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# THE SIMILARITY OF METRIC DIMENSION AND LOCAL METRIC DIMENSION OF ROOTED PRODUCT GRAPH

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## Abstract

Let $G$ be a connected graph with vertex set $V(G)$ and $W = \{w_1, w_2, ..., w_k\} \subset V(G)$. The representation of a vertex $v \in V(G)$ with respect to $W$ is the ordered $k$-tuple $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$, where $d(v, w)$ represents the distance between vertices $v$ and $w$. The set $W$ is called a resolving set for $G$ if every vertex of $G$ has a distinct representation. A resolving set containing a minimum

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number of vertices is called basis for $G$. The metric dimension of $G$, denoted by $\dim(G)$, is the number of vertices in a basis of $G$. If every two adjacent vertices of $G$ have a distinct representation with respect to $W$, then the set $W$ is called a local resolving set for $G$ and the minimum local resolving set is called a local basis of $G$. The cardinality of a local basis of $G$ is called local metric dimension of $G$, denoted by $\dim_l(G)$. In this paper, we study the local metric dimension of rooted product graph and the similarity of metric dimension and local metric dimension of rooted product graph.

## 1. Introduction

Let $G$ be a finite and simple connected graph. The vertex and edge sets of the graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. The distance between vertices $v$ and $w$ in $G$, denoted by $d(v, w)$, is the length of a shortest path between them. For the ordered set $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and $v$ is a vertex on the graph $G$, then the representation of $v$ with respect to $W$ is $k$-tuple, $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set $W$ is called a *resolving set* of $G$ if every vertex of $G$ has a distinct representation and minimum resolving set is called *basis* of $G$. The cardinality of basis is called *metric dimension* of $G$, denoted by $\dim(G)$ [1].

The $W$ set is called a *local resolving set* of $G$ if every two adjacent vertices of $G$ have a distinct representation with respect to $W$, that is, if $u, v \in V(G)$ such that $uv \in E(G)$, then $r(u|W) \neq r(v|W)$. The local resolving set of $G$ with minimum cardinality is called *local basis* of $G$, the cardinality of basis local of $G$ is called *local metric dimension* of $G$, denoted by $\dim_l(G)$. In [5], Rodriguez-Velazquez and Fernau observed the relationship between local metric dimension and metric dimension of a graph $G$, that is,

**Observation 1.1** [5]. $\dim_l(G) \leq \dim(G)$.

Godsil and McKay [3] defined the *rooted product* graph as follows. Let $G$ be a graph on $n$ vertices and $\mathcal{H}$ be a sequence of $n$ rooted graphs $H_1$, $H_2$,

$H_3, \ldots, H_n$. The rooted product graph of $G$ by $\mathcal{H}$ denoted by $G \circ \mathcal{H}$ is a graph obtained by grafting the root of $H_i$ with the $i$th vertex of $G$ [3]. If $H_1, H_2, H_3, \ldots, H_n$ are isomorphic to a graph of order $n'$, Saputro et al. called this notion by *comb product* [7]. Rodriguez-Velazquez et al. [6] observed the local metric dimension of rooted product graph as follows:

**Theorem 1.2** [6]**.** *Let G be a connected graph of order* $n \geq 2$ *and let* $\mathcal{H}$ *be a sequence of n connected bipartite graphs* $H_1, H_2, H_3, \ldots, H_n$. *Then for any rooted product graph* $G \circ \mathcal{H}$, $\dim_l(G \circ \mathcal{H}) = \dim_l(G)$.

**Theorem 1.3** [6]**.** *Let G be a connected graph of order* $n \geq 2$ *and let* $\mathcal{H}$ *be a sequence of n connected non-bipartite graphs* $H_1, H_2, H_3, \ldots, H_n$. *Then for any rooted product graph* $G \circ \mathcal{H}$,

$$\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{n} (\dim_l(H_j) - \alpha_j),$$

*where* $\alpha_j = 1$ *if the root of* $H_j$ *belongs to a local basis of* $H_j$ *and* $\alpha_j = 0$ *otherwise.*

The known results on metric dimension and local metric dimension of some particular classes of graphs have been discovered by Chartrand et al. [1] and Okamoto et al. [4] as given below.

**Theorem 1.4** [1]**.** *Let G be a connected graph of order* $n \geq 2$. *Then*:

(i) $\dim(G) = 1$ *if and only if* $G = P_n$.

(ii) $\dim(G) = n - 1$ *if and only if* $G = K_n$.

