludes that the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) = \frac{n+1}{2}$.

Case 2. For n even, we will show that the lower bound of the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) \geq \frac{n+2}{2}$. We assume that non-isolated resolving set W of $C_n \odot K_1$ with $|W| < \frac{n+2}{2}$. Without loss of generality, we choose $W = \{u_1, u_2, \ldots, u_{\frac{n}{2}}\}$ then there are at least two vertices $v_{\frac{n}{2}}$ and $u_{\frac{n+2}{2}} \in V(C_n \odot K_1)$ such that have the same representation, namely $r(v_{\frac{n}{2}}|W) = (t'_{\frac{n-2}{2}}, ..., t'_1, 1) = r(u_{\frac{n+2}{2}}|W)$ with $t'_l = l+1$, $1 \leq l \leq \frac{n-2}{2}$, a contradiction. Hence, the lower bound of the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) \geq \frac{n+2}{2}$.

Furthermore, we will show that the upper bound of the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) \leq \frac{n+2}{2}$. We choose $W \subset V(C_n \odot K_1)$ with $W = \{u_1, u_2, \ldots, u_{\frac{n+2}{2}}\}$ is a non-isolated resolving set of $C_n \odot K_1$ and the cardinality of non-isolated resolving set is $|W| = |\{u_1, u_2, \ldots, u_{\frac{n+2}{2}}\}| = \frac{n+2}{2}$. Thus, the representation of vertices $v \in V(C_n \odot K_1) - W$ respect to W are as follows.

$r(v_i|W) = (t'_{i-1}, ..., t'_1, 1, t_1, ..., t_{\frac{n+2}{2}-i})$ for $t_k = k+1$, $1 \leq k \leq \frac{n+2}{2} - i$ and $t'_l = l+1$, $1 \leq l \leq i-1$, $1 \leq i \leq \frac{n+2}{2}$,

$r(u_i|W) = (t'_{i-\frac{n+2}{2}}, ..., t'_1, \frac{n}{2}, \frac{n-2}{2}, t_1, ..., t_{n-i})$ for $t_k = \frac{n-2}{2} - k$, $1 \leq k \leq n-i$ and $t'_l = \frac{n-2}{2} - l + 1$, $1 \leq l \leq i - \frac{n+2}{2}$, $\frac{n+4}{2} \leq i \leq n$,

$r(u_i|W) = (t'_{i-\frac{n+2}{2}}, ..., t'_1, \frac{n+2}{2}, \frac{n}{2}, t_1, ..., t_{n-i})$ for $t_k = \frac{n-2}{2} - k + 1$, $1 \leq k \leq n-i$ and $t'_l = \frac{n-2}{2} - l + 2$, $1 \leq l \leq i - \frac{n+2}{2}$, $\frac{n+4}{2} \leq i \leq n$.

It is clear that every vertex $v \in V(C_n \odot K_1) - W$ has a distinct representation with respect to W. Furthermore, we need to shown that all vertices in W without isolated vertex. All vertices in vertex set $W = \{u_1, u_2, \ldots, u_{\frac{n+2}{2}}\}$ are connected by the edge set $\{u_i u_{i+1}; 1 \leq i \leq \frac{n}{2}\}$ which all vertices in W induces subgraph $P_{\frac{n+2}{2}}$. Hence, $\langle W \rangle$ has no isolated vertices. So, the upper bound non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) \leq \frac{n+2}{2}$ for n even. It concludes that the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) = \frac{n+2}{2}$ for n even.

In both cases, the non-isolated resolving number of $C_n \odot K_1$ is $nr(C_n \odot K_1) = \frac{n+1}{2}$ for n odd and $nr(C_n \odot K_1) = \frac{n+2}{2}$ for n even. It concludes the proof.

The wheel graph W_n can be defined as the graph $K_1 + C_n$ for $n \geq 3$, the non-isolated resolving number of $W_n \odot K_1$ in the following.

Theorem 2.6. Consider the wheel graph W_n for $n \geq 3$. Then

$$nr(W_n \odot K_1) = \begin{cases} \frac{n-1}{2} + 1 & \text{if } n \text{ odd and } n \neq 3 \\ \frac{n}{2} + 1 & \text{if } n \text{ even} \end{cases}$$

Proof: Let $W_n \odot K_1$ be a connected graph with vertex set $V(W_n \odot K_1) = \{u, u_i; 1 \leq i \leq n\} \cup \{v, v_i; 1 \leq i \leq n\}$ and edge set $E(W_n \odot K_1) = \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \{u_1 u_n\} \cup$

$\{uu_i; 1 \le i \le n\} \cup \{uv, u_iv_i; 1 \le i \le n\}$ where u, u_i is vertex in wheel graph W_n and v, v_i is a pendant vertices of u, u_i with the cardinality of vertices $|V(W_n \odot K_1)| = 2n + 2$ and the cardinality of edges $|E(W_n \odot K_1)| = 3n + 1$.

