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ON COMMUTATIVE CHARACTERIZATION OF GENERALIZED COMB AND CORONA PRODUCTS OF GRAPHS WITH RESPECT TO THE LOCAL METRIC DIMENSION

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Abstract

Let G be a connected graph with vertex set $V(G)$ and $W = \{w_1, w_2, ..., w_m\} \subset V(G)$. The representation of a vertex $v \in V(G)$, with respect to W is the ordered m-tuple $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_m))$, where $d(v, w)$ represents the distance

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between vertices v and w. This set W is called a local resolving set for G if every two adjacent vertices have a distinct representation and a minimum local resolving set is called a local basis of G. The cardinality of a local basis of G is called the local metric dimension of G, denoted by $\dim_l(G)$. In general, comb product and corona product are non-commutative operations in graphs. However, these operations can be made commutative with respect to local metric dimension for some graphs with certain conditions. In this paper, we determine the local metric dimension of generalized comb and corona products of graphs and obtain necessary and sufficient conditions of graphs in order that comb and corona products be commutative operations with respect to the local metric dimension.

1. Introduction

Let G be a finite, simple, and connected graph. The vertex and edge sets of the graph G are denoted by $V(G)$ and $E(G)$, respectively. The distance between vertices v and w in G, denoted by $d(v, w)$, is the length of the shortest path between them. For the ordered set

$$W = \{w_1, w_2, ..., w_m\} \subseteq V(G),$$

and a vertex $v \in V(G)$, the representation of v with respect to W is the m-tuple, $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_m))$. The set W is called a *local resolving set* of G if every two adjacent vertices of G have a distinct representation with respect to W, that is, if $u, v \in V(G)$ such that $uv \in E(G)$, then $r(u|W) \neq r(v|W)$. A local resolving set of G with minimum cardinality is called a *local basis* of G, and the cardinality of a local basis of G is called the *local metric dimension* of G, denoted by $\dim_l(G)$.

Godsil and McKay [3] defined the rooted product graph as follows. Let G be a graph on n vertices and $\mathcal{H}$ be a sequence of n rooted graphs $H_1, H_2, H_3, ..., H_n$. The rooted product graph of G by $\mathcal{H}$ denoted by $G \circ \mathcal{H}$ is a graph obtained by identifying the root of H_i with the ith vertex of G. Saputro et al. [7] studied the metric dimension of the comb product graph $G \circ H$, which is a special case of a rooted product graph.

Rodriguez-Velazquez et al. [6] and Susilowati et al. [8] observed the local metric dimension of rooted product graph as follows:

Theorem 1.1 [6]**.** *Let G be a connected labelled graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n connected bipartite graphs $H_1, H_2, H_3, \ldots, H_n$. Then, for any rooted product graph $G \circ \mathcal{H}$,* $\dim_l(G \circ \mathcal{H}) = \dim_l(G)$.

Theorem 1.2 [6]**.** *Let G be a connected labelled graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n connected non-bipartite graphs $H_1, H_2, H_3, \ldots, H_n$. Then, for any rooted product graph $G \circ \mathcal{H}$,* $\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{n}(\lim_l(H_j) - \alpha_j)$, *where $\alpha_j = 1$ if the root of H_j belongs to a local basis of H_j and $\alpha_j = 0$ otherwise.*

Theorem 1.3 [8]**.** *Let G be a connected labelled graph of order $n \geq 2$. Let $\mathcal{H}$ be a sequence of the combined n connected non-bipartite $H_1, H_2, \ldots, H_s$ and bipartite graphs $H_{s+1}, H_{s+2}, \ldots, H_n$. Let o_j be the root of H_j for $j \in \{1, 2, \ldots, n\}$. Then*

$$\dim_l(G \circ \mathcal{H})$$

$$\begin{cases} = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j), & \text{for } G = C_n, n \text{ odd}, s > 1 \text{ or} \\ & G \text{ bipartite or } G = K_n, s = n-1, \\ = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + 1, & \text{for } G = C_n, n \text{ odd}, s = 1, \\ = \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + \dim_l(G) - s, & \text{for } G = K_n, s < n-1, \\ < \sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + n - s - 1, & \text{otherwise,} \end{cases}$$

where $\alpha_j = 1$ if o_j belongs to a local basis of H_j and $\alpha_j = 0$ otherwise.

Frucht and Harary [2] defined the corona product graph. The corona graph, $G \odot H$, of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H and then joining by edge the ith vertex of G to every vertex in the ith copy of H. In [6], Rodriguez-Velazquez et al. generalized the corona product $G \odot \mathcal{H}$, where $\mathcal{H}$ is a sequence of n graphs H_1, H_2, H_3, ..., H_n, where H_i and H_j may not be isomorphic.