(iii) *For* $n \geq 4$, $\dim(G) = n - 2$ *if and only if* $G = K_{r,s}$; $(r;\, s \geq 1)$, $G = K_r + \overline{K}_s$, $(r \geq 1;\, s \geq 2)$, *or* $G = K_r + (K_1 \cup K_s)$, $(r, s \geq 1)$.

(iv) *For* $n \geq 3$, $\dim(C_n) = 2$.

**Theorem 1.5** [1]**.** *If G is a connected graph of order* $n \geq 2$ *and diameter k, then* $\dim(G) \leq n - k$.

**Theorem 1.6** [4]**.** *Let G be a connected graph of order* $n \geq 2$. *Then*:

(i) $\dim_l(G) = n - 1$ *if and only if* $G = K_n$,

(ii) $\dim_l(G) = 1$ *if and only if G is bipartite graph*.

**Theorem 1.7** [4]**.** *Let G be a connected graph of order n and diameter k*. *Then* $\dim_l(G) \leq n - k$.

In this paper, we study the local metric dimension of rooted product graph to complete the results of Rodriguez-Velazquez et al. presented in [6]. In Theorem 1.2 and Theorem 1.3 of the paper, they observed that the local metric dimension of rooted product graph $G \circ \mathcal{H}$, for $\mathcal{H}$, is a sequence of *n* connected bipartite and non-bipartite graphs, respectively, as a consequence of the theorem of local metric dimension of point attaching graph. Rodriguez-Velazquez et al. [6] presented those theorems as corollary without the proofs. The detail of the proofs will be shown in this paper. We also show the local metric dimension of rooted product graph $G \circ \mathcal{H}$, where $\mathcal{H}$ is a sequence of the combined of *n* connected bipartite and non-bipartite graphs. Furthermore, we observe the similarity of metric dimension and local metric dimension of rooted product graph. Before presenting the main results of this paper, we present diameter and twin equivalence class of graph and their relation with metric and local metric dimension of graph, as described in the following section.

## 2. The Similarity of Metric Dimension and Local Metric Dimension of Graph

Two distinct vertices *u* and *v* of graph *G* are called *twin* if *u* and *v* have the same neighbourhood in $V(G) - \{u, v\}$, and they are called *true twin* or *false twin* if *u* and *v* are adjacent and twin or *u* and *v* are not adjacent and twin, respectively, [4]. The following two lemmas describe the properties of twin that are discovered by Hernando et al. [2].

**Lemma 2.1** [2]**.** *If u and v are twin in graph G*, *then* $d(u, x) = d(v, x)$ *for every vertex in* $V(G) - \{u, v\}$.

**Lemma 2.2** [2]**.** *Let* $u$, $v$ *and* $w$ *be distinct vertices in graph* $G$. *If* $u$ *and* $v$ *are twin*, $v$ *and* $w$ *are twin*, *then* $u$ *and* $w$ *are also twin*.

In other words, *twin* is an equivalence relation on $V(G)$. The twin vertices produce the equivalence twin class.

In general, the twin relation divides the vertex set $V(G)$ into the partition of twin equivalence classes. There are three types of twin equivalence classes, namely, true twin equivalence class, false twin equivalence class, and singleton.

In this paper, we say that graph $G$ has *twin equivalence classes* if $G$ has true twin equivalence classes or false twin equivalence classes without singleton. Also, we say that graph $G$ has *true twin equivalence classes* if $G$ has true twin equivalence classes only.

**Lemma 2.3.** *Let* $G$ *be a connected graph*. *If* $G$ *has true twin equivalence classes or false twin equivalence classes* $B_1, B_2, B_3, \ldots, B_m$, *then* $\dim(G) = \sum_{i=1}^{m}(|B_i| - 1)$.

**Proof.** Let $B_i$ for $i = 1, 2, \ldots, m$ be equivalence classes of connected graph $G$. Take $B_i - \{u_i\}$, $u_i \in B_i$ for every $i = 1, 2, \ldots, m$. We see that every vertex in $G$ has the distinct representation with respect to $B = \bigcup_{i=1}^{m} B_i - \{u_i\}$. Thus, $B$ is resolving set of $G$. Suppose that there is $B_i$ for some $i = 1, 2, \ldots, m$ such that two elements of $B_i$ are not element $B$. By Lemma 2.1, $B$ is not resolving set. This means that $B = \bigcup_{i=1}^{m} B_i - \{u_i\}$ is the minimum resolving set or basis of $G$. Therefore, $\dim(G) = \sum_{i=1}^{m}(|B_i| - 1)$.

