Labeling of Chordal Rings

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Abstract

The chordal ring of order n, denoted by $CR_n(x, y, z)$, is the graph with vertex set Z_n, an additive group of integers modulo n, and adjacencies given by $i \sim i+x, i \sim i+y, i \sim i+z$ for all even vertex i and distinct odd integers x, y, z in $[1, n-1]$. In this paper, we provide super vertex-magic total labeling of $CR_n(1,3,5)$, $n \equiv 0 \pmod 4$ and (a,d)-antimagic labeling of $CR_n(1,3,7)$, $n \equiv 0 \pmod 4$.

Keywords : Chordal rings, super vertex-magic total labeling, (a,d)-antimagic labeling, crossing number.

1 Introduction

Throughout this paper, we let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$; we write v for $|V(G)|$ and e for $|E(G)|$. For a general reference to graph-theoretic definitions and notions, see [14].

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The notion of super vertex-magic total labeling was introduced in [9]. This is an assignment of the integers from 1 to $e+v$ to the vertices and the edges of G so that label on a vertex and labels on the edges incident at that vertex add to a fixed constant with the condition that the labels of the vertices are from 1 to v. More formally, a one-to-one map λ from $V \cup E$ onto the set $\{1, 2, ..., e+v\}$ is a super vertex-magic total labeling if there is a constant C so that for every vertex x,

$$\lambda(x) + \sum \lambda(xy) = C$$

where the sum is taken over all the vertices y adjacent to x and $\lambda(V(G)) = \{1, 2, \ldots, v\}$, $\lambda(E(G)) = \{v+1, v+2, \ldots, v+e\}$. Constant C is called a magic constant for λ.

J. MacDougall et al. [10] proved some results on super vertex-magic total labeling of some graphs. These results are: cycle C_n has a super vertex-magic total labeling if and only if n is odd, and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjectured that no tree has a super vertex-magic total labeling and that K_{4n} has a super vertex-magic total labeling when $n > 1$. Gómez [6] proved the conjecture: If $n \equiv 0 \pmod 4$, $n > 4$, then K_n has a super vertex-magic total labeling. Swaminathan and Jeyanthi [11] proved the following: no super vertex-magic total graph has two or more isolated vertices or an isolated edge; a tree with n internal edges and tn leaves is not super vertex-magic total if $t > (n+1)/n$; if Δ is the largest degree of any vertex in a tree T with p vertices and $\Delta > (-3 + \sqrt{1+16p})/2$, then T is not super vertex-magic total; the graph obtained from a comb by appending a pendant edge to each vertex of degree 2 is super vertex-magic total; the graph obtained by attaching a path with t edges to a vertex of an n-cycle is super vertex-magic total if and only if $n+t$ is odd. For more results concerning super vertex-magic total labelings, see a nice survey paper by Gallian [5].

A one-to-one mapping $\lambda : E \rightarrow \{1, 2, \ldots e\}$ is called an (a, d)-antimagic labeling of G such that the induced mapping $g_\lambda : V \rightarrow N$, defined by $g_\lambda(v) = \sum \lambda(vu), vu \in E(G)$ is injective and $g_\lambda(V) = \{a, a+d, a+2d, \ldots a+(v-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a, d)-antimagic graph. Bodendiek and Walther [3] showed that the theory of linear Diophantine equations and other concepts of number theory can be applied to determine the set of all connected (a, d)-antimagic graphs.

Bodendiek and Walther [2] proved that some graphs(including even cycles, paths of even order, stars, $C_3^{(k)}$, $C_4^{(k)}$, $K_{3,3}$, tree with odd order $n \geq 5$

and having a vertex that is adjacent to at least three end vertices) are not (a, d)-antimagic. They also proved that P_{2k+1} is $(k, 1)$-antimagic, C_{2k+1} is $(k+2, 1)$-antimagic, a tree of odd order $2k+1 (k > 1)$ is $(k, 1)$-antimagic, if $K_{4k} (k \geq 2)$ is (a, d)-antimagic then d is odd and $d \leq (2k+1)(4k-1)+1$, if $K_{2k+1} (k \geq 2)$ is (a, d)-antimagic then $d \leq (2k+1)(k-1)+1$. In [8], Ivančo and Semaničová proved that a 2-regular graph is super edge-magic if and only if it is $(a, 1)$-antimagic. As a corollary they proved that each of the following graphs are (a, 1)-antimagic: kC_n for n odd and at least 3; $k(C_3 \cup C_n)$ for n even and at least 6; $k(C_4 \cup C_n)$ for n odd and at least 5; $k(C_5 \cup C_n)$ for n even and at least 4; $k(C_m \cup C_n)$ for m even and at least 6, n odd, and $n \geq m/2+2$. Vilfred and Florida [12] proved the following: the one-sided infinite path is $(1, 2)$- antimagic; P_{2n} is not (a, d)-antimagic for any a and d and that a 2-regular graph G is (a, d)-antimagic if and only if $|V(G)| = 2n+1$ and $(a, d) = (n+2, 1)$. They also proved that for a graph with an (a, d)-antimagic labeling, q edges, minimum degree δ and maximum degree Δ, the vertex labels lie between $\delta(\delta+1)/2$ and $\Delta(2q-\Delta+1)/2$. For more results concerning (a, d)-antimagic labelings, see a nice survey paper by Gallian [5].

