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CONJECTURES AND OPEN PROBLEMS ON FACE ANTIMAGIC EVALUATIONS OF GRAPHS

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Abstract. Variations on antimagic labelings, including vertex antimagic, edge antimagic and (a,d) antimagic have been studied since antimagic labelings were developed in 1990. Face antimagic labelings are a relatively recent innovation. In this paper we survey results in face antimagic labelings and provide a summary of current conjectures and open problems.

1. INTRODUCTION

All graphs, $G = G(V, E, F)$ considered in this paper are simple, finite, undirected and planar. In all cases, a labeling will refer to a mapping from some combination of vertices, edges and faces into the positive integers.

Let $|V| = v$, $|E| = e$ and $|F| = f$. Assume that $a, b, c \in \{0, 1\}$. A labeling of type (a, b, c) assigns labels from the set $\{1, 2, 3, ..., av + be + cf\}$ to the vertices, edges and faces of G in such a way that each vertex receives a labels, each edge receives b labels, and each face receives c labels and each number is used exactly once as a label.

Labelings of types $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are also called *vertex*, *edge* and *face* labelings, respectively. Labelings of type $(1, 1, 0)$ are traditionally referred to as *total* labelings. A $(1, 1, 1)$ labeling is a bijection from the set $\{1, 2, ..., v + e + f\}$ into

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the vertices, the edges and the faces of $G = (V, E, F)$. This labeling is sometimes referred to as *supertotal.*

The *weight* of a face under a labeling is the sum of labels (if present) carried by that face and the edges and vertices surrounding it.

Definition 1.1. [21]: A labeling of type (a, b, c) is said to be *face-magic* if for every number s, all s-sided faces have the same weight.

Definition 1.2. [19]: A labeling of type (a, b, c) of plane graph G is called d-*antimagic* if for every number s the set of s-sided face weights is $W_s{=}\{a_s, a_s + d, a_s + 2d, ..., a_s + (f_s - 1)d\}$ for some integers a_s and d, where f_s is the number of s-sided faces.

Note that in the two definitions above we allow different sets W_s for different s. If s is the same for each face, then there is just one arithmetic sequence comprising the set of face weights and we may speak of a graph being (a, d)-*face antimagic.* Many common types of plane graphs have “almost” all faces the same, for example, the prism which consists of all-but-two 4-sided faces; or the antiprism which consists of all-but-two 3-sided faces. Such graphs are easily modified so that they contain all the same faces and so that we can consider (a, d)-face antimagic labeling on them. This is the topic of Section 2 of this paper, while in Section 3 we consider the more general d-antimagic labeling on various graphs with faces of more than one size.

For the following let $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, m\}$ be index sets.

2. (a, d)-FACE ANTIMAGIC EDGE LABELING

2.1. The Plane Graph $\mathcal{D}_n^m$ Based on m-prism D_n^m

The *m-prism* D_n^m, $n \geq 3$, $m \geq 1$, is a trivalent graph of a convex polytope which can be defined as the Cartesian product of a path on $m + 1$ vertices with a cycle on n vertices $(P_{m+1} \times C_n)$, embedded in the plane.

Let us denote the vertex set of m-prism D_n^m by $V(D_n^m) = \{x_{j,i} : i \in I \text{ and } j \in J \cup \{m + 1\}\}$ and the edge set by $E(D_n^m) = \{x_{j,i}x_{j,i+1} : i \in I \text{ and } j \in J \cup \{m + 1\}\} \cup \{x_{j,i}x_{j+1,i} : i \in I \text{ and } j \in J\}$. We make the convention that $x_{j,n+1} = x_{j,1}$ and $x_{j,n+2} = x_{j,2}$ for $j \in J \cup \{m + 1\}$.

The face set $F(D_n^m)$ contains nm 4-sided faces, an internal n-sided face and an external n-sided face. We will create a new graph from D_n^m by adding two vertices and appropriate edges to obtain a plane graph $\mathcal{D}_n^m$ which contains 4-sided faces only: We insert exactly one vertex y (respectively, z) into the internal (respectively, external) n-sided face of D_n^m.

Suppose that n is even, $n \geq 4$, and consider the graph $\mathcal{D}_n^m$ with vertex set $V(\mathcal{D}_n^m) = V(D_n^m) \cup \{y, z\}$ and

(i) if m is odd, the edge set $E(\mathcal{D}_n^m) = E(D_n^m) \cup \{x_{1,2k-1}y : \; k = 1, 2, \ldots, \frac{n}{2}\} \cup \{x_{m+1,2k}z : \; k = 1, 2, \ldots, \frac{n}{2}\}$; and

(ii) if m is even, the edge set $E(\mathcal{D}_n^m) = E(D_n^m) \cup \{x_{1,2k-1}y : k = 1, 2, \ldots, \frac{n}{2}\} \cup \{x_{m+1,2k-1}z : k = 1, 2, \ldots \frac{n}{2}\}$.

Then $\mathcal{D}_n^m$, $n \geq 4$, $m \geq 1$, is the plane graph of the convex polytope on $|V(\mathcal{D}_n^m)| = n(m+1) + 2$ vertices, $|E(\mathcal{D}_n^m)| = 2n(m+1)$ edges and consisting of $|F(\mathcal{D}_n^m)| = n(m+1)$ 4-sided faces. See Fig. 1.