□

**Lemma 2.4.** *Let* $G$ *be a connected graph*. *If* $G$ *has true twin equivalence classes* $B_1, B_2, B_3, \ldots, B_m$, *then* $\dim(G) = \dim_l(G)\sum_{i=1}^{m}(|B_i| - 1)$.

By Theorem 1.4 and Theorem 1.6, we obtain

**Corollary 2.5.** (a) $\dim_l(G) = \dim(G) = n - 1$ *if and only if* $G = K_n$,

(b) $\dim_l(G) = \dim(G) = 1$ *if and only if* $G = P_n$.

**Lemma 2.6.** *Let G be a connected graph with diameter k having l twin equivalence classes. Then* $k \leq l$.

**Proof.** Suppose that $l < k$. By Lemma 2.1, $d(x, y) \leq l < k$, for every $x, y$ in $G$. This contradicts with maximum distance of $G$ which is $k$. □

**Theorem 2.7.** *Let G be a connected graph of order* $n \geq 3$ *having twin equivalence classes and diameter k. If* $l = k$, *then* $k = 1$ *or* 2.

**Proof.** Let $G$ be a connected graph of order $n \geq 3$ and diameter $k$. The number of twin equivalence classes is $l = k$. There exist two vertices $u, v$ in $G$ such that $d(u, v) = k$. This leads to the two possibilities, either $u$ and $v$ are in the same class or $u$ and $v$ are in the distinct classes.

Suppose that $u$ and $v$ are in the distinct classes. Then $l > 1$ and $k = l > 1$ and there is path $u, v_1, v_2, v_3, ..., v_{k-1}, v_k = v$. Since the diameter is $k$, each $u, v_1, v_2, v_3, ..., v_{k-1}, v_k = v$ is in the $k + 1$ distinct twin equivalence classes. Thus, $G$ has $l = k + 1$ twin equivalence classes, contradiction with $l + k$. Therefore, the only chance is that $u$ and $v$ are in the same twin equivalence class. This leads to the two possibilities, either $u$ and $v$ are adjacent or $u$ and $v$ are non-adjacent.

a. If vertices $u$ and $v$ are adjacent, then $k = 1$ and every vertex in $G$ is adjacent. In other words, $G = K_n$, and every vertex in $G$ forms one true twin equivalence class.

b. If vertices $u$ and $v$ are non-adjacent, then $d(u, v) = k > 1$. If vertices $u$ are $v$ are the same false twin equivalence class, then, by Lemma 2.1, $u$ and $v$ have the same neighbourhood. So $d(u, v) = k = 2$. □

**Corollary 2.8.** *There is no connected graph with diameter k having k twin equivalence classes for* $k \geq 3$.

**Theorem 2.9.** *Let G be a connected graph of order $n \geq 3$ and diameter k having k twin equivalence classes. Then $\dim(G) = n - k$ if and only if $k = 1$ or $k = 2$.*

**Proof.** Let $G$ be a connected graph of order $n \geq 3$ and diameter $k$ having $k$ twin equivalence classes. If $\dim(G) = n - k$, then, by Theorem 2.7, $k = 1$ or $k = 2$. Conversely, let the diameter of $G$ be $k = 1$ or $k = 2$, and has $k$ twin equivalence classes. Thus:

For $k = 1$, then $G = K_n$, so $\dim(G) = n - 1 = n - k$.

For $k = 2$, then there are two vertices, say $u$ and $v$, in $G$ such that $d(u, v) = 2$. Suppose that $u$ and $v$ in the distinct twin equivalence class. Then $d(u, v) = 1$, a contradiction. So $u$ and $v$ must be in one twin equivalence class. Let $S_1, S_2$ be the twin equivalence classes in $G$. By Lemma 2.3, $\dim(G) = |S_1| - 1 + |S_2| - 1 = n - 2 = n - k$. □

Consequently, we have

**Corollary 2.10.** *Let G be a connected graph of order $n \geq 4$ and diameter k. Then G has k twin equivalence classes if and only if $G = K_n$ or $G = K_{n,m}$ or $G = K_s + \overline{K}_t$.*

**Theorem 2.11.** *Let G be a connected graph of order $n \geq 3$ without end vertex, diameter k and $G \neq K_s + (K_t \cup K_1)$, where $s, t \geq 1$. If G has $k + 1$ true twin equivalence classes or true twin equivalence classes and singleton, then $\dim(G) = \dim_l(G) = n - (k + 1)$.*

