Structural Properties and Labeling of Graphs

DAFIK

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of the requirement for the degree of PhD.

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Statement of Authorship

Except where explicit reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma. No other person's work has been relied upon or used without due acknowledgment in the main text and bibliography of the thesis.

Dafik

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List of Publications Arising from this Thesis

1. Dafik, M. Miller, J. Ryan and M. Bača, On super (a, d)-edge antimagic total labeling of disconnected graphs, *Discrete Math.*, in press.

2. Dafik, M. Miller, J. Ryan and M. Bača, Super edge-antimagic total labelings of $mK_{n,n,n}$, *Ars Combin.*, in press.

3. Dafik, M. Miller, J. Ryan and M. Bača, Antimagic total labeling of disjoint union of complete s-partite graphs, *J. Combin. Math. Combin. Comput.*, in press.

4. Dafik, M. Miller, C. Iliopoulos and Z. Ryjacek, On diregularity of digraphs of defect two, *Proceedings of International Workshop on Combinatorial Algorithms* (IWOCA'07) (2007) 39-48.

5. Dafik, M. Miller, J. Ryan and M. Bača, Super edge-antimagicness for a class of disconnected graphs, *Proceedings of Seventeenth Australasian Workshop on Combinatorial Algorithms* (AWOCA'06) (2006) 67-77.

6. Dafik, M. Miller, J. Ryan and Slamin, Open problems in the construction of large directed graphs, *Proceedings of Sixteenth Australasian Workshop on Combinatorial Algorithms* (AWOCA'05) (2005) 95-104.

7. Dafik, M. Miller, C. Iliopoulos and Z. Ryjacek, On the diregularity of digraphs of defect at most two, *J. Combin. Math. Combin. Comput.*, in press.

8. Dafik, M. Miller, J. Ryan and M. Bača, Antimagic labeling of union of stars, *Austral. J. Combin.*, in press.

9. Dafik, M. Miller and Slamin, On diregularity of digraphs of defect two and out-degree three, preprint.

Further Publications Produced During my PhD Candidature

1. M. Bača, Dafik, M. Miller and J. Ryan, Antimagic labeling of disjoint union of s-crowns, *Utilitas Math.*, in press.
2. M. Bača, Dafik, M. Miller and J. Ryan, On super (a, d)-edge antimagic total labeling of caterpillars, *J. Combin. Math. Combin. Comput.*, in press.

Abstract

The complexity in building massive scale parallel processing systems has resulted in a growing interest in the study of interconnection networks design. Network design affects the performance, cost, scalability, and availability of parallel computers. Therefore, discovering a good structure of the network is one of the basic issues.

From modeling point of view, the structure of networks can be naturally studied in terms of graph theory. Several common desirable features of networks, such as large number of processing elements, good throughput, short data communication delay, modularity, good fault tolerance and diameter vulnerability correspond to properties of the underlying graphs of networks, including large number of vertices, small diameter, high connectivity and overall balance (or regularity) of the graph or digraph.

The first part of this thesis deals with the issue of interconnection networks addressing system. From graph theory point of view, this issue is mainly related to a graph labeling. We investigate a special family of graph labeling, namely antimagic labeling of a class of disconnected graphs. We present new results in super (a, d)-edge antimagic total labeling for disjoint union of multiple copies of special families of graphs.

The second part of this thesis deals with the issue of regularity of digraphs with the number of vertices close to the upper bound, called the Moore bound, which is unobtainable for most values of out-degree and diameter. Regularity of the underlying graph of a network is often considered to be essential since the flow of messages and exchange of data between processing elements will be on average faster if there is a similar number of interconnections coming in and going out of each processing element. This means that the in-degree and out-degree of each processing element must be the same or almost the same. Our new results show that digraphs of order two less than Moore bound are either diregular or almost diregular.

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INTRODUCTION

In the early days of computer networks, interprocessor communication and scalability of applications was hampered by the high latency and the lack of bandwidth of the network. The IBM supercomputing project, which was begun in 1999, has proposed a new solution to the problem and built a new family of supercomputers optimizing the bandwidth, scalability and the ability to handle large amounts of transferring data. One of the world's fastest supercomputers was officially inaugurated at IBM's Zurich Research Laboratory (ZRL). The so-called BlueGene system, which is the IBM supercomputing project solution, has the same performance as the computer ranked 21st on the current list of the world's top 500 supercomputers (for more detail, see [68]). It will be used to address some of the most demanding problems faced by scientists regarding the future of information technology, such as, how computer chips can be made even smaller and more powerful. However, in massive parallel computers, the robustness of supercomputers is not the only factor. One of the most significant factors is the design of parallel processing systems circuits and, more precisely, the construction of their interconnection networks. Therefore, there has been a growing interest in the study of the design of large interconnection networks.

In communication network design, Fiol and Lladó [56] identified several factors which should be considered. Some of these factors seem fundamental, for instance, there must always exist a path from any processing element to another. Also, the data communication delay during processing must be as short as possible. Another factor we may consider in the design of an interconnection network is a modularity, a good fault tolerance, a diameter vulnerability and a vertex-symmetric interconnection network. We may also require an overall balance (or regularity) of the system.

A communication network can be modelled as a *graph* or a *directed graph* (*digraph*, for short), where each processing element is represented by a *vertex* and the connection between two processing elements is represented by an *edge* (or, in the case of a digraph, by a directed *arc*). The number of vertices is called the *order* of the graph or digraph. The number of connections incident to a vertex is called the *degree* of the vertex. If the connections are one way only then we distinguish between in-coming and out-going connections and we speak of the *in-degree* and the *out-degree* of a vertex. The *distance* between two vertices is the length of the shortest path, measured by the number of edges or arcs that need to be traversed in order to reach one vertex from another vertex. In either case, the largest distance between any two vertices, called the *diameter* of the graph or digraph, represents the maximum data communication delay in a communication network.

In the first part of this thesis we deal with graph labeling. Graph labelings provide useful mathematical models for a wide range of applications, such as radar and communication network addressing systems and circuit design, bioinformatics, various coding theory problems, automata, x-ray crystallography and data security. More detailed discussions about applications of graph labelings can be found in Bloom and Golomb's papers [27] and [28].

Many studies in graph labeling refer to Rosa's research in 1967 [104] and Golomb's research in 1972 [61]. Rosa introduced a kind of labeling, called *β-valuation* and Golomb independently studied the same type of labeling and called this labeling *graceful labeling*. Surprisingly, in 1963 Sedláček [105] had already published a paper which introduced another type of graph labeling, namely, *magic labeling*. Stewart [112] called magic labeling *supermagic* if the set of edge labels consists of consecutive integers.

Motivated by Sedláček and Stewart's research, many other labelings of graphs have been studied since then, and many new results have been published. However, there still exist many interesting open problems and conjectures. No polynomial time bounded algorithm is known for determining whether or not the various types of graph labelings exist for particular classes of graphs. Therefore, the question of whether a specific family of graphs admits a property of a specific labeling is still widely open. In the first part of this thesis we present new results in super graph labeling for disjoint unions of multiple copies of special families of graphs.

In the second part of this thesis we mainly consider the topology of networks. There are two aspects which should be considered in communication networks design. Firstly, the number of processing elements in an interconnection network should be as large as possible, given that each processing element can be connected only to a limited number of other processing elements. Secondly, the data communication delay among processing elements should be as short as possible.

For undirected case of networks, translating the above required conditions in terms of the underlying graphs, the problem is to find large graphs with given maximum degree and diameter. This naturally leads to the well-known fundamental problem called the $N(\Delta, D)$*-problem*: For given numbers Δ and D, construct graphs of maximum degree Δ and diameter $\leq D$, with the largest possible number of vertices $n_{\Delta,D}$. The $N(\Delta, D)$-problem is also known as the *degree/diameter problem*. The directed version of the problem differs only in that 'degree' is replaced by 'out-degree' in the statement of the problem, namely, $N(d, k)$*-problem*: For given numbers d and k, construct digraphs of maximum out-degree d and diameter $\leq k$, with the largest possible number of vertices $n_{d,k}$.

In the degree/diameter problem, the values of $N(\Delta, D)$ and $N(d, k)$ are not known for most values of Δ, D and d, k, respectively. Therefore, it is useful to investigate the lower and upper bounds on $N(\Delta, D)$ and $N(d, k)$. A natural number $n_{l_{\Delta,D}}$ (respectively, $n_{l_{d,k}}$) is a *lower bound* of $N(\Delta, D)$ (respectively, $N(d, k)$) if we can prove the existence of a graph of maximum degree at most Δ, diameter D and exactly $n_{l_{\Delta,D}}$ vertices; or alternatively, the existence of a digraph of maximum out-degree at most d, diameter k and exactly $n_{l_{d,k}}$ vertices. A natural number $n_{u_{\Delta,D}}$ (respectively, $n_{u_{d,k}}$) is an *upper bound* of $N(\Delta, D)$ (respectively, $N(d, k)$) if we can prove that there are no graphs of maximum degree at most Δ, diameter D, and with the number of vertices more than $n_{u_{\Delta,D}}$; or that there are no digraphs of maximum out-degree at most d, diameter k, and with the number of vertices more than $n_{u_{d,k}}$.

A natural general upper bound on the order $n_{\Delta,D}$ (respectively, $n_{d,k}$) of a graph (respectively, a digraph) is the *Moore bound*. However, there are very few graphs or digraphs of order attaining the Moore bound. This gives rise to two directions of research connected to the $N(\Delta, D)$-problem and $N(d, k)$-problem:

(i) Proving the non-existence of graphs or digraphs of order ‘close’ to the Moore bound and so lowering the upper bound $n_{u_{\Delta,D}}$ or $n_{u_{d,k}}$;

(ii) Constructing large graphs or digraphs and so incidentally obtaining better lower bounds $n_{l_{\Delta,D}}$ or $n_{l_{d,k}}$.

To prove the non-existence of digraphs of order close to the Moore bound, we may first wish to establish some useful structural properties of such potential digraphs. Knowing structural properties of potential digraphs can also be helpful in the construction of such digraphs. The second part of this thesis makes a contribution concerning one such property, namely, the diregularity of digraphs of order close to the Moore bound.

The thesis is organised as follows. We provide basic terminology in Chapter 1. In Chapter 2 we present an introduction to graph labeling. In Chapters 3, 4 and 5, we present new results on super edge antimagic total labeling of disjoint union of cycles and paths, stars, complete tripartite graphs and complete s-partite graphs. In Chapters 6 and 7, we provide literature review of the degree/diameter problem. In Chapters 8 and 9 we present new results on the diregularity of digraphs of defect at most two. Finally, we conclude the thesis in the last chapter.

The main contributions of this thesis are to be found in Chapters 3, 4, 5, 8, 9. All original results (lemmas, theorems and corollaries) are flagged by $\diamond$.

Chapter 1

Basic Terminology

1.1 Undirected graphs

By an *undirected graph*, or a *graph*, we mean a structure $G = (V(G), E(G))$, where $V(G)$ is a finite nonempty set of elements called *vertices*, and $E(G)$ is a set (possibly empty) of unordered pairs $\{u, v\}$ of vertices $u, v \in V(G)$, called *edges*. The number of vertices of a graph G is the *order* of G, commonly denoted by $|V(G)|$. The number of edges is the *size* of G, often denoted by $|E(G)|$. A graph G that has order $p = |V(G)|$ and size $q = |E(G)|$ is sometimes called a (p, q)-*graph*.

Let $u, v \in V(G)$. Vertex u is said to be *adjacent* to v if there is an edge e between u and v, that is, $e = uv$. Vertex v is then called a *neighbour* of u. The set of all neighbours of u is called the *neighbourhood* of u and is denoted by $N(u)$. We also say that u and v are *incident* with edge e. For example, in Figure 1.1, vertex v_1 is adjacent to vertex v_4; vertex v_4 is incident with edges v_4v_5 and v_4v_6; and the neighbours of vertex v_4 are v_1, v_3, v_5 and v_6.

The number of neighbours of v is called the *degree* of a vertex v of G. If a vertex v has degree 0, that is, v is not adjacent to any other vertex, then v

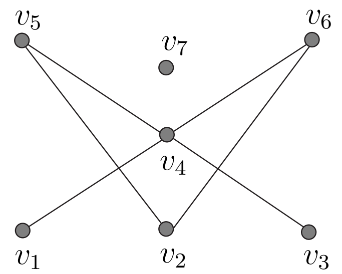


Figure 1.1: Example of a graph with isolated vertex.

is called an *isolated vertex*. A vertex of degree 1 is called an *end vertex*, or a *leaf*. If all the vertices of a graph G have the same degree d then G is said to be *regular* of degree d, or *d-regular*.

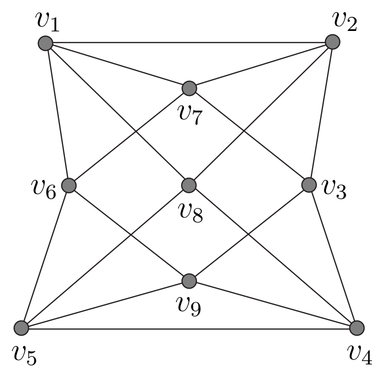


Figure 1.2: Example of a graph

A $v_0 - v_k$ *walk* of a graph G is a finite alternating sequence $v_0, e_1, v_1, e_2, ..., e_k, v_k$ of vertices and edges in G such that $e_i = v_{i-1}v_i$ for each i, $1 \leq i \leq k$. Such a walk may also be denoted by $v_0v_1...v_k$. We note that there may be repetition of vertices and edges in a walk. The *length* of a walk is the number of edges in the walk. A *closed* walk has $v_0 = v_k$. If all the vertices of a $v_0 - v_k$ walk are distinct, then the walk is called a *path.* A *cycle* C_k of length k is a closed walk of length $k > 2$ with all vertices are distinct (except $v_0 = v_k$). In Figure 1.2, $v_1v_2v_7v_6v_5v_8v_2v_3v_9$ is a walk of length 8 which is not a path, $v_1v_8v_4v_3v_7v_6v_9$ is a path of length 6, and $v_5v_6v_7v_3v_9v_5$ is a cycle of length 5.