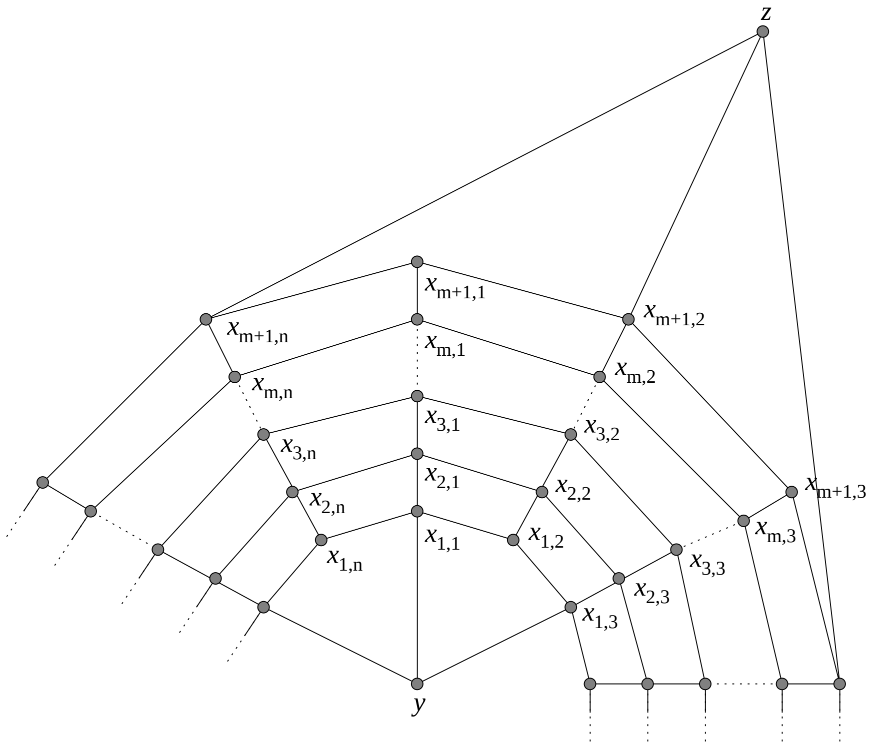

Figure 1: The plane graph $\mathcal{D}_n^m$.

The following theorems provide the necessary conditions for the graph $\mathcal{D}_n^m$ to bear an (a, d)-face antimagic edge labeling.

Theorem 2.1. *[12]: If $\mathcal{D}_n^m$ has (a, d)-face antimagic edge labeling then either $d = 2$ and $a = 3n(m+1) + 3$, or $d = 4$ and $a = 2n(m+1) + 4$, or $d = 6$ and $a = n(m+1) + 5$.*

For $m = 1$ the following results are known:

Theorem 2.2. *[7]: For $n \geq 4, n \equiv 0 \pmod 2$, the plane graph $\mathcal{D}_n^1$ has a $(6n+3, 2)$-face antimagic edge labeling.*

Theorem 2.3. *[7]: If n is even, $n \geq 4$, then the plane graph $\mathcal{D}_n^1$ has a $(4n+4,4)$-face antimagic edge labeling.*

For $m = 2$ in [14] it is proved

Theorem 2.4. *[14]: For $n \geq 4$, $n \equiv 0 \pmod 2$, the convex polytope $\mathcal{D}_n^2$ has a $(9n+3,2)$-face antimagic edge labeling.*

Theorem 2.5. *[14]: If n is even, $n \geq 4$, then the convex polytope $\mathcal{D}_n^2$ has a $(6n+4,4)$-face antimagic edge labeling.*

and conjectured

Conjecture 2.6. *For $n \geq 4$, $n \equiv 0 \pmod 2$, the convex polytope $\mathcal{D}_n^2$ has a $(3n+5,6)$-face antimagic edge labeling.*

If $m \geq 3$, then we have

Theorem 2.7. *[12]: If $n \equiv 0 \pmod 2$, $n \geq 4$ and $m \equiv 1 \pmod 2$, $m \geq 3$, or if $n \equiv 2 \pmod 4$, $n \geq 6$ and $m \equiv 0 \pmod 2$, $m \geq 4$, then the graph of the convex polytope $\mathcal{D}_n^m$ has $(3n(m+1)+3,2)$-face antimagic edge labeling.*

Theorem 2.8. *[12]: If n is even, $n \geq 4$, and m is odd, $m \geq 3$, or if $n \equiv 2 \pmod 4$, $n \geq 6$, and m is even, $m \geq 4$, then the plane graph $\mathcal{D}_n^m$ has $(2n(m+1)+4,4)$-face antimagic edge labeling.*

Although such labelings have yet to be found, we believe, as indicated in the following conjectures, that labeling schema exist conforming to the necessary conditions described in the previous three theorems.

Conjecture 2.9. *There are $(3n(m+1)+3,2)$-face antimagic edge labeling and $(2n(m+1)+4,4)$-face antimagic edge labelings for the plane graph $\mathcal{D}_n^m$ for $n \equiv 0 \pmod 4$, $n \geq 4$, and $m \equiv 0 \pmod 2$, $m \geq 4$.*

Then to completely characterize the graphs of $\mathcal{D}_n^m$ supporting a (a,d)-face antimagic edge labeling, it only remains to consider the case of $(n(m+1)+5,6)$-face antimagic labeling. This prompts us to propose the following conjecture.

Conjecture 2.10. *If n is even, $n \geq 4$, $m \geq 1$, then the plane graph $\mathcal{D}_n^m$ has a $(n(m+1)+5,6)$-face antimagic edge labeling.*

2.2. The Plane Graph $\mathcal{A}_n^m$ Obtained From a m-Antiprism A_n

The *antiprism* A_n, $n \geq 3$, is a regular graph of degree $r = 4$ also known as an Archimedean convex polytope. For $n = 3$, A_n is the octahedron.

For $n \geq 3$ and $m \geq 1$, we denote by A_n^m the plane graph of a convex polytope, which is obtained as a combination of m antiprisms A_n. Let us denote the vertex set of A_n^m by $V(A_n^m) = \{y_{j,i} : i \in I \text{ and } j \in J \cup \{m+1\}\}$ and the edge set by $E(A_n^m) = \{y_{j,i}y_{j,i+1} : i \in I \text{ and } j \in J \cup \{m+1\}\} \cup \{y_{j,i}y_{j+1,i} : i \in I \text{ and } j \in J\} \cup \{y_{j,i+1}y_{j+1,i} : i \in I \text{ and } j \in J,\ j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1} : i \in I \text{ and } j \in J,\ j \text{ even}\}$. We make the convention that $y_{j,n+1} = y_{j,1}$ for $j \in J \cup \{m+1\}$.

