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Research Article

On H-Supermagic Labelings of m-Shadow of Paths and Cycles

Ika Hesti Agustin,[1,2] **F. Susanto**,[2] **Dafik**,[1,3] **R. M. Prihandini**,[1,4]
R. Alfarisi,[1,4] **and I. W. Sudarsana**[5]

[1] *CGANT Research Group, University of Jember, Indonesia*
[2] *Department of Mathematics, University of Jember, Indonesia*
[3] *Department of Mathematics Education, University of Jember, Indonesia*
[4] *Department of Elementary School Teacher Education, University of Jember, Indonesia*
[5] *Combinatorial and Applied Mathematics Research Group, University of Tadulako, Indonesia*

Correspondence should be addressed to Dafik; d.dafik@unej.ac.id

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A simple graph $G = (V, E)$ is said to be an H-covering if every edge of G belongs to at least one subgraph isomorphic to H. A bijection $f : V \cup E \longrightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ is an (a,d)-H-antimagic total labeling of G if, for all subgraphs H' isomorphic to H, the sum of labels of all vertices and edges in H' form an arithmetic sequence $\{a, a + d, \ldots, (k-1)d\}$ where $a > 0, d \geq 0$ are two fixed integers and k is the number of all subgraphs of G isomorphic to H. The labeling f is called *super* if the smallest possible labels appear on the vertices. A graph that admits (super) (a, d)-H-antimagic total labeling is called (super) (a, d)-H-antimagic. For a special $d = 0$, the (super) $(a, 0)$-H-antimagic total labeling is called H-(super)magic labeling. A graph that admits such a labeling is called H-(super)magic. The m-shadow of graph G, $D_m(G)$, is a graph obtained by taking m copies of G, namely, $G_1, G_2, \ldots, G_m$, and then joining every vertex u in G_i, $i \in \{1, 2, \ldots, m-1\}$, to the neighbors of the corresponding vertex v in G_{i+1}. In this paper we studied the H-supermagic labelings of $D_m(G)$ where G are paths and cycles.

1. Introduction

Graph theory is a branch of discrete mathematics that has been grown rapidly. There are many applications of graph theory in other fields such as computer science, physics, chemistry, biology, engineering, and sociology [1]. A graph G is a pair of two sets, i.e., $V(G)$ and $E(G)$. These two sets, respectively, represent a vertex set of G and an edge set of G. The number of vertices in G is denoted by $|V(G)|$ and the number of edges in G is denoted by $|E(G)|$. Other basic terminologies about graph theory that are not mentioned in this paper can be seen in [2]. Note that all graphs considered in this paper are simple, finite, and undirected. By notation $[a, b]$ with a, b integers we mean $\{x \in \mathbb{N} \mid a \leq x \leq b\}$.

One of important topics in graph theory is graph labeling. A graph labeling can be defined as a mapping from some set of graph elements to a set of positive integers. A graph labeling whose domain is vertex set or edge set is called a vertex labeling or an edge labeling, respectively. Moreover, if domain is both vertex set and edge set, then we call such labelings as a total labeling.

An *edge-covering* of a graph G is a collection of subgraphs $H_1, H_2, \ldots, H_k$ such that every edge of G belongs to at least one of subgraphs H_i, $i \in [1, k]$. In this case, G is said to be an $(H_1, H_2, \ldots, H_k)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then G is said to be an H-*covering*.

For a graph G admitting an H-*covering*, an (a, d)-H-antimagic total labeling of G is a bijection $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that, for all subgraphs H' isomorphic to H, the H'-*weight*$wt(H')$, which is defined by $wt(H') = \sum_{v \in V(H')} f(v) + \sum_{uv \in E(H')} f(uv)$, forms an arithmetic sequence $\{a, a + d, \ldots, (k-1)d\}$ where $a > 0$, $d \geq 0$ are two fixed integers and k is the number of all subgraphs of G isomorphic to H. The labeling f is called *super* if the smallest possible labels appear on the vertices. A graph that admits (super) (a, d)-H-antimagic total labeling is called *(super)*(a, d)-H-*antimagic*. For a special $d = 0$, the (super) $(a, 0)$-H-antimagic total labeling is called *H-(super)magic labeling*. A graph that admits H-(super)magic labeling is called *H-(super)magic*.