Theorem 1.4 [6]. *Let G be a connected labelled graph of order* $n \geq 2$ *and let* $\mathcal{H}$ *be a sequence of n nonempty graphs* $H_1, H_2, H_3, \ldots, H_n$. *Then, for any corona product graph* $G \odot \mathcal{H}$,

$$\dim_l(G \odot \mathcal{H}) = \sum_{j=1}^{n} (\dim_l(K_1 + H_j) - \alpha_j),$$

where $\alpha_j = 1$ *if the vertex of* K_1 *belongs to a local basis of* $K_1 + H_j$ *and* $\alpha_j = 0$ *otherwise.*

A particular case of corona product graph, where the sequence H_1, H_2, H_3, ..., H_n consists of n graphs isomorphic to graph H, has been discovered by Rodriguez-Velazquez et al. [5].

Theorem 1.5 [5]. *Let H be a nonempty graph. The following statements hold*:

(i) *If the vertex of* K_1 *does not belong to any local basis for* $K_1 + H$, *then for any connected graph G of order n*,

$$\dim_l(G \odot H) = n \dim_l(K_1 + H).$$

(ii) *If the vertex of* K_1 *belongs to a local basis for* $K_1 + H$, *then for any connected graph G of order* $n \geq 2$,

$$\dim_l(G \odot H) = n(\dim_l(K_1 + H) - 1).$$

Okamoto et al. [4] discovered the characterization of local metric dimension for some graphs, presented below.

Theorem 1.6 [4]. *Let G be a connected graph of order* $n \geq 2$. *Then*

(i) $\dim_l(G) = n - 1$ *if and only if* $G = K_n$,

(ii) $\dim_l(G) = 1$ *if and only if G is bipartite graph.*

In this paper, we analyze the local metric dimension of generalized comb product graph, rooted product graph, and corona product graph relative to Rodriguez-Velazquez et al. [6] and Susilowati et al. [8] results. Furthermore, we formulate the necessary and sufficient conditions such that the local metric dimension of the generalized comb product graphs possess same value, even though the position of graph that operated is exchanged. Likewise, results for the generalized corona product graphs.

2. The Local Metric Dimension of Generalized Comb Product Graph

The generalized rooted product graphs that are presented in this paper are *k*-comb product graphs and *k*-rooted product graphs. The *k*-comb product is charged to a graph with respect to a graph and *k*-rooted product is charged to a graph with respect to a sequence of graphs, which is defined by Susilowati et al. [9] as below.

Let *G* and *H* be connected graphs, where the order of *G* is *n*, and *o* is a vertex of *H*. The *k-comb* product graph $G \circ_k H$ is obtained from *G* and *H* by taking one copy of *G* and *nk* copies of *H* that are $H_{11}, H_{12}, H_{13}, ..., H_{1k}, H_{21}, H_{22}, H_{23}, ..., H_{2k}, ..., H_{n1}, H_{n2}, H_{n3}, ..., H_{nk}$ and grafting the vertex o_{ij}, $j = 1, 2, 3, ..., k$ with *i*th vertex of *G*.

Let *G* be a labelled graph on *n* vertices and $\mathcal{H}$ be a sequence of *n* rooted graphs $H_1, H_2, H_3, ..., H_n$. The *k-rooted* product graph of *G* by $\mathcal{H}$ denoted by $G \circ_k \mathcal{H}$ is obtained by taking *k* copies of H_i for every $i = 1, 2, ..., n$, that are $H_{11}, H_{12}, H_{13}, ..., H_{1k}, H_{21}, H_{22}, H_{23}, ..., H_{2k}, H_{31}, H_{32}, H_{33}, ..., H_{3k}, ..., H_{n1}, H_{n2}, H_{n3}, ..., H_{nk}$ and grafting the root of H_{ij}, $j = 1, 2, 3, ..., k$ with the *i*th vertex of *G*. If o_{js} is the root of H_{js}, for $s = 1, 2, ..., k$, then $o_{js} = o_j$ in the graph $G \circ_k \mathcal{H}$, for $s = 1, 2, ..., k$.

Susilowati et al. [8] described the properties of rooted product graphs as the following lemma and observation.

Observation 2.1 [8]**.** Let G be a labelled graph of order $n \geq 2$ and $\mathcal{H}$ be a sequence of n connected graphs H_j, $j = 1, 2, 3, ..., n.$ In the rooted product graph $G \circ \mathcal{H}$, if every H_j is a connected bipartite graph, then every two adjacent vertices in H_j are at different distances from the root of H_j and all vertices in $G \circ \mathcal{H}$.

Lemma 2.2 [8]**.** *Let G be a labelled graph of order* $n \geq 2$ *and* $\mathcal{H}$ *be a sequence of n connected graphs* H_j, $j = 1, 2, 3, ..., n$. *In the rooted product graph* $G \circ \mathcal{H}$, *if* o_j *is the root of* H_j, *and* U_j *is a local basis of* H_j, *then the following statements hold*:

(i) *If* $o_j \in U_j$, *then there are two adjacent vertices x, y in* H_j *such that* $r(x|S) = r(y|S)$ *for every* $S \subset V(H_j), |S| \leq |U_j| - 2.$

(ii) *If* $o_j \notin U_j$, *then there are two adjacent vertices x, y in* H_j *such that* $r(x|S) = r(y|S)$ *for every* $S \subset V(H_j), |S| \leq |U_j| - 1.$

Using Theorems 1.1 and 1.2, respectively, we get Corollaries 2.3 and 2.4, respectively, as below.