**Proof.** Let $G$ be a connected graph of order $n \geq 3$ without end vertex, diameter $k$ and $G \neq K_s + (K_t \cup K_1)$, $s, t \geq 1$. Let $G$ has $k + 1$ true twin equivalence classes or has the combination of $k + 1$ true twin equivalence classes and singleton. Let $B_1, B_2, B_3, \ldots, B_k, B_{k+1}$ be true twin equivalence classes or singleton. Let the distance of vertices in $B_i$ to vertices in $B_{i+1}$ be one for $i = 1, 2, \ldots, k$, and $|B_1| + |B_2| + |B_3| + \cdots + |B_k| + |B_{k+1}| = n$. There are two cases:

(i) There are $|B_i| = 1$ for some $i$, $G$ has no end vertex, so $i \neq 1, k+1$. Without loss of generality, let $i = 2$ and 4. Choose $|B_i| - 1$ vertices in $B_i$, for $i \neq 2, 4$ as elements of set $W$. Thus,

$$|W| = \sum_{i \neq 2,4}^{k+1} iB_i - (k+1-2) = \sum_{i \neq 2,4}^{k+1} iB_i + 2 - (k+1) = n - (k+1).$$

By Lemma 2.3 and Lemma 2.4, we get $W$ is basis and local basis of $G$. Thus, $\dim(G) = \dim_l(G) = n - (k+1)$.

(ii) If $|B_i| > 1$ for all $i$, choose $|B_i| - 1$ vertices in $B_i$, for all $i$ as elements of set $W$, so $|W| = n - (k+1)$. By Lemma 2.3 and Lemma 2.4, we get $W$ is basis and local basis of $G$. Thus, $\dim(G) = \dim_l(G) = n - (k+1)$.

□

## 3. The Similarity of Metric Dimension and Local Metric Dimension of Rooted Product Graph

Before presenting the main results, we first present local metric dimensions of cycle graph and properties of rooted product graphs, that are used to prove the main theorems as described in lemmas and observations below.

**Lemma 3.1.** *Let $C_n$ be a cycle on $n \geq 3$ vertices. Then*

$$\dim_l(C_n) = \begin{cases} 1, & \text{for even } n \\ 2, & \text{for odd } n. \end{cases}$$

**Proof.** For even $n$, $C_n$ is bipartite graph, by Theorem 1.6(ii), we get $\dim_l(C_n) = 1$. For odd $n$, $C_n$ is not bipartite graph. Choose $W = \{x, y\}$, $xy \in E(C_n)$. It easy to see that every two adjacent vertices have the distinct representation with respect $W$. By Theorem 1.6(ii), $W$ is a local basis of $C_n$ and $\dim_l(C_n) = 2$. □

**Observation 3.2.** Every two adjacent vertices in $C_n$ for odd $n$, form local basis of $C_n$.

**Observation 3.3.** Let $G$ be a graph of order $n \geq 2$ and $\mathcal{H}$ be a sequence of $n$ connected graphs $H_j$, $j = 1, 2, 3, \ldots, n$. In the rooted product graph $G \circ \mathcal{H}$, if every $H_j$ is connected bipartite graph, then every two adjacent vertices in $H_j$ have distinct distance to the root of $H_j$ and to all vertices in $G \circ \mathcal{H}$.

**Lemma 3.4.** *Let $G$ be a graph of order $n \geq 2$ and $\mathcal{H}$ be a sequence of $n$ connected graphs $H_j$, $j = 1, 2, \ldots, n$. In the rooted product graph $G \circ \mathcal{H}$, if $o_j$ is the root of $H_j$, and $U_j$ is a local basis of $H_j$, then*:

(i) *if $o_j \in U_j$, then there are two adjacent vertices $x, y$ in $H_j$ such that $r(x|S) = r(y|S)$ for every $S \subset H_j$, $|S| \leq |U_j| - 2$,*

(ii) *if $o_j \notin U_j$, then there are two adjacent vertices $x, y$ in $H_j$ such that $r(x|S) = r(y|S)$ for every $S \subset H_j$, $|S| \leq |U_j| - 1$.*

The following two theorems are similar with Theorem 1.2 and Theorem 1.3 presented by Rodriguez-Velazquez et al. [6], but the proofs shall be completed in this paper.

**Theorem 3.5.** *Let $G$ be a connected graph of order $n \geq 2$, and let $\mathcal{H}$ be a sequence of the connected bipartite graphs $H_1, H_2, \ldots, H_n$ and $o_j$ is the root of $H_j$. Then* $\dim_l(G \circ \mathcal{H}) = \dim_l(G)$.

