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On distance-irregular labelings of cycles and wheels

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In memory of Mirka Miller

Abstract

Distance-irregular labeling was introduced by Slamin, and in his paper he determined the distance-irregular labeling for cycles and wheels of length $\{0, 1, 2, 5\} \mod 8$. In this paper, we complete his results for cycles and wheels in general and prove the conjecture regarding the distance irregularity strength on wheels. We also show the general relation of the distance irregularity strength between a distance-irregular graph G and the graph $G + K_1$. Finally, we determine the distance irregularity strength of m-book graphs B_m.

1 Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph with vertex set V and edge set E. The order of the graph is $|V| = n$ and the size of the graph is $|E| = m$. A labeling is a mapping from the set of elements in a graph (vertices, edges, or both) to a set of numbers (usually positive integers). There are many types of labelings that have been studied (see [2] for a complete survey on labelings).

One of the labelings that was introduced by Miller, Rodger and Simanjuntak [3] is the 1-*vertex-magic labeling*, combining both magic labelings and distance labelings. The weight of a vertex v in this labeling is counted as the sum of all labels of vertices of distance 1 from v, i.e. the sum of all labels of vertices in the open neighbourhood of v.

Furthermore, the notion of irregular labeling was introduced by Chartrand et al. [1] in 1988. The aim of this labeling is to find the minimum largest label that we can assign to the edges of a graph, such that the weights (the sum of edge labels incident to a vertex) of each vertex are distinct. The minimum largest label amongst all the possible irregular assignments of a graph is called the *irregularity strength*.

Slamin [4] combined the distance and the irregular labeling to be the *distance-irregular vertex labeling*. Let k be a positive integer. A *distance-irregular vertex labeling* of the graph G with vertex set V is an assignment $\lambda : V \to \{1, 2, \ldots, k\}$ such that the weights at each vertex are distinct. In this labeling, the *weight* $wt(x)$ of a vertex x in G is defined as the sum of the labels of all the vertices adjacent to x, i.e. vertices at distance 1 from x. Let $N(x)$ denote the set of neighbours of x, so $N(x) = \{v \in V : d(x, v) = 1\}$. Formally,

$$wt(x) = \sum_{y \in N(x)} \lambda(y).$$

The *distance irregularity strength* of G, denoted by $\mathrm{dis}(G)$, is the minimum value of the largest label k over all such irregular assignments. In the paper, the author solved the distance-irregular labeling for complete graphs, paths, cycles C_n and wheels W_n where $n \geq 5, n \in \{0, 1, 2, 5\} \mod 8$.

Fig. 1 shows a distance-irregular labeling of a cycle on 12 vertices with $\mathrm{dis}(G) = 7$. The number outside the cycle shows the weight of the given vertex.

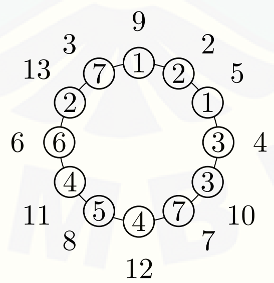

Figure 1: Distance irregular labeling on C_{12} with $\mathrm{dis}(C_{12}) = 7$.

2 Known Results

Some important observations in [4] regarding the distance-irregular labeling are:

(i) Let u and w be any two distinct vertices in a connected graph G. If u and w

have identical neighbours, i.e., $N(u) = N(w)$, then G has no distance-irregular vertex labeling.

This condition shows that not all graphs have distance-irregular labelings. Some graphs that do not have distance-irregular labelings are complete bipartite graphs $K_{m,n}$, complete multipartite graphs, and trees T_n $(n \geq 3)$ that contain a vertex with at least two leaves.

(ii) Let u and w be two adjacent vertices in a connected graph G. If $N(u) - w = N(w) - u$, then the labels of u and w must be distinct, that is, $\lambda(u) \neq \lambda(w)$.

The aim of a distance-irregular labeling is to determine the distance irregularity strength, which is the minimum value of the largest label k over all such irregular assignments. The lower bound for distance irregularity strength for connected graphs in general is given by Slamin [4] in Lemma 2.1.