Corollary 2.3. *Let G be a labelled connected graph of order* $n \geq 2$ *and let* $\mathcal{H}$ *be a sequence of n connected bipartite rooted graphs* H_1, H_2, $H_3, ..., H_n$. *Then*

$$\dim_l(G \circ_k \mathcal{H}) = \dim_l(G \circ \mathcal{H}) = \dim_l(G).$$

Proof. Let G be a labelled connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n connected bipartite rooted graphs $H_1, H_2, H_3, ..., H_n$. Let $H_{11}, H_{12}, H_{13}, ..., H_{1k}, H_{21}, H_{22}, H_{23}, ..., H_{2k}, H_{31}, H_{32}, H_{33}, ..., H_{3k}, ..., H_{n1}, H_{n2}, H_{n3}, ..., H_{nk}$ be k copies of $\mathcal{H}$. Let o_{js} be the root of H_{js}, for $s = 1, 2, ..., k$, choose $W =$ a local basis of G.

Take any two adjacent vertices x, y in H_{js}, $j = 1, 2, ..., n$; $s = 1, 2, ..., k$. Because H_{js} is connected bipartite, by Observation 2.1, we get $d(x|z) \neq d(y|z)$ for every $z \in V(G \circ_k \mathcal{H})$. Therefore, $r(x|W) = r(y|W)$. Take any two adjacent roots o_{is}, o_{js} in $G \circ_k \mathcal{H}$. Because $W =$ a local basis of G, then $r(o_{is}|W) \neq r(o_{js}|W)$ and W is a local basis of $G \circ_k \mathcal{H}$. So $\dim_l(G \circ_k \mathcal{H}) = \dim_l(G)$. The same reason for $\dim_l(G \circ \mathcal{H}) = \dim_l(G)$. □

Corollary 2.4. *Let G be a connected labelled graph of order n and let $\mathcal{H}$ be a sequence of n connected non-bipartite rooted graphs of order at least two $H_1, H_2, H_3, ..., H_n$. If o_j is the root of H_j for every $j =$ 1, 2, ..., n, then*

$$\dim_l(G \circ_k \mathcal{H}) = k\sum_{j=1}^{n}(\dim_l(H_j) - \alpha_j),$$

where $\alpha_j = 1$ if o_j belongs to a basis of H_j and $\alpha_j = 0$ otherwise.

Proof. Let G be a connected labelled graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of the connected non-bipartite rooted graphs $H_1, H_2, H_3, ..., H_n$. Let $H_{11}, H_{12}, H_{13}, ..., H_{1k}, H_{21}, H_{22}, H_{23}, ..., H_{2k}, H_{31}, H_{32}, H_{33}, ..., H_{3k}, ..., H_{n1}, H_{n2}, H_{n3}, ..., H_{nk}$ be k copies of $H_1, H_2, H_3, ..., H_n$. Let o_{js} be the root of H_{js} and W_{js} be a local basis of H_{js} for $n =$ 1, 2, ..., n; $s = 1, 2, ..., k$. Choose $W = \bigcup_{j=1}^{n}(\bigcup_{s=1}^{k}(W_{js} - \{o_{js}\}))$.

Take any two adjacent vertices x, y in H_{js}, $j = 1, 2, ..., n$; $s = 1, 2, ..., k$. Because W_{js} is a local basis of H_{js}, then $r(x|W) \neq r(y|W)$. Take any two adjacent roots o_{is}, o_{js} in $G \circ_k \mathcal{H}$, then $d(o_{is}|z) \neq d(o_{js}|z)$ for every $z \in V(H_{js})$. Therefore, $r(o_{is}|W) \neq r(o_{js}|W)$ and W is a local resolving set of $G \circ_k \mathcal{H}$. Take any set $S \subseteq V(G \circ_k \mathcal{H})$ with $|S| < |W|$. It means that there is H_{js} such that maximum $(\dim_l(H_j) - \alpha_j - 1)$ vertices of H_{js}

which be elements of S. By Lemma 2.2, there are two adjacent vertices in H_{js} that have the same representation with respect to S. So W is a minimum local resolving set of $G \circ_k \mathcal{H}$ and $|W| = k\sum_{j=1}^{n}(\dim_l(H_j) - \alpha_j)$, where $\alpha_j = 1$ if o_j belongs to a basis of H_j and $\alpha_j = 0$ otherwise. □

By Corollary 2.3, we get Corollary 2.5 as below.