The notion of super (a, d)-H-antimagic total labeling was firstly introduced by Inayah, Salman, and Simanjuntak [3]. In 2013, Inayah *et al.* [4] studied super (a, d)-H-antimagic total labeling of a shackle graph *shack*(H, v, k). Dafik *et al.* [5] introduced a generalized shackle of graph denoted by *gshack*$(H, K \subset H, k)$. They showed the existence of super (a, d)-H-antimagic total labeling of *gshack*$(H, K \subset H, k)$ when $K \cong K_2$. Furthermore, Dafik *et al.* [6, 7] studied about H-super antimagicness of disconnected graphs as well as constructions of H-antimagic graphs using smaller edge antimagic graphs. More results about super (a, d)-H-antimagic total labeling can be seen in [8–11].

The notion H-supermagic labeling was firstly introduced by Gutiérrez and Lladó [12]. In their paper, they investigated *star*-(super)magic and *path*-(super)magic labelings of some classes of connected graphs. Maryati *et al.* [13] studied P_h-supermagic labeling of some classes of trees, i.e., shrubs and banana trees. For more results about H-supermagic labeling can be seen in [14].

In this paper, we investigate the H-supermagic labeling of graphs, namely, m-shadow of graphs which is a generalization of a shadow graph introduced by [15]. The m-shadow of graph G denoted by $D_m(G)$ is a graph obtained by taking m copies of G, namely, $G_1, G_2, \ldots, G_m$, and then joining every vertex u in G_i, $i \in [1, m-1]$, to the neighbors of the corresponding vertex v in G_{i+1}. We have proved that $D_m(G)$ admits H-supermagic labelings for some classes of graph G, namely, paths and cycles.

2. Main Results

2.1. $D_m(P_t)$-Supermagic Labeling of $D_m(P_n)$. In this part, we present the H-supermagic labeling of m-shadow of paths. Let $D_m(P_n)$ be the m-shadow of paths with vertex set $V(D_m(P_n)) = \{v_i^j \mid i \in [1, n],\ j \in [1, m]\}$ and edge set $E(D_m(P_n)) = \{v_i^j v_{i+1}^j \mid i \in [1, n-1],\ j \in [1, m]\} \cup \{v_i^j v_{i+1}^{j+1} \mid i \in [1, n-1],\ j \in [1, m-1]\} \cup \{v_{i+1}^j v_i^{j+1} \mid i \in [1, n-1],\ j \in [1, m-1]\}$. Next, we will show the existence of H-supermagic labeling of $D_m(P_n)$ in the following theorem.

Theorem 1. *$D_m(P_n)$ is $D_m(P_t)$-supermagic for any integer $m \geq 2$, $n \geq 4$ and $t \in [3, n-1]$.*

Proof (let $G \cong D_m(P_n)$). Define a total labeling $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$. In constructing the total labeling f, we distinguish between the vertices labeling and the edges labeling. First, label every vertex of $D_m(P_n)$ in the following way.

$$f\left(v_i^j\right) = \gamma_1^i + (j-1)n, \quad \text{for } j \in [1, m] \tag{1}$$

with

$$\gamma_1^i = \begin{cases} \dfrac{i-k}{t} + (k-1)\left(\dfrac{n}{t}\right) + 1 & \text{for } n \equiv 0 \pmod t; \\ & i = k, t+k, 2t+k, \ldots, n-t+k; \\ & k \in [1, t] \\ \dfrac{i-k}{t} + (k-1)\left(\dfrac{n-r}{t} + 1\right) + 1 & \text{for } n \equiv r \pmod t,\ r \in [1, t-1]; \\ & i = k, t+k, 2t+k, \ldots, n-r+k; \\ & k \in [1, r] \\ \dfrac{i-k}{t} + (k-1)\left(\dfrac{n-r}{t}\right) + r + 1 & \text{for } n \equiv r \pmod t,\ r \in [1, t-1]; \\ & i = k, t+k, 2t+k, \ldots, n-r-t+k; \\ & k \in [r+1, t] \end{cases} \tag{2}$$