The face set $F(A_n^m)$ contains $2mn$ 3-sided faces, an internal n-sided face and an external n-sided face. We shall modify A_n^m to obtain a graph $\mathcal{A}_n^m$ containing only 3-sided faces: We insert exactly one vertex x (z) into the internal (external) n-sided face of A_n^m and connect the vertex x (z) with the vertices $y_{1,i}$ ($y_{m+1,i}$), $i \in I$. Thus, we obtain the plane graph $\mathcal{A}_n^m$, consisting of 3-sided faces with the vertex set $V(\mathcal{A}_n^m) = V(A_n^m) \cup \{x, z\}$ and the edge set $E(\mathcal{A}_n^m) = E(A_n^m) \cup \{xy_{1,i} : i \in I\} \cup \{y_{m+1,i}z : i \in I\}$ where $|V(\mathcal{A}_n^m)| = (m+1)n + 2$, $|E(\mathcal{A}_n^m)| = 3n(m+1)$ and $|F(\mathcal{A}_n^m)| = 2n(m+1)$. See Fig. 2.

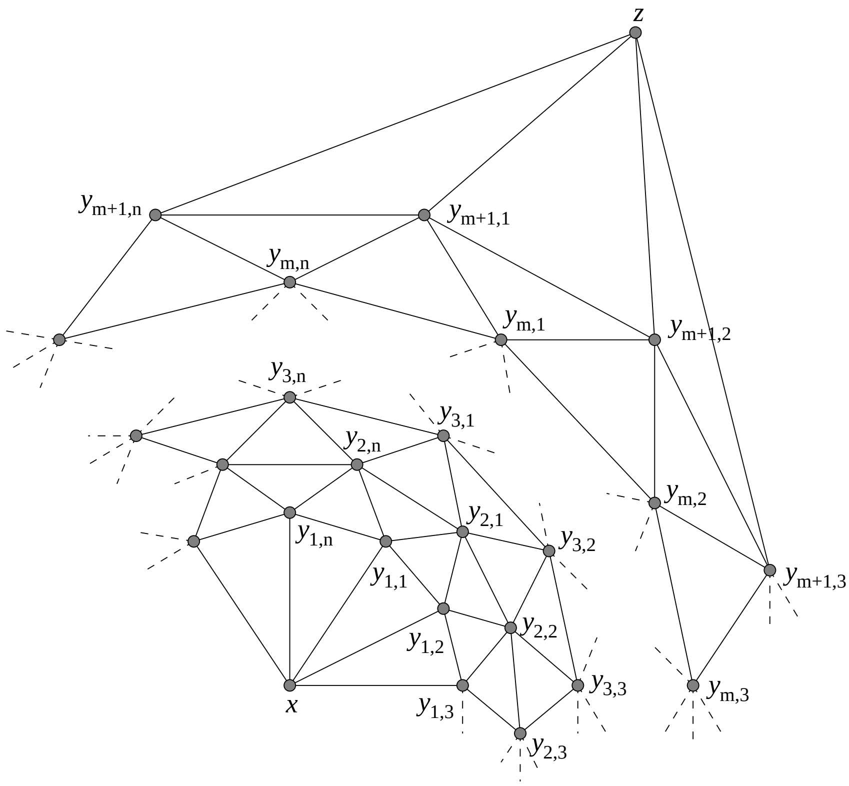

Figure 2: The plane graph $\mathcal{A}_n^m$.

Necessary conditions for $\mathcal{A}_n^m$ to support a (a, d)-face antimagic edge labeling are given in [15] and summarised below.

If $\mathcal{A}_n^m$ is (a, d)-face antimagic, then

(i) for n even, $n \geq 4$ and $m \geq 1$, or for n odd, $n \geq 3$ and m odd, $m \geq 1$, d

is odd, and we have exactly two possibilities: $(a, d) = \left(\frac{7n(m+1)}{2} + 2, 1\right)$ and $(a, d) = \left(\frac{3n(m+1)}{2} + 3, 3\right)$.

(ii) for n odd, $n \geq 3$ and m even, $m \geq 2$, d is even, and we have exactly two possibilities: $(a, d) = \left(\frac{5n(m+1)+5}{2}, 2\right)$ and $(a, d) = \left(\frac{n(m+1)+7}{2}, 4\right)$.

If $m = 1$ then in [9] is shown:

Theorem 2.11. *[9]: For $n \geq 3$, the plane graph $\mathcal{A}_n^1$ has $(7n + 2, 1)$-face antimagic edge labeling.*

For $m = 2$ it was proved:

Theorem 2.12. *[6]: If n is even, $n \geq 4$, then the graph of the convex polytope $\mathcal{A}_n^2$ has a $\left(\frac{21n}{2} + 2, 1\right)$-face antimagic edge labeling.*

The paper [6] proposes the following two conjectures:

Conjecture 2.13. *For $n \equiv 0 \pmod 2), \mathrm{n} \geq 4$, the convex polytope $\mathcal{A}_n^2$ has a $\left(\frac{9n}{2} + 3, 3\right)$-face antimagic edge labeling.*

Conjecture 2.14. *If n is odd, $n \geq 3$, then the convex polytope $\mathcal{A}_n^2$ bears a $\left(\frac{15n+5}{2}, 2\right)$-face antimagic edge labeling and a $\left(\frac{3n+7}{2}, 4\right)$-face antimagic edge labeling.*

For $m \geq 3$ in [15] are proved the following results

Theorem 2.15. *[15]: If m is odd, $m \geq 3$, $n \geq 3$, then the plane graph $\mathcal{A}_n^m$ has $\left(\frac{7n(m+1)}{2} + 2, 1\right)$-face antimagic edge labeling.*

Theorem 2.16. *[15]: If n and m are even, $n \geq 4$, $m \geq 4$, then the graph of the convex polytope $\mathcal{A}_n^m$ has $(\frac{7n(m+1)}{2} + 2, 1)$-face antimagic edge labeling.*

In addition to the labeling schema given in the previous two theorems, we offer the following conjectures.