Corollary 2.5. *Let G and H be connected graphs. If H is a bipartite graph, then*

$$\dim_l(G \circ_k H) = \dim_l(G).$$

Corollary 2.6. *Let G be a connected graph of order n, H be a connected non-bipartite graph of order at least* 2, *and o is a grafting vertex. Then*

$$\dim_l(G \circ_k H) = \begin{cases} nk\dim_l(H) - 1, & \text{if } o \text{ belongs to a local basis of } H, \\ nk\dim_l(H), & \text{otherwise.} \end{cases}$$

Proof. Let G be a connected graph of order n, H be a connected non-bipartite graph of order at least 2, and o is a grafting vertex. Let $H_{11}, H_{12}, H_{13}, \ldots, H_{1k}, H_{21}, H_{22}, H_{23}, \ldots, H_{2k}, \ldots, H_{n1}, H_{n2}, H_{n3}, \ldots, H_{nk}$ be the nk copies of H and W_{js} is a local basis of H_{js}. If o is an element of local basis of H, then o_{js} is the element of local basis of H_{js} for all $j = 1, 2, \ldots, n$; $s = 1, 2, \ldots, k$, choose $W = \bigcup_{j=1}^{n}(\bigcup_{s=1}^{k}(W_{js} - \{o_{js}\}))$ so $|W| = nk(\dim_l(H) - 1)$. If o is not an element of local basis of H, then o_{js} is not an element of H_{js} for all $j = 1, 2, \ldots, n$; $s = 1, 2, \ldots, k$. Choose $W = \bigcup_{j=1}^{n}(\bigcup_{s=1}^{k}(W_{js}))$ so $|W| = nk(\dim_l(H))$. In these two conditions, we can prove that W is a local basis of $G \circ_k H$. □

Theorem 2.7. *Let G be a connected labelled graph of order $n \geq 2$, and let $\mathcal{H}$ be a sequence of the combined n connected non-bipartite graphs $H_1, H_2, \ldots, H_s$ and bipartite graphs $H_{s+1}, H_{s+2}, \ldots, H_n$, and o_j is the* *root of* H_j, *then*

$$\dim_l(G \circ_k \mathcal{H})$$

$$\begin{cases} = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j), & \textit{for } G = C_n, n \textit{ odd}, s > 1 \textit{ or} \\ & G \textit{ bipartite or } G = K_n, s = n-1, \\ = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + 1, & \textit{for } G = C_n, n \textit{ odd}, s = 1, \\ = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + \dim_l(G) - s, & \textit{for } G = K_n, s < n-1, \\ < k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + n - s - 1, & \textit{otherwise}, \end{cases}$$

where $\alpha_j = 1$ *if the root of* H_j *belongs to a local basis of* H_j *and* $\alpha_j = 0$ *otherwise.*

Proof. Let G be a connected labelled graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of the combined n connected non-bipartite graphs $H_1, H_2, H_3, \ldots, H_s$ and bipartite graphs $H_{s+1}, H_{s+2}, H_{s+3}, \ldots, H_n$. Let T be a local basis of G and U_j be a local basis of H_j, $j = 1, 2, \ldots, n$, and o_j is the root of H_j. Let $H_{11}, H_{12}, H_{13}, \ldots, H_{1k}, H_{21}, H_{22}, H_{23}, \ldots, H_{2k}, H_{31}, H_{32}, H_{33}, \ldots, H_{3k}, \ldots, H_{s1}, H_{s2}, H_{s3}, \ldots, H_{sk}$ be k copies of $H_1, H_2, H_3, \ldots, H_s$ and $H_{(s+1)1}, H_{(s+1)2}, H_{(s+1)3}, \ldots, H_{(s+1)k}, H_{(s+2)1}, H_{(s+2)2}, H_{(s+2)3}, \ldots, H_{(s+2)k}, H_{(s+3)1}, H_{(s+3)2}, H_{(s+3)3}, \ldots, H_{(s+3)k}, \ldots, H_{n1}, H_{n2}, H_{n3}, \ldots, H_{nk}$ be k copies of $H_{s+1}, H_{s+2}, H_{s+3}, \ldots, H_n$. Let o_{jl} be the root of H_{jl}, for $l = 1, 2, \ldots, k$.

Case 1. For $G = C_n$, n odd, $s > 1$ or G bipartite or $G = K_n$, $s = n - 1$. Choose $W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k} (U_{jl} - \{o_{jl}\}))$, so $|W| = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j)$.

Take any two adjacent roots o_{il}, o_{jl} in $G \circ_k \mathcal{H}$, if $G = C_n$, n odd, $s > 1$. Because G is bipartite, so $r(o_{il} | W) \neq r(o_{jl} | W)$. Take any two adjacent vertices x, y in H_{jl}, $j = 1, 2, ..., s$, $l - 1, 2, ..., k$. Because U_{jl} is a local basis of H_{jl}, then $r(x|U_{jl}) \neq r(y|U_{jl})$, so $r(x|W) \neq r(y|W)$. Take any two adjacent vertices x, y in H_{jl}, $j = s+1, s+2, ..., n$. Because H_{jl} is bipartite, we get $r(x|W) \neq r(y|W)$. Therefore, W is a local resolving set of $G \circ_k \mathcal{H}$. By Lemma 2.2, we can see that $W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k}(U_{jl} - \{o_{jl}\}))$ is a minimum local resolving set of $G \circ_k \mathcal{H}$ and

$$\dim_l(G \circ_k \mathcal{H}) = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j).$$