Lemma 2.1 *[4] Let G be a connected graph on v vertices with minimum degree δ and maximum degree Δ and such that there is no vertex having identical neighbours. Then $dis(G) \geq \lceil \frac{(v+\delta-1)}{\Delta} \rceil$.*

For a small cycle C_3, $\text{dis}(C_3) = 3$, while $\text{dis}(C_4)$ does not exist, since there are two vertices having identical neighbours. For cycles of other sizes, the distance irregularity strength has been proved by Slamin [4] for $n \equiv 0, 1, 2, 5 \mod 8$. The results are given in Theorem 2.2.

Theorem 2.2 *[4] Let C_n be a cycle with $n \geq 5$ vertices for $n \equiv 0, 1, 2, 5 \mod 8$; then $dis(C_n) = \lceil \frac{n+1}{2} \rceil$.*

A *wheel* graph, denoted by W_n, is a graph constructed from a cycle C_n by adding a vertex and connecting this added vertex to all vertices of the cycle. The distance irregularity strength of W_n for $n \equiv 0, 1, 2, 5 \mod 8$ is given in Theorem 2.3 [4].

Theorem 2.3 *[4] Let W_n be a wheel with $n \geq 5$ rim vertices for $n \equiv 0, 1, 2, 5 \mod 8$; then $dis(W_n) = \lceil \frac{n+1}{2} \rceil$.*

Slamin conjectures that the distance irregularity strengths of cycles and wheels are the same.

Conjecture 2.4 *[4] The distance irregularity strength of wheels is equal to the distance irregularity strength of cycles, that is, $dis(W_n) = dis(C_n)$ for $n \geq 5$.*

3 Main Results

In this section, we complete the distance-irregular labeling on cycles C_n and we determine the distance irregularity strength of the join graph $G + K_1$ where G is a graph that admit a distance-irregular labeling and K_1 is a single vertex. Recall that the join graph $G + H$ is a new graph obtained by connecting each vertex of G to each vertex of H by a new edge, where G and H are two vertex-disjoint graphs. The result of the join graph implies the conjecture of the distance irregularity strength on wheels given in [4].

Theorem 3.1 *Let C_n be a cycle on n vertices, $n \geq 6$, then*

$$dis(C_n) = \begin{cases} \frac{n+3}{2} & \text{if } n \equiv 3,7 \mod 8 \\ \lceil\frac{n+1}{2}\rceil & \text{if } n \equiv 4,6 \mod 8 \end{cases}$$

PROOF: From Lemma 2.1, we know that for the lower bound for $\mathrm{dis}(C_n) \geq \lceil\frac{n+1}{2}\rceil$.

Case 1: When $n \equiv 3,7 \mod 8$, this is equivalent to $n \equiv 3 \mod 4$.

Let λ be a distance-irregular labeling on cycles C_n. Note that, in the cycles, each label contributes twice to the weight. So

$$\sum_{v \in C_n} wt(v) = 2 \sum_{v \in C_n} \lambda(v).$$

In this case, if the largest label is $\frac{n+1}{2}$, then the largest possible weight is $n+1$. Since the smallest possible weight is 2, all vertex weights have to be distinct and there are exactly n vertices in the cycles, so it follows that the weights are $\{2, 3, \ldots, n+1\}$. Summing up these weights gives an odd number, contradicting the fact that $2\sum_{v \in C_n} \lambda(v)$ is even. Thus, the largest label is at least $\frac{n+3}{2}$.

We show that $\mathrm{dis}(C_n) = \frac{n+3}{2}$ by constructing the following labeling. Observe the solid and dashed cycle in Fig. 2.

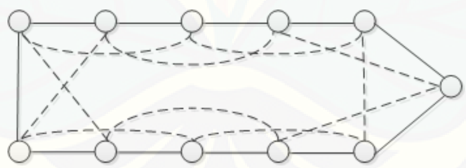

Figure 2: Solid and dashed cycles of C_{11}.

Define the edge weight of the dashed cycle to be the sum of the labels of vertices incident to it. If we label the vertices of the cycle, then the weights of distance-irregular labeling at each vertex of the solid cycle is equivalent to the weight of the corresponding edge in the dashed cycle. Hence, if we are able to construct an irregular labeling on the dashed cycle, with all edge weights distinct, then we obtain a distance-irregular labeling of the solid cycle. The example in Fig. 3 shows the relation between the labeling in solid and dashed cycles.