Conjecture 2.17. *If n is odd, $n \geq 3$, and m is even, $m \geq 2$, then the plane graph $\mathcal{A}_n^m$ has $\left(\frac{5n(m+1)+5}{2}, 2\right)$-face antimagic edge labeling and $\left(\frac{n(m+1)+7}{2}, 4\right)$-face antimagic edge labeling.*

Conjecture 2.18. *If n is even, $n \geq 4$, and $m \geq 1$, or if n is odd, $n \geq 3$, and m is odd, $m \geq 1$, then the graph of the convex polytope $\mathcal{A}_n^m$ has $\left(\frac{3n(m+1)}{2} + 3, 3\right)$-face antimagic edge labeling.*

Open Poblem 2.19. *Investigate (a, d)-face antimagic edge labelings for other regular polytopes.*

3. d-ANTIMAGIC TYPE $(1, 1, 1)$ LABELINGS

3.1. The Prism D_n

The prism D_n, $n \geq 3$, is a cubic graph which can be defined as the cartesian product of a path on two vertices with a cycle on n vertices ($P_2 \times C_n$), embedded in the plane. See Fig. 3.

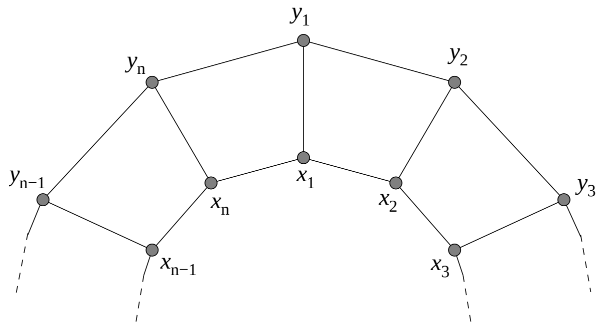

Figure 3: The prism D_n.

It was proved in [16] that for $n \geq 3$, the prism D_n is 1-antimagic of type $(1, 1, 1)$ and for $n \equiv 3 \pmod 4$ and $d = 2, 3, 4, 6$ there exist d-antimagic labelings of type $(1, 1, 1)$. Subsequently in [19] it is proved that

Theorem 3.1. *[19]: For $n \geq 3, n \neq 4$, the prism D_n has a 3-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.2. *[24]: For $n \geq 3$ and $d \in \{2, 4, 5, 6\}$, the prism D_n has a d-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.3. *[26]: For $n \geq 5$, the prism D_n has a d-antimagic labeling of type $(1, 1, 1)$ for $d \in \{7, 8, 9, 10\}$.*

Theorem 3.4. *[26]: For $n \geq 6$ the prism D_n has a 15-antimagic labeling of type $(1, 1, 1)$. For $n \geq 7$ the prism D_n has a 18-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.5. *[26]: For $n \geq 7$, n odd, the prism D_n has a 12-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.6. *[26]: For $n \geq 7$, n odd, and $d \in \{14, 17, 20\}$, the prism D_n has a d-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.7. *[26]: For $n \geq 9$, n odd, and $d \in \{16, 26\}$, the prism D_n has a d-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.8. *[26]: For $n \geq 7$, n odd, and $d \in \{21, 24, 27, 30, 36\}$, the prism D_n has a d-antimagic labeling of type $(1, 1, 1)$.*

These theorems may not be a complete characterisation of d-antimagic labelings of type $(1, 1, 1)$ for D_n, and so we propose the following open problem.

Open Problem 3.9. *Find other possible values of the parameter d and the corresponding d-antimagic labeling of type $(1, 1, 1)$ for prisms D_n.*

3.2. The Antiprism A_n

Recall that the antiprism A_n, $n \geq 3$, is a 4-regular graph and, for $n = 3$, it is the octahedron. *Antiprism A_n, $n \geq 3$,* consists of an outer n-cycle $y_1\ y_2\ ...\ y_n$, an inner n-cycle $x_1\ x_2\ ...\ x_n$, and a set of n spokes $x_i y_i$ and $x_{i+1} y_i$, $i = 1, 2, ..., n$ with indices taken modulo n. $|V(A_n)| = 2n$, $|E(A_n)| = 4n$, $|F(A_n)| = 2n+2$. We define the 3-sided face $f_{1,i}$ as the face bounded by the edges $x_{i+1} y_{i+1}$, $x_{i+1} y_i$, $y_i y_{i+1}$, and we define the 3-sided face $f_{0,i}$ as the face bounded by the edges $x_i y_i$, $x_i x_{i+1}$ and $y_i x_{i+1}$. We denote the inner face by $z_{n,1}$ and the outer face by $z_{n,2}$ (see Figure 4).

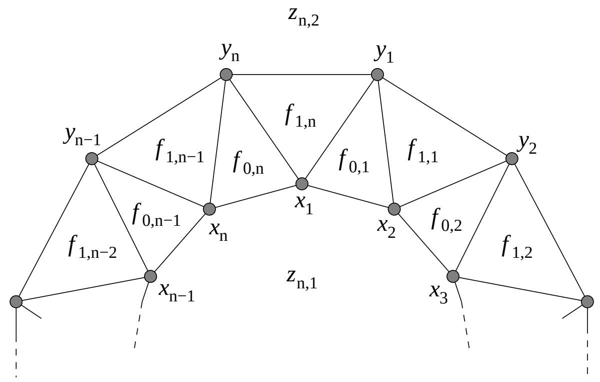

Figure 4: The antiprism A_n.

Theorem 3.10. *[19]: For $n \geq 4$, the antiprism A_n has a d-antimagic labeling of type $(1, 1, 1)$ for $d \in \{1, 2, 4\}$.*

Theorem 3.11. *[25]: For $n \geq 5$, the antiprism A_n has a 3-antimagic and 6-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.12. *[25]: For $n \geq 3$, the antiprism A_n has a 5-antimagic labeling of type* $(1,1,1)$.