The *distance* from vertex u to v, denoted by $\delta(u, v)$, is the length of a shortest path from vertex u to vertex v. For any vertices u, v, w in G, we have $\delta(u, w) \leq \delta(u, v) + \delta(v, w)$ and if $\delta(u, v) \geq 2$ then there is a vertex z in G such that $\delta(u, v) = \delta(u, z) + \delta(z, v)$. For example, the distance from vertex v_1 to v_4 of the graph in Figure 1.2 is 2. The *eccentricity* of v, denoted by $e(v)$, is defined by $e(v) = max\{\delta(u, v) : u \in V, u \neq v\}$ and the radius of G, denoted by $rad\ G$, is defined by $rad\ G = min\{e(v) : v \in V\}$. The *diameter* of a graph G is the longest distance between any two vertices in G and is denoted $diam\ G = max\{e(v) : v \in V\}$ and the *girth* of a graph G is the length of the shortest cycle in G. For example, the graph in Figure 1.2 has diameter 2 and girth 3.

A graph H is a *subgraph* of G if every vertex of H is a vertex of G, and every edge of H is an edge of G. In other words, $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We say that a subgraph H is a *spanning subgraph* of G if H contains all the vertices of G. Let F be a nonempty subset of the vertex set $V(G)$. The *induced subgraph* $G[F]$ is a subgraph of G consisting of the vertex-set F together with all the edges uv of G, where $u, v \in F$. In Figure 1.3, F_1 is a spanning subgraph of G, F_2 is an induced subgraph of G, and F_3 is a subgraph of G but not an induced subgraph (because in F_3, $v_2, v_6 \in V(G)$ but there is no edge between v_2 and v_6, while $v_2v_6 \in E(G)$).

Let $G(V, E)$ be a graph. An *automorphism* of the graph G is a one-to-one mapping f from V onto itself which preserves vertex adjacency, that is, $\{f(u), f(v)\} \in E(G)$ if and only if $\{u, v\} \in E(G)$. Two graphs G_1 and G_2, each with n vertices, are said to be *isomorphic* if there exists a one-to-one mapping $f : V(G_1) \to V(G_2)$ which preserves vertex adjacency, that is, $\{f(u), f(v)\} \in E(G_2)$ if and only if $\{u, v\} \in G_1$. In Figure 1.4, graphs G_1 and G_2 are isomorphic under the mapping $f(u_i) = v_i$, for every $i = 1, 2, ..., 12$. However, graphs G_1 and G_3 are not isomorphic because G_1 contains cycles of

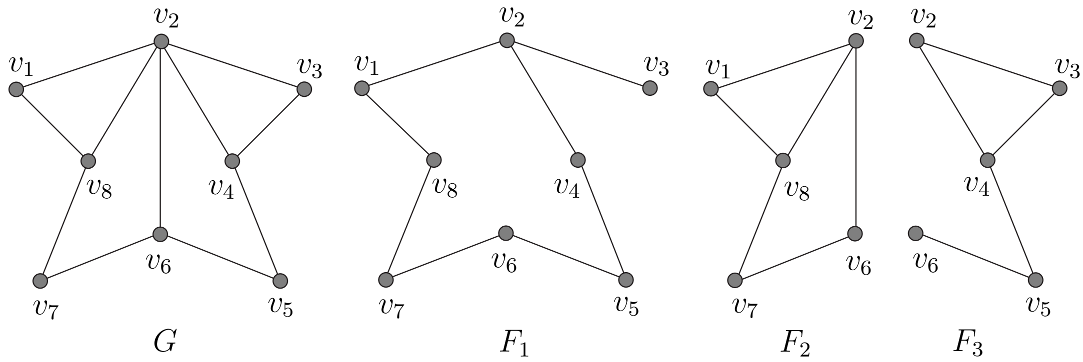


Figure 1.3: Graph and three of its subgraphs

length three while G_3 does not and consequently there cannot be any one-to-one mapping preserving adjacencies.

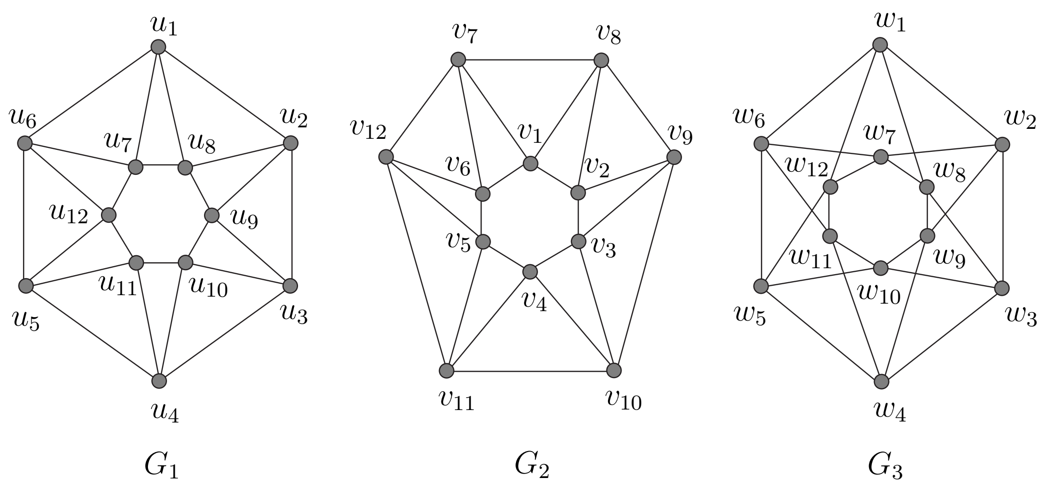
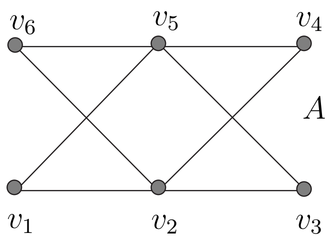


Figure 1.4: Isomorphism in graphs

The *adjacency matrix* of a graph G and vertex-set $V(G) = \{v_1, v_2, ..., v_n\}$ is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1.5 shows a graph of order 5 with its adjacency matrix.



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 1.5: Graph and its adjacency matrix

A graph G is *connected* if for any two distinct vertices u and v of G there is a path between u and v. Otherwise, G is *disconnected*. The *disjoint union* (or *union*, for short) of two or more graphs $G_1, \dots G_m$, denoted by $G_1 \cup \cdots \cup G_m$, is defined as the graph with vertex set $V_1 \cup \cdots \cup V_m$ and edge set $E_1 \cup \cdots \cup E_m$. This type of a graph is disconnected and often referred to as a graph with m components. Figure 1.4 also shows an example of a disjoint union of three graphs $G_1 \cup G_2 \cup G_3$.

1.2 Directed graphs

By a *directed graph*, or a *digraph*, we mean a structure $G = (V(G), A(G))$, where $V(G)$ is a finite nonempty set of distinct elements called *vertices*, and $A(G)$ is a set of ordered pairs (u, v) of distinct vertices $u, v \in V(G)$, called *arcs*.

The *order* of a digraph G is the number of vertices in G. An *in-neighbour* (respectively, *out-neighbour*) of a vertex v in G is a vertex u (respectively, w) such that $(u, v) \in A(G)$ (respectively, $(v, w) \in A(G)$). The set of all in-neighbours (respectively, out-neighbours) of a vertex v is called the *in-neighbourhood* (respectively, the *out-neighbourhood*) of v and denoted by $N^-(v)$ (respectively, $N^+(v)$).

The *in-degree* (respectively, *out-degree*) of a vertex v is the number of its in-neighbours (respectively, out-neighbours), and is denoted by $d^-(v)$ (respectively, $d^+(v)$). If every vertex of a digraph G has the same in-degree (respectively, out-degree) then G is said to be *in-regular* (respectively, *out-regular*). A digraph G is called a *diregular* digraph of degree d if G is in-regular of in-degree d and out-regular of out-degree d. For example, the digraph G_1 in Figure 1.6 is diregular of degree 2, but the digraph G_2 is not diregular (G_2 is out-regular but not in-regular).

A digraph H is a *subdigraph* of G if every vertex of H is a vertex of G, and every arc of H is an arc of G. In other words, $V(H) \subset V(G)$ and $A(H) \subset A(G)$. We say that a subdigraph H is a *spanning subdigraph* of G if H contains all the vertices of G. Let F be a nonempty subset of the vertex set $V(G)$. The *induced subdigraph* $G[F]$ is a subdigraph of G consisting of the vertex-set F together with all the arcs uv of G, where $u, v \in F$.

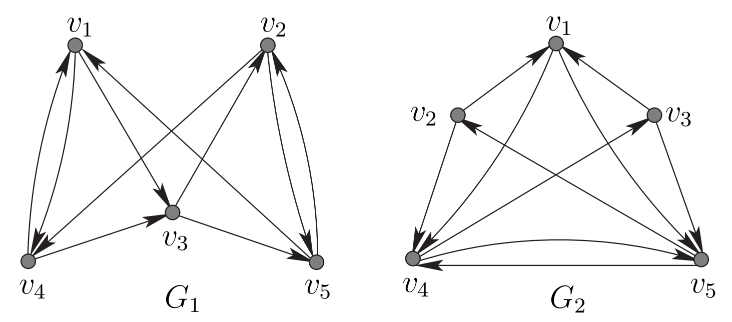


Figure 1.6: Diregular and non-diregular digraphs.

An alternating sequence $v_0a_1v_1a_2...a_kv_k$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$, for each i, $1 \leq i \leq k$, is called a *walk* of length k in G. A walk is *closed* if $v_0 = v_k$. If all the vertices of a $v_0 - v_k$ walk are distinct, then such a walk is called a *path*. A *cycle* is a closed walk of length $k > 1$ with all vertices are distinct (except $v_0 = v_k$). A *digon* is a cycle of length 2.

The *distance* from vertex u to vertex v, denoted by $\delta(u, v)$, is the length of a shortest path from u to v, if any; otherwise, $\delta(u, v) = \infty$. Note that, in general, $\delta(u, v)$ is not necessarily equal to $\delta(v, u)$. The *in-eccentricity* of v, denoted by $e^-(v)$, is defined as $e^-(v) = max\{\delta(u, v) : u \in V\}$ and *out-eccentricity* of v, denoted by $e^+(v)$, is defined as $e^+(v) = max\{\delta(v, u) : u \in V\}$. The *radius* of G, denoted by rad(G), is defined as rad(G)= $min\{e^-(v) : v \in V\}$. The *diameter* of G, denoted by diam(G), is defined as diam(G)= $max\{e^-(v) : v \in V\}$. Note that, equivalently, we could have defined the radius and the diameter of a digraph in terms of out-eccentricity instead of in-eccentricity. The *girth* of a digraph G is the length of the shortest cycle in G. For example, both digraphs in Figure 1.6 have radius 1, girth 2 and diameter 2.

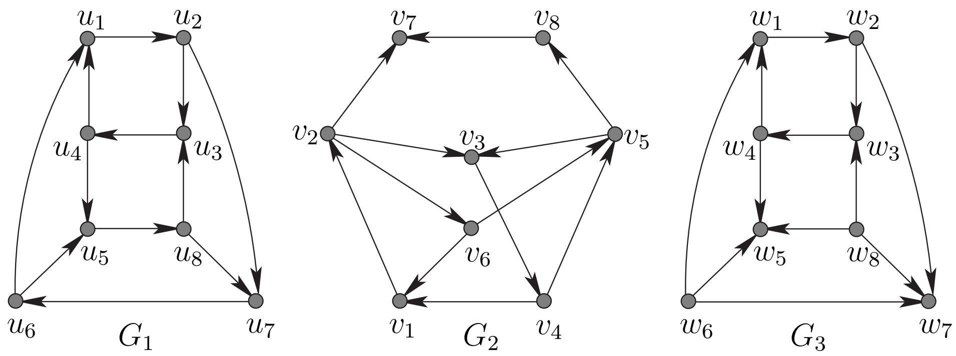


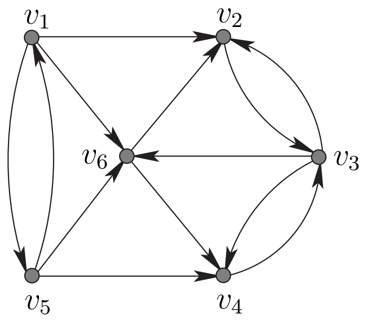
Figure 1.7: Isomorphism in digraphs.

Let $G(V, A)$ be a digraph. An *automorphism* of the digraph G is a one-to-one mapping f from V onto itself which preserves all the adjacencies, that is, $(f(u), f(v)) \in A(G)$ if and only if $(u, v) \in A(G)$. Two digraphs G_1 and G_2, each with n vertices, are said to be *isomorphic* if there exists a one-to-one mapping $f : V(G_1) \to V(G_2)$ which preserves all the adjacencies, that is, $(f(u), f(v)) \in A(G_2)$ if and only if $(u, v) \in A(G_1)$. In Figure 1.7, digraphs G_1 and G_2 are isomorphic under the mapping $f(u_i) = v_i$, for every $i = 1, 2, ..., 8$. However, digraphs G_1 and G_3 are not isomorphic since G_3 contains two vertices of in-degree 3 while G_1 does not, and consequently, a one-to-one mapping preserving adjacencies cannot exist.