In general, we name the vertices of the dashed cycle as in Fig. 4. Label the vertices as follow:

$$\lambda(u_i) = \begin{cases} 1 & \text{if } i = 1 \\ i+1 & \text{if } i \geq 3 \text{ and is odd} \\ i & \text{if } i \text{ is even.} \end{cases}$$

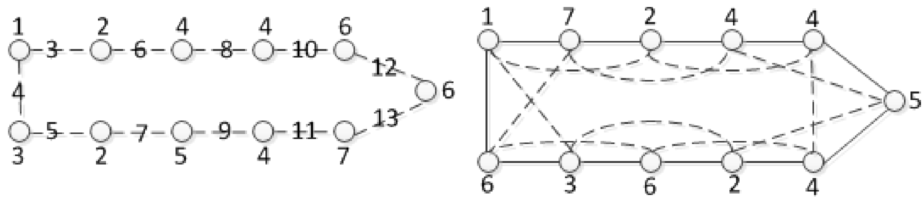

Figure 3: The labeling of dashed C_{11} corresponds to a distance-irregular labeling of solid C_{11}.

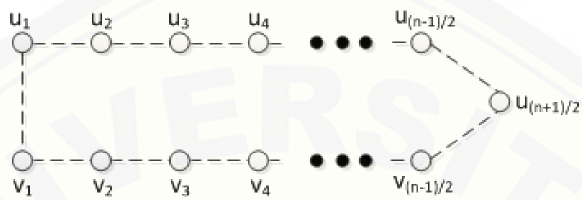

Figure 4: Vertices in the dashed cycle.

$$\lambda(v_j) = \begin{cases} j+2 & \text{if } j \text{ is odd} \\ j & \text{if } j \text{ is even.} \end{cases}$$

The largest label is obtained in $\lambda(v_{\frac{n-1}{2}}) = \frac{n+3}{2}$. We need to check that all the edge weights are distinct. This is clear because $w(u_1u_2) = 3, w(u_1v_1) = 4, w(u_{i-1}u_i) = 2u_i$ for $i \geq 3$ (even weights), $w(v_{j-1}v_j) = 2j+1$ (odd weights) and the largest weight is $w(v_{\frac{n-1}{2}}u_{\frac{n+1}{2}}) = n+2$.

Case 2: When $n \equiv 4, 6 \mod 8$.

The lower bound of $\text{dis}(C_n)$ is achieved in this case and the labels are obtained using a similar technique as in Case 1. However, for n even, we will obtain two disjoint even cycles of the same size. Observe the dashed and dotted disjoint cycles in Fig. 5.

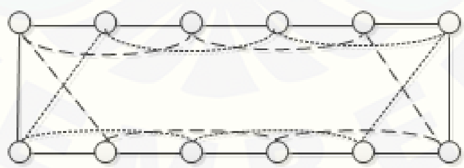

Figure 5: Dotted and dashed disjoint cycles of an even cycle.

The weights of a distance-irregular labeling at each vertex of the solid cycle is equivalent to the weight of the corresponding edge in the dashed or dotted cycles. Hence, if we are able to construct an irregular labeling on dashed and dotted cycles, with all edge weights distinct, then we obtain a distance-irregular labeling of the solid cycle.

When $n \equiv 4 \mod 8, n \geq 12, \lceil \frac{n+1}{2} \rceil = \frac{n+2}{2}$. Name the vertices of the dashed and dotted cycles as in Figure 6. Label the vertices as follow:

$$\lambda(u_i) = \begin{cases} 1 & \text{if } i = 1, 2 \\ i+1 & \text{if } i = 3, 4, \ldots, \frac{n}{4} \end{cases}$$

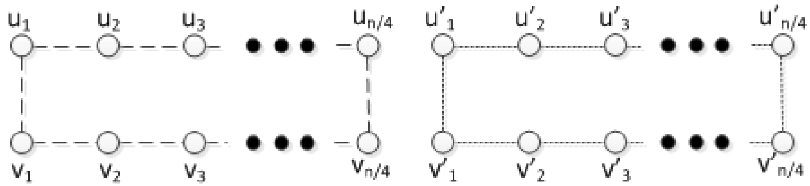

Figure 6: Vertices of dashed and dotted even cycles.