Theorem 3.13. *[22]: For $n \geq 3$, the antiprism A_n has a 7-antimagic labeling of type* $(1,1,1)$.

Theorem 3.14. *[22]: For $n \geq 11$, the antiprism A_n has a 12-antimagic labeling of type* $(1,1,1)$.

As in the previous subsection, we note that these theorems may not be a complete characterisation of d-antimagic labelings of type $(1,1,1)$ for A_n, and so, in a similar vein, we propose the following open problem.

Open Problem 3.15. *Find other possible values of the parameter d and the corresponding d-antimagic labeling of type $(1,1,1)$ for antiprisms A_n.*

3.3. The Pumpkin Graph P_a^b

Let a and b be integers, $a \geq 3$ and $b \geq 2$. Let $y_1, y_2, \ldots, y_a$ be fixed vertices, we connect the vertices y_i and y_{i+1} by means of b internally disjoint paths p_i^j of length $i+1$ each, $1 \leq i \leq a-1$, $1 \leq j \leq b$. Let $y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,i}, y_{i+1}$ be the vertices of path p_i^j. The resulting graph embedded in the plane is denoted by P_a^b (*pumpkin graph*), where $V(P_a^b) = \{y_i : 1 \leq i \leq a\} \bigcup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} : 1 \leq k \leq i\}$ and $E(P_a^b) = \bigcup_{i=1}^{a-1} \{y_i x_{i,j,1} : 1 \leq j \leq b\} \bigcup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i-1\} \bigcup \bigcup_{i=1}^{a-1} \{x_{i,j,i} y_{i+1} : 1 \leq j \leq b\}$. Fig. 5 gives an example of P_4^5.

The face set $F(P_a^b)$ contains $b-1$ (2i+2)-sided faces, $1 \leq i \leq a-1$, and one external infinite face. Let $v = |V(P_a^b)| = \frac{ab(a-1)}{2} + a$, $e = |E(P_a^b)| = \frac{b(a-1)(a+2)}{2}$ and $f = |F(P_a^b)| = (a-1)(b-1)+1$.

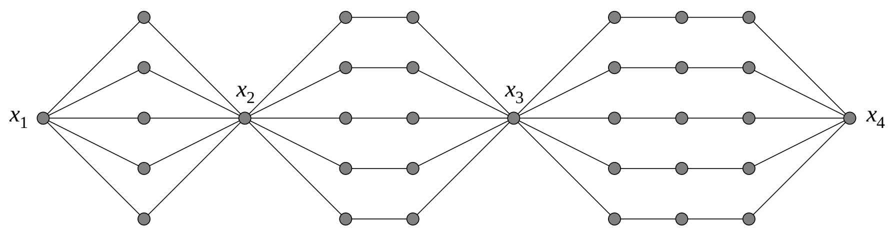

Figure 5: The pumpkin graph P_4^5.

Kathiresan and Ganesan [20] have proved

Theorem 3.16. *[20]: For $a \geq 3, b \geq 2$, and $d \in \{0,1,2,3,4,6\}$, the plane graph P_a^b has a d-antimagic labeling of type* $(1,1,1)$.

The vertex labelings and edge labelings defined by Kathiresan and Ganesan ([20]) can be used to proving

Theorem 3.17. *For $a \geq 3, b \geq 2$, and $d \in \{a, a-2, a+1, a-3, a+4, |a-6|\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

The existence of d-antimagic labeling of type $(1,1,1)$ for P_b^a for many other values of parameter d can be found in [23].

Theorem 3.18. *[23]: For $a \geq 3, b \geq 2$, and $d \in \{5, 7, |a-7|, a+5\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Theorem 3.19. *[23]: For $a \geq 3, b \geq 2$, the plane graph P_a^b has $|a-4|$-antimagic and $(a+2)$-antimagic labelings of type $(1,1,1)$.*

Theorem 3.20. *[23]: For $a \geq 3, b \geq 2$, and $d \in \{2a-3, 2a-1, a-1, 3a-3\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Theorem 3.21. *[23]: For $a \geq 3, b \geq 2$, and $d \in \{a+3, 2a+1, 2a+3, 3a+1\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Theorem 3.22. *[23]: For $a \geq 3, b \geq 2$, and $d \in \{4a-1, 4a-3, 5a-3, 3a-1\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Theorem 3.23. *[23]: For $a \geq 3, b \geq 2$, and $d \in \{6a-5, 6a-7, 7a-7, 5a-5\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Theorem 3.24. *Find other possible values of the parameter d and the corresponding d-antimagic labeling of type $(1,1,1)$ for plane graphs P_a^b .*

3.4. The Generalized Petersen Graph $P(n,2)$

Let n, m be integers such that $n \geq 3$, $1 \leq m < n$ and $n \neq 2m$. For such n, m, the *generalized Petersen graph* $P(n,m)$ is defined by $V(P(n,m)) = \{x_i,\ y_i : 1 \leq i \leq n\}$ and $E(P(n,m)) = \{y_i y_{i+1},\ x_i x_{i+m},\ x_i y_i : 1 \leq i \leq n\}$ (subscripts are to be read modulo n). The standard Petersen graph is the instance $P(5,2)$. Fig. 6 shows graph $P(10,2)$. By definition, $P(n,m)$ is a 3-regular graph which has $2n$ vertices and $3n$ edges. Generalized Petersen graphs were first defined by Watkins [27]. Note that $P(n, m_1) \cong P(n, m_2)$ if $m_1 + m_2 = n$ or $m_1 m_2 \equiv \pm 1 \pmod{n}$.

If $m = 1$ and $n \geq 3$ or $m = 2$ and n is even, $n \geq 6$, then the generalized Petersen graph $P(n,m)$ is plane. Note that $P(n,1)$ is the prism D_n.

Necessary conditions for $P(n,2)$ to possess a d-antimagic labeling of type $(1,1,1)$ are given in [11] and listed below.