Many definitions of chordal rings have been proposed in literature. Chordal rings of degree 3, proposed by Arden and Lee [1], are obtained from an even-order undirected cycle by adding chords in a regular manner. All the new chords have the same length and connect an even vertex to an odd vertex [1, 4]. Let $n \geq 3$ be an even integer, and let x, y and z be three distinct odd integers in $[1, n-1]$. The chordal ring of order n and chords x, y and z, is denoted by $CR_n(x, y, z)$ and is defined as the graph with vertex set Z_n, an additive group of integers modulo n, and adjacencies given by $i \sim i+x, i \sim i+y$ and $i \sim i+z$ for all even vertex i.

From the definition, chordal rings are 3-regular. They are also bipartite since even vertices are pairwise independent and so are the odd vertices. Note that every odd vertex i of $CR_n(x, y, z)$ is adjacent to $i-x, i-y$ and $i-z$. In this paper, we consider two chordal rings $CR_n(1, 3, 5)$ and $CR_n(1, 3, 7)$. There is an isomorphism $\phi : x \mapsto x$ if x is even and $\phi : x \mapsto x-2$ if x is odd between the chordal rings $CR_n(1, 3, 5)$, $CR_n(1, 3, 7)$ and the chordal rings $CR_n(1, 3, n-1)$, $CR_n(1, 5, n-1)$, respectively, which shows that $CR_n(1, 3, 5) \cong CR_n(1, 3, n-1)$ and $CR_n(1, 3, 7) \cong CR_n(1, 5, n-1)$.

In this paper, we provide super vertex-magic total labeling of $CR_n(1, 3, 5)$, $n \equiv 0 \pmod 4$ by providing super vertex-magic total labeling of $CR_n(1, 3, n-1)$, $n \equiv 0 \pmod 4$ and (a, d)-antimagic labeling of $CR_n(1, 3, 7)$, $n \equiv 0 \pmod 4$ by providing (a, d)-antimagic labeling of $CR_n(1, 5, n-1)$, $n \equiv 0 \pmod 4$.

One can see that the vertex set Z_n of $CR_n(1,5,n-1)$ can be partitioned into two sets $V = \{v_{\frac{i}{2}} : even\ i \in Z_n\}$, $U = \{u_{\frac{i+1}{2}} : odd\ i \in Z_n\}$, subscripts taken modulo $\frac{n}{2}$. Hence we have a graph with vertex set $V \cup U$ and edge set $\bigcup_{i=0}^{\frac{n}{2}-1}\{v_iu_i, v_iu_{i+1}, v_iu_{i+3}\}$ which is a 3-regular Knödel graph [13]. This graph has n vertices and $\frac{3n}{2}$ edges. Similarly, the vertex set of $CR_n(1,3,n-1)$ can be also partitioned and we have a graph with vertex set $V \cup U$ and edge set $\bigcup_{i=0}^{\frac{n}{2}-1}\{v_iu_i, v_iu_{i+1}, v_iu_{i+2}\}$. Again, this graph has n vertices and $\frac{3n}{2}$ edges.

In [13], it was shown that the crossing number $cr(CR_8(1,5,7)) = 0$, $cr(CR_{10}(1,5,9)) = 1$ and $cr(CR_n(1,5,n-1)) = \lfloor\frac{n}{6}\rfloor + (n\ mod\ 6)/2$ (*even* $n > 10$). In order to show that $CR_n(1,3,n-1)$ and $CR_n(1,5,n-1)$ are not isomorphic, we prove that $CR_n(1,3,n-1)$ is a planar graph for $n \equiv 0 \pmod 4$ and has crossing number equal to 1 for $n \equiv 2 \pmod 4$ (See Appendix).

2 Main Results

In this section, we show that $CR_n(1,3,n-1)$ is super vertex-magic for $n \equiv 0 \pmod 4), n \geq 8$. Also, we provide (a,d)-antimagic labeling of $CR_n(1,5,n-1)$, $n \equiv 0 \pmod 4$. The chordal graph for $x = 1, y = 3,\ z = n-1$ and $n = 8$ is shown in Figure 1.

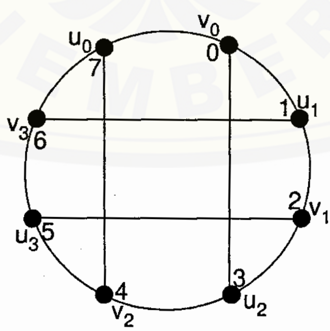

Figure 1: The chordal graph $CR_8(1,3,7)$

Theorem 1 *The chordal ring $CR_n(1,3,n-1)$ is super vertex-magic for $n \equiv 0 \pmod 4), n \geq 8$.*

Proof. For $n = 8$, $CR_8(1,3,7) \cong CR_8(1,5,7)$ since there is a bijection $\psi : V(CR_8(1,3,7)) \to V(CR_8(1,5,7))$ defined by $\psi(x) = x$ if x is even and $\psi(x) = 5x + 2$ if x is odd, which preserves adjacencies and non-adjacencies. Hence the chordal ring $CR_8(1,3,7)$ is super vertex-magic [15]. We define the edge labeling λ of $CR_n(1,3,n-1)$, $n \geq 12$ and $n \equiv 0 \pmod 4$ as follows:

$$\lambda(v_iu_i) = \begin{cases} (2n+4-i)/2, & 0 \leq i \leq 2 \wedge i \equiv 0 \pmod 2, \\ n+1+i/2, & 4 \leq i \leq n/2-2 \wedge i \equiv 0 \pmod 2, \\ 2n-1, & i = 1, \\ (7n-2i-2)/4, & 3 \leq i \leq n/2-3 \wedge i \equiv 1 \pmod 2, \\ 7n/4-1, & i = n/2-1. \end{cases}$$

$$\lambda(v_iu_{i+1}) = \begin{cases} 5n/4+1+i/2, & 0 \leq i \leq n/2-4 \wedge i \equiv 0 \pmod 2, \\ 2n-(1+i)/2, & 3 \leq i \leq n/2-5 \wedge i \equiv 1 \pmod 2, \\ 3n/2, & i = 1, \end{cases}$$

$$\lambda(v_iu_{i+1}) = \begin{cases} 7n/4, & i = n/2-3, \\ 7n/4+1, & i = n/2-2, \\ 2n+1, & i = n/2-1. \end{cases}$$

$$\lambda(v_iu_{i+2}) = \begin{cases} (5n-i)/2, & 0 \leq i \leq n/2-4 \wedge i \equiv 0 \pmod 2, \\ 2n+(3+i)/2, & 1 \leq i \leq n/2-5 \wedge i \equiv 1 \pmod 2, \\ 9n/4+1, & i = n/2-3, \\ 9n/4, & i = n/2-2, \\ 2n, & i = n/2-1. \end{cases}$$

Now we verify that λ is a bijection from the edge set $E(CR_n(1,3,n-1))$ onto $\{n+1, n+2, ..., 5n/2\}$. Denoted by

$$\begin{aligned} S_1 &= \{\lambda(v_iu_i) | 0 \leq i \leq n/2-1\}, \\ S_2 &= \{\lambda(v_iu_{i+1}) | 0 \leq i \leq n/2-1\}, \\ S_3 &= \{\lambda(v_iu_{i+2}) | 0 \leq i \leq n/2-1\}. \end{aligned}$$

Then

$$\begin{aligned} S_1 &= S_{11} \cup S_{12} S_{13} \cup S_{14} \cup S_{15}, \\ S_{11} &= \{\lambda(v_iu_i) | 0 \leq i \leq 2 \wedge i \equiv 0 \pmod 2\} \\ &= \{(2n+4-i)/2 | 0 \leq i \leq 2 \wedge i \equiv 0 \pmod 2\} \\ &= \{n+2, n+1\} \\ &= \{n+1, n+2\}, \end{aligned}$$

$$
\begin{aligned}
S_{12} &= \{\lambda(v_iu_i)|4 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\}\\
&= \{(n+1+i/2|4 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\}\\
&= \{n+3, n+4, ..., 5n/4\},\\
S_{13} &= \{\lambda(v_iu_i)|3 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\}\\
&= \{(7n-2i-2)/4|3 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\}\\
&= \{7n/4-2, 7n/4-3, ..., 3n/2+1\}\\
&= \{3n/2+1, 3n/2+2, ..., 7n/4-2\},\\
S_{14} &= \{\lambda(v_iu_i)|i=1\} = \{2n-1\},\\
S_{15} &= \{\lambda(v_iu_i)|i=n/2-1\} = \{7n/4-1\},\\
S_2 &= S_{21} \cup S_{22} \cup S_{23} \cup S_{24} \cup S_{25} \cup S_{26},\\
S_{21} &= \{\lambda(v_iu_{i+1})|0 \le i \le n/2-4 \wedge i \equiv 0 \pmod 2\}\\
&= \{5n/4+1+i/2|0 \le i \le n/2-4 \wedge i \equiv 0 \pmod 2\}\\
&= \{5n/4+1, 5n/4+2, ..., 3n/2-1\},\\
S_{22} &= \{\lambda(v_iu_{i+1})|i=1\} = \{3n/2\},\\
S_{23} &= \{\lambda(v_iu_{i+1})|3 \le i \le n/2-5 \wedge i \equiv 1 \pmod 2\}\\
&= \{2n-(1+i)/2|3 \le i \le n/2-5 \wedge i \equiv 1 \pmod 2\}\\
&= \{2n-2, 2n-3, ..., 7n/4+2\}\\
&= \{7n/4+2, 7n/4+3, ..., 2n-2\},\\
S_{24} &= \{\lambda(v_iu_{i+1})|i=n/2-3\} = \{7n/4\},\\
S_{25} &= \{\lambda(v_iu_{i+1})|i=n/2-2\} = \{7n/4+1\},\\
S_{26} &= \{\lambda(v_iu_{i+1})|i=n/2-1\} = \{2n+1\},\\
S_3 &= S_{31} \cup S_{32} \cup S_{33} \cup S_{32} \cup S_{33} \cup S_{34} \cup S_{35},\\
S_{31} &= \{\lambda(v_iu_{i+2})|0 \le i \le n/2-4 \wedge i \equiv 0 \pmod 2\}\\
&= \{(5n-i)/2|0 \le i \le n/2-4 \wedge i \equiv 0 \pmod 2\}\\
&= \{5n/2, 5n/2-1, ..., 9n/4+2\}\\
&= \{9n/4+2, 9n/4+3, ..., 5n/2\},\\
S_{32} &= \{\lambda(v_iu_{i+2})|1 \le i \le n/2-5 \wedge i \equiv 1 \pmod 2\}\\
&= \{2n+(3+i)/2|1 \le i \le n/2-5 \wedge i \equiv 1 \pmod 2\}\\
&= \{2n+2, 2n+3, ..., 9n/4-1\},\\
S_{33} &= \{\lambda(v_iu_{i+2})|i=n/2-3\} = \{9n/4+1\},\\
S_{34} &= \{\lambda(v_iu_{i+2})|i=n/2-2\} = \{9n/4\},\\
S_{35} &= \{\lambda(v_iu_{i+2})|i=n/2-1\} = \{2n\},
\end{aligned}
$$

