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Domination Number of Vertex Amalgamation of Graphs

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Domination Number of Vertex Amalgamation of Graphs

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Abstract. For a graph *G* = (*V*, *E*), a subset *S* of *V* is called a dominating set if every vertex *x* in *V* is either in *S* or adjacent to a vertex in *S*. The domination number $\boldsymbol{\gamma(G)}$ is the minimum cardinality of the dominating set of *G*. The dominating set of *G* with a minimum cardinality denoted by $\boldsymbol{\gamma(G)}$-set. Let G_1, G_2, ... , G_t be subgraphs of the graph *G*. If the union of all these subgraphs is *G* and their intersection is $\{v\}$, then we say that *G* is the vertex-amalgamation of G_1, G_2, ... , G_t at vertex *v*. Based on the membership of the common vertex *v* in the $\boldsymbol{\gamma(G_i)}$-set, there exist three conditions to be considered. First, if *v* elements of every $\boldsymbol{\gamma(G_i)}$-set, second if there is no $\boldsymbol{\gamma(G_i)}$-set containing *v*, and third if either *v* is element of $\boldsymbol{\gamma(G_i)}$-set for $\boldsymbol{1 \le i \le p}$ or there is no $\boldsymbol{\gamma(G_i)}$-set containing *v* for $\boldsymbol{p < i \le t}$. For these three conditions, the domination number of *G* as vertex-amalgamation of G_1, G_2, ... , G_t at vertex *v* can be determined.

1. Introduction

Several results about domination number $\gamma(G)$ and operation of graphs have been explored by some researchers. Some of these results were obtained by Pavlic and Zerovnik [6], Go and Canoy [2], and Kuziak, Lemanska, and Yero [5]. In this paper, we determined the domination number of vertex amalgamation of graphs.

Suppose G_1 and G_2 are subgraphs of a graph $G = (V, E)$ and $v \in V$. The graph G is called vertex amalgamation of G_1 and G_2 at vertex v, denoted by $G = G_1 \vee^1_{\{v\}} G_2$, if $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$. This vertex amalgamation definition proposed by Yang and Kong [9] and can be generated over more than two subgraphs. Suppose $G_1, G_2, \dots, G_t$ are subgraphs of G and $v \in V$. If $G = \bigcup_{i=1}^{t} G_i$ and $\bigcap_{i=1}^{t} G_i = \{v\}$, then G is a vertex amalgamation of $G_1, G_2, \dots, G_t$ at vertex v, denoted by $G = \vee^1_{\{v\}} \{G_1, G_2, \dots, G_t\}$. If $G_1, G_2, \dots, G_t$ has $n_1, n_2, \dots, n_t$ vertices respectively, then $\vee^1_{\{v\}} \{G_1, G_2, \dots, G_t\}$ has $\sum_{i=1}^{t} n_i - t + 1$ vertices.

For a survey of some family of graphs, see [1]. The open neighborhood of $v \in V$ is the set $N(v) = \{w \in V; vw \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined as $\bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set S of vertices of a graph G is called a *dominating set* if each vertex of $V - S$ is adjacent to a member of S. It is equivalent to that $N[S] = V$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of the dominating set of G [3].

Some known results on domination number of some graphs are $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ for $n > 1$ and $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$ for $n > 3$ by Klobucar [4], $\gamma(K_n) = 1$, $\gamma(K_{m,n}) = \gamma(K_{n_1,n_2,\dots,n_k}) = 2$ for $m, n > 1$ and $n_i > 1$ by Snyder [7], $\gamma(F_{n,k}) = n$ for $k > 1$, and $\gamma(B_{n,k}) = n + 1$, for $n, k > 1$ by Wardani [8], $\gamma(L_n) = \left\lceil \frac{n+1}{2} \right\rceil$, and $\gamma(P_{n,f}) = 1$. The bounds of domination number was given by Berge [3], that is, for a graph with order n and the maximum degree $\Delta(G)$ holds $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \le \gamma(G) \le n - \Delta(G)$.