Case 2. For $G = C_n$, n odd, $s = 1$. Choose $W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k}(U_{jl} - \{o_{jl}\})) \cup \{z\} = \bigcup_{l=1}^{k}(U_{1l} - \{o_{1l}\}) \cup \{z\}$, $z \in H_{jl}$ for any $j = s+1, s+2, ..., n$; $l = 1, 2, ..., k$ and $z \neq o_{1l}$, without loss of generality, let $z \in H_{2l}$.

Take any two adjacent roots o_{il}, o_{jl} in $G \circ_k \mathcal{H}$. Because G is bipartite, then $d(o_{il}, o_{1l}) \neq d(o_{jl}, o_{1l})$. Therefore, $r(o_{il}|U_{1l}) \neq r(o_{jl}|U_{1l})$ and $r(o_{il}|W) \neq r(o_{jl}|W)$. Take any two adjacent vertices x, y in H_{jl}, for $j = 1$. Because U_{1l} is a local basis of H_{1l}, then $r(x|U_{1l}) \neq r(y|U_{1l})$, so $r(x|W) \neq r(y|W)$. Take any two adjacent vertices x, y in H_{jl}, for $j = 2, 3, ..., n$. Because H_{jl} is bipartite, then $d(x, o_{jl}) \neq d(y, o_{jl})$, so $d(x, o_{2l}) \neq d(y|o_{2l})$. Therefore, $r(x|W) \neq r(y|W)$.

So, $W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k}(U_{jl} - \{o_{jl}\})) \cup \{z\} = \bigcup_{l=1}^{k}(U_{1l} - \{o_{1l}\}) \cup \{z\}$ for $z \in H_{2l}$, is a local resolving set of $G \circ_k \mathcal{H}$. Take any set $S \subset V(G \circ_k \mathcal{H})$, where $|S| < |W|$, there are two conditions on S. First, the number of elements H_{1l} which lie in S is less than $|U_{1l} - \{o_{1l}\}|$. Because U_{1l} is a local basis of H_{1l}, so there are two adjacent vertices in H_{1l} that have the same representation with respect to S. Second, no element in H_{jl} for all $j = s+1, s+2, ..., n$; $l = 1, 2, ..., k$ lie in S. Because G is bipartite, there are two adjacent root vertices that have the same representation with respect to S. So, W is a local basis of $G \circ_k \mathcal{H}$ and

$$\dim_l(G \circ_k \mathcal{H}) = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + 1,$$

where $\alpha_j = 1$ if o_j belongs to a local basis of H_j and $\alpha_j = 0$ otherwise.

Case 3. For $G = K_n$, $s < n - 1$. Choose $W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k}(U_{jl} - \{o_{jl}\})) \bigcup \{u_{jl} | u_{jl} \neq o_{jl}, j = s+1, s+2, ..., n-1, l = 1, 2, 3, ..., k\}$. Without loss of generality, let $s = n - 2$, it means that H_{jl}, $j = 1, 2, ..., n-2$ are non-bipartite graphs and $H_{(n-1)l}$, H_{nl} are bipartite graphs and

$$W = \bigcup_{j=1}^{n-2}(\bigcup_{l=1}^{k}(U_{jl} - \{o_{jl}\})) \bigcup \{u_{(n-1)l} | u_{(n-1)l} \neq o_{(n-1)l}, l = 1, 2, 3, ..., k\}.$$

Take any two adjacent roots in $G \circ_k \mathcal{H}$, there are three possibilities:

First, two adjacent roots are $o_{(n-1)l}$, o_{nl}. We get $d(o_{(n-1)l}, o_{jl}) = d(o_{nl}, o_{jl})$ for all $j = 1, 2, ..., n-2$; $l = 1, 2, 3, ..., k$. It implies that $r(o_{(n-1)l} | U_{jl}) = r(o_{nl} | U_{jl})$, but $r(o_{(n-1)l} | o_{(n-1)l}) \neq r(o_{nl} | o_{(n-1)l})$. Therefore, $r(o_{(n-1)l} | W) \neq r(o_{nl} | W)$. Second, one of the roots is an element of H_{n-1} or H_n and one of the roots is an element of H_j, $j = 1, 2, ..., n-2$. Without loss of generality, let o_{nl} and o_{jl} for some j. We get $d(o_{jl}, o_{nl}) \neq d(o_{jl}, o_{jl})$, so $r(o_{nl} | W) \neq r(o_{jl} | W)$. Third, two adjacent roots are elements of H_j, $j = 1, 2, ..., n-2$. It is obvious that $r(o_{il} | W) \neq r(o_{tl} | W)$.