**Proof.** Let $G$ be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of the bipartite graphs $H_1, H_2, H_3, \ldots, H_n$. Let $o_j$ be the root of $H_j$. Choose $W$ as a local basis of $G$. Take any two adjacent vertices $x, y$ in $H_j$, $j = 1, 2, \ldots, n$. Since $H_j$ bipartite, by Observation 3.3, we get $d(x, z) \neq d(x, z)$ for every $z \in G \circ \mathcal{H}$, so $r(x|W) \neq r(y|W)$.

Take any two adjacent roots $o_i, o_j$ in $G \circ \mathcal{H}$. Since $W$ is a local basis of $G$, $r(o_i|W) \neq (o_j|W)$, and $W$ is a local basis of $G \circ \mathcal{H}$. Thus, $\dim_l(G \circ \mathcal{H}) = \dim_l(G)$. □

**Theorem 3.6.** *Let G be a connected graph of order* $n \geq 2$ *and let* $\mathcal{H}$ *be a sequence of n connected non-bipartite graphs* $H_1, H_2, H_3, ..., H_n$, *and* $o_j$ *is the root of* $H_j$. *Then*

$$\dim_l(G \circ \mathcal{H}) = \begin{cases} \sum_{j=1}^{n} (\dim_l(H_j) - 1), & \text{if } o_j \text{ is element of local basis of } H_j, \\ \sum_{j=1}^{n} \dim_l(H_j), & \text{otherwise}. \end{cases}$$

**Proof.** Let $G$ be a connected graph of order $n \geq 2$ and $\mathcal{H}$ be a sequence of the connected non-bipartite graphs $H_1, H_2, H_3, ..., H_n$. Let $o_j$ be the root of $H_j$, $j = 1, 2, 3, ..., n$. First, let $o_j$ be an element of a local basis of $H_j$. Choose $W = \bigcup_{j=1}^{n}(W_j - \{o_j\})$, where $W_j$ is a local basis of $W_j$ and $o_j \in W_j$. Then $|W| = \sum_{j=1}^{n}(\dim_l(H_j) - 1)$.

Take any two adjacent vertices $x, y$ in $H_j$, $j = 1, 2, ..., n$. There are two possibilities, that is, either $d(x, o_j) = d(y, o_j)$ or $d(x, o_j) \neq d(y, o_j)$. Since $W_j$ is a local basis of $H_j$ and $o_j \in W_j$, for $d(x, o_j) = d(y, o_j)$, there exist $u_j \in W_j - \{o_j\}$ such that $d(x, u_j) \neq d(y, u_j)$ which implies that $r(x|W) \neq r(y|W)$.

For $d(x, o_j) \neq d(y, o_j)$, then $d(x, s) \neq d(y, s)$ for every

$$s \in V(G \circ \mathcal{H})/(V(H_j) - \{o_j\}),$$

implies $r(x|W) \neq r(y|W)$.

Take any two adjacent roots $o_i, o_j$ in $G \circ \mathcal{H}$, then $d(o_i, z) \neq d(o_j, z)$ for every $z \in V(H_j)$. Since $W_j \subseteq H_j$ and $W_j \subseteq W$, $r(o_i|W) \neq r(o_j|W)$. Thus, $W$ is a local resolving set of $G \circ \mathcal{H}$.

To show that $W$ is a minimum local resolving set of $G \circ \mathcal{H}$, take any set $S \subseteq V(G \circ \mathcal{H})$ with $|S| < |W|$. This means that there is $H_j$ such that $(\dim_l(H_j) - 2)$ vertices of that be elements of $S$. By Lemma 3.4(i), we get that there are two adjacent vertices $x$, $y$ in $H_j$ such that $r(x|S) = r(y|S)$. So $W$ is a minimum local resolving set of $G \circ \mathcal{H}$ and $\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{n}(\dim_l(H_j) - 1)$.

Second, let $o_j$ be not element of a local basis of $H_j$. Choose $W = \bigcup_{i=1}^{n} W_i$, where $W_j$ is a local basis of $H_j$ and $o_j \in W_j$. Then $|W| = \sum_{j=1}^{n}\dim_l(H_j)$. Take any two adjacent vertices $x$, $y$ in $H_j$, $j = 1, 2, \ldots, n$. Since $W_j$ is a local basis of $H_j$, $r(x|W_j) \neq r(y|W_j)$. Thus, $r(x|W) \neq r(y|W)$, and $W = \bigcup_{i=1}^{n} W_i$ is a local resolving of $G \circ \mathcal{H}$.