Hence $S_1 \cup S_2 \cup S_3$ is the set of labels of all edges, and
$S_1 \cup S_2 \cup S_3 = S_{11} \cup S_{12} \cup S_{13} \cup S_{14} \cup S_{15} \cup S_{21} \cup S_{22} \cup S_{23} \cup S_{24} \cup S_{25} \cup$

$$S_{26} \cup S_{31} \cup S_{32} \cup S_{33} \cup S_{34} \cup S_{35}$$
$$= S_{11} \cup S_{14} \cup S_{21} \cup S_{22} \cup S_{13} \cup S_{15} \cup S_{25} \cup S_{24} \cup S_{26} \cup S_{12} \cup S_{33} \cup S_{23} \cup S_{32} \cup S_{34} \cup S_{35} \cup S_{31}$$
$$= \{n+1, n+2\} \cup \{n+3, n+4, ..., 5n/4\} \cup \{5n/4+1, 5n/4+2, ..., 3n/2-1\} \cup \{3n/2\} \cup \{3n/2+1, 3n/2+2, ..., 7n/4-2\} \cup \{7n/4-1\} \cup \{7n/4\} \cup \{7n/4+1\} \cup \{7n/4+2, 7n/4+3, ..., 2n-2\} \cup \{2n-1\} \cup \{2n\} \cup \{2n+1\} \cup \{2n+2, 2n+3, ..., 9n/4-1\} \cup \{9n/4\} \cup \{9n/4+1\} \cup \{9n/4+2, 9n/4+3, ..., 5n/2\}$$
$$= \{n+1, n+2, ..., 5n/2\}.$$

Therefore we conclude that λ is a bijection from $E(G)$ onto $\{n+1, n+2, ..., 5n/2\}$. Define $g_\lambda : V(G) \to N$ as

$$g_\lambda(v) = C - \sum \lambda(vu), vu \in E(G) \text{ and}$$
$$W = \{g_\lambda(v) | v \in V(G)\}.$$

Now we show that g_λ is a bijective mapping from $V(G)$ onto W. Let us denote the sets of the weights under an edge labeling λ of vertices v_i and u_i of $CR_n(1, 3, n-1)$ by

$$\begin{aligned} W_1 &= \{g_\lambda(v_i) | 0 \le i \le n/2-1\} \\ &= \{C - (\lambda(v_i u_i) + \lambda(v_i u_{i+1}) + \lambda(v_i u_{i+2})) | 0 \le i \le n/2-1\}, \\ W_2 &= \{g_\lambda(u_i) | 0 \le i \le n/2-1\} \\ &= \{C - (\lambda(v_i u_i) + \lambda(v_{i-1} u_i) + \lambda(v_{i-2} u_i)) | 0 \le i \le n/2-1\}. \end{aligned}$$

Where

$$\begin{aligned} W_1 &= W_{11} \cup W_{12} \cup W_{13} \cup W_{14} \cup W_{15} \cup W_{16}, \\ W_{11} &= \{C - (\lambda(v_i u_i) + \lambda(v_i u_{i+1}) + \lambda(v_i u_{i+2})) | 0 \le i \le 2 \wedge i \equiv 0 \pmod 2\} \\ &= \{23n/4 + 2 - (19n/4 + 3 - i/2) | 0 \le i \le 2 \wedge i \equiv 0 \pmod 2\} \\ &= \{n-1, n\}, \\ W_{12} &= \{C - (\lambda(v_i u_i) + \lambda(v_i u_{i+1}) + \lambda(v_i u_{i+2}) | 4 \le i \le n/2 - 4 \wedge i \equiv 0 \pmod 2\} \\ &= \{23n/4 + 2 - (19n/4 + 2 + i/2) | 4 \le i \le n/2 - 4 \wedge i \equiv 0 \pmod 2\} \\ &= \{n-2, n-3, ..., 3n/4+2\} \\ &= \{3n/4+2, 3n/4+3, ..., n-2\}, \\ W_{13} &= \{C - (\lambda(v_i u_i) + \lambda(v_i u_{i+1}) + \lambda(v_i u_{i+2})) | 3 \le i \le n/2 - 3 \wedge i \bmod 2 = 1\} \\ &= \{23n/4 + 2 - (23n + 2 - 2i)/4 | 3 \le i \le n/2 - 3 \wedge i \equiv 1 \pmod 2\} \\ &= \{3, 4, ..., n/4\}, \\ W_{14} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-3})) | i = 1\} = \{n/4 + 1\}, \\ W_{15} &= \{C - (\lambda(v_i u_i) + \lambda(v_i u_{i+1}) + \lambda(v_i u_{i+2})) | i = n/2 - 2\} = \{n/2 + 1\}, \\ W_{16} &= \{C - (\lambda(v_i u_i) + \lambda(v_i u_{i+1}) + \lambda(v_i u_{i+2})) | i = n/2 - 1\} = \{2\}, \end{aligned}$$