2. Results

We start this section by the observation concerning on the domination number of vertex amalgamation of some complete graphs.

Observation 1. If $G = \bigvee_{\{v\}}^{1} \{K_{n_1}, K_{n_2}, \dots, K_{n_t}\}$ then $\gamma(G) = 1$.

The domination number of vertex amalgamation of a complete k-partite graphs is presented in the following theorem.

Theorem 1.

a. For a complete bipartite graph $K_{mn,}$ where $m, n \geq 2$, if $G = \bigvee_{\{v\}}^{1} \{K_{m_1,n_1}, K_{m_2,n_2}, \dots, K_{m_t,n_t}\}$ then $\gamma(G) = t + 1$.
b. Let a complete multipartite graph H_i has cardinality for each partite be more than one. If $G = \bigvee_{\{v\}}^{1} \{H_1, H_2, \dots, H_t\}$ then $\gamma(G) = t + 1$.

Proof.

a. If $G = \bigvee_{\{v\}}^{1} \{K_{m_1,n_1}, K_{m_2,n_2}, \dots, K_{m_t,n_t}\}$ where $m_i, n_i \geq 2$ and $m_i \leq n_i$ then the order of G is $|V(G)| = \sum_{i=1}^{t}(m_i + n_i) - t + 1$ and the maximum degree is $\sum_{i=1}^{t} m_i \leq \Delta(G) \leq \sum_{i=1}^{t} n_i$. Using Berge, the upper bound of the domination number of G is $(\sum_{i=1}^{t}(m_i + n_i) - t + 1) - \sum_{i=1}^{t} n_i = \sum_{i=1}^{t} m_i - t + 1$. Because $m_i \geq 2$ then $\sum_{i=1}^{t} m_i - t + 1 \geq t + 1$ so that the least upper bound is $\gamma(G) \leq t + 1$. Suppose there is $T \subseteq V(G)$ which $|T| = t$. Let $V(K_{m_i,n_i}) = V_i \cup V_{v_i}$ where V_{v_i} is the vertex partite which consists of v. If $v \in T$ then there is V_j such that $x_j \notin T$ for every $x_j \in V_j$. So, for every $y_j \in V_{v_j}$ which $y_j \neq v$ holds $y_j \in V(G) - T$, $y_j \sim x_j$ but $y_j \nsim v$. If $v \notin T$ then there are two conditions. First, there exists a $z_i \in V(K_{m_i,n_i})$ for every i such that $z_i \in T$. In this case, every vertex which belongs to the same partite of z adjacent with no element of T. Second, there exists K_{m_i,n_i} such that no vertex of K_{m_i,n_i} belongs to T. In this case, every vertiex of K_{m_i,n_i} except v is not adjacent to the element of T. It's means that T is not dominating set.
b. Let $S = \{v, a_1, a_2, \dots, a_t\}$ where $a_i \in H_i$ and a_i belongs to the diffferent partite with v. For every $x \in (V(G) - S)$ holds: (i) if x belongs to the same partite with v then $x \sim a_i$ for some i; (ii) if x belongs to the same partite with a_i then $x \sim v$: and (iii) if x belongs to the diferrent partite with neither v and a_i then both $x \sim v$ and $x \sim a_i$ sor some i. So S is a dominating set. Suppose there is $T \subseteq V(G)$ which $|T| = t$. If $v \in T$ then there is H_i such that no element of this set belongs to T except v. It implies that the elements of H_i which belong to the same partite with v adjacent just with the vertex in the partite that not consist of v. If $v \notin T$ then there are two cases. First, there is exactly a vertex x of H_i belongs to T. In this case, every vertex belongs to the same partite with x has not adjacent with x. Second, there exists H_i such that no vertex of H_i belongs to T. So, every vertex of this H_i was not adjacent with any element of T. ♦

The following theorem presents the domination number of vertex amalgamation of some cycles.