Take any two adjacent vertices x, y in H_{jl}, $j = n-1, n$, $l = 1, 2, 3, ..., k$. Because H_{n-1} and H_n are bipartite, we get $r(x | W) \neq r(y | W)$. Take any two adjacent vertices in H_{jl}, $j = 1, 2, ..., n-2$; $l = 1, 2, 3, ..., k$. Because U_{jl}, $j = 1, 2, ..., n-2$ is a basis of H_{jl}, then $r(x | W) \neq r(y | W)$.

So $W = \bigcup_{j=1}^{n-2}(\bigcup_{l=1}^{k}(U_{jl} - \{o_{jl}\})) \cup \{u_{(n-1)l} \mid u_{(n-1)l} \neq o_{(n-1)l}, l = 1, 2, 3, ..., k\}$ is a local resolving set of $G \circ_k \mathcal{H}$. Take any set $S \subset V(G \circ_k \mathcal{H})$, where $|S| < |W|$, there are two conditions for *S*. First, there is H_{jl}, for some $j = 1, 2, ..., n-2$; $l = 1, 2, 3, ..., k$ such that the number of elements H_{jl} which lie in *S* is less than $|U_{jl} - \{o_{jl}\}|$. Because U_{jl} is a local basis of H_{jl}, so there are two adjacent vertices in H_{jl} that have the same representation with respect to *S*. Second, there is $u_{(n-1)l}$ for some $l = 1, 2, 3, ..., k$ is not element of *S*. Because $G = K_n$, so there are two adjacent root vertices o_{jl}, o_{sl}; $j, s \neq n, n-1$ that have the same representation with respect to *S*. So *W* is a local basis of $G \circ_k \mathcal{H}$ and $|W| = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + n - 1 - s$. Because *G* is a complete graph K_n and $\dim_l(K_n) = n - 1$, then $\dim_l(G \circ_k \mathcal{H}) = k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + \dim_l(G) - s$, where $\alpha_j = 1$ if o_j belongs to a local basis of H_j and $\alpha_j = 0$ otherwise.

Case 4. For *G* is otherwise, $\dim_l(G \circ_k \mathcal{H}) < k\sum_{j=1}^{s}(\dim_l(H_j) - \alpha_j) + n - s - 1$, it is obvious because K_n is the graph with the biggest local metric dimension. □

3. The Local Metric Dimension of Generalized Corona Product Graph

Susilowati et al. [9] defined the generalized corona product graph. Let *G* be a graph of order *n*, *k*-corona graph $G \odot_k H$ of two graphs *G* and *H* is obtained by taking one copy of *G* and *nk* copies of *H*, that is, $H_{11}, H_{12}, H_{13}, ..., H_{1k}, H_{21}, H_{22}, H_{23}, ..., H_{2k}, H_{31}, H_{32}, H_{33}, ..., H_{3k}, ..., H_{n1}, H_{n2}, H_{n3}, ..., H_{nk}$ and then joining by edge the *i*th vertex of *G* to every vertex in *ij*th copy of *H*. Similarly, let *G* be a labelled graph on *n* vertices and $\mathcal{H}$ be a sequence of *n* connected graphs $H_1, H_2, H_3, ..., H_n$, then k-corona product $G \odot_k \mathcal{H}$ is obtained by taking k copies of H_i for every i, that is, $H_{11}, H_{12}, H_{13}, \ldots, H_{1k}, H_{21}, H_{22}, H_{23}, \ldots, H_{2k}, H_{31}, H_{32}, H_{33}, \ldots, H_{3k}, \ldots, H_{n1}, H_{n2}, H_{n3}, \ldots, H_{nk}$ and then joining by edge the ith vertex of G to every vertex in ijth copy of H_i, $j = 1, 2, 3, \ldots, k$.

Lemma 3.1. *Let G be a connected nontrivial labelled graph. If H is an empty graph and $\mathcal{H}$ is a sequence of n empty graphs $H_1, H_2, H_3, \ldots, H_n$, then* $\dim_l(G \odot_k H) = \dim_l(G \odot_k \mathcal{H}) = \dim_l(G)$.

Proof. Let H be an empty graph and $\mathcal{H}$ be a sequence of n empty graphs $H_1, H_2, H_3, \ldots, H_n$. Then there are no edges in H and $\mathcal{H}$. In graph $G \odot_k H$ and $G \odot_k \mathcal{H}$, respectively, every vertex in H and $\mathcal{H}$ is adjacent to one vertex only in G. Therefore, the local metric dimension of $G \odot_k H$ and $G \odot_k \mathcal{H}$ depend on local metric dimension of G only. □

Using Theorems 1.4 and 1.5, respectively, we get Corollaries 3.2 and 3.3, respectively, as below.

Corollary 3.2. *Let G be a connected labelled graph of order $n \geq 2$, and $\mathcal{H}$ be a sequence of n nonempty graphs $H_1, H_2, H_3, \ldots, H_n$. Then*

$$\dim_l(G \odot_k \mathcal{H}) = k \sum_{i=1}^{n} (\dim_l(K_1 + H_i) - \alpha_i),$$

where $\alpha_i = 1$ if the vertex of K_1 belongs to a local basis of $K_1 + H_j$ and $\alpha_j = 0$ otherwise.