$$
\begin{aligned}
W_2 &= W_{21} \cup W_{22} \cup W_{23} \cup W_{24} \cup W_{25} \cup W_{26} \cup W_{26},\\
W_{21} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-2}))|i = 0\} = \{n/2 - 1\},\\
W_{22} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-2}))|i = 1\} = \{n/2 + 2\},\\
W_{23} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-2}))|i = 2\} = \{3n/4 + 1\},\\
W_{24} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-2}))|3 \leq i \leq n/2 - 3 \wedge i \bmod 2 = 1\}\\
&= \{23n/4 + 2 - (5n + 1/2 + i/2)|3 \leq i \leq n/2 - 3 \wedge i \equiv 1 \pmod 2\}\\
&= \{3n/4, 3n/4 - 1, ..., n/2 + 3\}\\
&= \{n/2 + 3, n/2 + 4, ..., 3n/4\},\\
W_{25} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-2}))|4 \leq i \leq n/2 - 4 \wedge i \equiv 0 \pmod 2\}\\
&= \{23n/4 + 2 - (22n + 8 - 2i)/4|4 \leq i \leq n/2 - 4 \wedge i \equiv 0 \pmod 2\}\\
&= \{n/4 + 2, n/4 + 3, ..., n/2 - 2\},\\
W_{26} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-2}))|i = n/2 - 2\} = \{n/2\},\\
W_{27} &= \{C - (\lambda(u_i v_i) + \lambda(u_i v_{i-1}) + \lambda(u_i v_{i-2}))|i = n/2 - 1\} = \{1\}.
\end{aligned}
$$

Hence $W = W_1 \cup W_2$ is the set of the weights of all vertices, and

$$
\begin{aligned}
W &= W_1 \cup W_2\\
&= W_{11} \cup W_{12} \cup W_{13} \cup W_{14} \cup W_{15} \cup W_{16} \cup W_{21} \cup W_{22} \cup W_{23} \cup W_{24} \cup W_{25} \cup W_{26} \cup W_{27}\\
&= W_{23} \cup W_{13} \cup W_{15} \cup W_{14} \cup W_{27} \cup W_{21} \cup W_{24} \cup W_{12} \cup W_{22} \cup W_{26} \cup W_{25} \cup W_{16} \cup W_{11}\\
&= \{1\} \cup \{2\} \cup \{3, 4, ..., n/4\} \cup \{n/4 + 1\} \cup \{n/4 + 2, n/4 + 3, ..., n/2 - 2\} \cup \{n/2 - 1\} \cup \{n/2\} \cup \{n/2 + 1\} \cup \{n/2 + 2\} \cup \{n/2 + 3, n/2 + 4, ..., 3n/4\} \cup \{3n/4 + 1\}\{3n/4 + 2, 3n/4 + 3, ..., n - 2\} \cup \{n - 1, n\}\\
&= \{1, 2, ..., n - 1, n\}.
\end{aligned}
$$

We can see that the labels of each vertex

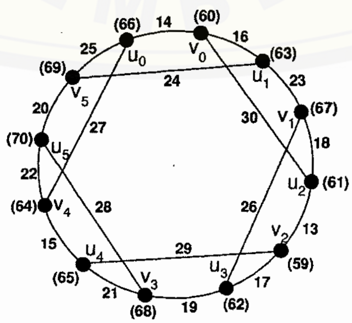

Figure 2: Super vertex-magic total labeling of $CR_{12}(1, 3, 11)$

are distinct, and the vertex labels are $\{1, 2, ..., n\}$. According to the definition of super vertex-magic total labeling, we thus conclude that the graph $CR_n(1, 3, n-1)$ is super vertex-magic for $n \equiv 0 \pmod 4, n \geq 12$ and magic constant C is $\frac{23n}{4} + 2$. □

In the following theorem, we show that the chordal ring $CR_n(1, 5, n-1)$ is $(a, 1)$-antimagic for $n \equiv 0 \pmod 4$. The chordal graph for $x = 1, y = 5, z = n - 1$ and $n = 8$ is shown in Figure 3.

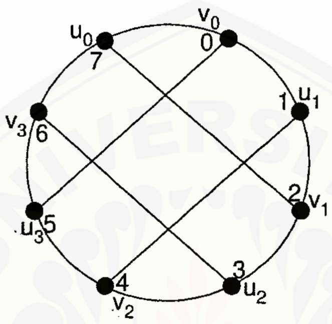

Figure 3: The Chordal graph $CR_8(1, 5, 7)$

Theorem 2 *The chordal ring* $CR_n(1, 5, n-1)$, $n \equiv 0 \pmod 4$, *is* $(\frac{7n+8}{4}, 1)$-*antimagic.*

Proof. We define the edge labeling λ of $CR_n(1, 5, n-1)$, $n \equiv 0 \pmod 4$ as follows:

$$\lambda(v_i u_i) = \begin{cases} i/2 + 1, & 0 \leq i \leq n/2 - 2 \wedge i \equiv 0 \pmod 2, \\ (n + i - 1)/2, & 1 \leq i \leq n/2 - 1 \wedge i \equiv 1 \pmod 2, \end{cases}$$

$$\lambda(v_i u_{i+1}) = \begin{cases} n/4 + 1 + i/2, & 0 \leq i \leq n/2 - 4 \wedge i \equiv 0 \pmod 2, \\ n - (1 + i)/2, & 1 \leq i \leq n/2 - 3 \wedge i \equiv 1 \pmod 2, \\ 3n/4, & i = n/2 - 2, \\ n + 1, & i = n/2 - 1, \end{cases}$$

$$\lambda(v_i u_{i+3}) = \begin{cases} (3n - i)/2, & 0 \leq i \leq n/2 - 2 \wedge i \equiv 0 \pmod 2, \\ n + (3 + i)/2, & 1 \leq i \leq n/2 - 3 \wedge i \equiv 1 \pmod 2, \\ n, & i = n/2 - 1. \end{cases}$$