$$\lambda(v_j) = \begin{cases} j+1 & \text{if } j \text{ is odd} \\ j+2 & \text{if } j \text{ is even} \end{cases}$$

$$\lambda(u'_i) = \begin{cases} i+1 & \text{if } i = 1,2 \\ \frac{n}{2}+4-i & \text{if } i = 3,5,\ldots,\frac{n}{4}, \\ \frac{n}{2}+3-i & \text{if } i = 4,6,\ldots,\frac{n}{4}-1. \end{cases}$$

$$\lambda(v'_j) = \frac{n}{2}+2-j \text{ for } j = 1,2,\ldots,\frac{n}{4}$$

The largest label is $\frac{n+2}{2} = \frac{n}{2}+1$ and it is achieved at vertices u'_3 and v'_1. From the dashed edges, we obtain weight set $\{2,3,\ldots,\frac{n}{2}+2\}\backslash\{5\}$, and from the dotted edges we obtain weight set $\{5\} \cup \{\frac{n}{2}+3, \frac{n}{2}+4, \ldots, n+1\}$. Hence we obtain n distinct weights for all n vertices. Combining the labels of dotted and dashed cycles into the original solid cycle gives the distance-irregular labeling for $C_n, n \equiv 4 \mod 8$ with largest label $\lceil \frac{n+1}{2} \rceil = \frac{n+2}{2}$.

When $n \equiv 6 \mod 8$, $\lceil \frac{n+1}{2} \rceil = \frac{n+2}{2}$. The general dashed and dotted cycles are given by Fig. 7. And the labels are given by:

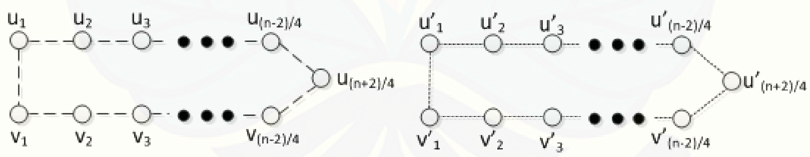

Figure 7: The dashed and dotted cycles when $n \equiv 6 \mod 8$.

$$\lambda(u_i) = \begin{cases} 1 & \text{if } i = 1,2 \\ i+1 & \text{if } i \text{ odd} \\ i & \text{if } i \text{ even} \end{cases}$$

$$\lambda(v_j) = \begin{cases} 2 & \text{if } j = 1 \\ j+2 & \text{if } j > 1 \text{ is odd} \\ j & \text{if } j \text{ is even} \end{cases}$$

$$\lambda(u'_i) = \begin{cases} 3 & \text{if } i = 1,2 \\ \frac{n}{2}+4-i & \text{if } i = 3,5,\ldots,\frac{n-2}{4}, \\ \frac{n}{2}+5-i & \text{if } i = 4,6,\ldots,\frac{n+2}{4}. \end{cases}$$

$$\lambda(v'_j) = \begin{cases} \frac{n}{2} + 1 - j & \text{for } j = 1, 2, \ldots, \frac{n-6}{4} \\ \frac{n+2}{4} + 2 & \text{if } j = \frac{n-2}{4}. \end{cases}$$

The largest label is $\frac{n+2}{2} = \frac{n}{2} + 1$ and is achieved at vertices u'_3 and u'_4. From the dashed edges, we obtain weight set $\{2, 3, \ldots, \frac{n}{2} + 2\} \backslash \{6\}$, and from the dotted edges we obtain weight set $\{6\} \cup \{\frac{n}{2} + 3, \frac{n}{2} + 4, \ldots, n + 2\} \backslash \{n + 1\}$. Hence we obtain n distinct weights for all n vertices. Combining the labels of dotted and dashed cycles into the original solid cycle gives the distance-irregular labeling for $C_n, n \equiv 6 \mod 8$ with largest label $\lceil \frac{n+1}{2} \rceil = \frac{n+2}{2}$. □

Theorem 3.2 *Let G be a graph admitting a distance-irregular labeling and let K_1 be a single vertex. Then dis$(G + K_1) =$ dis(G).*

PROOF: Suppose we have a distance-irregular labeling on a graph join $G + K_1$ with $\text{dis}(G+K_1) = s$; then taking away the vertex K_1 will give a distance-irregular labeling on G with $\text{dis}(G) \leq s$, and thus $\text{dis}(G + K_1) \geq \text{dis}(G)$. On the other hand, suppose we have a distance-irregular labeling on a graph G with $\text{dis}(G) = t$. Join a graph K_1 to the graph G to obtain a new graph $G + K_1$. Label the vertex in K_1 with 1; then the weight of each vertex in G increases by 1 and the weight of the vertex in K_1 is the sum of all labels in G, and hence all the weights are different. The largest label used in the labeling is the same as the largest label of G, since we only introduced one new vertex of label 1. Thus, $\text{dis}(G + K_1) = \text{dis}(G)$. □