To show that $W$ is a minimum local resolving set of $G \circ \mathcal{H}$, take any set $S \subseteq V(G \circ \mathcal{H})$ with $|S| < |W|$. This means that there is $H_j$ such that $(\dim_l(H_j) - 1)$ vertices of $H_j$ be elements of $S$. By Lemma 3.4(ii), we get that there are two adjacent vertices $x$, $y$ in $H_j$ such that $r(x|S) = r(y|S)$. So $W$ is a minimum local resolving set of $G \circ \mathcal{H}$ and $\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{n}(\dim_l(H_j))$. □

**Theorem 3.7.** *Let $G$ be a connected graph of order $n \geq 2$, and let $\mathcal{H}$ be a sequence of the combined $n$ connected non-bipartite $H_1, H_2, \ldots, H_s$ and bipartite graphs $H_{s+1}, H_{s+2}, \ldots, H_n$, and $o_j$ is the root of $H_j$. Then*

$$\dim_l(G \circ \mathcal{H})$$

$$\begin{cases} = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j), & \text{for } G = C_n,\ n \text{ odd},\ s > 1 \text{ or} \\ & G \text{ bipartite or } G = K_n,\ s = n-1 \\ = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + 1, & \text{for } G = C_n,\ n \text{ odd},\ s = 1 \\ = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + \dim_l(G) - s, & \text{for } G = K_n,\ s < n-1 \\ < \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + n - s - 1, & \text{otherwise,} \end{cases}$$

*where $\alpha_j = 1$ if $o_j$ belongs to a local basis of $H_j$ and $\alpha_j = 0$ otherwise.*

**Proof.** Let $G$ be a connected graph of order $n \geq 2$ and $\mathcal{H}$ be a sequence of the combined $n$ connected non-bipartite $H_1, H_2, H_3, \ldots, H_s$ and bipartite graphs $H_{s+1}, H_{s+2}, H_{s+3}, \ldots, H_n$. Let $T$ be the local basis of $G$ and $U_j$ is local basis of $H_j$, $j = 1, 2, \ldots, s$, and $o_j$ is the root of $H_j$.

**Case 1.** For $G = C_n$, $n$ odd, $s > 1$ or $G$ bipartite or $G = K_n$, $s = n - 1$, choose $W = \bigcup_{j=1}^{s}(U_j - \{o_j\})$. Take any two adjacent roots $o_i, o_j$ in $G \circ \mathcal{H}$. If $G = C_n$, for $n$ odd and $s > 1$, by Observation 3.2, we get $r(o_i | W) \neq r(o_j | W)$. If $G$ bipartite, by Theorem 1.6(ii), we get $r(o_i | W) \neq r(o_j | W)$. If $G = K_n$, $s = n - 1$, by Theorem 1.6(i), we obtain $r(o_i | W) \neq r(o_j | W)$.

Take any two adjacent vertices $x, y$ in $H_j, j = 1, 2, \ldots, s$. Then $r(x | U_j) \neq r(y | U_j)$, so $r(x | W) \neq r(y | W)$, for $G = C_n$, $n$ odd, $s > 1$ or $G$ bipartite or $G = K_n$, $s = n - 1$.

Take any two adjacent vertices $x, y$ in $H_j$, $j = s + 1, s + 2, \ldots, n$, by Observation 3.3, we get $r(x | W) \neq r(y | W)$, for $G = C_n$, $n$ odd, $s > 1$ or $G$ bipartite or $G = K_n$, $s = n - 1$.

So $W = \bigcup_{j=1}^{s}(U_j - \{o_j\})$ is a local resolving set of $G \circ \mathcal{H}$, by Lemma 3.4, we get $W = \bigcup_{j=1}^{s}(U_j - \{o_j\})$ is a local basis of $G \circ \mathcal{H}$, and $\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j)$, where $\alpha_j = 1$ if $o_j$ belongs to a local basis of $H_j$ and $\alpha_j = 0$ otherwise.

**Case 2.** For $G = C_n$, $n$ odd, $s = 1$ choose $W = \bigcup_{j=1}^{s}(U_j - \{o_j\}) \cup \{z\} = (U_1 - \{o_1\}) \cup \{z\}$, $z \in H_i$ for any $i = s + 1, s + 2, \ldots, n$ and $x \neq o_i$. Without loss of generality, let $z \in H_2$. Take any two adjacent roots $o_i, o_j$ in $G \circ \mathcal{H}$. Then $d(o_i, o_1) \neq d(o_j, o_1)$ so that $r(o_i | U_1) \neq r(o_j | U_1)$ and $r(o_i | W) \neq r(o_j | W)$.