The *adjacency matrix* of a digraph G with vertex-set $V(G) = \{v_1, v_2, ..., v_n\}$ is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in A(G), \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1.8 shows a digraph of order 5 with its adjacency matrix.



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 1.8: Digraph and its adjacency matrix.

In the second part of this thesis, we will use two special digraphs, namely, *line digraphs* and *Kautz digraphs*. The *line digraph* of a digraph $G = G(A, V)$ is, $L(G) = (A, N)$, where N is the set of walks of length 2. The set of vertices of $L(G)$ is equal to the set of arcs of G. This means that a vertex uv of $L(G)$ is adjacent to a vertex wx if and only if $v = w$. The order of the line digraph $L(G)$ is equal to the number of arcs in the digraph G. For a diregular digraph G of out-degree $d \geq 2$, the sequence of line digraph iterations $L(G), L^2(G) = L(L(G)), ..., L^i(G) = L(L^{i-1}(G)), ...$ is an infinite sequence of diregular digraphs of degree d. One family of line digraphs which is very important in the degree/diameter problem is the Kautz digraph. We denote Kautz digraph of degree d and diameter k by $Ka(d, k)$. Let K_n be a *complete digraph* of order n. A Kautz digraph $Ka(d, k)$ can be defined as the line digraph $L^k(K_{d+1})$.

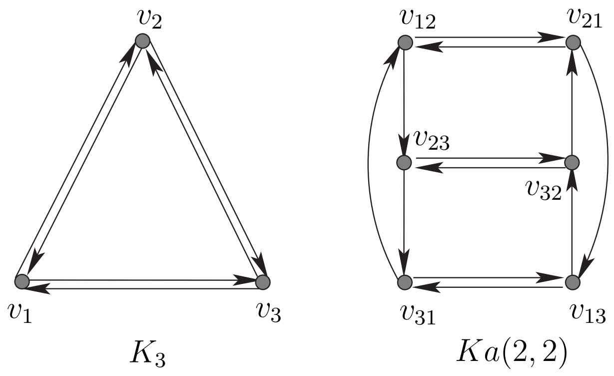


Figure 1.9: An example of Kautz digraph $Ka(2,2)$ obtained from K_3.

For example, Figure 1.9(b) shows the Kautz digraph $Ka(2,2)$ of degree 2, diameter 2 and order 6 obtained by applying the line digraph technique (once) to the complete digraph K_3.

PART I

Chapter 2

Super Edge-antimagic Total Graphs

2.1 Motivation

A labeling for a graph G is a mapping that sends some set of graph elements to a set of non-negative integers. If the domain is the vertex-set or the edge-set, the labeling is called a *vertex labeling* or an *edge labeling*, respectively. If the domain is the set of all vertices and edges then the labeling is called a *total labeling*.

Graph labelings provide useful mathematical models for a wide range of applications. Qualitative labelings of graph elements have inspired research in diverse fields of human enquiry such as conflict resolution in social psychology, electrical circuit theory, and energy crisis. Quantitative labelings of graphs have led to quite intricate fields of applications such as radar and communication network addressing system and circuit design, bioinformatics, various coding theory problems, automata and x-ray crystallography. More detailed discussions about applications of graph labelings can be found in [27] and [28].

Many studies in graph labeling refer to Rosa's research in 1967 [104]. Rosa introduced a function f from a set of vertices in a graph G to the set of integers $\{0, 1, 2, \ldots, |E(G)|\}$ so that each edge xy is assigned the label $|f(x) - f(y)|$, with all labels distinct. Rosa called this labeling *β-valuation*. Independently, Golomb [61] studied the same type of labeling and called this labeling *graceful labeling*.

Surprisingly, in 1963 Sedláček [105] had already published a paper which introduced another type of graph labeling, namely, *magic labeling*. His definition was motivated by the magic square notion in number theory. *A magic labeling* is a mapping from the set of edges of graph G into non-negative real numbers, so that the sums of the edge labels around any vertex in G are all the same. Note that Sedláček's definition allowed for any real numbers to be used but usually only integers are used. Stewart [112] called magic labeling *supermagic* if the set of edge labels consisted of consecutive integers.

Motivated by Sedláček and Stewart's research, many other labelings of graphs have been studied, including labeling of faces of planar graphs, and many new results have been found. Unaware of the work done by each other, similar concepts have been reintroduced a few times, and some results have been rediscovered independently. For example, Enomoto *et al.* [46] call edge-magic total labelings of a graph G *super edge-magic total labelings* if the set of vertex labels is $\{1, 2, \ldots, |V(G)|\}$. Wallis [119] calls these labelings *strongly edge-magic*.

Although there is a large number of publications on magic-type graph labelings, there still exist many interesting open problems and conjectures. This is due to the fact that to decide whether G admits a vertex-magic or an edge-magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution. No polynomial time bounded algorithm is known for determining whether G is vertex-magic or edge-magic. Therefore, the question of whether a particular family of graphs admits a particular labeling is still open.

2.2 Magic and antimagic labelings

As mentioned in the previous section, Sedláček introduced the magic labeling concept in 1963. The notion of an antimagic graph was introduced by Hartsfield and Ringel in 1989 [63]. Subsequently, as mentioned by Nicholas *et al.* [97], Bodendiek and Walther in 1996 [30] were the first to introduce the concept of (a, d)-vertex-antimagic edge labeling; they called this labeling (a, d)-vertex-antimagic labeling.

All graphs in this chapter are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A (p, q)-graph G is a graph such that $|V(G)| = p$ and $|E(G)| = q$. In both magic and antimagic labelings, the sum of all labels associated with a graph element is called a 'weight'.

Formal definitions of (a, d)-vertex antimagic edge labeling and (a, d)-vertex antimagic total labeling of graphs are as follows.

A bijective function $f : E(G) \rightarrow \{1, 2, ..., q\}$ is called an (a, d)-*vertex-antimagic edge labeling*, if the set of *vertex weights under edge labeling*, $w(u) = \Sigma_{v \in N(u)} f(uv)$, of all the vertices in G is $\{a, a + d, ..., a + (p - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p + q\}$ is called an (a, d)-*vertex-antimagic total labeling*, if the set of *vertex weights under total labeling*, $w(u) = f(u) + \Sigma_{v \in N(u)} f(uv)$, of all the vertices in G is $\{a, a + d, ..., a + (p - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

A total labeling f is called a *super* (a, d)-*vertex-antimagic total* if $f(V) = \{1, 2, \ldots, p\}$. If $d = 0$ then (a, d)-*vertex-antimagic total labeling* is called similarly a *vertex-magic total* labeling.

Other types of antimagic labelings are (a, d)-edge-antimagic vertex labeling and super (a, d)-edge-antimagic total labeling. These notions were introduced by Simanjuntak, Bertault and Miller in [106]. They are natural extensions of the notions of edge magic labeling which was introduced by Kotzig and Rosa [76, 77], and super edge magic labeling which was defined by Enomoto *et al.* in [46]. Hegde (1989) in his thesis also introduced the concept of a *strongly* (k, d)*-indexable labeling* which is equivalent to (a, d)-edge-antimagic vertex labeling (see [1]) and Wallis *et al.* [119, 118, 80, 81] use the term *strongly edge magic total labeling* in place of super edge magic total labeling. Many other researchers investigated different forms of antimagic graphs. For example, see Bodendiek and Walther [31] and [32], and Hartsfield and Ringel [64]. Sugeng [114] studied properties of (a, d)-edge-antimagic total labeling and found many families of connected graphs which admit (a, d)-edge-antimagic total labeling.

Formal definition of (a, d)-edge antimagic vertex labeling and (a, d)-edge antimagic total labeling of graphs are as follows.

A bijective function $f : V(G) \rightarrow \{1, 2, ..., p\}$ is called an (a, d)*-edge-antimagic vertex labeling* if the set of *edge weights under vertex labeling*, $w(uv) = f(u) + f(v)$, of all the edges in G is $\{a, a + d, ..., a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p + q\}$ is called an (a, d)*-edge-antimagic total labeling*, if the set of *edge weights under total labeling*, $w(uv) = f(u) + f(v) + f(uv)$, of all the edges in G is $\{a, a + d, ..., a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

We use the term 'weight' to denote any of the weights defined in this section, whenever it is clear from the context.

A total labeling f is called a *super* (a, d)*-edge-antimagic total* if $f(V) = \{1, 2, \ldots, p\}$. If $d = 0$ then the labeling is called similarly a *super edge-magic* *total* labeling.

For brevity's sake, we often refer to an edge-antimagic vertex labeling as an EAV labeling, super edge-antimagic total labeling as a SEAT labeling, and super edge-magic total labeling as a SEMT labeling, see [114] for details.

Furthermore, a graph G is called (a, d)-*vertex-antimagic total* or *super* (a, d)-*vertex-antimagic total* if there exists an (a, d)-vertex-antimagic total labeling or super (a, d)-vertex-antimagic total labeling of G. A graph G is called (a, d)-*edge-antimagic total* or *super* (a, d)-*edge-antimagic total* if there exists an (a, d)-edge-antimagic total labeling or super (a, d)-edge-antimagic total labeling of graph G.

In this part of the thesis we investigate the super edge-antimagicness of disconnected graphs. We are studying the following problem: if a graph G is super (a, d)-edge-antimagic total, is the disjoint union of multiple copies of the graph G super (a, d)-edge-antimagic total as well?

2.3 Known results on super edge-antimagic total graphs

The study of super (a, d)-EAT labelings is relatively new. As mentioned above, these notions were introduced by Simanjuntak, Bertault and Miller in 2000. Since 2000, there have been many related publications. In [50], Figueroa-Centeno, Ichishima and Muntaner-Batle gave a necessary and sufficient condition for a graph to be super $(a, 0)$-EAT. Bača, Lin, Miller and Simanjuntak [9] also give a necessary condition for a graph to be super (a, d)-EAT: if (p, q) graph G has an (a, d)-EAV labeling then G has a super $(a + p + 1, d + 1)$-EAT labeling and a super $(a+p+q, d-1)$-EAT labeling. By using the adjacency matrices of EAV graphs, Sugeng and Miller [115] studied the relationship between

EAV labeling and super (a, d)-EAT labeling.

Many SEAT graphs have been found. Bača, Baskoro, Simanjuntak and Sugeng [5] proved the following: cycle C_n is SEAT if and only if either $d \in \{0, 2\}$ and $n \geq 3$ odd or $d = 1$ and $n \geq 3$; generalised Petersen graph $P(n, m)$ is also SEAT for certain values of m, n, and they conjectured that $P(n, m)$ is SEAT if $n \geq 9$ odd and $d \in \{0, 2\}$. In [7], Bača, Lin, Miller and Youssef proved that several families of graphs admit SEAT: friendship graph F_n is SEAT if $n \in \{1, 3, 4, 5, 7\}$ and $d \in \{0, 2\}$, and if $n \geq 1$ and $d = 1$; fan $\mathbb{F}_n$ is SEAT if $2 \leq n \leq 6$ and $d \in \{0, 1, 2\}$; wheel W_n is SEAT if and only if $d = 1$ and $n \not\equiv 1 \pmod 4$; K_n is SEAT if and only if either $d = 0$ and $n = 3$, or $d = 1$ and $n \geq 3$, or $d = 2$ and $n = 3$; and $K_{n,n}$ is SEAT if and only if $d = 1$ and $n \geq 2$. MacDougall and Wallis [81] investigated the existence of super $(a, 0)$-edge antimagic total labeling of graphs C_n^t derived from cycles by adding one chord.

Furthermore, Bača and Murugan [10] obtained the values of t for which there exists a SEAT labeling of C_n^t and they conjectured that C_n^t is a super $(a, 1)$-EAT if $n \equiv 0 \pmod 4$ for $t \equiv 0 \pmod 4$, and if $n \equiv 2 \pmod 4$ for t even. Bača, Lin and Muntaner-Batle [8] found a necessary and sufficient condition for path and path-like tree to be SEAT: path P_n is SEAT if and only if $d \in \{0, 1, 2, 3\}$ and $n \geq 2$; and path-like tree T is SEAT if and only if $d \in \{0, 1, 2, 3\}$;

Sugeng, Miller and Bača [116] proved that ladder L_n is SEAT for certain values of n, and they conjectured that L_n is SEAT if $n \geq 2$ even and $d \in \{0, 2\}$; triangular ladder $\mathbb{L}_n$ is SEAT if and only if $d \in \{0, 1, 2\}$ and $n \geq 2$; generalised prism $C_m \times P_n$ is SEAT if $m \geq 3$ odd and $n \geq 2$ for $d \in \{0, 1, 2\}$, and if $n \geq 4$ even, $n \geq 2$ for $d = 1$, and they conjectured that $C_m \times P_n$ is SEAT if $m \geq 4$ even, $n \geq 3$ and $d \in \{0, 2\}$; and generalised antiprism A_m^n is SEAT if and only if $d = 1$ and $m \geq 3, n \geq 2$. Furthermore, in [117], Sugeng, Miller, Slamin and Bača proved that star S_n is SEAT if and only if either $d \in \{0, 1, 2\}$ and

$n \geq 1$, or $d = 3$ and $1 \leq n \leq 2$; and caterpillar $S_{n_1,n_1,\ldots,n_r}$ is also SEAT for special value of parameter r. For the latest results in super edge-antimagic total labelings, see the dynamic survey by Gallian [59].