Proof. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$, B_i is a local basis of H_i and B_{ij} is a basis of $\langle v_i \rangle + H_{ij}$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k$. So $B_{ij} = B_i$ for $j = 1, 2, \ldots, k$. Choose $W = \bigcup_{i=1}^{n} (\bigcup_{j=1}^{k} (B_{ij} - \{v_i\}))$. Because $\langle v_i \rangle + H_{ij} \approx K_1 + H_i$ so $|W| = k\sum_{i=1}^{n} (\dim_l(K_1 + H_i) - 1)$ if v_i is an element of a local basis of $K_1 + H_i$ and $|W| = k\sum_{i=1}^{n} \dim_l(K_1 + H_i)$ if v_i is not an element of a local basis of $K_1 + H_i$. Using the similar logic to Corollary 2.6, we can prove that W is a local basis of $G \odot_k \mathcal{H}$. □

Corollary 3.3. *Let H be a nonempty graph. The following statements hold*:

(i) *If the vertex of* K_1 *does not belong to any local basis for* $K_1 + H$, *then for any connected graph G of order n*,

$$\dim_l(G \odot_k H) = nk(\dim_l(K_1 + H)).$$

(ii) *If the vertex of* K_1 *belongs to a local basis for* $K_1 + H$, *then for any connected graph G of order* $n \geq 2$,

$$\dim_l(G \odot_k H) = nk(\dim_l(K_1 + H) - 1).$$

4. Commutative Characterization of Generalized Comb and Corona Products Graphs with Respect to Local Metric Dimension

An operation $*$ defined on two graphs is said to be *commutative* if $A * B \cong B * A$ for every graphs A and B. An operation $*$ defined on two graphs G and H is said to be *commutative* with respect to local metric dimension if $\dim_l(G * H) = \dim_l(H * G)$, denoted by $(G * H) \cong_{\dim l} (H * G)$.

In this section, we present commutative characterization of generalized comb and corona products graphs with respect to local metric dimension. Because of the commutativity of operation applies only to the two graphs, then the commutative characterization of generalized comb and corona products graphs is only addressed to $G \circ_k H$ and $G \odot_k H$.

Theorem 4.1. *Let G and H be connected bipartite graphs. Then*

$$(G \circ_k H) \cong_{\dim l} (H \circ_k G) \text{ if and only if } \dim_l(H) = \dim_l(G).$$

Proof. Let G and H be connected bipartite graphs. Let $(G \circ_k H) \cong_{\dim l} (H \circ_k G)$. It means that $\dim_l(G \circ_k H) = \dim_l(H \circ_k G)$. By Corollary 2.5, we get $\dim_l(G) = \dim_l(H)$.

Conversely, let $\dim_l(G) = \dim_l(H)$. By Corollary 2.5, we get $\dim_l(G) = \dim_l(G \circ_k H)$ and $\dim_l(H) = \dim_l(H \circ_k G)$. Therefore, $\dim_l(G \circ_k H) = \dim_l(H \circ_k G)$. So $(G \circ_k H) \cong_{\dim l} (H \circ_k G)$. □

For the case of non-bipartite graphs, the formula of commutative characterization of generalized comb product with respect to local metric dimension is presented based on existence of grafting vertex, whether element of a local basis of graph operated.

Theorem 4.2. *Let G and H be connected graphs of order at least three. Let G and H be non-bipartite graphs. If the grafting vertex of $G \circ_k H$ belongs to a local basis of H and the grafting vertex of $H \circ_k G$ belongs to a local basis of G, then*

$$(G \circ_k H) \cong_{\dim l} (H \circ_k G) \text{ if and only if}$$

$$|V(G)|(\dim_l(H) - 1) = |V(H)|(\dim_l(G) - 1).$$

Proof. Let G and H be connected graphs of order at least three. Let G and H be non-bipartite graphs. Let the grafting vertex of $G \circ_k H$ belong to a local basis of H and the grafting vertex of $H \circ_k G$ belong to a local basis of G. Let $(G \circ_k H) \cong_{\dim l} (H \circ_k G)$. By Corollary 2.6, we get $\dim_l(G \circ_k H) = k|V(G)|(\dim_l(H) - 1)$ and $\dim_l(H \circ_k G) = k|V(H)|(\dim_l(G) - 1)$. Therefore, $k|V(G)|(\dim_l(H) - 1) = k|V(H)|(\dim_l(G) - 1)$. So $|V(G)|(\dim_l(H) - 1) = |V(H)|(\dim_l(G) - 1)$.

Conversely, let $|V(G)|(\dim_l(H) - 1) = |V(H)|(\dim_l(G) - 1)$. Then

$$k|V(G)|(\dim_l(H) - 1) = k|V(H)|(\dim_l(G) - 1).$$

Therefore, $\dim_l(G \circ_k H) = \dim_l(H \circ_k G)$. □

The case when grafting vertex does not belong to a local basis of graph operated is dealt with below.