To label the edges of $D_m(P_n)$, first, let $q = n - 1$. Next, label every edge as follows:

$$\begin{aligned} f\left(v_i^j v_{i+1}^j\right) &= \gamma_2^i + mn + (j-1)(n-1), \quad \text{for } j \in [1, m] \\ f\left(v_i^j v_{i+1}^{j+1}\right) &= \gamma_3^i + mn + m(n-1) + (j-1)(n-1), \quad \text{for } j \in [1, m-1] \\ f\left(v_{i+1}^j v_i^{j+1}\right) &= \gamma_2^i + mn + m(n-1) + (m-1)(n-1) + (j-1)(n-1), \quad \text{for } j \in [1, m-1] \end{aligned} \tag{3}$$

with

$$\gamma_2^i = \begin{cases} -\frac{i-k}{t-1} + (t-k)\left(\frac{n-1}{t-1}\right) & \text{for } q \equiv 0 \,(\text{mod}\,(t-1));\; i = k, t+k-1, 2(t-1)+k, \ldots, n-t+k;\; k \in [1, t-1] \\ -\frac{i-k}{t-1} + (s-k+1)\left(\frac{n-s-1}{t-1} + 1\right) & \text{for } q \equiv s \,(\text{mod}\,(t-1)),\; s \in [1, t-2];\; i = k, t+k-1, 2(t-1)+k, \ldots, n-1-s+k;\; k \in [1, s] \\ -\frac{i-k}{t-1} + (t-k)\left(\frac{n-s-1}{t-1}\right) + s\left(\frac{n-s-1}{t-1} + 1\right) & \text{for } q \equiv s \,(\text{mod}\,(t-1)),\; s \in [1, t-2];\; i = k, t+k-1, 2(t-1)+k, \ldots, n-s-t+k;\; k \in [s+1, t-1] \end{cases}$$

$$\gamma_3^i = \begin{cases} \frac{i-k}{t-1} + (k-1)\left(\frac{n-1}{t-1}\right) + 1 & \text{for } q \equiv 0 \,(\text{mod}\,(t-1));\; i = k, t+k-1, 2(t-1)+k, \ldots, n-t+k;\; k \in [1, t-1] \\ \frac{i-k}{t-1} + (k-1)\left(\frac{n-s-1}{t-1} + 1\right) + 1 & \text{for } q \equiv s \,(\text{mod}\,(t-1)),\; s \in [1, t-2];\; i = k, t+k-1, 2(t-1)+k, \ldots, n-1-s+k;\; k \in [1, s] \\ \frac{i-k}{t-1} + (k-1)\left(\frac{n-s-1}{t-1}\right) + s + 1 & \text{for } q \equiv s \,(\text{mod}\,(t-1)),\; s \in [1, t-2];\; i = k, t+k-1, 2(t-1)+k, \ldots, n-s-t+k;\; k \in [s+1, t-1] \end{cases} \tag{4}$$

It can be checked that $f(V(D_mP_n)) \in [1, mn]$. For $z \in [1, n-t+1]$, let $D_m(P_t^{(z)})$ be sub-m-shadow of paths with $V(D_m(P_t^{(z)})) = \{v_i^j \mid i \in [z, z+t-1],\, j \in [1, m]\}$ and $E(D_m(P_t^{(z)})) = \{v_i^j v_{i+1}^j \mid i \in [z, z+t-2],\, j \in [1, m]\} \cup \{v_i^j v_{i+1}^{j+1} \mid i \in [z, z+t-2],\, j \in [1, m-1]\} \cup \{v_{i+1}^j v_i^{j+1} \mid i \in [z, z+t-2],\, j \in [1, m-1]\}$. It can be shown that

$$\begin{aligned} f\left(v_i^j\right) &= f\left(v_{i+t}^j\right) - 1, \quad \text{for } i \in [1, n-t] \text{ and } j \in [1, m] \\ f\left(v_i^j v_{i+1}^j\right) &= f\left(v_{i+t-1}^j v_{i+t}^j\right) + 1, \quad \text{for } i \in [1, n-t] \text{ and } j \in [1, m] \\ f\left(v_i^j v_{i+1}^{j+1}\right) &= f\left(v_{i+t-1}^j v_{i+t}^{j+1}\right) - 1, \quad \text{for } i \in [1, n-t] \text{ and } j \in [1, m-1] \\ f\left(v_{i+1}^j v_i^{j+1}\right) &= f\left(v_{i+t}^j v_{i+t-1}^{j+1}\right) + 1, \quad \text{for } i \in [1, n-t] \text{ and } j \in [1, m-1] \end{aligned} \tag{5}$$