For disconnected graphs, there are only a few families of SEAT graphs known so far. This problem is considered quite difficult as the number of nodes which must be assigned a label is much larger than in each connected component graph separately, and there is no guarantee that if a graph G is super (a, d)-EAT then the disjoint union of multiple copies of the graph G is super (a', d')-EAT. Therefore, more research is required and even partial solutions would be significant contributions in this area.

Sudarsana, Ismaimuza, Baskoro and Assiyatun [113] proved that $P_n \cup P_{n+1}$ is SEAT if $n \geq 2$ and $d \in \{1, 3\}$, and if $n \geq 3$ odd and $d = 2$. Bača and Barrientos [3] proved that mK_n is SEAT if and only if either $d \in \{0, 2\}$ and $n \in \{2, 3\}, m \geq 3$ odd, or $d = 1$ and $m, n \geq 2$, or $d \in \{3, 5\}$ and $n = 2, m \geq 2$, or $d = 4$ and $n = 2, m \geq 3$ odd. In [6], Bača and Brankovic proved that $mK_{n,n}$ is SEAT for (i) $d \in \{3, 5\}$ if and only if $n = 1$ and $m \geq 2$, (ii) $d = 4$ if and only if $n = 1$ and $m \geq 3$ odd, (iii) $d = 1$ and for every $n \geq 1$ and $m \geq 2$, (iv) $d = 2$ if $n = 1$ and $m \geq 3$ odd.

We summarise the known results in super (a, d)-*edge-antimagic total* labeling for connected and disconnected graphs in Tables 2.1 and 2.2 for all feasible values of d.

Table 2.1: Summary of super edge (a, d)-antimagic total labelings of connected graphs.

Graph	d	Notes
C_n	$d \leq 2$	iff either (i) $d \in \{0, 2\}$ and $n \geq 3$ odd, or (ii) $d = 1$ and $n \geq 3$ [5]
$P(n, m)$	$d \leq 2$	(i) $d \in \{0, 2\}$ and $n \geq 3$ odd, $m \in \{1, 2, \frac{n-1}{2}\}$ [5]

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Table 2.1 – Continued.

Graph	d	Notes
		(ii) $d = 1$ and $n \geq 3$, $1 \leq m < \frac{n}{2}$ [98]
		Conjecture:
		• $d \in \{0, 2\}$, $n \geq 9$ odd, $3 \leq m \leq \frac{n-3}{2}$ [5]
F_n	$d \leq 2$	(i) $d \in \{0, 2\}$ and $n \in \{1, 3, 4, 5, 7\}$
		(ii) $d = 1$ and $n \geq 1$ [7]
		Open problem:
		• $d \in \{0, 2\}$ for $n > 7$
$\mathbb{F}_n$	$d \leq 2$	for $2 \leq n \leq 6$ [7]
W_n	$d = 1$	for $n \not\equiv 1 \pmod 4$ [7]
K_n	$d \leq 2$	iff either
		(i) $d = 0$ and $n = 3$, or
		(ii) $d = 1$ and $n \geq 3$, or
		(iii) $d = 2$ and $n = 3$ [7]
$K_{n,n}$	$d = 1$	for $n \geq 2$ [7]
C_n^t	$d \leq 2$	(i) if $d \in \{0, 2\}$
		(a) for $n \geq 5$ odd and for all possible values t
		(b) for $n \equiv 0 \pmod 4$ and for all $t \equiv 2 \pmod 4$
		(c) for $n = 10$ and $n \equiv 2 \pmod 4$, $n \geq 18$,
		and for all $t \equiv 3 \pmod 4$ and $t = 2, 6$
		(ii) if $d = 1$ [81][10]
		(a) for $n \geq 5$ odd and for all possible values t
		(b) for $n \geq 6$ even and for all $t \geq 3$ odd
		(c) for $n \equiv 0 \pmod 4$ and for $t \equiv 2 \pmod 4$ [10]
P_n	$d \leq 3$	for $n \geq 2$ [8]
Path-like tree	$d \leq 3$	for $n \geq 4$ [8]
L_n	$d \leq 2$	(i) $d \in \{0, 1, 2\}$ $n \geq 1$ odd
		(ii) $d = 1$ $n \geq 1$ even
		Conjecture:
		• $d \in \{0, 2\}$ $n \geq 1$ even [114], [116]
$\mathbb{L}_n$	$d \leq 2$	for $n \geq 2$ [114], [116]
$C_m \times P_n$	$d \leq 2$	(i) $d \in \{0, 1, 2\}$ and $m \geq 3$ odd, $n \geq 2$
		(ii) $d = 1$ and $m \geq 4$ even, $n \geq 2$

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Table 2.1 – Continued.

Graph	d	Notes
		Conjecture:
		• $d \in \{0,2\}$ and $m \geq 4$ even and $n \geq 3$ [114], [116]
A_m^n	$d = 1$	for $m \geq 3$ and $n \geq 2$ [114], [116]
S_n	$d \leq 3$	iff either
		(i) $d \in \{0,1,2\}$ and $n \geq 1$, or
		(ii) $d = 3$ and $1 \leq n \leq 2$ [114], [117]
$S_{n_1,n_1,\ldots,n_r}$	$d \leq 3$	(i) $d \in \{0,1,2\}$
		(ii) for $d = 3$ if r is even and $N_1 = N_2$ or $\lvert N_1 - N_2 \rvert = 1$
		(iii) for $d = 3$ if r is odd and $N_1 = N_2$ or $N_1 = N_2 + 1$
		where $N_1 = \sum_{i=1}^{\lceil \frac{r}{2} \rceil} n_{2i-1}$ and $N_2 = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} n_{2i}$ [114], [117]
		Open problem:
		• $d = 3$ for odd r and $N_2 = N_1 + 1$

Table 2.2: Summary of super (a, d)-edge-antimagic total labelings of disconnected graphs.

Graph	d	Notes
$P_n \cup P_{n+1}$	$1 \leq d \leq 3$	(i) $d \in \{1,3\}$ and $n \geq 2$
		(ii) $d = 2$ and $n \geq 3$ is odd [113]
		Open problem:
		• $d = 2$ for even n
$nP_2 \cup P_n$	$1 \leq d \leq 3$	$d \in \{1,2\}$ and $n \geq 2$ [113]
		Open problem:
		• $d = 3$ for $n \geq 2$
$nP_2 \cup P_{n+2}$	$1 \leq d \leq 4$	$d \in \{1,2\}$ and $n \geq 1$ [113]
		Open problem:
		• $d \in \{3,4\}$ for $n \geq 1$
mK_n	$d \leq 5$	iff either
		(i) $d \in \{0,2\}$ and $n \in \{2,3\}, m \geq 3$ odd, or
		(ii) $d = 1$ and $m, n \geq 2$, or
		(iii) $d \in \{3,5\}$ and $n = 2, m \geq 2$, or
		(iv) $d = 4$ and $n = 2, m \geq 3$ odd [3]

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Table 2.2 – Continued.

Graph	d	Notes
$mK_{n,n}$	$d \leq 5$	(i) if $d = 1$ for all m and n (ii) if $d \in \{0, 2\}$ for $n = 1$ and $m \geq 3$ odd (iii) iff $d \in \{3, 5\}$ for $n = 1$ and all $m \geq 2$ (iv) iff $d = 4$ for $n = 1$ and all $m \geq 3$ odd [3] Open problem: • if $d \in \{0, 2\}$ for $n = 3$ and $m \geq 3$ odd

Chapter 3

SEATL of Disconnected Graphs

In this chapter we present new results in super edge antimagic total labeling for some particular families of disconnected graphs. As mentioned in the previous chapter, our main problem is the following: if a graph G is super (a, d)-edge-antimagic total, is the disjoint union of multiple copies of the graph G super (a, d)-edge-antimagic total as well? We will answer this question for the case when the graph G is either a cycle or a path. We start this chapter by providing a necessary condition for a graph to be super (a, d)-edge-antimagic total which will provide a least upper bound for the feasible value of d.

Lemma 3.0.1 *[116] If a (p, q)-graph is super (a, d)-edge-antimagic total then $d \leq \frac{2p+q-5}{q-1}$.*

Proof. Assume that a (p, q)-graph has a super (a, d)-edge-antimagic total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$. The minimum possible edge weight in the labeling f is at least $1+2+p+1 = p+4$. Thus, $a \geq p+4$. On the other hand, the maximum possible edge weight is at most $(p-1)+p+(p+q) = 3p+q-1$. Hence $a+(q-1)d \leq 3p+q-1$. From the last inequality, we obtain the desired upper bound for the difference d. □

The following lemma, proved by Figueroa-Centeno *et al.* in [50], gives a necessary and sufficient condition for a graph to be super $(a, 0)$-edge-antimagic total or super edge-magic total labeling.

Lemma 3.0.2 [50] *A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{f(u)+f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p+q+s$, where $s = min(S)$ and $S = \{a-(p+1), a-(p+2), \ldots, a-(p+q)\}$.*

In our terminology, the previous lemma states that a (p, q)-graph G is super $(a, 0)$-edge-antimagic total if and only if there exists an $(a-p-q, 1)$-edge-antimagic vertex labeling.

Next, we restate the following lemma which is mentioned in [117]. This lemma is very useful especially for finding a super $(a, 1)$-edge-antimagic total labeling.

Lemma 3.0.3 [117] *Let $\mathfrak{A}$ be a sequence $\mathfrak{A} = \{c, c+1, c+2, \ldots c+k\}$, k even. Then there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $\mathfrak{A}+\Pi(\mathfrak{A}) = \{2c+\frac{k}{2}, 2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, \ldots, 2c+\frac{3k}{2}-1, 2c+\frac{3k}{2}\}$.*

Proof. Let $\mathfrak{A}$ be a sequence $\mathfrak{A} = \{a_i|\ a_i = c+(i-1),\ 1 \le i \le k+1\}$ and k be even. Define a permutation $\Pi(\mathfrak{A}) = \{b_i|\ 1 \le i \le k+1\}$ of the elements of $\mathfrak{A}$ as follows:

$$b_i = \begin{cases} c+\frac{k}{2}+\frac{1-i}{2} & \text{if } i \text{ is odd}, 1 \le i \le k+1 \\ c+k+\frac{2-i}{2} & \text{if } i \text{ is even}, 2 \le i \le k. \end{cases}$$

By direct computation, we obtain that

$\mathfrak{A}+\Pi(\mathfrak{A}) = \{a_i+b_i|\ 1 \le i \le k+1\} =$

$\{2c+\frac{k}{2}+\frac{i-1}{2}|\ i \text{ odd}, 1 \le i \le k+1\} \cup \{2c+k+\frac{i}{2}|\ i \text{ even}, 2 \le i \le k\} =$

$\{2c+\frac{k}{2}, 2c+\frac{k}{2}+1, \ldots, 2c+\frac{3k}{2}-1, 2c+\frac{3k}{2}\}$,

and we arrive at the desired result. □

3.1 Cycles

In [5], it is proved that the cycle C_n has super (a,d)-edge-antimagic total labeling if and only if either (i) $d \in \{0,2\}$ and n is odd, $n \geq 3$; or (ii) $d = 1$ and $n \geq 3$. Now, we will study super edge-antimagicness of a disjoint union of m copies of C_n, denoted by mC_n. mC_n is the disconnected graph with vertex set $V(mC_n) = \{x_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and edge set $E(mC_n) = \{x_i^j x_{i+1}^j : 1 \leq i \leq n-1,\ 1 \leq j \leq m\} \cup \{x_n^j x_1^j : 1 \leq j \leq m\}$.

If the disjoint union of m copies of C_n is super (a,d)-edge-antimagic total then, for $p = q = mn$, it follows from Lemma 3.0.1 that $d \leq 3 - \frac{2}{mn-1}$. If $m \geq 2$ and $n \geq 3$ then $\frac{2}{mn-1} > 0$ and thus $d < 3$.

◇ **Theorem 3.1.1** *The graph mC_n has an $(\frac{mn+3}{2},1)$-edge-antimagic vertex labeling if and only if m and n are odd, $m,n \geq 3$.*

Proof. Assume that mC_n has an $(a,1)$-edge-antimagic vertex labeling $\alpha : V(mC_n) \to \{1,2,\ldots,mn\}$ and $W = \{w(uv) : uv \in E(mC_n)\} = \{a, a+1, a+2, \ldots, a+mn-1\}$ is the set of edge-weights. The sum of the edge-weights in the set W is

$$\sum_{uv \in E(mC_n)} w(uv) = mna + \frac{mn(mn-1)}{2}. \tag{3.1}$$

In the computation of the edge-weights of mC_n, the label of every vertex is used twice. The sum of all vertex labels used to calculate the edge-weights is equal to

$$2 \sum_{u \in V(mC_n)} \alpha(u) = mn(mn+1). \tag{3.2}$$

Since (3.1) and (3.2) gives the following equation

$$\sum_{uv \in E(mC_n)} w(uv) = 2 \sum_{u \in V(mC_n)} \alpha(u), \tag{3.3}$$

it immediately follows that

$$a = \frac{mn+3}{2}.$$

The minimum edge weight a is an integer if and only if m and n are odd.