Case 1: For n odd, we will show that the lower bound of the non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) \ge \frac{n-1}{2} + 1$. We assume that non-isolated resolving set W of $W_n \odot K_1$ with $|W| < \frac{n-1}{2} + 1$. Without loss of generality, we can describe the reasons as follows:

(a) If we choose $W = \{u_i; 1 \le i \le n-1, i \text{ odd}\}$ then all vertices $u' \in V(W_n \odot K_1) - W$ has the distinct representation but based on edge set uu_i for $1 \le i \le n-1$, i odd then every vertices in W are not connected. Hence, $\langle W \rangle$ has isolated vertices, it is a contradiction.

(b) If we choose $W = \{u_i; 1 \le i \le n-3, i \text{ odd}\} \cup \{u\}$ then based on edge set uu_i for $1 \le i \le n-1$ then every vertices in W are connected. Hence, $\langle W \rangle$ has no isolated vertices but there are at least two vertices u_{n-1} and $v_{n-1} \in V(W_n \odot K_1)$ such that have the same representation, namely $r(u_{n-1}|W) = (\underbrace{2, ..., 2}_{\frac{n-3}{2}\ times}) = r(u_{n-2}|W)$, it is a contradiction.

Furthermore, we will show that the upper bound of the non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) \le \frac{n-1}{2} + 1$. We choose $W \subset V(W_n \odot K_1)$ with $W = \{u_i; 1 \le i \le n-1, i \text{ odd}\} \cup \{u\}$ is a non-isolated resolving set of $W_n \odot K_1$ and the cardinality of non-isolated resolving set is $|W| = |\{u_i; 1 \le i \le n-1, i \text{ odd}\} \cup \{u\}| = \frac{n-1}{2} + 1$. Thus, the representation of vertices $v \in V(W_n \odot K_1) - W$ respect to W are as follows.

$$r(u_i|W) = (\underbrace{2, ..., 2}_{\frac{i-2}{2}\ times}, 1, 1, \underbrace{2, ..., 2}_{\frac{n-i-3}{2}\ times}, 1) \text{ for } 1 \le i \le n-3, i \text{ even.}$$
$$r(v_i|W) = (\underbrace{3, ..., 3}_{\frac{i-2}{2}\ times}, 2, 2, \underbrace{3, ..., 3}_{\frac{n-i-3}{2}\ times}, 2) \text{ for } 1 \le i \le n-3, i \text{ even.}$$
$$r(v_i|W) = (\underbrace{3, ..., 3}_{\frac{i-1}{2}\ times}, 1, \underbrace{3, ..., 3}_{\frac{n-i-2}{2}\ times}, 2) \text{ for } 1 \le i \le n-1, i \text{ odd.}$$
$$r(u_{n-1}|W) = (\underbrace{2, ..., 2}_{\frac{n-3}{2}\ times}, 1, 1)$$
$$r(v_{n-1}|W) = (\underbrace{3, ..., 3}_{\frac{n-3}{2}\ times}, 2, 2)$$
$$r(u_n|W) = (1, \underbrace{2, ..., 2}_{\frac{n-3}{2}\ times}, 1)$$
$$r(v_n|W) = (2, \underbrace{3, ..., 3}_{\frac{n-3}{2}\ times}, 2)$$

$$r(v|W) = (\underbrace{2, ..., 2}_{\frac{n-1}{2}\ times}, 1)$$

It is clear that every vertex $v \in V(W_n \odot K_1) - W$ has a distinct representation with respect to W. Furthermore, we need to shown that all vertices in W without isolated vertex. All vertices in vertex set $W = \{u_i; 1 \leq i \leq n-1, i \text{ odd}\} \cup \{u\}$ are connected by the edge set $\{uu_i; 1 \leq i \leq n\}$ which all vertices in W induces subgraph S_n. Hence, $\langle W \rangle$ has no isolated vertices. So, the upper bound non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) \leq \frac{n-1}{2} + 1$.

Thus, the lower bound and upper bound of the non-isolated resolving number of $W_n \odot K_1$ are $\frac{n-1}{2} + 1 \leq nr(W_n \odot K_1) \leq \frac{n-1}{2} + 1$. It concludes that the non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) = \frac{n-1}{2} + 1$.