Theorem 2. If $G = \bigvee_{\{v\}}^{1} \{C_{n_1}, C_{n_2}, \dots, C_{n_t}\}$ then $\gamma(G) = \sum_{i=1}^{t} \left\lceil \frac{n_i}{3} \right\rceil - t + 1$.

Proof. Let $V(C_{n_i}) = \{v_{i,j}\,; i = 1, \dots, t, \text{ and } j = 1, \dots, n_i\}$. Without loss of generality let $v = v_{i,1}$ be the common vertex. Let $S = \{v_{i,1+3j}\,; i = 1, \dots, t \text{ and } j = 0, 1, \dots, \left\lceil \frac{n_i}{3} \right\rceil - 1\}$. For every $x \in (V(G) - S)$ then $x = v_{i,k}$ where $i = 1, \dots, t$ and $k \in \{2, 3, \dots, n_i\,;\ k \not\equiv 1 (mod\ 3)\}$ such that there is $y \in S$ with $x \sim y$. So S is a dominating set. It is clear that $|S| = 1 + \sum_{i=1}^{t}\left(\left\lceil \frac{n_i}{3} \right\rceil - 1\right) = \sum_{i=1}^{t} \left\lceil \frac{n_i}{3} \right\rceil - t + 1$. Suppose there exist $T \subseteq V(G)$ which $|T| = \sum_{i=1}^{t}\left(\left\lceil \frac{n_i}{3} \right\rceil - 1\right)$. If $v \in T$ then there is C_{n_i} such that $\gamma(C_{n_i}) = \left\lceil \frac{n_i}{3} \right\rceil - 1$. If $v \notin T$ then $\gamma(C_{n_i}) = \left\lceil \frac{n_i}{3} \right\rceil - 1$ for every C_{n_i}. ♦

A specially case of Theorem 2, if the cycles are isomorphic, that is, if $n_1 = n_2 = \cdots = n_t = n$, then $\gamma(G) = t\left\lceil\frac{n}{3}\right\rceil - t + 1$.

The following theorem presents the domination number of vertex amalgamation of some paths.

Theorem 3. If $G = \vee^1_{\{v\}}\{P_{n_1}, P_{n_2}, \dots, P_{n_t}\}$ then

$$\gamma(G) = \begin{cases} \sum_{i=1}^{t}\left\lceil\frac{n_i}{3}\right\rceil - t + 1, \text{if } v \text{ belongs to every } \gamma\left(P_{n_i}\right) - \text{set} \\ \sum_{i=1}^{t}\left\lceil\frac{n_i}{3}\right\rceil \quad\quad , \text{ if } v \text{ has not belongs to any } \gamma\left(P_{n_i}\right) - \text{set} \end{cases}.$$

Proof. Let $U = \{v\} \cup \{w \in V(G); w \sim v\}$. For induced subgraph $\langle U \rangle$, it holds $G - \langle U \rangle = \cup_{i=1}^{t}\left(P_{n_i} - \langle U \rangle\right)$ such that $\gamma(G - \langle U \rangle) = \gamma\left(\cup_{i=1}^{t}\left(P_{n_i} - \langle U \rangle\right)\right) = \sum_{i=1}^{t}\gamma\left(P_{n_i} - \langle U \rangle\right)$. If v belongs to every $\gamma\left(P_{n_i}\right)$-set then $\gamma\left(P_{n_i} - \langle U \rangle\right) = \gamma\left(P_{n_i}\right) - 1 = \left\lceil\frac{n_i}{3}\right\rceil - 1$. It is clear that $\gamma(\langle U \rangle) = 1$. So we have $\gamma(G) = \gamma(G - \langle U \rangle) + \gamma(\langle U \rangle) = \sum_{i=1}^{t}\left(\left\lceil\frac{n_i}{3}\right\rceil - 1\right) + 1 = \sum_{i=1}^{t}\left\lceil\frac{n_i}{3}\right\rceil - t + 1$. If v has not belongs to any $\gamma\left(P_{n_i}\right)$-set then $v \sim z_i$ which z_i belongs to $\gamma\left(P_{n_i}\right)$-set. In this case v is independent to the $\gamma\left(P_{n_i}\right)$, so for $\gamma(G)$ too. It implies $\gamma(G) = t\gamma\left(P_{n_i}\right)$. ♦

The domination number of vertex amalgamation of some ladders is presented in the following theorem.