Now, define the vertex labeling $\alpha_1 : V(mC_n) \to \{1, 2, \dots, mn\}$ in the following way:

$$\alpha_1(x_i^j) = \begin{cases} \frac{j+1}{2} + \frac{i-1}{2}m & \text{if } i \text{ is odd, } 1 \le i \le n-2, \text{and } j \text{ is odd} \\ \frac{mi+1+j}{2} & \text{if } i \text{ is odd, } 1 \le i \le n-2, \text{and } j \text{ is even} \\ \frac{n+i+1}{2}m - j + 1 & \text{if } i \text{ is even and } 1 \le j \le m \\ \frac{mn+j}{2} & \text{if } i = n \text{ and } j \text{ is odd} \\ \frac{m(n-1)+j}{2} & \text{if } i = n \text{ and } j \text{ is even.} \end{cases}$$

We can see that the vertex labeling α_1 is a bijective function. The edge-weights of mC_n, under the labeling α_1, constitute the sets

$W^1_{\alpha_1} = \{w^1_{\alpha_1}(x_i^j x_{i+1}^j) = \frac{m(n+2i+1)+3-j}{2} : \text{ if } 1 \le i \le n-2, \text{ and } j \text{ is odd}\}$,
$W^2_{\alpha_1} = \{w^2_{\alpha_1}(x_i^j x_{i+1}^j) = \frac{m(n+2i+2)+3-j}{2} : \text{ if } 1 \le i \le n-2, \text{ and } j \text{ is even}\}$,
$W^3_{\alpha_1} = \{w^3_{\alpha_1}(x_{n-1}^j x_n^j) = \frac{3mn+2-j}{2} : \text{ if } j \text{ is odd}\}$,
$W^4_{\alpha_1} = \{w^4_{\alpha_1}(x_{n-1}^j x_n^j) = \frac{m(3n-1)+2-j}{2} : \text{ if } j \text{ is even}\}$,
$W^5_{\alpha_1} = \{w^5_{\alpha_1}(x_n^j x_1^j) = \frac{mn+1}{2} + j : \text{ if } 1 \le j \le m\}$.

Hence, the set $\bigcup_{r=1}^5 W^r_{\alpha_1} = \{\frac{mn+3}{2}, \frac{mn+5}{2}, \dots, \frac{3mn+1}{2}\}$ consists of consecutive integers. Thus α_1 is a $(\frac{mn+3}{2}, 1)$-edge-antimagic vertex labeling. □

Let $\alpha : V(mC_n) \cup E(mC_n) \to \{1, 2, \dots, 2mn\}$ be a super (a, d)-edge-antimagic total labeling of mC_n. The sum of all vertex and edge labels used to calculate the edge-weights is equal to the sum of the edge-weights:

$$2 \sum_{u \in V(mC_n)} \alpha(u) + \sum_{uv \in E(mC_n)} \alpha(uv) = \sum_{uv \in E(mC_n)} w(uv)$$

which is equivalent to the equation

$$5mn + 3 = 2a + (mn-1)d. \tag{3.4}$$

If $d = 0$ then, from (3.4), it follows that $a = \frac{5mn+3}{2}$. The value a is an integer if and only if m and n are odd.

In the previous theorem we proved that the vertex labeling α_1 is a $(\frac{mn+3}{2}, 1)$-edge-antimagic vertex labeling. With respect to Lemma 3.0.2, the labeling α_1 extends to a super $(a, 0)$-edge-antimagic total labeling, where, for $p = q = mn$, the value $a = \frac{5mn+3}{2}$. Thus the following theorem holds.

◇ **Theorem 3.1.2** *The graph mC_n has a super $(\frac{5mn+3}{2}, 0)$-edge-antimagic total labeling if and only if m and n are odd, $m, n \geq 3$.*

◇ **Theorem 3.1.3** *The graph mC_n has a super $(\frac{3mn+5}{2}, 2)$-edge-antimagic total labeling if and only if m and n are odd, $m, n \geq 3$.*

Proof. Suppose that mC_n has a super $(a, 2)$-edge-antimagic total labeling $\alpha : V(mC_n) \cup E(mC_n) \to \{1, 2, \ldots, 2mn\}$ and $W = \{w(uv) : uv \in E(mC_n)\} = \{a, a+2, a+4, \ldots, a+(mn-1)2\}$ is the set of the edge-weights. For $d = 2$, Equation (3.4) gives $a = \frac{3mn+5}{2}$. Since a is an integer, it follows that m and n must be odd.

We construct a total labeling α_2 as follows:

$$\begin{aligned}
\alpha_2(x_i^j) &= \alpha_1(x_i^j), \text{ for every } i \text{ and } j \text{ with } 1 \leq i \leq n,\ 1 \leq j \leq m \\
\alpha_2(x_n^j x_1^j) &= mn + j, \text{ if } 1 \leq j \leq m
\end{aligned}$$

$$\alpha_2(x_i^j x_{i+1}^j) = \begin{cases} \frac{m(2n+2i+1)+2-j}{2} & \text{if } 1 \leq i \leq n-2, \text{and } j \text{ is odd} \\ (n+i+1)m + \frac{2-j}{2} & \text{if } 1 \leq i \leq n-2, \text{and } j \text{ is even} \end{cases}$$

$$\alpha_2(x_{n-1}^j x_n^j) = \begin{cases} 2mn + \frac{1-j}{2} & \text{if } j \text{ is odd} \\ m(2n-1) + \frac{m-j+1}{2} & \text{if } j \text{ is even.} \end{cases}$$

The total labeling α_2 is a bijective function from mC_n onto the set $\{1, 2, \ldots, 2mn\}$. The edge-weights of mC_n, under the labeling α_2, constitute the sets

$$
\begin{aligned}
W^1_{\alpha_2} &= \{w^1_{\alpha_2}(x^j_i x^j_{i+1}) = w^1_{\alpha_1}(x^j_i x^j_{i+1}) + \alpha_2(x^j_i x^j_{i+1}) : \text{ if } 1 \le i \le n-2, \text{ and } j \text{ is odd}\} \\
&= \{\tfrac{3mn+4im+2m-2j+5}{2} : \text{ if } 1 \le i \le n-2, \text{ and } j \text{ is odd}\}, \\
W^2_{\alpha_2} &= \{w^2_{\alpha_2}(x^j_i x^j_{i+1}) = w^2_{\alpha_1}(x^j_i x^j_{i+1}) + \alpha_2(x^j_i x^j_{i+1}) : \text{ if } 1 \le i \le n-2, \text{ and } j \text{ is even}\} \\
&= \{\tfrac{3mn+4im+4m-2j+5}{2} : \text{ if } 1 \le i \le n-2, \text{ and } j \text{ is even}\}, \\
W^3_{\alpha_2} &= \{w^3_{\alpha_2}(x^j_{n-1} x^j_n) = w^3_{\alpha_1}(x^j_{n-1} x^j_n) + \alpha_2(x^j_{n-1} x^j_n) : \text{ if } j \text{ is odd}\} \\
&= \{\tfrac{7mn-2j+3}{2} : \text{ if } j \text{ is odd}\}, \\
W^4_{\alpha_2} &= \{w^4_{\alpha_2}(x^j_{n-1} x^j_n) = w^4_{\alpha_1}(x^j_{n-1} x^j_n) + \alpha_2(x^j_{n-1} x^j_n) : \text{ if } j \text{ is even}\} \\
&= \{\tfrac{7mn-2m-2j+3}{2} : \text{ if } j \text{ is even}\}, \\
W^5_{\alpha_2} &= \{w^5_{\alpha_2}(x^j_n x^j_1) = w^5_{\alpha_1}(x^j_n x^j_1) + \alpha_2(x^j_n x^j_1) : \text{ if } 1 \le j \le m\} \\
&= \{\tfrac{3mn+4j+1}{2} : \text{ if } 1 \le j \le m\}.
\end{aligned}
$$

It is not difficult to see that the set $\bigcup_{r=1}^{5} W^r_{\alpha_2} = \{\frac{3mn+5}{2}, \frac{3mn+9}{2}, \ldots, \frac{7mn+1}{2}\}$ contains an arithmetic sequence with the first term $\frac{3mn+5}{2}$ and common difference 2. Thus α_2 is a super $(\frac{3mn+5}{2}, 2)$-edge-antimagic total labeling. This concludes the proof. □

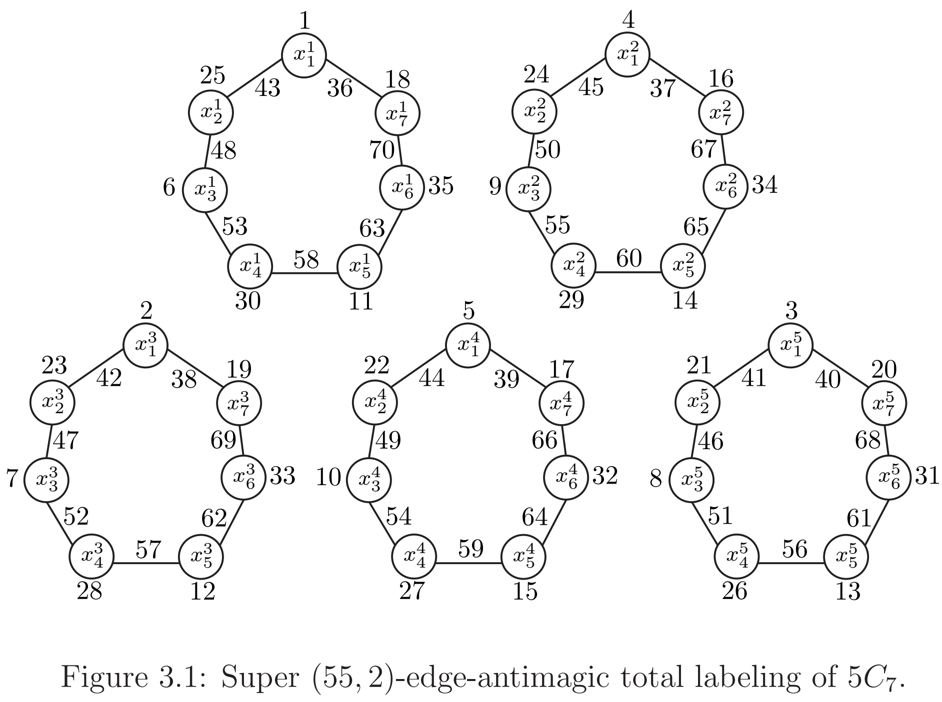


Figure 3.1: Super $(55, 2)$-edge-antimagic total labeling of $5C_7$.

◇ **Theorem 3.1.4** *The graph mC_n has a super $(2mn+2,1)$-edge-antimagic total labeling for every $m \geq 2$ and $n \geq 3$.*

Proof. Assume that mC_n has a super $(a,1)$-edge-antimagic total labeling $\alpha : V(mC_n) \cup E(mC_n) \to \{1,2,\ldots,2mn\}$ and $W = \{w(uv) : uv \in E(mC_n)\} = \{a, a+1, a+2, \ldots, a+mn-1\}$ is the set of edge-weights. Putting $d=1$, Equation (3.4) gives $a = 2mn+2$ and this is an integer for all m and n, $m \geq 2$ and $n \geq 3$.

Construct the bijection $\alpha_3 : V(mC_n) \cup E(mC_n) \to \{1,2,\ldots,2mn\}$ as follows:

$$\begin{aligned}
\alpha_3(x_i^j) &= j + (i-1)m, \text{ if } 1 \leq i \leq n \text{ and } 1 \leq j \leq m \\
\alpha_3(x_i^j x_{i+1}^j) &= (2n-i+1)m + 1 - j, \text{ if } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m \\
\alpha_3(x_n^j x_1^j) &= (n+1)m + 1 - j, \text{ if } 1 \leq j \leq m.
\end{aligned}$$

The edge-weights of mC_n, under the labeling α_3, constitute the sets

$W_{\alpha_3}^1 = \{w_{\alpha_3}^1(x_i^j x_{i+1}^j) = 2mn + im + 1 + j : \text{ if } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m\}$,
$W_{\alpha_3}^2 = \{w_{\alpha_3}^2(x_n^j x_1^j) = 2mn + j + 1 : \text{ if } 1 \leq j \leq m\}$.

Hence, the set $\bigcup_{r=1}^{2} W_{\alpha_3}^r = \{2mn+2, 2mn+3, \ldots, 3mn+1\}$ consists of consecutive integers. Thus α_3 is a super $(2mn+2,1)$-edge-antimagic total labeling.
□

Independently, Ngurah, Baskoro and Simanjuntak in [99] also gave an alternative proof of Theorem 3.1.4. Finally, we summarise the results presented in this subsection in the following theorem.

◇ **Theorem 3.1.5** *The graph mC_n has a super (a,d)-edge-antimagic total labeling if and only if either*
(i) $d \in \{0,2\}$ and m, n are odd, $m,n \geq 3$; or
(ii) $d = 1$, for every $m \geq 2$ and $n \geq 3$.

3.2 Paths

In [8], it is shown that the path P_n, $n \geq 2$, has a super (a, d)-edge-antimagic total labeling if and only if $d \in \{0, 1, 2, 3\}$. Let us now consider a disjoint union of m copies of P_n and denote it by mP_n. The graph mP_n is disconnected with vertex set $V(mP_n) = \{x_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and $E(mP_n) = \{x_i^j x_{i+1}^j : 1 \leq i \leq n-1,\ 1 \leq j \leq m\}$.

From Lemma 3.0.1, it follows that if mP_n is super (a, d)-edge-antimagic total, $p = mn$ and $q = (n-1)m$, then

$$d \leq 3 + \frac{2m-2}{mn-m-1}.$$

If $n = 2$ and $m \geq 2$ then $\frac{2m-2}{mn-m-1} = 2$ and thus $d \leq 5$. If $n \geq 3$ and $m \geq 2$ then $0 < \frac{2m-2}{mn-m-1} < 1$ and thus $d < 4$.