Now we verify that λ is a bijection from the edge set $E(G)$ onto $\{1,2,...,e\}$. Denoted by

$$\begin{aligned} S_1 &= \{\lambda(v_iu_i)|0 \le i \le n/2-1\}, \\ S_2 &= \{\lambda(v_iu_{i+1})|0 \le i \le n/2-1\}, \\ S_3 &= \{\lambda(v_iu_{i+3})|0 \le i \le n/2-1\}. \end{aligned}$$

Then

$$\begin{aligned} S_1 &= S_{11} \cup S_{12}, \\ S_{11} &= \{\lambda(v_iu_i)|0 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\} \\ &= \{i/2+1|0 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\} \\ &= \{1,2,...,n/4\}, \\ S_{12} &= \{\lambda(v_iu_i)|1 \le i \le n/2-1 \wedge i \equiv 1 \pmod 2\} \\ &= \{(n+i-1)/2|1 \le i \le n/2-1 \wedge i \equiv 1 \pmod 2\} \\ &= \{n/2, n/2+1, ..., 3n/4-1\}, \\ S_2 &= S_{21} \cup S_{22} \cup S_{23} \cup S_{24}, \\ S_{21} &= \{\lambda(v_iu_{i+1})|0 \le i \le n/2-4 \wedge i \equiv 0 \pmod 2\} \\ &= \{n/4+1+i/2|0 \le i \le n/2-4 \wedge i \equiv 0 \pmod 2\} \\ &= \{n/4+1, n/4+2, ..., n/2-1\}, \\ S_{22} &= \{\lambda(v_iu_{i+1})|1 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\} \\ &= \{n-(1+i)/2|1 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\} \\ &= \{n-1, n-2, ..., 3n/4+1\} \\ &= \{3n/4+1, 3n/4+2, ..., n-1\}, \\ S_{23} &= \{\lambda(v_iu_{i+1})|i = n/2-2\} = \{3n/4\}, \\ S_{24} &= \{\lambda(v_iu_{i+1})|i = n/2-1\} = \{n+1\}, \\ S_3 &= S_{31} \cup S_{32} \cup S_{33}, \\ S_{31} &= \{\lambda(v_iu_{i+3})|0 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\} \\ &= \{(3n-i)/2|0 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\} \\ &= \{3n/2, 3n/2-1, ..., 5n/4+1\} \\ &= \{5n/4+1, 5n/4+2, ..., 3n/2\}, \\ S_{32} &= \{\lambda(v_iu_{i+3})|1 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\} \\ &= \{n+(3+i)/2|1 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\} \\ &= \{n+2, n+3, ..., 5n/4\}, \\ S_{33} &= \{\lambda(v_iu_{i+3})|i = n/2-1\} = \{n\}. \end{aligned}$$

Hence $S_1 \cup S_2 \cup S_3$ is the set of labels of all edges, and
$S = S_1 \cup S_2 \cup S_3$

$= S_{11} \cup S_{12} \cup S_{21} \cup S_{22} \cup S_{23} \cup S_{24} \cup S_{31} \cup S_{32} \cup S_{33}$
$= S_{11} \cup S_{21} \cup S_{12} \cup S_{23} \cup S_{22} \cup S_{33} \cup S_{24} \cup S_{32} \cup S_{31}$
$= \{1, 2, ..., n/4\} \cup \{n/4+1, n/4+2, ..., n/2-1\} \cup \{n/2, n/2 + 1, ..., 3n/4-1\} \cup \{3n/4\} \cup \{3n/4+1, 3n/4+2, ..., n-1\} \cup \{n\} \cup \{n+1\} \cup \{n+2, n+3, ..., 5n/4\} \cup \{5n/4+1, 5n/4+2, ..., 3n/2\}$
$= \{1, 2, ..., 3n/2\}$.

Therefore we conclude that λ is a bijection from $E(G)$ onto $\{1, 2, ..., 3n/2\}$. Denoted by

$g_\lambda(v) = \sum \lambda(vu), vu \in E(G),$
$W = \{g_\lambda(v) | v \in V(G)\}.$

Now we show that g_λ is a bijective mapping from $V(G)$ onto W. Let us denote the sets of the weights under an edge labeling λ of vertices v_i and u_i of $CR_n(1, 5, n-1)$ by

$$\begin{aligned} W_1 &= \{g_\lambda(v_i) | 0 \le i \le n/2-1\} \\ &= \{\lambda(v_iu_i) + \lambda(v_iu_{i+1}) + \lambda(v_iu_{i+3}) | 0 \le i \le n/2-1\}, \\ W_2 &= \{g_\lambda(u_i) | 0 \le i \le n/2-1\} \\ &= \{\lambda(v_iu_i) + \lambda(v_{i-1}u_i) + \lambda(v_{i-3}u_i) | 0 \le i \le n/2-1\}. \end{aligned}$$