Furthermore, it can be shown that $V(D_m(P_t^{(z)})) \cap V(D_m(P_t^{(z+1)})) = \{v_i^j \mid i \in [z+1, z+t-1],\, j \in [1, m]\}$ and $E(D_m(P_t^{(z)})) \cap E(D_m(P_t^{(z+1)})) = \{v_i^j v_{i+1}^j \mid i \in [z+1, z+t-2],\, j \in [1, m]\} \cup \{v_i^j v_{i+1}^{j+1} \mid i \in [z+1, z+t-2],\, j \in [1, m-1]\} \cup \{v_{i+1}^j v_i^{j+1} \mid i \in [z+1, z+t-2],\, j \in [1, m-1]\}$. By combining these pieces of information, we obtain

$$\begin{aligned} wt\left(D_m\left(P_t^{(z)}\right)\right) &= \sum_{i=z}^{z+t-1}\left(\sum_{j=1}^{m} f\left(v_i^j\right)\right) + \sum_{i=z}^{z+t-2}\left(\sum_{j=1}^{m} f\left(v_i^j v_{i+1}^j\right)\right) + \sum_{i=z}^{z+t-2}\left(\sum_{j=1}^{m-1} f\left(v_i^j v_{i+1}^{j+1}\right)\right) + \sum_{i=z}^{z+t-2}\left(\sum_{j=1}^{m-1} f\left(v_{i+1}^j v_i^{j+1}\right)\right) \\ &= \sum_{i=z+1}^{z+t}\left(\sum_{j=1}^{m} f\left(v_i^j\right)\right) - m + \sum_{i=z+1}^{z+t-1}\left(\sum_{j=1}^{m} f\left(v_i^j v_{i+1}^j\right)\right) + m \\ &\quad + \sum_{i=z+1}^{z+t-1}\left(\sum_{j=1}^{m-1} f\left(v_i^j v_{i+1}^{j+1}\right)\right) - (m-1) + \sum_{i=z+1}^{z+t-1}\left(\sum_{j=1}^{m-1} f\left(v_{i+1}^j v_i^{j+1}\right)\right) \\ &\quad + (m-1) = wt\left(D_m\left(P_t^{(z+1)}\right)\right) \end{aligned} \tag{6}$$

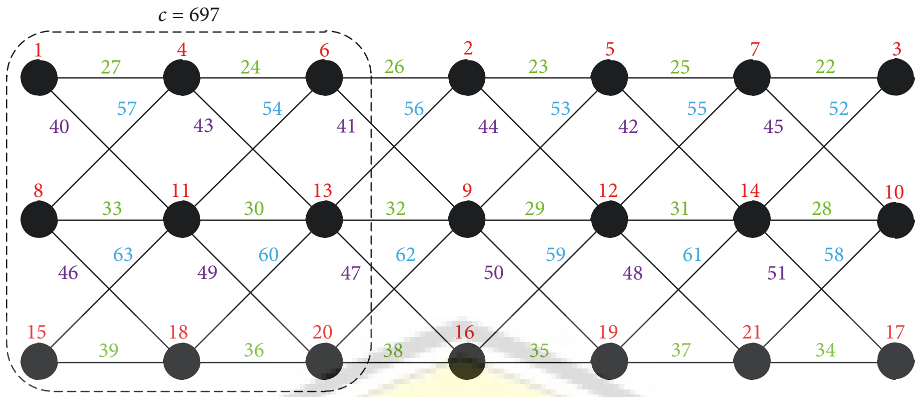

FIGURE 1: $D_3(P_3)$-supermagic labeling of $D_3(P_7)$ with magic constant $c = 697$.

Let $wt(D_m(P_t^{(1)})) = c$ be the magic constant. For the value of c, consider the following cases:

(i) For $n \equiv 0 \pmod t$ and $q \equiv 0 \pmod{(t-1)}$, it can be verified that $c = -8ntm + 2nt + 8ntm^2 - 3t + 8mt - (9/2)m^2t + 8mn + 3 - (15/2)m - 2n - (15/2)nm^2 + (9/2)m^2$.

(ii) For $n \equiv 0 \pmod t$ and $q \equiv s \pmod{(t-1)}$, $s \in [1, t-2]$, it can be verified that $c = -(9/2)m^2t - 8ntm + (3/2)smt + 8mt + 2nt - st + 8ntm^2 - 3t - 2n + 8mn - (15/2)nm^2 - (3/2)sm + 3 - (3/2)s^2\mathrm{m} + (9/2)m^2 + s^2 - (15/2)m + s$.

(iii) For $n \equiv r \pmod t$, $r \in [1, t-1]$, and $q \equiv 0 \pmod{(t-1)}$, it can be verified that $c = 2nt - 3t + 8mt + 8ntm^2 + (1/2)trm - 8ntm - (9/2)m^2t + (9/2)m^2 + 3 - 2n + 8mn - (1/2)r^2m - (15/2)m - (15/2)nm^2$.