◇ **Theorem 3.2.1** *If m is odd, $m \geq 3$, and $n \geq 2$, then the graph mP_n has an $(a, 1)$-edge-antimagic vertex labeling.*

Proof.
Case 1. n odd
We construct a vertex labeling β_1 of mP_n, $m \geq 3$ and $n \geq 3$, in the following way:

$$\beta_1(x_i^j) = \begin{cases} \frac{mn+j}{2} & \text{if } i = 1 \text{ and } j \text{ is odd} \\ \frac{m(n-1)+j}{2} & \text{if } i = 1 \text{ and } j \text{ is even} \\ \frac{m(i-2)+1+j}{2} & \text{if } i \text{ is even and } j \text{ is odd} \\ \frac{m(i-1)+1+j}{2} & \text{if } i \text{ is even and } j \text{ is even} \\ \frac{m(n+i)}{2} + 1 - j & \text{if } i \text{ is odd, } 3 \leq i \leq n, \text{ and } 1 \leq j \leq m. \end{cases}$$

We can see that the vertex labeling β_1 is a bijective function from $V(mP_n)$ onto the set $\{1, 2, \ldots, mn\}$. The edge-weights of mP_n under the labeling β_1 constitute the sets

$W^1_{\beta_1} = \{w^1_{\beta_1}(x^j_1 x^j_2) = \frac{mn+1}{2} + j : \text{ if } 1 \leq j \leq m\},$
$W^2_{\beta_1} = \{w^2_{\beta_1}(x^j_i x^j_{i+1}) = \frac{m(n+2i-1)+3-j}{2} : \text{ if } 2 \leq i \leq n-1, \text{ and } j \text{ is odd}\},$
$W^3_{\beta_1} = \{w^3_{\beta_1}(x^j_i x^j_{i+1}) = \frac{m(n+2i)+3-j}{2} : \text{ if } 2 \leq i \leq n-1, \text{ and } j \text{ is even}\}.$

Hence, the set $\bigcup_{r=1}^{3} W^r_{\beta_1} = \{\frac{mn+3}{2}, \frac{mn+5}{2}, \ldots, \frac{3mn-2m+1}{2}\}$ consists of consecutive integers. Thus β_1 is a $(\frac{mn+3}{2}, 1)$-edge-antimagic vertex labeling.

Case 2. n even
For $m \geq 3$ and $n \geq 2$, define the bijection $\beta_2 : V(mP_n) \rightarrow \{1, 2, \ldots, mn\}$ as follows:

$$\beta_2(x^j_i) = \begin{cases} \frac{m(n+1)+j}{2} & \text{if } i = 1 \text{ and } j \text{ is odd} \\ \frac{mn+j}{2} & \text{if } i = 1 \text{ and } j \text{ is even} \\ \frac{m(i-2)+1+j}{2} & \text{if } i \text{ is even and } j \text{ is odd} \\ \frac{m(i-1)+1+j}{2} & \text{if } i \text{ is even and } j \text{ is even} \\ \frac{m(n+i+1)}{2} + 1 - j & \text{if } i \text{ is odd, } 3 \leq i \leq n-1, \text{ and } 1 \leq j \leq m. \end{cases}$$

Then for the edge-weights of mP_n we have:

$W^1_{\beta_2} = \{w^1_{\beta_2}(x^j_1 x^j_2) = \frac{m(n+1)+1}{2} + j : \text{ if } 1 \leq j \leq m\},$
$W^2_{\beta_2} = \{w^2_{\beta_2}(x^j_i x^j_{i+1}) = \frac{m(n+2i)+3-j}{2} : \text{ if } 2 \leq i \leq n-1, \text{ and } j \text{ is odd}\},$
$W^3_{\beta_2} = \{w^3_{\beta_2}(x^j_i x^j_{i+1}) = \frac{m(n+2i+1)+3-j}{2} : \text{ if } 2 \leq i \leq n-1, \text{ and } j \text{ is even}\}$

and $\bigcup_{r=1}^{3} W^r_{\beta_2} = \{\frac{m(n+1)+3}{2}, \frac{m(n+1)+5}{2}, \ldots, \frac{m(3n-1)+1}{2}\}$ consists of consecutive integers. This implies that β_2 is a $(\frac{m(n+1)+3}{2}, 1)$-edge-antimagic vertex labeling.

□

We utilize the vertex labelings β_1 and β_2 from the proof of Theorem 3.2.1 to prove the following theorem.

◇ **Theorem 3.2.2** *If m is odd, $m \geq 3$, and $n \geq 2$, then the graph mP_n has a super $(a, 0)$-edge-antimagic total labeling and a super $(a', 2)$-edge-antimagic total labeling.*

Proof.

First, let us turn to $d = 0$.

For m and n odd, we consider the vertex labeling β_1 which is a $(\frac{mn+3}{2}, 1)$-edge-antimagic vertex labeling. According to Lemma 3.0.2, by completing the edge labels $p+1, p+2, \ldots, p+q$, we are able to extend labeling β_1 to a super $(a, 0)$-edge-antimagic total labeling, where, for $p = mn$ and $q = nm - m$, the value $a = \frac{5mn-2m+3}{2}$.

For m odd and n even, the vertex labeling β_2 is a $(\frac{m(n+1)+3}{2}, 1)$-edge-antimagic vertex labeling. It follows from Lemma 3.0.2 that the labeling β_2 can be extended, by completing the edge labels $p+1, p+2, \ldots, p+q$, to a super $(a, 0)$-edge-antimagic total labeling, where, in the case $p = mn$ and $q = nm - m$, the value $a = \frac{m(5n-1)+3}{2}$. Thus for m odd, $m \geq 3$ and $n \geq 2$, mP_n has a super $(a, 0)$-edge-antimagic total labeling.

For $d = 2$, we distinguish two cases.

Case 1. n odd

Label the vertices and edges of mP_n in the following way:

$$\beta_3(x_i^j) = \beta_1(x_i^j), \text{ for every } i \text{ and } j \text{ with } 1 \leq i \leq n,\ 1 \leq j \leq m$$

$$\beta_3(x_i^j x_{i+1}^j) = \begin{cases} mn + j & \text{if } i = 1 \text{ and } 1 \leq j \leq m \\ \frac{m(2n+2i-1)+2-j}{2} & \text{if } 2 \leq i \leq n-1 \text{ and } j \text{ is odd} \\ \frac{2-j}{2} + mn + im & \text{if } 2 \leq i \leq n-1 \text{ and } j \text{ is even.} \end{cases}$$

The total labeling β_3 is a bijective function from $V(mP_n) \cup E(mP_n)$ onto the set $\{1, 2, \ldots, 2mn - m\}$. For the edge-weights of mP_n, under the total labeling β_3, we have:

$$\begin{aligned} W^1_{\beta_3} &= \{w^1_{\beta_3}(x_1^j x_2^j) = w^1_{\beta_1}(x_1^j x_2^j) + \beta_3(x_1^j x_2^j) : \text{ if } 1 \leq j \leq m\} \\ &= \{\tfrac{3mn+1}{2} + 2j : \text{ if } 1 \leq j \leq m\}, \\ W^2_{\beta_3} &= \{w^2_{\beta_3}(x_i^j x_{i+1}^j) = w^2_{\beta_1}(x_i^j x_{i+1}^j) + \beta_3(x_i^j x_{i+1}^j) : \text{ if } 2 \leq i \leq n-1, \text{ and } j \text{ is odd}\} \\ &= \{\tfrac{m(3n+4i-2)+5}{2} - j : \text{ if } 2 \leq i \leq n-1, \text{ and } j \text{ is odd}\}, \end{aligned}$$

$$W^3_{\beta_3} = \{w^3_{\beta_3}(x^j_i x^j_{i+1}) = w^3_{\beta_1}(x^j_i x^j_{i+1}) + \beta_3(x^j_i x^j_{i+1}) : \text{ if } 2 \le i \le n-1, \text{ and } j \text{ is even}\}$$
$$= \{\tfrac{m(3n+4i)+5}{2} - j : \text{ if } 2 \le i \le n-1, \text{ and } j \text{ is even}\}.$$

Hence, $\bigcup_{r=1}^{3} W^r_{\beta_3} = \{\frac{3mn+5}{2}, \frac{3mn+9}{2}, \ldots, \frac{7mn-4m+1}{2}\}$ and this implies that β_3 is a super $(\frac{3mn+5}{2}, 2)$-edge-antimagic total labeling.

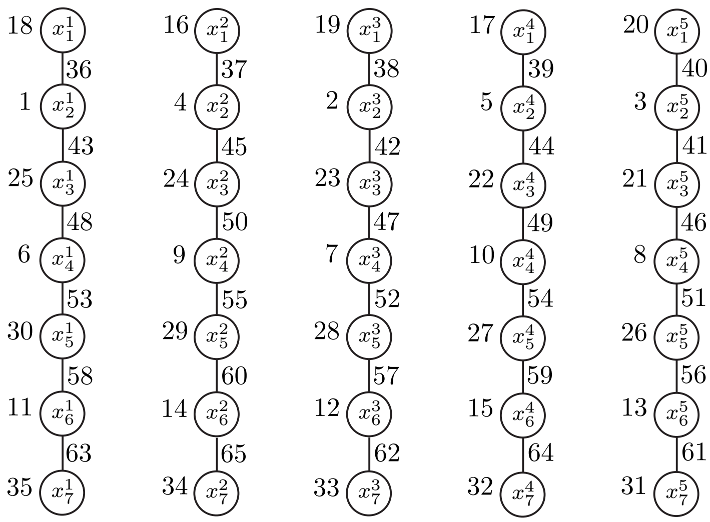


Figure 3.2: Super $(55, 2)$-edge-antimagic total labeling of $5P_7$.

Case 2. n even

Label the vertices and edges of mP_n as follows:

$$\beta_4(x^j_i) = \beta_2(x^j_i), \text{ for every } i \text{ and } j \text{ with } 1 \le i \le n,\ 1 \le j \le m$$
$$\beta_4(x^j_i x^j_{i+1}) = \beta_3(x^j_i x^j_{i+1}), \text{ for every } i \text{ and } j \text{ with } 1 \le i \le n-1,\ 1 \le j \le m.$$

Clearly, the total labeling $\beta_4 : V(mP_n) \cup E(mP_n) \to \{1, 2, \ldots, 2mn - m\}$ is a bijection. The edge-weights of mP_n under the labeling β_4 constitute the sets

$$W^1_{\beta_4} = \{w^1_{\beta_4}(x^j_1 x^j_2) = w^1_{\beta_2}(x^j_1 x^j_2) + \beta_4(x^j_1 x^j_2) : \text{ if } 1 \le j \le m\}$$
$$= \{\tfrac{m(3n+1)+1}{2} + 2j : \text{ if } 1 \le j \le m\},$$
$$W^2_{\beta_4} = \{w^2_{\beta_4}(x^j_i x^j_{i+1}) = w^2_{\beta_2}(x^j_i x^j_{i+1}) + \beta_4(x^j_i x^j_{i+1}) : \text{ if } 2 \le i \le n-1, \text{ and } j \text{ is odd}\}$$
$$= \{\tfrac{m(3n+4i-1)+5}{2} - j : \text{ if } 2 \le i \le n-1, \text{ and } j \text{ is odd}\},$$
$$W^3_{\beta_4} = \{w^3_{\beta_4}(x^j_i x^j_{i+1}) = w^3_{\beta_2}(x^j_i x^j_{i+1}) + \beta_4(x^j_i x^j_{i+1}) : \text{ if } 2 \le i \le n-1, \text{ and } j \text{ is even}\}$$

$= \{\frac{m(3n+4i+1)+5}{2} - j : \text{ if } 2 \leq i \leq n-1, \text{ and } j \text{ is even}\}.$

It can be seen that the total labeling β_4 is super $(\frac{m(3n+1)+5}{2}, 2)$-edge-antimagic total. □

◇ **Theorem 3.2.3** *The graph mP_n has a $(m+2,2)$-edge-antimagic vertex labeling, for every $m \geq 2$ and $n \geq 2$.*

Proof. Now, for $m \geq 2$ and $n \geq 2$, consider the following function $\beta_5 : V(mP_n) \to \{1, 2, \ldots, mn\}$, where if $1 \leq i \leq n$ and $1 \leq j \leq m$, then

$$\beta_5(x_i^j) = j + (i-1)m.$$

We conclude that β_5 is a bijective function and the edge-weights under this function constitute the set

$W_{\beta_5} = \{w_{\beta_5}(x_i^j x_{i+1}^j) = m(2i-1) + 2j : \text{ if } 1 \leq i \leq n-1, \text{ and } 1 \leq j \leq m\},$

which implies that β_5 is a $(m+2,2)$-edge-antimagic vertex labeling. □

◇ **Theorem 3.2.4** *The graph mP_n has a super $(2mn+2,1)$-edge-antimagic total labeling and a super $(mn+m+3,3)$-edge-antimagic total labeling, for every $m \geq 2$ and $n \geq 2$.*

Proof. Let $m \geq 2$ and $n \geq 2$. We distinguish two cases, according to whether $d = 1$ or $d = 3$.