Where

$$\begin{aligned} W_1 &= W_{11} \cup W_{12} \cup W_{13} \cup W_{14}, \\ W_{11} &= \{\lambda(v_iu_i) + \lambda(v_iu_{i+1}) + \lambda(v_iu_{i+3}) | 0 \le i \le n/2-4 \wedge i \equiv 0 \pmod 2\} \\ &= \{7n/4+2, 7n/4+3, ..., 2n\}, \\ W_{12} &= \{\lambda(v_iu_i) + \lambda(v_iu_{i+1}) + \lambda(v_iu_{i+3}) | 1 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\} \\ &= \{5n/2+1, 5n/2+2, ..., 11n/4-1\}, \\ W_{13} &= \{\lambda(v_iu_i) + \lambda(v_iu_{i+1}) + \lambda(v_iu_{i+3}) | i = n/2-2\} = \{9n/4+1\}, \\ W_{14} &= \{\lambda(v_iu_i) + \lambda(v_iu_{i+1}) + \lambda(v_iu_{i+3}) | i = n/2-1\} = \{11n/4\}, \\ W_2 &= W_{21} \cup W_{22} \cup W_{23} \cup W_{24} \cup W_{25} \cup W_{26}, \\ W_{21} &= \{\lambda(u_iv_i) + \lambda(u_iv_{i-1}) + \lambda(u_iv_{i-3}) | i = 0\} = \{9n/4+2\}, \\ W_{22} &= \{\lambda(u_iv_i) + \lambda(u_iv_{i-1}) + \lambda(u_iv_{i-3}) | i = 1\} = \{2n+2\}, \\ W_{23} &= \{\lambda(u_iv_i) + \lambda(u_iv_{i-1}) + \lambda(u_iv_{i-3}) | i = 2\} = \{2n+1\}, \\ W_{24} &= \{\lambda(u_iv_i) + \lambda(u_iv_{i-1}) + \lambda(u_iv_{i-3}) | 3 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\} \\ &= \{(9n+2i+6)/4 | 3 \le i \le n/2-3 \wedge i \equiv 1 \pmod 2\} \\ &= \{9n/4+3, 9n/4+4, ..., 5n/2\}, \\ W_{25} &= \{\lambda(u_iv_i) + \lambda(u_iv_{i-1}) + \lambda(u_iv_{i-3}) | 4 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\} \\ &= \{(4n+i+2)/2 | 4 \le i \le n/2-2 \wedge i \equiv 0 \pmod 2\} \end{aligned}$$

$$= \{2n+3, 2n+4, ..., 9n/4\},$$
$$W_{26} = \{\lambda(u_iv_i) + \lambda(u_iv_{i-1}) + \lambda(u_iv_{i-3}) | i = n/2 - 1\} = \{11n/4 + 1\}.$$

Hence $W = W_1 \cup W_2$ is the set of the weights of all vertices and

$$\begin{aligned} W &= W_1 \cup W_2 \\ &= W_{11} \cup W_{12} \cup W_{13} \cup W_{14} \cup W_{21} \cup W_{22} \cup W_{23} \cup W_{24} \cup W_{25} \cup W_{26} \\ &= W_{11} \cup W_{23} \cup W_{22} \cup W_{25} \cup W_{13} \cup W_{21} \cup W_{24} \cup W_{12} \cup W_{14} \cup W_{26} \\ &= \{7n/4+2, 7n/4+3, ..., 2n\} \cup \{2n+1\} \cup \{2n+2\} \cup \{2n+3, 2n+ \\ &\quad 4, ..., 9n/4\} \cup \{9n/4+1\} \cup \{9n/4+2\} \cup \{9n/4+3, 9n/4+4, ..., 5n/2\} \cup \\ &\quad \{5n/2+1, 5n/2+2, ..., 11n/4-1\} \cup \{11n/4\} \cup \{11n/4+1\} \\ &= \{7n/4+2, 7n/4+3, ..., 11n/4+1\}. \end{aligned}$$

We can see that each vertex of $CR_n(1,5,n-1)$ receives exactly one label of weight from W and each number from W is used exactly once as a label of weight and further that the set $W = \{a, a+d, ..., a+(|V(G)|-1)d\}$ where $a = \frac{7n+8}{4}$ and $d = 1$. According to the definition of (a,d)-antimagic labeling, we thus conclude that chordal ring is $(\frac{7n+8}{4}, 1)$-antimagic for $n \equiv 0 \pmod 4$ which completes the proof. □

In [15], it was shown that the chordal ring $CR_n(1,5,n-1)$, $n \equiv 0(\text{mod } 4)$ is super vertex-magic with magic constant C is $\frac{23n}{4}+2$. In [7], the (a,d)-antimagic labeling of the chordal ring $CR_n(1,3,n-1)$, $n \equiv 0(\text{mod } 4)$ has been shown with $a = \frac{7n+8}{4}$ and $d = 1$. By Theorem 1 and Theorem 2, we conclude that the chordal rings $CR_n(1,3,n-1)$ and $CR_n(1,5,n-1)$, $n \equiv 0(\text{mod } 4)$ both are super vertex-magic with magic constant C is $\frac{23n}{4}+2$, and (a,d)-antimagic with $a = \frac{7n+8}{4}$ and $d = 1$. From our observation, we tend to believe that chordal rings are super vertex-magic with magic constant $C = \frac{23n}{4} + 2$ and (a,d)-antimagic with $a = \frac{7n+8}{4}$ and $d = 1$. We make the following conjecture.