(iv) For $n \equiv r \pmod t$, $r \in [1, t-1]$, and $q \equiv s \pmod{(t-1)}$, $s \in [1, t-2]$, it can be verified that $c = -st + 2nt - (9/2)m^2t + 8ntm^2 + (1/2)trm - 8ntm + (3/2)smt - 3t + 8mt - (3/2)sm - (15/2)nm^2 + 8mn + (9/2)m^2 + s - 2n + 3 - (3/2)s^2m + s^2 - (1/2)mr^2 - (15/2)m$.

Therefore, $D_m(P_n)$ is $D_m(P_t)$-supermagic for each $m \geq 2$, $n \geq 4$, and $t \in [3, n-1]$. □

For an illustration, we give an example of H-supermagic labeling of $D_m(P_n)$ in Figure 1.

2.2. $D_t(C_n)$-Supermagic Labeling of $D_m(C_n)$. In this part, we focus on the H-supermagic labeling of m-shadow of cycles. Let $D_m(C_n)$ be the m-shadow of cycles with vertex set $V(D_m(C_n)) = \{v_i^j \mid i \in [1,n], j \in [1,m]\}$ and edge set $E(D_m(C_n)) = \{v_i^j v_{i+1}^j \mid i \in [1, n-1], j \in [1,m]\} \cup \{v_n^j v_1^j \mid j \in [1,m]\} \cup \{v_i^j v_{i+1}^{j+1} \mid i \in [1, n-1], j \in [1, m-1]\} \cup \{v_n^j v_1^{j+1} \mid j \in [1, m-1]\} \cup \{v_{i+1}^j v_i^{j+1} \mid i \in [1, n-1], j \in [1, m-1]\} \cup \{v_1^j v_n^{j+1} \mid j \in [1, m-1]\}$. Next, the H-supermagic labeling of $D_m(C_n)$ will be shown in the following theorem.

Theorem 2. *$D_m(C_n)$ is $D_t(C_n)$-supermagic for any integer $m \geq 4$, $n \geq 3$, and $t \in [3, m-1]$.*

Proof (let $G \cong D_m(C_n)$). Define a total labeling $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ in the following way:

(i) Label the sets $\{v_i^j \mid i \in [1,n], j \in [1,m]\} \cup \{v_i^j v_{i+1}^j \mid i \in [1, n-1], j \in [1,m]\} \cup \{v_n^j v_1^j \mid j \in [1,m]\}$ as follows:

$$f\left(v_i^j\right) = \gamma_1^j + (i-1)m, \quad \text{for } i \in [1,n]$$
$$f\left(v_i^j v_{i+1}^j\right) = \gamma_2^j + mn + (i-1)m, \quad \text{for } i \in [1, n-1] \tag{7}$$
$$f\left(v_n^j v_1^j\right) = \gamma_2^j + mn + m(n-1)$$

with

$$\gamma_1^j = \begin{cases} \dfrac{j-k}{t} + (k-1)\left(\dfrac{m}{t}\right) + 1 & \text{for } m \equiv 0 \pmod t; \\ & j = k, t+k, 2t+k, \ldots, m-t+k; \\ & k \in [1,t] \\ \dfrac{j-k}{t} + (k-1)\left(\dfrac{m-r}{t} + 1\right) + 1 & \text{for } m \equiv r \pmod t, \ r \in [1, t-1]; \\ & j = k, t+k, 2t+k, \ldots, m-r+k; \\ & k \in [1,r] \\ \dfrac{j-k}{t} + (k-1)\left(\dfrac{m-r}{t}\right) + r + 1 & \text{for } m \equiv r \pmod t, \ r \in [1, t-1]; \\ & j = k, t+k, 2t+k, \ldots, m-r-t+k; \\ & k \in [r+1, t] \end{cases}$$

$$\gamma_2^j = \begin{cases} -\dfrac{j-k}{t} + (t-k+1)\left(\dfrac{m}{t}\right) & \text{for } m \equiv 0 \,(\mathrm{mod}\, t)\,; \\ & j = k, t+k, 2t+k, \ldots, m-t+k; \\ & k \in [1,t] \\ -\dfrac{j-k}{t} + (r-k+1)\left(\dfrac{m-r}{t}+1\right) & \text{for } m \equiv r \,(\mathrm{mod}\, t),\ r \in [1,t-1]\,; \\ & j = k, t+k, 2t+k, \ldots, m-r+k; \\ & k \in [1,r] \\ -\dfrac{j-k}{t} + (t-k+1)\left(\dfrac{m-r}{t}\right) + r\left(\dfrac{m-r}{t}+1\right) & \text{for } m \equiv r \,(\mathrm{mod}\, t),\ r \in [1,t-1]\,; \\ & j = k, t+k, 2t+k, \ldots, m-r-t+k; \\ & k \in [r+1,t] \end{cases} \tag{8}$$