Case 1. $d = 1$

Define $\beta_6 : V(mP_n) \cup E(mP_n) \to \{1, 2, \ldots, 2mn - m\}$ to be the bijective function such that

$$\begin{aligned} \beta_6(x_i^j) &= \beta_5(x_i^j), \text{ for every } i \text{ and } j \text{ with } 1 \leq i \leq n,\ 1 \leq j \leq m \\ \beta_6(x_i^j x_{i+1}^j) &= m(2n-i) + 1 - j, \text{ if } 1 \leq i \leq n-1, \text{ and } 1 \leq j \leq m. \end{aligned}$$

Thus $W_{\beta_6} = \{w_{\beta_6}(x_i^j x_{i+1}^j) = \beta_6(x_i^j) + \beta_6(x_{i+1}^j) + \beta_6(x_i^j x_{i+1}^j) = w_{\beta_5}(x_i^j x_{i+1}^j) + \beta_6(x_i^j x_{i+1}^j) :$ if $1 \leq i \leq n-1,$ and $1 \leq j \leq m\} = \{m(2n+i-1)+1+j :$ if $1 \leq i \leq n-1,$ and $1 \leq j \leq m\}$ is a set of $m(n-1)$ consecutive integers $2mn+2, 2mn+3, \ldots, 3mn-m+1$. It follows that the total labeling β_6 is a super $(2mn+2, 1)$-edge-antimagic total.

Case 2. $d = 3$

Consider the labeling $\beta_7 : V(mP_n) \cup E(mP_n) \to \{1, 2, \ldots, 2mn-m\}$ such that

$$\begin{aligned} \beta_7(x_i^j) &= \beta_5(x_i^j), \text{ for every } i \text{ and } j \text{ with } 1 \leq i \leq n,\ 1 \leq j \leq m \\ \beta_7(x_i^j x_{i+1}^j) &= m(n+i-1)+j, \text{ if } 1 \leq i \leq n-1, \text{ and } 1 \leq j \leq m. \end{aligned}$$

It suffices to observe that β_7 is a bijection and the set
$W_{\beta_7} = \{w_{\beta_7}(x_i^j x_{i+1}^j) = \beta_7(x_i^j) + \beta_7(x_{i+1}^j) + \beta_7(x_i^j x_{i+1}^j) = w_{\beta_5}(x_i^j x_{i+1}^j) + \beta_7(x_i^j x_{i+1}^j) :$ if $1 \leq i \leq n-1,$ and $1 \leq j \leq m\} = \{m(n+3i-2)+3j :$ if $1 \leq i \leq n-1,$ and $1 \leq j \leq m\}$ consists of an arithmetic sequence with the first term $a = m(n+1)+3$ and common difference $d = 3$. This completes the proof. □

Let mP_2, $m \geq 2$, be super (a, d)-edge-antimagic total with a super (a, d)-edge-antimagic total labeling $\beta : V(mP_2) \cup E(mP_2) \to \{1, 2, \ldots, 3m\}$. Thus $\{w(uv) = \beta(u) + \beta(uv) + \beta(v) : uv \in E(mP_2)\} = \{a, a+d, a+2d, \ldots, a+(m-1)d\}$ is the set of edge-weights.

In the computation of the edge-weights of mP_2, the label of each vertex and each edge is used once. The sum of all vertex and edge labels used to calculate the edge-weights is equal to the sum of edge-weights. Thus the equation

$$\sum_{u \in V(mP_2)} \beta(u) + \sum_{uv \in E(mP_2)} \beta(uv) = \sum_{uv \in E(mP_2)} w(uv)$$

is equivalent to the equation

$$9m + 3 = 2a + (m-1)d. \tag{3.5}$$

◇ **Theorem 3.2.5** *The graph mP_2, $m \geq 3$, has a super $\left(\frac{5m+7}{2}, 4\right)$-edge-antimagic total labeling if and only if m is odd.*

Proof. Let $\beta : V(mP_2) \cup E(mP_2) \to \{1, 2, \ldots, 3mn\}$ be a super $(a, 4)$-edge-antimagic total labeling. If $d = 4$ then from (3.5) it follows that

$$a = \frac{5m+7}{2}.$$

The minimum edge weight a is an integer if and only if m is odd. We define the required super $\left(\frac{5m+7}{2}, 4\right)$-edge-antimagic total labeling in the following way:

$$\beta_8(x_i^j) = \begin{cases} j & \text{if } i = 1 \text{ and } 1 \leq j \leq \frac{m+1}{2} \\ 2j - \frac{m+1}{2} & \text{if } i = 1 \text{ and } \frac{m+3}{2} \leq j \leq m \\ \frac{m+1}{2} + 2j - 1 & \text{if } i = 2 \text{ and } 1 \leq j \leq \frac{m+1}{2} \\ m + j & \text{if } i = 2 \text{ and } \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$\beta_8(x_1^j x_2^j) = 2m + j, \text{ if } 1 \leq j \leq m.$$ □

◇ **Theorem 3.2.6** *The graph mP_2 has a super $(2m + 4, 5)$-edge-antimagic total labeling, for every $m \geq 2$.*

Proof. Assume that mP_2 has a super $(a, 5)$-edge-antimagic total labeling. Then, for $d = 5$, Equation (3.5) gives

$$a = 2m + 4$$

and this is an integer for all $m \geq 2$. The required super $(2m + 4, 5)$-edge-antimagic total labeling can be defined as follows:

$$\beta_9(x_i^j) = 2j + i - 2, \text{ if } i = 1, 2 \text{ and } 1 \leq j \leq m$$

$$\beta_9(x_1^j x_2^j) = \beta_7(x_1^j x_2^j), \text{ for every } j \text{ with } 1 \leq j \leq m.$$ □

The graph mP_n has a super (a, d)-edge-antimagic total labeling for almost all feasible values of the parameters m, n and d. The only unsolved problem is to answer whether or not the graph mP_n has a super (a,d)-edge-antimagic total labeling for $d \in \{0,2\}$ and m even. Therefore, we propose the following open problem.

Open Problem 3.2.1 *For mP_n, $m \geq 2$ even, $n > 2$, determine if there is a super (a,d)-edge-antimagic total labeling with $d \in \{0,2\}$.*

3.3 Paths and cycles

In the previous sections, we studied super edge-antimagicness of a disjoint union of m copies of paths and m copies of cycles. Now we will consider super edge-antimagicness of a disjoint union of a combination of them, denoted by $mP_n \cup \mu C_n$. It is the disconnected graph with the vertex set $V(mP_n \cup \mu C_n) = \{x_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m+\mu\}$ and the edge set $E(mP_n \cup \mu C_n) = \{x_i^j x_{i+1}^j : 1 \leq i \leq n-1,\ 1 \leq j \leq m+\mu\} \cup \{x_n^j x_1^j : m+1 \leq j \leq m+\mu\}$. Thus $p = |V(mP_n \cup \mu C_n)| = (m+\mu)n$ and $q = |E(mP_n \cup \mu C_n)| = (m+\mu)n - m$.

If the disjoint union of $mP_n \cup \mu C_n$ is super (a,d)-edge-antimagic total then, from Lemma 3.0.1, it follows that $d \leq 3 + \frac{2m-2}{(m+\mu)n-m-1}$. If $m \geq 1, \mu \geq 1$ and $n \geq 3$ then $\frac{2m-2}{(m+\mu)n-m-1} < 1$ and thus $d < 4$.

◇ **Theorem 3.3.1** *If $(m+\mu)$ and n are odd, $m \geq 1, \mu \geq 1$ and $n \geq 3$, then the graph $mP_n \cup \mu C_n$ has an $\left(\frac{(m+\mu)n+2m+3}{2}, 1\right)$-edge-antimagic vertex labeling.*

Proof. Let $1 \leq j \leq m+\mu$ and define $\chi_1 : V(mP_n \cup \mu C_n) \to \{1, 2, \ldots, (m+$

$\mu)n\}$ to be the vertex labeling such that

$$\chi_1(x_i^j) = \begin{cases} \frac{j+1}{2} + \frac{i-1}{2}(m+\mu) & \text{if } i \text{ is odd, } 1 \le i \le n-2, \text{and } j \text{ is odd} \\ \frac{(m+\mu)i+1+j}{2} & \text{if } i \text{ is odd, } 1 \le i \le n-2, \text{and } j \text{ is even} \\ \frac{n+i+1}{2}(m+\mu) - j + 1 & \text{if } i \text{ is even, } 2 \le i \le n-1 \\ \frac{(m+\mu)n+j}{2} & \text{if } i = n \text{ and } j \text{ is odd} \\ \frac{(m+\mu)(n-1)+j}{2} & \text{if } i = n \text{ and } j \text{ is even.} \end{cases}$$

We can see that the vertex labeling χ_1 is a bijective function. The edge-weights of $mP_n \cup \mu C_n$, under the labeling χ_1, constitute the sets

$W^1_{\chi_1} = \{w^1_{\chi_1}(x_i^j x_{i+1}^j) = \chi_1(x_i^j) + \chi_1(x_{i+1}^j) = \frac{(m+\mu)(n+2i+1)}{2} + \frac{3-j}{2} :$ if $1 \le i \le n-2,$ and j is odd$, 1 \le j \le m+\mu\}$,

$W^2_{\chi_1} = \{w^2_{\chi_1}(x_i^j x_{i+1}^j) = \chi_1(x_i^j) + \chi_1(x_{i+1}^j) = \frac{(m+\mu)(n+2i+2)+3-j}{2} :$ if $1 \le i \le n-2,$ and j is even$, 2 \le j \le m+\mu-1\}$,

$W^3_{\chi_1} = \{w^3_{\chi_1}(x_{n-1}^j x_n^j) = \chi_1(x_{n-1}^j) + \chi_1(x_n^j) = \frac{3(m+\mu)n+2-j}{2} :$ if j is odd$, 1 \le j \le m+\mu\}$,

$W^4_{\chi_1} = \{w^4_{\chi_1}(x_{n-1}^j x_n^j) = \chi_1(x_{n-1}^j) + \chi_1(x_n^j) = \frac{(m+\mu)(3n-1)+2-j}{2} :$ if j is even$, 2 \le j \le m+\mu-1\}$,

$W^5_{\chi_1} = \{w^5_{\chi_1}(x_n^j x_1^j) = \chi_1(x_n^j) + \chi_1(x_1^j) = \frac{(m+\mu)n+1}{2} + j :$ if $m+1 \le j \le m+\mu\}$.

The set $\bigcup_{r=1}^5 W^r_{\chi_1} = \{\frac{(m+\mu)n+2m+3}{2}, \frac{(m+\mu)n+2m+5}{2}, \dots, \frac{3(m+\mu)n+1}{2}\}$ consists of consecutive integers, which implies that χ_1 is a $\left(\frac{(m+\mu)n+2m+3}{2}, 1\right)$-edge-antimagic vertex labeling. □

Bača, Lin, Miller and Simanjuntak (see [9], Theorem 5) have proved that if (p,q)-graph G has an (a,d)-edge-antimagic vertex labeling then G has a super $(a+p+q, d-1)$-edge-antimagic total labeling and a super $(a+p+1, d+1)$-edge-antimagic total labeling. With the Theorem 3.3.1 in hand, and using Theorem 5 from [9], we obtain the following result.

◇ **Theorem 3.3.2** *If $(m+\mu)$ and n are odd, $m, \mu \ge 1$ and $n \ge 3$, then the* *graph* $mP_n \cup \mu C_n$ *has a super* $\left(\frac{5(m+\mu)n+3}{2}, 0\right)$*-edge-antimagic total labeling and a super* $\left(\frac{3(m+\mu)n+2m+5}{2}, 2\right)$*-edge-antimagic total labeling.*

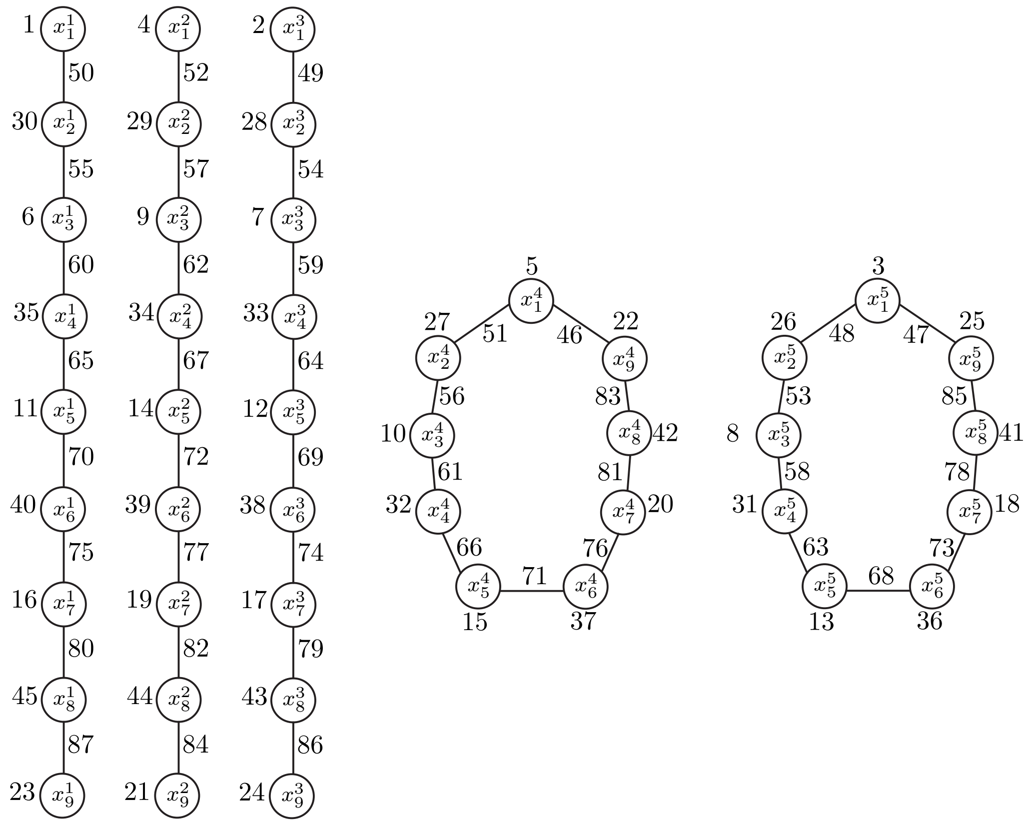


Figure 3.3: Super $(73, 2)$-edge-antimagic total labeling of $3P_9 \cup 2C_9$.