Theorem 3.25. *[11]: For every generalized Petersen graph $P(n,2)$, $n \geq 6$, there is no d-antimagic vertex labeling with $d \geq 10$.*

Theorem 3.26. *[11]: For every graph $P(n,2)$, $n \geq 6$, there is no d-antimagic edge labeling with $d \geq 15$.*

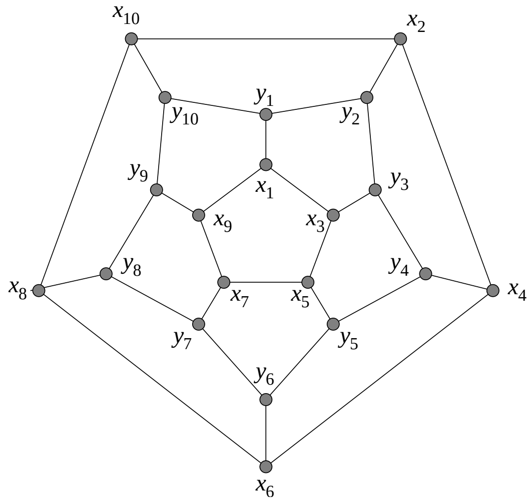

Figure 6: The generalised Petersen graph $P(10, 2)$.

Theorem 3.27. *[11]: Let $P(n, 2)$, $n \geq 6$, be a generalized Petersen graph which admits d_1-antimagic vertex labeling λ_1, d_2-antimagic edge labeling λ_2 and 1-antimagic face labeling λ_3, $d_1 \geq 0$, $d_2 \geq 0$. If the labelings λ_1, $v + \lambda_2$ and $v + e + \lambda_3$ combine to a d-antimagic labeling of type $(1, 1, 1)$ then the parameter $d \leq 24$.*

Theorem 3.28. *[11]: If n is even, $n \geq 6$, then the generalized Petersen graph $P(n, 2)$ has an 1-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.29. *[11]: If $n \equiv 2 \pmod 4$, $n \geq 6$, $n \neq 10$, then the generalized Petersen graph $P(n, 2)$ has a 0-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.30. *[11]: The graph of the dodecahedron has a 2-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.31. *[11]: If $n \equiv 2 \pmod 4$, $n \geq 6$, $n \neq 10$ and $d \in \{2, 3\}$, then the generalized Petersen graph $P(n, 2)$ has a d-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.32. *[11]: For $n \equiv 0 \pmod 4$, $n \geq 8$ and $d \in \{2, 3\}$, the generalized Petersen graph $P(n, 2)$ has a d-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.33. *[11]: If $n \equiv 0 \pmod 4$, $n \geq 8$ and $d \in \{6, 9\}$, then the graph $P(n, 2)$ has a d-antimagic labeling of type $(1, 1, 1)$.*

The last theorem states that $P(n, 2)$ has 6-antimagic and 9-antimagic labelings of type $(1, 1, 1)$ when $n \equiv 0 \pmod 4$ but does not mention the case when $n \equiv 2 \pmod 4$. We conjecture

Theorem 3.34. *There is a d-antimagic labeling of type* $(1, 1, 1)$ *for the generalized Petersen graph* $P(n, 2)$ *for* $n \equiv 2 \pmod 4$*,* $n \geq 6$ *and* $d \in \{6, 9\}$*.*

We conclude this subsection with the following

Theorem 3.35. *Find other possible values of the parameter d and the corresponding d-antimagic labeling of type* $(1, 1, 1)$ *for the generalized Petersen graph* $P(n, 2)$.

3.5. The Honeycomb H_n^m

For $n \geq 1$, $m \geq 1$ we denote by H_n^m (*honeycomb*) the hexagonal plane map with m rows and n columns of hexagons (see Figure 7 for n odd). The face set $F(H_n^m)$ contains mn 6-sided faces and one external infinite face.

$|V(H_n^m)| = 2mn + 2(m + n)$, $|E(H_n^m)| = |V(H_n^m)| + mn - 1$.

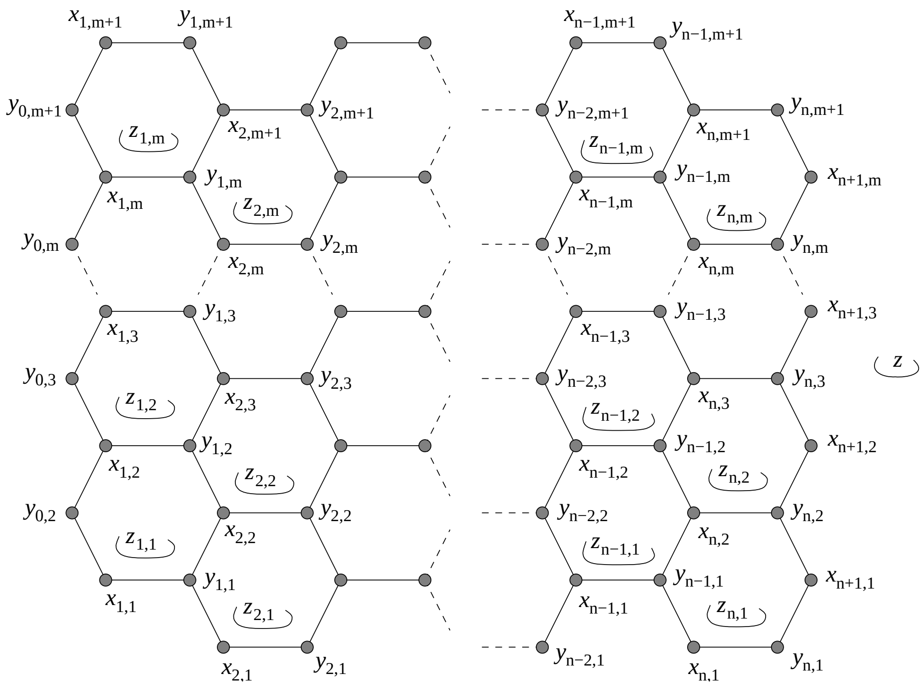

Figure 7: The honeycomb H_n^m.