(ii) Label the sets $\{v_i^j v_{i+1}^{j+1} \mid i \in [1,n-1], j \in [1,m-1]\} \cup \{v_n^j v_1^{j+1} \mid j \in [1,m-1]\} \cup \{v_{i+1}^j v_i^{j+1} \mid i \in [1,n-1], j \in [1,m-1]\} \cup \{v_1^j v_n^{j+1} \mid j \in [1,m-1]\}$. To label these sets, first, let $q = m-1$. Next, label the sets in the following way:

$$f\left(v_i^j v_{i+1}^{j+1}\right) = \gamma_3^j + 2mn + (i-1)(m-1), \quad \text{for } i \in [1,n-1]$$

$$f\left(v_n^j v_1^{j+1}\right) = \gamma_3^j + 2mn + (m-1)(n-1)$$

$$f\left(v_{i+1}^j v_i^{j+1}\right) = \gamma_4^j + 2mn + (m-1)n + (i-1)(m-1), \quad \text{for } i \in [1,n-1]$$

$$f\left(v_1^j v_n^{j+1}\right) = \gamma_4^j + 2mn + (m-1)n + (m-1)(n-1) \tag{9}$$

with

$$\gamma_3^j = \begin{cases} \dfrac{j-k}{t-1} + (k-1)\left(\dfrac{m-1}{t-1}\right) + 1 & \text{for } q \equiv 0 \,(\mathrm{mod}\,(t-1))\,; \\ & j = k, t+k-1, 2(t-1)+k, \ldots, m-t+k; \\ & k \in [1,t-1] \\ \dfrac{j-k}{t-1} + (k-1)\left(\dfrac{m-s-1}{t-1}+1\right) + 1 & \text{for } q \equiv s \,(\mathrm{mod}\,(t-1)),\ s \in [1,t-2]\,; \\ & j = k, t+k-1, 2(t-1)+k, \ldots, m-1-s+k; \\ & k \in [1,s] \\ \dfrac{j-k}{t-1} + (k-1)\left(\dfrac{m-s-1}{t-1}\right) + s + 1 & \text{for } q \equiv s \,(\mathrm{mod}\,(t-1)),\ s \in [1,t-2]\,; \\ & j = k, t+k-1, 2(t-1)+k, \ldots, m-s-t+k; \\ & k \in [s+1,t-1] \end{cases} \tag{10}$$

$$\gamma_4^j = \begin{cases} -\dfrac{j-k}{t-1} + (t-k)\left(\dfrac{m-1}{t-1}\right) & \text{for } q \equiv 0 \,(\mathrm{mod}\,(t-1))\,; \\ & j = k, t+k-1, 2(t-1)+k, \ldots, m-t+k; \\ & k \in [1,t-1] \\ -\dfrac{j-k}{t-1} + (s-k+1)\left(\dfrac{m-s-1}{t-1}+1\right) & \text{for } q \equiv s \,(\mathrm{mod}\,(t-1)),\ s \in [1,t-2]\,; \\ & j = k, t+k-1, 2(t-1)+k, \ldots, m-1-s+k; \\ & k \in [1,s] \\ -\dfrac{j-k}{t-1} + (t-k)\left(\dfrac{m-s-1}{t-1}\right) + s\left(\dfrac{m-s-1}{t-1}+1\right) & \text{for } q \equiv s \,(\mathrm{mod}\,(t-1)),\ s \in [1,t-2]\,; \\ & j = k, t+k-1, 2(t-1)+k, \ldots, m-s-t+k; \\ & k \in [s+1,t-1] \end{cases}$$