Theorem 4. If $G = \vee^1_{\{v\}}\{L_{n_1}, L_{n_2}, \dots, L_{n_t}\}$ for ladder graph L_n then

$$\gamma(G) = \begin{cases} \sum_{i=1}^{t}\left\lceil\frac{n_i+1}{2}\right\rceil - t + 1, \text{if } v \text{ belongs to every } \gamma\left(L_{n_i}\right) - \text{set} \\ \sum_{i=1}^{t}\left\lceil\frac{n_i+1}{2}\right\rceil \quad\quad , \text{ if } v \text{ has not belongs to any } \gamma\left(L_{n_i}\right) - \text{set} \end{cases}. \quad ♦$$

Before proving the dominating number of a vertex amalgamation of some graphs, we present a lemma concerning on the dominating number of the graph G obtained by joining all vertices of a graph H to a vertex K_1, that is, $G = K_1 + H$.

Lemma 1. For every graph G, $\gamma(G) = 1$ if and ony if $G = K_1 + H$ for some graph H.

As the diameter of $G = K_1 + H$ is one, it is very easy to see that any one vertex in G can dominate other vertices in G. We now have the following theorem.

Theorem 5. Let graph G_i which $\gamma(G_i) = 1$ and $G = \vee^1_{\{v\}}\{G_1, G_2, \dots, G_t\}$.

a. If v belongs to every $\gamma(G_i)$-set then $\gamma(G) = 1$.
b. If v has not belongs to any $\gamma(G_i)$-set then $\gamma(G) = t$.
c. If v has not belongs to any $\gamma(G_i)$-set for $1 \le i \le p$ and v belongs to every $\gamma(G_i)$-set for $p < i \le t$ then $\gamma(G) = p + 1$.

Proof. From Lemma 1, we have $G_i = K_1 + H_i$. Let $|V(H_i)| = n_i$ so $|V(G_i)| = n_i + 1$ and $|V(G)| = \sum_{i=1}^{t} n_i + 1$. The degree of every vertex in $\gamma(G_i)$–set is n_i and less than n_i for the others.

a. If v belongs to every $\gamma(G_i)$-set then v has $\sum_{i=1}^{t} n_i$ degree so $\{v\}$ is dominating set.
b. If v has not belong to every $\gamma(G_i)$-set then v adjacent with t vertices which degrees are $n_1, n_2, \dots, n_t$ respectively. These t vertices span the dominating set of G. The cardinality of this dominating set is minimum, if there exist a set with t-1 cardinality then there exists G_i where the n_i vertices which less than n_i degree is not adjacent to these t-1 vertices.
c. We combine (b) condition for p first subgraphs G_i and (a) condition for the others then we have $\gamma(G) = p + 1$. ♦

The next lemma describes the dominating set of vertex amalgamation of some graphs.

Lemma 2. Let $G = \vee_{\{v\}}^{1} \{G_1, G_2, \dots, G_t\}$. If S_i is a dominating set of G_i for every $i = 1,2,..,t$, then $\bigcup_{i=1}^{t} S_i$ is a dominating set of G.

In the theorems below we notice S_i as a dominating set of G_i, then the set P_i is as $\gamma(G_i)$-set, and the set P is as $\gamma(G)$-set.

Theorem 6. Let $G = \vee_{\{v\}}^{1} \{G_1, G_2, \dots, G_t\}$ and v belongs to every $\gamma(G_i)$-set, then $\gamma(G) = \sum_{i=1}^{t} \gamma(G_i) - t + 1$.