Case 2: For n even, we will show that the lower bound of the non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) \geq \frac{n}{2} + 1$. We assume that non-isolated resolving set W of $W_n \odot K_1$ with $|W| < \frac{n}{2} + 1$. Without loss of generality, we can describe the reasons as follows:

(a) If we choose $W = \{u_i; 1 \leq i \leq n-1, i \text{ odd}\}$ then all vertices $u' \in V(W_n \odot K_1) - W$ has the distinct representation but based on edge set uu_i for $1 \leq i \leq n-1$, i odd then there is a isolated vertex in W. Hence, $\langle W \rangle$ has isolated vertices, it is a contradiction.

(b) If we choose $W = \{u_i; 1 \leq i \leq n-3, i \text{ odd}\} \cup \{u\}$ then based on edge set uu_i for $1 \leq i \leq n-1$ then every vertices in W are connected. Hence, $\langle W \rangle$ has no isolated vertices but there are at least two vertices u_{n-1} and $v_{n-1} \in V(W_n \odot K_1)$ such that have the same representation, namely $r(u_{n-1}|W) = (\underbrace{2, ..., 2}_{\frac{n-2}{2}\ times}, 1) = r(v|W)$, it is a contradiction.

Furthermore, we will show that the upper bound of the non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) \leq \frac{n}{2} + 1$. We Choose $W \subset V(W_n \odot K_1)$ with $W = \{u_i; 1 \leq i \leq n-1, i \text{ odd}\} \cup \{u\}$ is a non-isolated resolving set of $W_n \odot K_1$ and the cardinality of non-isolated resolving set is $|W| = |\{u_i; 1 \leq i \leq n-1, i \text{ odd}\} \cup \{u\}| = \frac{n}{2} + 1$. Thus, the representation of vertices $v \in V(W_n \odot K_1) - W$ respect to W are as follows.

$$r(u_i|W) = (\underbrace{2, ..., 2}_{\frac{i-2}{2}\ times}, 1, 1, \underbrace{2, ..., 2}_{\frac{n-i-3}{2}\ times}, 1) \text{ for } 1 \leq i \leq n-2, i \text{ even.}$$

$$r(v_i|W) = (\underbrace{3, ..., 3}_{\frac{i-2}{2}\ times}, 2, 2, \underbrace{3, ..., 3}_{\frac{n-i-3}{2}\ times}, 2) \text{ for } 1 \leq i \leq n-2, i \text{ even.}$$

$$r(v_i|W) = (\underbrace{3, ..., 3}_{\frac{i-1}{2}\ times}, 1, \underbrace{3, ..., 3}_{\frac{n-i-1}{2}\ times}, 2) \text{ for } 1 \leq i \leq n-1, i \text{ odd.}$$

$$r(u_n|W) = (1, \underbrace{2, ..., 2}_{\frac{n-4}{2}\ times}, 1, 1)$$
$$r(v_n|W) = (2, \underbrace{3, ..., 3}_{\frac{n-4}{2}\ times}, 2, 2)$$
$$r(v|W) = (\underbrace{2, ..., 2}_{\frac{n}{2}\ times}, 1)$$

It is clear that every vertex $v \in V(W_n \odot K_1) - W$ has a distinct representation with respect to W. Furthermore, we need to shown that all vertices in W without isolated vertex. All vertices in vertex set $W = \{u_i; 1 \leq i \leq n-1, i \text{ odd}\} \cup \{u\}$ are connected by the edge set $\{uu_i; 1 \leq i \leq n\}$ which all vertices in W induces subgraph S_n. Hence, $\langle W \rangle$ has no isolated vertices. So, the upper bound non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) \leq \frac{n}{2} + 1$.

Thus, the lower bound and upper bound of the non-isolated resolving number of $W_n \odot K_1$ are $nr(W_n \odot K_1) \geq \frac{n}{2} + 1$. It concludes that the non-isolated resolving number of $W_n \odot K_1$ is $nr(W_n \odot K_1) = \frac{n}{2} + 1$.

3. Conclusion

We have shown the non-isolated resolving number of graph with pendant edges. The results show that the non-isolated resolving number attain the best lower bound. However we have not found the sharpest lower bound for any connected graph, therefore we proposed the following open problem.

Open Problem 3.1. Find the non-isolated resolving number of $G \odot mK_1$, with G is connected graph of order n and $m \geq 2$.

Open Problem 3.2. Find the non-isolated resolving number of H, where H is a graph with pendant edge more general that of corona product $G \odot mK_1$ with G is connected graph of order n and $m \geq 1$.

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