Conjecture 1 *For each odd integer Δ, $3 \le \Delta \le n-3$ and $n \equiv 0(\text{mod } 4)$, the chordal ring $CR_n(1,\Delta,n-1)$ is super vertex-magic with magic constant $C = \frac{23n}{4} + 2$, and (a,d)-antimagic with $a = \frac{7n+8}{4}$ and $d = 1$.*

Appendix. Crossing number of $CR_n(1,3,n-1)$

Given a "good" graph G (*i.e.*, one for which all intersecting graph edges intersect in a single point and arise from four distinct vertices) the crossing number, denoted by $cr(G)$, is the minimum possible number of crossings with which the graph can be drawn. A graph with crossing number 0 is a planar graph.

Lemma 1 $cr(CR_{10}(1,3,9)) = 1$.

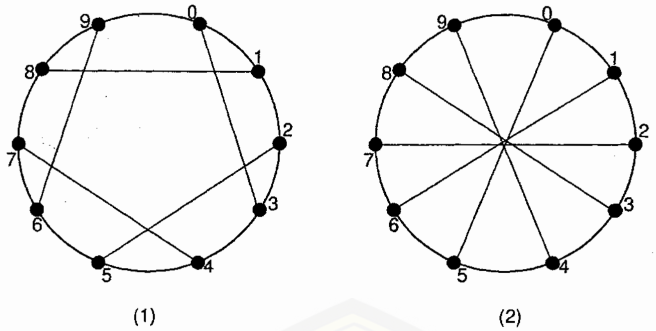

Figure 4: (1) $CR_{10}(1,3,9)$ and (2) $CR_{10}(1,5,9)$

Proof.

The chordal graph $CR_{10}(1,5,9)$ is a Knödel graph $W_{3,10}$ and in [13] it was shown that $cr(W_{3,10}) = 1$. Figure 4 shows that $CR_{10}(1,3,9) \cong CR_{10}(1,5,9)$ since there is a one-one correspondence $\psi : (0)(7)(2486)(1539)$ between the vertex set of $CR_{10}(1,3,9)$ and the vertex set of $CR_{10}(1,5,9)$ which preserves adjacencies and non-adjacencies. Hence $cr(CR_{10}(1,3,9)) = 1$. □

Theorem 3 *For all even $n \geq 8$, $CR_n(1,3,n-1) = 0$ is a planar graph with zero crossing number when $n \equiv 0 \pmod 4$, and $cr(CR_n(1,3,n-1)) = 1$ when $n \equiv 2 \pmod 4$.*

Proof. Figure 5 shows a planar drawing of $CR_n(1,3,n-1)$, $n \equiv 0 \pmod 4$.

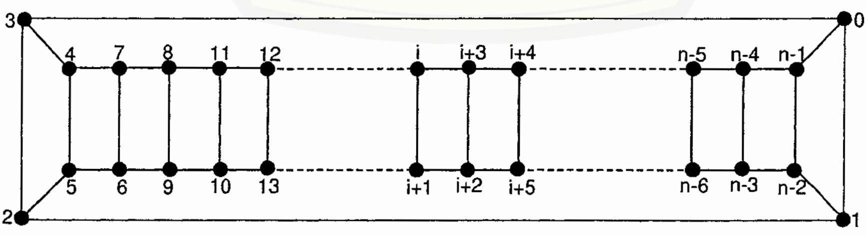

Figure 5: Optimal drawing of $CR_n(1,3,n-1)$, $n \equiv 0 \pmod 4$

So $cr(CR_n(1,3,n-1)) = 0$. Figure 6 shows a drawing of $CR_n(1,3,n-1)$, $n \equiv 2 \pmod 4$, with one crossing. So $cr(CR_n(1,3,n-1)) \leq 1$. We show

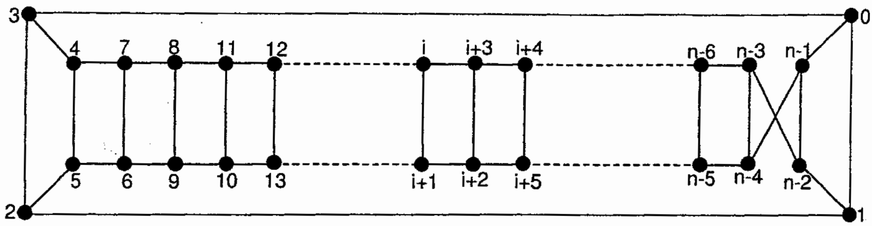

Figure 6: Optimal drawing of $CR_n(1,3,n-1)$, $n \equiv 2 \pmod 4$)

that $cr(CR_n(1,3,n-1)) \geq 1$. We prove this by applying induction on n. By deleting the edges (8,9) and (10,11) from $CR_{14}(1,3,13)$ we can get a subgraph homeomorphic to $CR_{10}(1,3,9)$, so $cr(CR_{14}(1,3,13)) \geq 1$, by Lemma 1. Now, we assume that for all $n = k$, $cr(CR_k(1,3,k-1)) \geq 1$. We prove that $cr(CR_{k+1}(1,3,k)) \geq 1$. By deleting the edges $(k-5, k-4)$ and $(k-3, k-2)$ from $CR_{k+1}(1,3,k)$ we can get a graph homeomorphic to $CR_k(1,3,k-1)$, so $cr(CR_{k+1}(1,3,k)) \geq 1$. Hence for all n, $cr(CR_n(1,3,n-1)) \geq 1$. □

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