Take any two adjacent vertices $x, y$ in $H_j$, $j = 1$, then $r(x|U_1) \neq r(y|U_1)$, so $r(x|W) \neq r(y|W)$.

Take any two adjacent vertices in $H_j$, $j = 2, 3, \ldots, n$, there are exactly two vertices $x, y$ in $H_j$, for some $j = 2, 3, \ldots, n$ such that $d(x, o_1) = d(y, o_1)$, but $d(x, o_2) \neq d(y, o_2)$, so $d(x, z) \neq d(y, z)$, implies $r(x|W) \neq r(y|W)$.

So $W = \bigcup_{j=1}^{s}(U_j - \{o_j\}) \cup \{z\} = (U_1 - \{o_1\}) \cup \{z\}$, for $z \in H_2$, is a local resolving set of $G \circ \mathcal{H}$. By Lemma 3.4, take any set $S \subset G \circ \mathcal{H}$, where $|S| < |W|$. Then there are two adjacent vertices in $H_1$ or two adjacent root vertices that have the same representation with respect to *S*. Thus, *W* is a local basis of $G \circ \mathcal{H}$ and $\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j)$, where $\alpha_j = 1$ if $o_j$ belongs to a local basis of $H_j$ and $\alpha_j = 0$ for otherwise.

**Case 3.** For $G = K_n$, $s < n-1$, choose $W = \bigcup_{j=1}^{s}(U_j - \{o_j\}) \cup \{u_i | u_i \neq o_i, i = s+1, s+2, \ldots, k < n\}$. Without loss of generality, let $s = n-2$. It means that $H_j$, $j = 1, 2, \ldots, n-2$ is non-bipartite graph and $H_{n-1}$ and $H_n$ are bipartite graphs, and $W = \bigcup_{j=1}^{n-2}(U_j - \{o_j\}) \cup \{u_{n-1}\}$, $u_{n-1} \neq o_{n-1}$.

Take any two adjacent roots in $G \circ \mathcal{H}$, there are three possibilities:

First, two adjacent roots are $o_{n-1}, o_n$, so $d(o_{n-1}, o_j) = d(o_n, o_j)$ for all $j = 1, 2, \ldots, n-2$. This implies that $r(o_{n-1}|U_j) = r(o_n|U_j)$. However, $r(o_{n-1}|o_{n-1}) \neq r(o_n|o_{n-1})$, so $r(o_{n-1}|W) \neq r(o_n|W)$. Second, one of the roots is element of $H_{n-1}$ or $H_n$ and one of the roots is element of $H_j$, $j = 1, 2, \ldots, n-2$. Without loss of generality, let $o_n$ and $o_j$ for some *j*, so that $d(o_j, o_n) \neq d(o_j, o_j)$. Then $r(o_n|W) \neq r(o_j|W)$. Third, two adjacent roots are $o_i, o_l$ in $H_j$, $j = 1, 2, \ldots, n-2$. It is obvious that $r(o_i|W) \neq r(o_j|W)$.

Take any two adjacent vertices $x, y$ in $H_j$, $j = n-1, n$. Since $H_{n-1}$ and $H_n$ are bipartite, by Observation 3.3, we get $r(x|W) \neq r(y|W)$.

Take any two adjacent vertices in $H_j$, $j = 2, 3, ..., n-2$. Since $U_j$, for $j = 2, 3, ..., n-2$, is basis of $H_j$, $r(x|W) \neq r(y|W)$.

So $W = \bigcup_{j=1}^{s}(U_j - \{o_j\}) \bigcup \{u_i | u_i \neq o_i, i = s+1, s+2, ..., k < n\}$ is a local resolving set of $G \circ \mathcal{H}$. By Lemma 3.4, take any set $S \subset G \circ \mathcal{H}$, where $|S| < |W|$. Then there are two adjacent vertices in $H_j$, $j = 2, 3, ..., n-2$ or two adjacent root vertices that have the same representation with respect to $S$. Thus, $W$ is a local basis of $G \circ \mathcal{H}$ and $|W| = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + n - 1 - s$. Since $G$ is complete graph $K_n$ and $\dim_l(K_n) = n-1$, $\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + \dim_l(G) - s$, where $\alpha_j = 1$ if $o_j$ belongs to a local basis of $H_j$, and $\alpha_j = 0$ otherwise.