Corollary 3.3 *Let W_n be a wheel on $n + 1$ vertices and f_n be a fan graph on $n + 1$ vertices. Then dis$(W_n) =$ dis(C_n) and dis$(f_n) =$ dis(P_n).*

PROOF: This follows from Theorem 3.2 and the fact that $W_n = C_n + K_1$ and $f_n = P_n + K_1$. □

This proof confirms the conjecture in [4], that $\text{dis}(W_n) = \text{dis}(C_n)$.

The *m-book graph* is the Cartesian product $S_{m+1} \square P_2$, where S_{m+1} is a star graph and P_2 is the path graph on two vertices. When $m = 1$, the graph B_1 is a square (C_4), and thus it does not have a distance-irregular labeling as previously mentioned in Section 2.

Theorem 3.4 *Let B_m be an m-book graph, $m \geq 2$; then dis$(B_m) = m + 1$.*

PROOF: In B_m, there are $2m$ vertices of degree 2, and for those vertices the smallest possible weight is 2, with minimum possible range of the weights being $\{2, 3, \ldots, 2m+1\}$, since all the weights have to be distinct. Thus the largest label of the graph is at least $m + 1$, which implies $\text{dis}(B_m) \geq m + 1$. For $m = 2$ and $m = 3$, the labeling of the book graph is given by Fig. 8, with the largest labels being 3 and 4, respectively.

For $m \geq 4$, the book graphs are seen as two copies of S_{m+1}. Let u and v denote the centre vertices of the two stars and let $u_i, i = 1, 2, \ldots, m$ be the vertices attached to

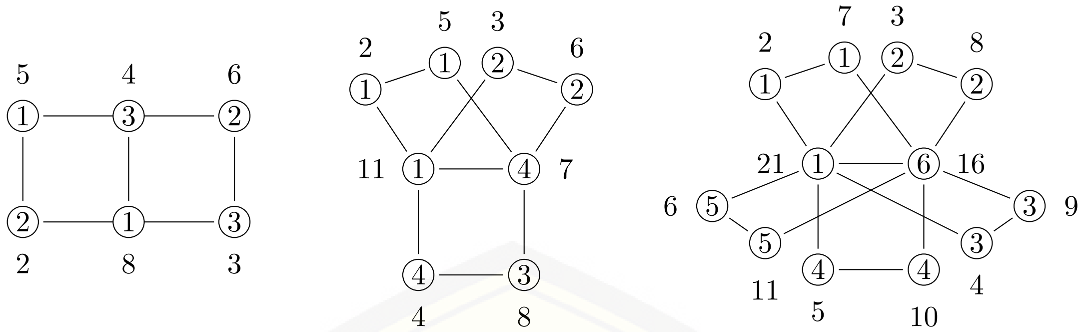

Figure 8: Examples of distance-irregular labeling on B_2, B_3 and B_5.

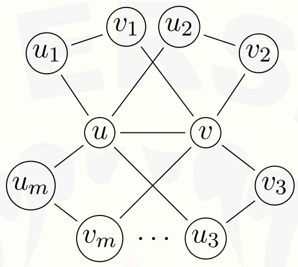

Figure 9: The naming of the vertices

the centre u (similarly, vertices $v_i, i = 1, 2, \ldots, m$ are attached to v). We connect both stars with the following rules. Vertices u and v are connected, and u_i is connected to v_i for all i; see Fig. 9 for the illustration. Let λ be a distance-irregular labeling on graph B_m.

$$\begin{aligned} \lambda(u_i) &= \lambda(v_i) = i, i = 1, 2, \ldots, m; \\ \lambda(u) &= 1; \\ \lambda(v) &= m + 1. \end{aligned}$$

The largest label is $m+1$. The weight set for the vertices are $w(u_i) = \{2, 3, \ldots, m+1\}$, $w(v_i) = \{m+2, m+3, \ldots, 2m+1\}$, $w(u) = \frac{(m+1)(m+2)}{2}$, and $w(v) = \frac{m(m+1)}{2} + 1$. Since $m \geq 4$, all these weights are different. This shows that $\text{dis}(B_m) = m + 1$. □

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