Proof. By Lemma 2, $\bigcup_{i=1}^{t} S_i$ is a dominating set of G such that $\gamma(G) = \gamma(\bigcup_{i=1}^{t} G_i) \leq \left|\bigcup_{i=1}^{t} S_i\right|$. If v belongs to every $\gamma(G_i)$-set then $v \in S_i$ for every i such that $\bigcap_{i=1}^{t} S_i = \{v\}$. It implies $\left|\bigcup_{i=1}^{t} S_i\right| = 1 + \sum_{i=1}^{t}(|S_i| - 1) = \sum_{i=1}^{t}|S_i| - t + 1$. We have $\gamma(G) = \gamma(\bigcup_{i=1}^{t} G_i) \leq \sum_{i=1}^{t}|S_i| - t + 1$. The least upper bound reached for $|S_i| = \gamma(G_i)$, so $\gamma(G) \leq \sum_{i=1}^{t} \gamma(G_i) - t + 1$. Suppose there exists $P \subset V(G)$ which $P = \bigcup_{i=1}^{t} P_i$ for $P_i \subset V(G_i)$ such that $|P| = \sum_{i=1}^{t} \gamma(G_i) - t$. If $v \in P$ then there exists P_i such that $|P_i| = \gamma(G_i) - 1$. However, there exists $x \in V(G_i) - P_i$ such that for every $y \in P_i$ holds $x \nsim y$. Because $V(G_i) \subset V(G)$ it mean that there is $x \in V(G) - P$ such that for every $y \in P$ holds $x \nsim y$. So, P have not a dominating set. If $v \notin P$ then $|P_i| = \gamma(G_i) - 1$ for every i. It means that P_i is not a dominating set with minimum cardinality. So does P. ♦

Theorem 7. If $G = \vee_{\{v\}}^{1} \{G_1, G_2, \dots, G_t\}$ and v has not belong to every $\gamma(G_i)$-set then $\gamma(G) = \sum_{i=1}^{t} \gamma(G_i)$.

Proof. By Lemma 2, we have $\gamma(G) = \gamma(\bigcup_{i=1}^{t} G_i) \leq \left|\bigcup_{i=1}^{t} S_i\right|$. If there is no $\gamma(G_i)$-set consist of v then $v \notin S_i$ for every i such that $\bigcap_{i=1}^{t} S_i = \emptyset$. It implies $\gamma(G) = \gamma(\bigcup_{i=1}^{t} G_i) \leq \left|\bigcup_{i=1}^{t} S_i\right| = \sum_{i=1}^{t}|S_i|$. The least upper bound reached for $|S_i| = \gamma(G_i)$, so $\gamma(G) \leq \sum_{i=1}^{t} \gamma(G_i)$. Suppose there is vertices subset $P = \bigcup_{i=1}^{t} P_i$ which $|P| = \sum_{i=1}^{t} \gamma(G_i) - 1$. There exist P_i such that $|P_i| = \gamma(G_i) - 1$ for every i. It means that P_i is not dominating set with minimum cardinality. So do P. ♦

Corollary 8. Let $G = \vee_{\{v\}}^{1} \{G_1, G_2, \dots, G_t\}$. If v belongs to every $\gamma(G_i)$-set for $1 \leq i \leq p$ and v has not belongs to any $\gamma(G_i)$-set for $p + 1 \leq i \leq t$, then $\gamma(G) = \sum_{i=1}^{t} \gamma(G_i) - p + 1$.

3. Conclusion

We conclude this paper with the domination number of vertex amalgamation of some graphs at a vertex v is like the order of these graphs especially if v belongs to every $\gamma(G_i)$-set, that is $\gamma\left(\vee_{\{v\}}^{1} \{G_1, G_2, \dots, G_t\}\right) = \sum_{i=1}^{t} \gamma(G_i) - t + 1$. The following open problems for future work.

Open problem. Find the distance domination number of particular classes of graphs and the graphs obtained from graph operations.

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