**Case 4.** For $G$ otherwise, $\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + n - s - 1$. It is obvious because $K_n$ is the graph with the biggest local metric dimension. □

**Observation 3.8.** Let $G$ be a connected graph of order $n$, $\mathcal{H}$ be a sequence of $n$ connected graphs $H_1, H_2, H_3, ..., H_n$. Then $G \circ \mathcal{H}$ is a path if and only if $G$ is a path of order $n \leq 2$, where $\mathcal{H}$ is a sequence of paths and the root of $H_j$ is element of basis of $H_j$.

The relationship between metric dimension and local metric dimension of rooted product of two connected graphs is given as follows.

**Theorem 3.9.** *Let G be a connected graph of order* $n \geq 3$. *If* $\mathcal{H}$ *is a sequence of nodd cycle graphs, then* $\dim(G \circ \mathcal{H}) = \dim_l(G \circ \mathcal{H}) = |V(G)|$.

**Proof.** Let $\mathcal{H}$ be a sequence of $n$ odd cycle graphs $H_1, H_2, H_3, ..., H_n$, and $\alpha_i$ is the root of $H_i$. Choose $W = \bigcup_{i=1}^{n}\{u_i | u_i\alpha_i \in E(Hi)\}$. Then there are two vertices $x, y$ in $H_i$ that are adjacent to $u_i$, and $d(x, \alpha_i) \neq$

$d(y, \alpha_i)$. This implies that $x$ and $y$ have distinct distance to all vertices in $V(G \circ \mathcal{H})/V(H_i)$. Thus, $W$ is a resolving set of $G \circ \mathcal{H}$. Suppose that there is $H_i$ such that no vertex in $H_i$ that belongs to $W$. Then there are two vertices $x, y$ in $V(H_i)$ that are adjacent to the root of $H_i$. Thus, $x$ and $y$ have the same distance to the root $H_i$. This implies that $x$ and $y$ have the same distance to all vertices in $V(G \circ \mathcal{H})/V(H_i)$. Therefore, $W$ is minimum resolving set of $G \circ \mathcal{H}$ and $\dim(G \circ \mathcal{H}) = |V(G)|$.

Since $W$ is a resolving set of $G \circ \mathcal{H}$, $W$ is a local resolving set of $G \circ \mathcal{H}$. Suppose that there is $H_i$ such that no vertex in $H_i$ that belongs to $W$. Since $H_i$ is odd cycle, there are exactly two adjacent vertices $u, v$ in $H_i$ such that $d(u, \alpha_i) = d(v, \alpha_i) = \frac{m-1}{2}$. Then $d(u, s) = d(v, s)$ for all $s \in V(G \circ \mathcal{H})/V(H_i)$, so $W$ is a minimum local resolving set of $G \circ \mathcal{H}$ and

$$\dim_l(G \circ \mathcal{H}) = n = |V(G)|.$$

So $\dim(G \circ \mathcal{H}) = \dim_l(G \circ \mathcal{H}) = |V(G)|$. □

As a consequence of Corollary 2.5(b) and Observation 3.8, we obtain sufficient and necessary condition of similarity metric dimension and local metric dimension of rooted product graph.

**Corollary 3.10.** *Let $G$ be a connected graph of order $n$, $\mathcal{H}$ be a sequence of $n$ connected graphs $H_1, H_2, H_3, \ldots, H_n$. Then* $\dim(G \circ \mathcal{H}) = \dim_l(G \circ \mathcal{H}) = 1$ *if and only if $G$ is a path of order $n \leq 2$, $\mathcal{H}$ is a sequence of $n$ paths and the root of $H_j$ is element of basis of $H_j$.*

**Proof.** Let $G$ be a path of order $n \leq 2$, $\mathcal{H}$ be a sequence of $n$ path graphs, and the root of $H_j$ is element of basis of $H_j$. Then $(G \circ \mathcal{H})$ is a path too. By Corollary 2.5(b), $\dim(G \circ \mathcal{H}) = \dim_l(G \circ \mathcal{H}) = 1$. Conversely, let $\dim(G \circ \mathcal{H}) = \dim_l(G \circ \mathcal{H}) = 1$. By Corollary 2.5(b), $G \circ \mathcal{H}$ is path. By Observation 3.8, $G$ is a path of order $n \leq 2$, $\mathcal{H}$ is a sequence of $n$ path graphs, and the root of $H_j$ is element of basis of $H_j$. □

As the consequence of Corollary 2.5(a) and Theorem 1.7, we get

**Corollary 3.11.** *If* $\mathcal{H}$ *is a sequence of n path graphs*, *then* $\dim(K_n \circ \mathcal{H}) = \dim_l(K_n \circ \mathcal{H}) = n-1.$


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