It can be seen that $f(V(D_m(C_n))) \in [1, mn]$. Let $D_t(C_n^{(1)})$, $D_t(C_n^{(2)}), \dots, D_t(C_n^{(m-t+1)})$ be sub-m-shadow of cycles with $V(D_t(C_n^{(z)})) = \{v_i^j \mid i \in [1,n],\ j \in [z, z+t-1]$ and $E(D_t(C_n^{(z)})) = \{v_i^j v_{i+1}^j \mid i \in [1, n-1], j \in [z, z+t-1] \cup \{v_n^j v_1^j \mid j \in [z, z+t-1] \cup \{v_i^j v_{i+1}^{j+1} \mid i \in [1, n-1], j \in [z, z+t-2] \cup \{v_n^j v_1^{j+1} \mid j \in [z, z+t-2] \cup \{v_{i+1}^j v_i^{j+1} \mid i \in [1, n-1]$, $j \in [z, z+t-2] \cup \{v_1^j v_n^{j+1} \mid j \in [z, z+t-2]$. It can be checked that

$$\begin{aligned}
f\left(v_i^j\right) &= f\left(v_i^{j+t}\right) - 1, \\
&\qquad \text{for } i \in [1,n] \text{ and } j \in [1, m-t] \\
f\left(v_i^j v_{i+1}^j\right) &= f\left(v_i^{j+t} v_{i+1}^{j+t}\right) + 1, \\
&\qquad \text{for } i \in [1,n-1] \text{ and } j \in [1, m-t] \\
f\left(v_n^j v_1^j\right) &= f\left(v_n^{j+t} v_1^{j+t}\right) + 1, \quad \text{for } j \in [1, m-t] \\
f\left(v_i^j v_{i+1}^{j+1}\right) &= f\left(v_i^{j+t-1} v_{i+1}^{j+t}\right) - 1, \\
&\qquad \text{for } i \in [1,n-1] \text{ and } j \in [1, m-t] \\
f\left(v_n^j v_1^{j+1}\right) &= f\left(v_n^{j+t-1} v_1^{j+t}\right) - 1, \quad \text{for } j \in [1, m-t] \\
f\left(v_{i+1}^j v_i^{j+1}\right) &= f\left(v_{i+1}^{j+t-1} v_i^{j+t}\right) + 1, \\
&\qquad \text{for } i \in [1,n-1] \text{ and } j \in [1, m-t] \\
f\left(v_1^j v_n^{j+1}\right) &= f\left(v_1^{j+t-1} v_n^{j+t}\right) + 1, \quad \text{for } j \in [1, m-t]
\end{aligned} \tag{11}$$

Furthermore, for $z \in [1, m-t+1]$, it can be shown that $V(D_t(C_n^{(z)})) \cap V(D_t(C_n^{(z+1)})) = \{v_i^j \mid i \in [1,n], j \in [z+1, z+t-1]\}$ and $E(D_t(C_n^{(z)})) \cap E(D_t(C_n^{(z+1)})) = \{v_i^j v_{i+1}^j \mid i \in [1, n-1]$, $j \in [z+1, z+t-1]\} \cup \{v_n^j v_1^j \mid j \in [z+1, z+t-1]\} \cup \{v_i^j v_{i+1}^{j+1} \mid i \in [1, n-1], j \in [z+1, z+t-2]\} \cup \{v_n^j v_1^{j+1} \mid j \in [z+1, z+t-2]\} \cup \{v_{i+1}^j v_i^{j+1} \mid i \in [1, n-1], j \in [z+1, z+t-2]\} \cup \{v_1^j v_n^{j+1} \mid j \in [z+1, z+t-2]\}$. By combining these pieces of information, we have

$$\begin{aligned}
&wt\left(D_t\left(C_n^{(z)}\right)\right) \\
&= \sum_{i=1}^{n}\left(\sum_{j=z}^{z+t-1} f\left(v_i^j\right)\right) + \sum_{i=1}^{n-1}\left(\sum_{j=z}^{z+t-1} f\left(v_i^j v_{i+1}^j\right)\right) \\
&+ \sum_{j=z}^{z+t-1} f\left(v_n^j v_1^j\right) + \sum_{i=1}^{n-1}\left(\sum_{j=z}^{z+t-2} f\left(v_i^j v_{i+1}^{j+1}\right)\right) \\
&+ \sum_{j=z}^{z+t-2} f\left(v_n^j v_1^{j+1}\right) + \sum_{i=1}^{n-1}\left(\sum_{j=z}^{z+t-2} f\left(v_{i+1}^j v_i^{j+1}\right)\right) \\
&+ \sum_{j=z}^{z+t-2} f\left(v_1^j v_n^{j+1}\right) \\
&= \sum_{i=1}^{n}\left(\sum_{j=z+1}^{z+t} f\left(v_i^j\right)\right) - n + \sum_{i=1}^{n-1}\left(\sum_{j=z+1}^{z+t} f\left(v_i^j v_{i+1}^j\right)\right) \\
&+ (n-1) + \sum_{j=z+1}^{z+t} f\left(v_n^j v_1^j\right) + 1 \\
&+ \sum_{i=1}^{n-1}\left(\sum_{j=z+1}^{z+t-1} f\left(v_i^j v_{i+1}^{j+1}\right)\right) - (n-1) \\
&+ \sum_{j=z+1}^{z+t-1} f\left(v_n^j v_1^{j+1}\right) - 1 \\
&+ \sum_{i=1}^{n-1}\left(\sum_{j=z+1}^{z+t-1} f\left(v_{i+1}^j v_i^{j+1}\right)\right) + (n-1) \\
&+ \sum_{j=z+1}^{z+t-1} f\left(v_1^j v_n^{j+1}\right) + 1 = wt\left(D_t\left(C_n^{(z+1)}\right)\right)
\end{aligned} \tag{12}$$