Magic (that is, 0-antimagic) type $(1, 1, 1)$ labelings for honeycomb are given in [4]. It was proved in [10] that if n is even, $n \geq 2$ and $m \geq 1$, then the plane map H_n^m supports 2-antimagic and 4-antimagic labelings of type $(1, 1, 1)$.

Theorem 3.36. *[8]: If n is odd,* $n \geq 1$*,* $m \geq 1$*,* $mn > 1$ *and* $d \in \{1, 3\}$*, then the hexagonal plane map* H_n^m *has a d-antimagic supertotal labeling.*

Theorem 3.37. *[8]: If n is odd, $n \geq 1$, $m \geq 1$, $mn > 1$ and $d \in \{2, 4\}$, then the plane map H_n^m has a d-antimagic supertotal labeling.*

Open Problem 3.38. *Find other possible values of the parameter d and the corresponding d-antimagic supertotal labelings for the hexagonal plane map H_n^m.*

3.6. The Grid G_n^m

For $n \geq 1$ and $m \geq 1$, let G_n^m be the *grid graph* which can be defined as the Cartesian product $P_{m+1} \times P_{n+1}$ of a path on $m+1$ vertices with a path on $n+1$ vertices embedded in the plane and labeled as in Figure 8.

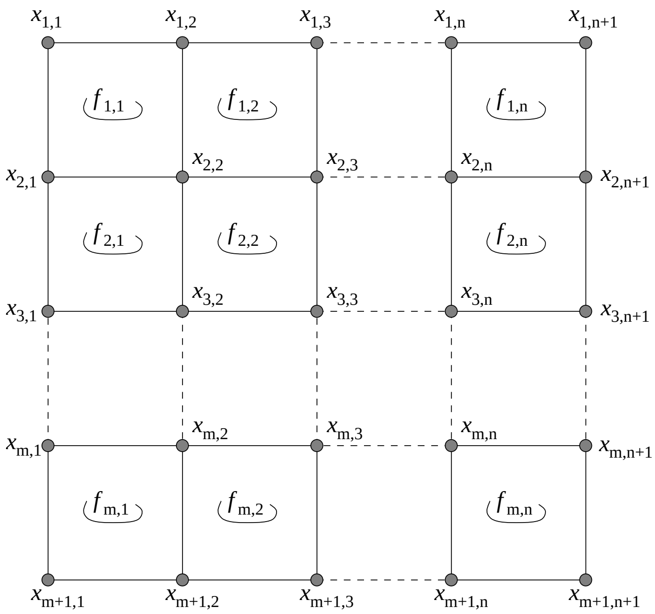

Figure 8: The grid G_n^m.

Magic (i.e., 0-antimagic) labelings of type $(1, 1, 1)$ for grid graphs are given in [5].

Necessary conditions for grids to bear d-antimagic labelings of types $(1, 0, 0)$ and $(0, 1, 0)$ as listed in [13] are given in the following propositions.

Theorem 3.39. *For every grid graph G_n^m, $m, n > 7$, there is no d-antimagic vertex labeling with $d \geq 5$.*

Theorem 3.40. *For every grid graph G_n^m, $m, n > 7$, there is no d-antimagic edge labeling with $d \geq 9$.*

Applying previous two theorems, and the fact that under d-antimagic face labeling $F(G_n^m) \to \{1, 2, 3, \ldots, |F(G_n^m)|\}$, the parameter d is no more than 1, we obtain

Theorem 3.41. *Let G_n^m, $m, n > 7$, be a graph which admits d_1-antimagic vertex labeling g_1, d_2-antimagic edge labeling g_2 and 1-antimagic face labeling g_3, $d_1 \geq 0$, $d_2 \geq 0$. If the labelings g_1, $v + g_2$ and $v + e + g_3$ combine to a d-antimagic labeling of type $(1, 1, 1)$ then the parameter $d \leq 13$.*

Theorem 3.42. *[13]: For $m \geq 1$, $n \geq 1$ and $n + m \neq 2$, the grid graph G_n^m has a 1-antimagic labeling and 3-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.43. *[13]: For $m \geq 1$, $n \geq 1$ and $n + m \neq 2$, the grid graph G_n^m has a 4-antimagic labeling of type $(1, 1, 1)$.*

Theorem 3.44. *[13]: For $m \geq 1$, $n \geq 1$ and $n + m \neq 2$, the graph G_n^m has a 2-antimagic labeling and 6-antimagic labeling of type $(1, 1, 1)$.*

The last three theorems above give results for $d = 1, 2, 3, 4$ and 6 which lead us to propose

Theorem 3.45. *There is a 5-antimagic labeling of type $(1, 1, 1)$ for the plane graph G_n^m and for all $m \geq 1$, $n \geq 1$, $m + n \neq 2$.*

From the necessary conditions we have a bound for the feasible values of the parameter $d \leq 13$. Therefore we formulate the following open problem.

Theorem 3.46. *Find other possible values of the parameter d and corresponding d-antimagic labelings of type $(1, 1, 1)$ for G_n^m.*

3.7. The Mőbius Grid M_n^m

For $n \geq 1$ and $m \geq 1$, let $P_{n+1} \times P_m$ be the Cartesian product of a path P_{n+1} on $n + 1$ vertices with a path P_m on m vertices embedded in the plane. Let vertices $x_{i,j}$, $i \in I \cup \{n + 1\}$ and $j \in J$ of $P_{n+1} \times P_m$, be labeled so that $x_{i,1}\ x_{i,2}\ x_{i,3}\ \ldots\ x_{i,m-2}\ x_{i,m-1}\ x_{i,m}$ are vertices of the path $P_m(i)$, $i \in I \cup \{n + 1\}$ and $x_{1,j}\ x_{2,j}\ x_{3,j}\ \ldots\ x_{n-1,j}\ x_{n,j}\ x_{n+1,j}$ are vertices of the path $P_{n+1}(j)$, $j \in J$.