Theorem 4.3. *Let G and H be connected graphs of order at least three. Let G and H be non-bipartite graphs. If the grafting vertex of* $G \circ_k H$ *does not belong to a local basis of H and the grafting vertex of* $H \circ_k G$ *does not belong to a local basis of G, then*

$$(G \circ_k H) \cong_{\dim l} (H \circ_k G) \text{ if and only if}$$

$$|V(G)|(\dim_l(H)) = |V(H)|(\dim_l(G)).$$

Proof. Let G and H be connected graphs of order at least three. Let G and H be non-bipartite graphs. Let the grafting vertex of $G \circ_k H$ does not belong to a local basis of H and the grafting vertex of $H \circ_k G$ do not belong to a local basis of G. Let $(G \circ_k H) \cong_{\dim l} (H \circ_k G)$. By Corollary 2.6, we get $\dim_l(G \circ_k H) = k|V(G)|(\dim_l(H))$ and

$$\dim_l(H \circ_k G) = k|V(H)|(\dim_l(G)).$$

Therefore, $k|V(G)|(\dim_l(H)) = k|V(H)|(\dim_l(G))$. So $|V(G)|\dim_l(H) = |V(H)|\dim_l(G)$. Conversely, let $|V(G)|(\dim_l(H)) = |V(H)|(\dim_l(G))$. Then

$$k|V(G)|(\dim_l(H)) = k|V(H)|(\dim_l(G)).$$

Therefore, $\dim_l(G \circ_k H) = \dim_l(H \circ_k G)$. □

In the next theorems, we present the commutative characterization of generalized corona product with respect to local metric dimension.

Theorem 4.4. *Let G and H be nonempty connected graphs. If the vertex of* K_1 *does not belong to a local basis of* $K_1 + H$ *and* $K_1 + G$, *then*

$$(G \odot_k H) \cong_{\dim l} (H \odot_k G) \text{ if and only if}$$

$$|V(G)|\dim_l(K_1 + H) = |V(H)|\dim_l(K_1 + H).$$

Proof. Let G and H be nonempty connected graphs. Let the vertex of K_1 does not belong to a local basis of $K_1 + H$ and $K_1 + G$. Let $(G \odot_k H)$

$\cong_{\dim l} (H \odot_k G)$. Based on Corollary 3.3(i), we get $\dim_l(G \odot_k H) = k|V(G)|(\dim_l(K_1 + H) - 1)$ and

$$\dim_l(G \odot_k H) = k|V(G)|(\dim_l(K_1 + H) - 1).$$

Therefore, $k|V(G)|(\dim_l(K_1 + H)) = k|V(H)|(\dim_l(K_1 + G))$. So

$$|V(G)|\dim_l(K_1 + H) = |V(H)|\dim_l(K_1 + H).$$

Conversely, let $|V(G)|\dim_l(K_1 + H) = |V(H)|\dim_l(K_1 + H)$. Then

$$k|V(G)|(\dim_l(K_1 + H)) = k|V(H)|(\dim_l(K_1 + G)).$$

Based on Corollary 3.3(i), we get $\dim_l(G \odot_k H) = \dim_l(H \odot_k G)$. In other words, $(G \odot_k H) \cong_{\dim l} (H \odot_k G)$. □

Theorem 4.5. *Let G and H be nonempty connected graphs of order at least two. If the vertex of K_1 belongs to a local basis of $K_1 + H$ and $K_1 + G$, then*

$$(G \odot_k H) \cong_{\dim l} (H \odot_k G) \text{ if and only if}$$

$$|V(G)|(\dim_l(K_1 + H) - 1) = |V(H)|(\dim_l(K_1 + H) - 1).$$

Proof. Let G and H be nonempty connected graphs of order at least two. Let the vertex of K_1 belong to a local basis of $K_1 + H$ and $K_1 + G$. Let $(G \odot_k H) \cong_{\dim l} (H \odot_k G)$. By Corollary 3.3(ii), we get $\dim_l(G \odot_k H) = k|V(G)|(\dim_l(K_1 + H) - 1)$ and

$$\dim_l(H \odot_k G) = k|V(H)|(\dim_l(K_1 + G) - 1).$$

Therefore, $|V(G)|(\dim_l(K_1 + H) - 1) = |V(H)|(\dim_l(K_1 + G) - 1)$.

Conversely, let $|V(G)|(\dim_l(K_1 + H) - 1) = |V(H)|(\dim_l(K_1 + G) - 1)$. Then $k|V(G)|(\dim_l(K_1 + H) - 1) = k|V(H)|(\dim_l(K_1 + G) - 1)$. From Corollary 3.3(ii), we get $\dim_l(G \odot_k H) = \dim_l(H \odot_k G)$. In other words, $(G \odot_k H) \cong_{\dim l} (H \odot_k G)$. □

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