Let $wt(D_t(C_n^{(1)})) = c$ be the magic constant. For the value of c, consider the following cases:

(i) For $m \equiv 0 \pmod t$ and $q \equiv 0 \pmod{(t-1)}$, it can be verified that $c = 8mn^2t - 2tn^2 + 2nt - n - 6mn^2 + 2n^2$.

(ii) For $m \equiv 0 \pmod t$ and $q \equiv s \pmod{(t-1)}$, $s \in [1, t-2]$, it can be verified that $c = 8mn^2t - 2tn^2 + nts + 2nt - n - ns - 6mn^2 - ns^2 + 2n^2$.

(iii) For $m \equiv r \pmod t$, $r \in [1, t-1]$, and $q \equiv 0 \pmod{(t-1)}$, it can be verified that $c = 8mn^2t + 2nt + nrt - 2tn^2 - n - 6mn^2 + 2n^2 - nr^2$.

(iv) For $m \equiv r \pmod t$, $r \in [1, t-1]$, and $q \equiv s \pmod{(t-1)}$, $s \in [1, t-2]$, it can be verified that $c = 8mtn^2 - 2tn^2 + nts + 2nt + nrt - ns^2 + 2n^2 - nr^2 - n - 6mn^2 - ns$.

Hence, $D_m(C_n)$ is $D_t(C_n)$-supermagic for each $m \geq 4$, $n \geq 3$, and $t \in [3, m-1]$. □

For an illustration, an example of H-supermagic labeling of $D_m(C_n)$ can be seen in Figure 2.

3. Conclusion

We have studied the H-supermagic labeling of one type of graph operation, namely, m-shadow of graphs. In this paper, we have only shown the existence of H-supermagic labeling of m-shadow of paths and cycles. Meanwhile, the H-supermagic labeling of m-shadow of other classes of graphs is still widely open. Therefore, we propose open problems as follows.

Open Problem 1. Determine H-supermagic labelings of m-shadow of other classes of graphs.

Open Problem 2. Determine H-supermagic labelings of m-shadow of any connected graph.

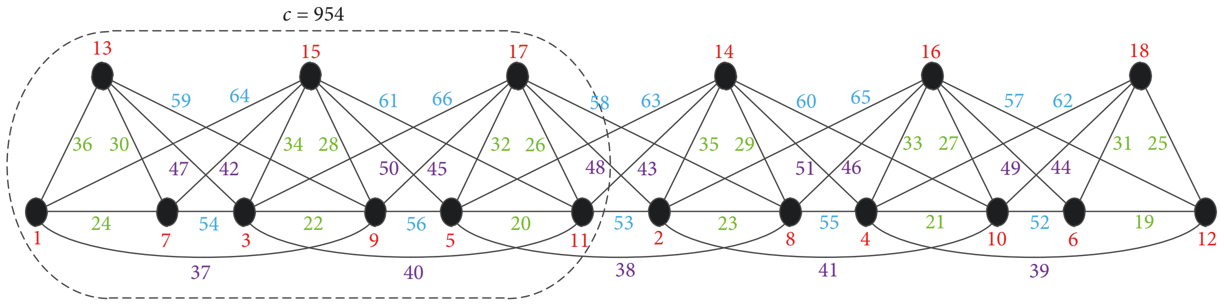

Figure 2: $D_3(C_3)$-supermagic labeling of $D_6(C_3)$ with magic constant $c = 954$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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