Now, for $n \geq 1$, $m \geq 1$, we denote by M_n^m (Mőbius grid) the graph with $V(M_n^m) = V(P_{n+1} \times P_m) = \{x_{i,j} : i \in I \cup \{n+1\},\ j \in J\}$ and $E(M_n^m) = \{x_{i,j}x_{i,j+1} : i \in I \cup \{n + 1\},\ j \in J - \{m\}\} \cup \{x_{i,j}x_{i+1,j} : i \in I, j \in J\} \cup \{x_{i,m}x_{n+2-i,1} : i \in I \cup \{n + 1\}\}$.

If we consider the Mőbius grid M_n^m drawn in Euclidean space and not on the Euclidean plane then the face set $F(M_n^m)$ is unambiguous and contains mn 4-sided faces.

We have proved [1] that if m is odd, $m \geq 3$ and $n \geq 1$, then the Mőbius grid M_n^m has a magic (0-antimagic) labeling of type $(1,1,1)$.

Theorem 3.47. *[17]: If m is odd, $m \geq 3$, $n \geq 1$ and $d \in \{1,2,4\}$, then the Mőbius grid M_n^m has a d-antimagic labeling of type* $(1,1,1)$.

We conclude with the following open problem.

Theorem 3.48. *Find other possible values of the parameter d and corresponding d-antimagic labelings of type $(1,1,1)$ for M_n^m.*

3.8. The Special Class L_n^m

For $n \geq 2$, $1 \leq m \leq 4$, let L_n^m be the graph with the vertex set $V(L_n^m) = \{x_{i,j} : i \in I \text{ and } j \in J \cup \{m+1\}\}$ and the edge set

$E(L_n^m) = \{x_{i,j}x_{i+1,j} : i \in I - \{n\} \text{ and } j \in J \cup \{m+1\}\}$
$\cup\{x_{i,j}x_{i,j+1} : i \in I \text{ and } j \in J\}$
$\cup\{x_{i+1,j}x_{i,j+1} : i \in I - \{n\}, j \in J \text{ and } j \text{ is odd }\}$
$\cup\{x_{i,j}x_{i+1,j+1} : i \in I - \{n\}, j \in J \text{ and } j \text{ is even }\}$,

embeded in the plane and labeled as in Figure 9 (if $m = 4$).

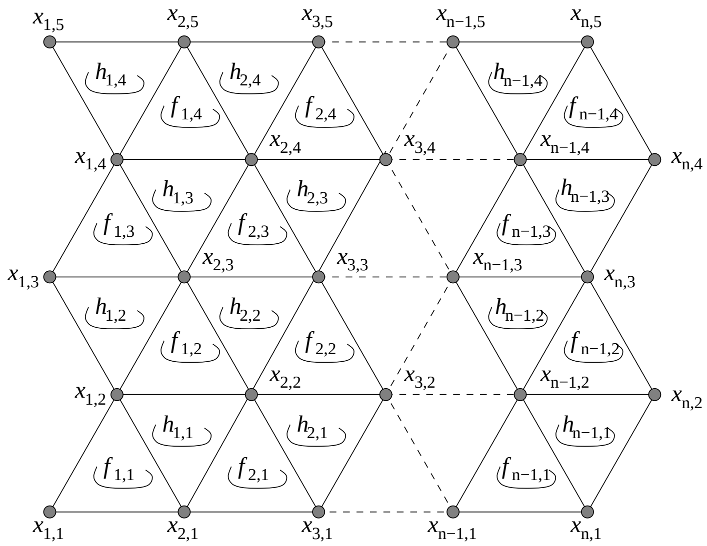

Figure 9: The graph L_n^m for $m = 4$.

The face set $F(L_n^m)$ contains $|F(L_n^m)| - 1 = 2(n-1)m$ 3-sided faces and one external infinite face. $|V(L_n^m)| = n(m+1)$, $|E(L_n^m)| = |V(L_n^m)| + |F(L_n^m)| - 2$.

Magic (0-antimagic) labelings of type $(1,1,1)$ of plane graphs L_n^m for $n \geq 2$, $m = 1$ are described in [2] and for $n \geq 2$, $2 \leq m \leq 3$ are given in [3].

In [18] are found bounds for a feasible value d for the vertex labeling and the edge labeling of L_n^m.

Theorem 3.49. *[18]: For every plane graph L_n^m, $n \geq 2$, $m \geq 1$, there is no d-antimagic vertex labeling with $d > 3$.*

Theorem 3.50. *[18]: For every plane graph L_n^m, $n \geq 2$, $m \geq 1$, there is no d-antimagic edge labeling whenever $d > 6$.*

Applying previous two theorems and the fact that under d-antimagic face labeling $F(L_n^m) \rightarrow \{1, 2, ..., |F(L_n^m)|\}$ the parameter d is no more than 1, we obtain

Theorem 3.51. *[18]: Let L_n^m, $n \geq 2$, $m \geq 1$, be a plane graph which admits d_1-antimagic vertex labeling h_1, d_2-antimagic edge labeling h_2 and 1-antimagic face labeling h_3, $d_1 \geq 0$, $d_2 \geq 0$. If the labelings h_1, $|V(L_n^m)| + h_2$ and $|V(L_n^m)| + |E(L_n^m)| + h_3$ combine into a d-antimagic labeling of type $(1,1,1)$ then the parameter $d \leq 10$.*

In [18] it is shown how to construct d-antimagic labelings of L_n^m.

Theorem 3.52. *[18]: If $n \geq 2$, $1 \leq m \leq 4$ and $d \in \{0, 2\}$, then the plane graph L_n^m has a d-antimagic labeling of type $(1,1,1)$.*

Theorem 3.53. *[18]: If $n \geq 2$, $1 \leq m \leq 4$, then the plane graph L_n^m has a 4-antimagic labeling of type $(1,1,1)$.*

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