Figure 3.3 gives an example of super (a, d)-edge-antimagic total labeling of $3P_9 \cup 2C_9$ for $d = 2$.

The result that $mP_n \cup \mu C_n$ has a super $\left(\frac{5(m+\mu)n+3}{2}, 0\right)$-edge-antimagic total labeling when $(m+\mu)n$ is odd, is not new. It follows as a corollary from a theorem of Ivančo and Lučkaničová (see Theorem 1 in [71]).

Directly from Theorem 3.3.1, with respect to Lemma 3.0.3, it follows that the graph $mP_n \cup \mu C_n$ has a super $\left(\frac{4(m+\mu)n+m+4}{2}, 1\right)$-edge-antimagic total labeling for $(m+\mu)n$ odd and m even. The following theorem also covers this case.

◇ **Theorem 3.3.3** *For m even, $m \geq 2$, and for every $\mu \geq 1$ and $n \geq 3$, the graph $mP_n \cup \mu C_n$ has a super $(2n(m+\mu)+2, 1)$-edge-antimagic total labeling.*

Proof. Let m be an even integer. Consider a bijective function $\chi_2 : V(mP_n \cup \mu C_n) \cup E(mP_n \cup \mu C_n) \rightarrow \{1, 2, \ldots, 2(m+\mu)n - m\}$ defined by

$$\chi_2(x_i^j) = \begin{cases} (m+\mu)(i-1)+j & \text{if } 1 \leq i \leq n, \text{ and } 1 \leq j \leq \frac{m}{2} \\ (m+\mu)(i-1)+\mu+j & \text{if } 1 \leq i \leq n, \text{ and } \frac{m}{2}+1 \leq j \leq m \\ (m+\mu)(i-1)-\frac{m}{2}+j & \text{if } 1 \leq i \leq n, \text{ and } m+1 \leq j \leq m+\mu \end{cases}$$

$$\chi_2(x_i^j x_{i+1}^j) =$$

$$\begin{cases} (m+\mu)(2n-i+1)-m+1-j & \text{if } 1 \leq i \leq n-1, \text{ and } 1 \leq j \leq \frac{m}{2} \\ (m+\mu)(2n-i)+1-j & \text{if } 1 \leq i \leq n-1, \text{ and } \frac{m}{2}+1 \leq j \leq m \\ (m+\mu)(2n-i+1)-\frac{m}{2}+1-j & \text{if } 1 \leq i \leq n-1, \text{ and } m+1 \leq j \leq m+\mu \end{cases}$$

$$\chi_2(x_n^j x_1^j) = (m+\mu)(n+1)+1-j, \text{ if } m+1 \leq j \leq m+\mu.$$

It is a matter for routine checking to see that the set of the edge-weights consists of the consecutive integers $\{2n(m+\mu)+2, 2n(m+\mu)+3, \ldots, 3n(m+\mu)-m+1\}$ and the labeling χ_2 is super $(2n(m+\mu)+2, 1)$-edge-antimagic total. □

Finally, we summarize that the graph $mP_n \cup \mu C_n$ has a super (a, d)-edge-antimagic total labeling for (i) $d \in \{0, 2\}$, if $m+\mu$ and n are odd; and (ii) $d = 1$, if m is even. In the case when $d \in \{0, 2\}$ and $(m+\mu)n$ is even; and when $d = 1$ and m is odd, we do not have the complete answer. We list here the following open problems.

Open Problem 3.3.1 *For the graph $mP_n \cup \mu C_n$, $(m+\mu)n$ is even, determine if there is a super (a, d)-edge-antimagic total labeling with $d \in \{0, 2\}$.*

Open Problem 3.3.2 *For the graph $mP_n \cup \mu C_n$, m is odd, determine if there is a super $(a, 1)$-edge-antimagic total labeling.*

In the case when $d = 3$ we do not have any answer. So, we present another problem for further investigation.

Open Problem 3.3.3 *For the graph $mP_n \cup \mu C_n$, $m, \mu \geq 1$ and $n \geq 3$, determine if there is a super $(a, 3)$-edge-antimagic total labeling.*

Chapter 4

SEATL of Disjoint Union of Stars

As mentioned in Chapter 2, given a graph G, the problem of deciding whether G admits a vertex-magic or an edge-magic labeling is equivalent to the problem of deciding whether or not a set of linear homogeneous Diophantine equations has a solution. For this, there is no polynomial time bounded algorithm. In the disjoint union of multiple copies of stars, the only vertices which have the highest degree are the centers of each star. In 2002, Lee and Kong [79] proposed a conjecture of a super edge-magicness of a disjoint union of multiple copies of stars. In this chapter we present new results on the super edge-antimagicness of disjoint union of multiple copies of stars and so settle this conjecture. In the first section we show that the disjoint union of two stars admits super (a, d)-edge-antimagic total labeling, and we generalise this result for m copies of stars in the second section.

4.1 Two stars

It is proved in [117] that the star $K_{1,n}$ has a super (a,d)-edge-antimagic total labeling if and only if either (i) $d \in \{0,1,2\}$ and $n \geq 1$, or (ii) $d = 3$ and $1 \leq n \leq 2$. Now, we study super edge antimagic of the disjoint union of two stars, denoted by $K_{1,m} \cup K_{1,n}$. The disjoint union of $K_{1,m}$ and $K_{1,n}$ is the disconnected graph with vertex set $V(K_{1,m} \cup K_{1,n}) = \{x_{1,j} : j = 0,1,\dots,m\} \cup \{x_{2,i} : i = 0,1,\dots,n\}$ and edge set $E(K_{1,m} \cup K_{1,n}) = \{x_{1,0}x_{1,j} : j = 1,2,\dots,m\} \cup \{x_{2,0}x_{2,i} : i = 1,2,\dots,n\}$.

If the graph $K_{1,m} \cup K_{1,n}$ is super (a,d)-edge-antimagic total then, according to Lemma 3.0.1, for $p = m+n+2$ and $q = m+n$, we have $d \leq 3 + \frac{2}{m+n-1}$. We can see that:

(i) if $m \geq 2$ and $n \geq 2$ then there is no super (a,d)-edge-antimagic total labeling of $K_{1,m} \cup K_{1,n}$ with $d > 3$;

(ii) if $m + n = 3$ then there is no super (a,d)-edge-antimagic total labeling of $K_{1,m} \cup K_{1,n}$ with $d > 4$;

(iii) if $m + n = 2$ then there is no super (a,d)-edge-antimagic total labeling of $K_{1,m} \cup K_{1,n}$ with $d > 5$.

If $m + n = 2$ then we have the graph $K_{1,1} \cup K_{1,1}$. Assume that $K_{1,1} \cup K_{1,1}$ has a super (a,d)-edge-antimagic total labeling. This means that $\sum_{k=1}^{6} k = 2a + d$. For $d = 0, 2$ and 4 the value a is not an integer. Therefore for the graph $K_{1,1} \cup K_{1,1}$, there is no super (a,d)-edge-antimagic total labeling.

For $d = 1, 3$ and 5 the requested super (a,d)-edge-antimagic total labeling δ_1 is given in the following.

d	$\delta_1(x_{1,0})$	$\delta_1(x_{2,0})$	$\delta_1(x_{1,1})$	$\delta_1(x_{2,1})$	$\delta_1(x_{1,0}x_{1,1})$	$\delta_1(x_{2,0}x_{2,1})$
1	2	1	3	4	5	6
3	1	2	3	4	5	6
5	2	4	1	3	5	6

◇ **Theorem 4.1.1** *For the graph $K_{1,m} \cup K_{1,n}$, $m+n=3$, there is no super $(a,4)$-edge-antimagic total labeling.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, for $m+n=3$, has a super $(a,4)$-edge-antimagic total labeling $\delta_2 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \to \{1,2,\ldots,8\}$, and $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a+4, a+8\}$ is the set of edge-weights. In the computation of the edge-weights of $K_{1,m} \cup K_{1,n}$ the label of a vertex of degree two is used twice but the labels of the remaining vertices are used once each. The label of every edge is used once. The sum of all vertex and edge labels used to calculate the edge-weights is equal to the sum of the edge-weights. If s_1 is the label of the vertex of degree two then

$$s_1 + \sum_{u \in V(K_{1,m} \cup K_{1,n})} \delta_2(u) + \sum_{uv \in E(K_{1,m} \cup K_{1,n})} \delta_2(uv) = \sum_{uv \in E(K_{1,m} \cup K_{1,n})} w(uv)$$

and

$$a = 8 + \frac{s_1}{3}.$$

Since a must be an integer, for s_1 we have only one possible value, namely, $s_1 = 3$, which gives $a = 9$.

The smallest value of edge weight $a = 9$ can be obtained only from the triple $(1,2,6)$, where 1 and 2 are values of adjacent vertices of degree one and 6 is the value of the edge. The remaining vertices of degree one must be labeled by the values 4 and 5. Thus, we have the triples $(3,4,7)$ and $(3,5,8)$ or $(3,4,8)$ and $(3,5,7)$. This contradicts the fact that $K_{1,m} \cup K_{1,n}$, for $m+n=3$, has a super $(a,4)$-edge-antimagic total labeling. □

If $m = t(n+1)$ (respectively, $n = t(m+1)$) then m is a multiple of $(n+1)$ (respectively, n is a multiple of $(m+1)$).

◇ **Theorem 4.1.2** *The graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a $(t+4,1)$-edge-antimagic vertex labeling if and only if either m is a multiple of $n+1$ or n is a multiple of $m+1$.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a $(a,1)$-edge-antimagic vertex labeling $\delta_3 : V(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, m+n+2\}$ and that $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a+1, a+2, \ldots, a+m+n-1\}$ is the set of the edge-weights. The sum of the elements of W is

$$\sum_{uv \in E(K_{1,m} \cup K_{1,n})} w(uv) = (m+n)a + \frac{(m+n)(m+n-1)}{2}.$$

In the computation of the edge-weights of $K_{1,m} \cup K_{1,n}$, the label of the central vertices, $\delta_3(x_{1,0})$ and $\delta_3(x_{2,0})$, is used m and n times, respectively, and the labels of the remaining vertices are used once each. Let $s_1 = \delta_3(x_{1,0})$ and $s_2 = \delta_3(x_{2,0})$. The sum of all vertex labels used to calculate the edge-weights is equal to

$$(m-1)\delta_3(x_{1,0}) + (n-1)\delta_3(x_{2,0}) + \sum_{k=1}^{m+n+2} k =$$

$$(m-1)s_1 + (n-1)s_2 + \frac{(m+n+3)(m+n+2)}{2}.$$

The sum of the vertex labels used to obtain the edge-weights is naturally equal to the sum of all the edge-weights. Thus,

$$(m+n)a = 3(m+n+1) + (m-1)s_1 + (n-1)s_2. \tag{4.1}$$

Clearly, $s_1 + s_2 \notin \{a, a+1, a+2, \ldots, a+m+n-1\}$ because exactly one endpoint of any edge belongs to $\{x_{1,0}, x_{2,0}\}$. Without loss of generality, we may assume that $s_1 + s_2 < a$. If $s_1 + s_2 > a+m+n-1$ then we consider

$(a', 1)$-edge-antimagic vertex labeling g, given by $g(v) = m + n + 3 - \delta_3(v)$, for all $v \in V(K_{1,m} \cup K_{1,n})$.

If $1 \notin \{s_1, s_2\}$ then $a > s_1 + s_2 > \min_{1\le j\le m} \delta_3(x_{1,j}) + s_2 \ge 1 + s_2 \ge a$ or $a > s_1 + s_2 > s_1 + \min_{1\le i\le n} \delta_3(x_{2,i}) \ge s_1 + 1 \ge a$, a contradiction.

Suppose $s_1 = 2$ and $s_2 = 1$. Then, from (4.1), it follows that

$$(m + n)(a - 4) = m,$$

which implies that m is a multiple of $m + n$, a contradiction.

Suppose $s_1 > 2$ and $s_2 = 1$. We can say that $a = s_1+2$ because if $\min_{1\le i\le n} \delta_3(x_{2,i}) = 2$ then $\min_{1\le i\le n} \delta_3(x_{2,i})+s_2 < s_1+s_2 < a$, thus the vertex labeled by 2 must belong to $K_{1,m}$. From (4.1), it follows that

$$\begin{aligned} (m + n)(s_1 + 2) &= 3(m + n + 1) + (m - 1)s_1 + (n - 1) \text{ and} \\ (s_1 - 2)(n + 1) &= m, \end{aligned}$$

which means that $m > n$ and m is a multiple of $n + 1$.

For the sake of completeness, we assume that $m = t(n + 1)$, and consider the vertex labeling δ_3, described by Ivančo and Lučkaničová in [71].

$$\delta_3(x_{1,j}) = \begin{cases} 2 + t & \text{if } \ j = 0 \\ \lceil \frac{j}{t} \rceil + j & \text{if } \ 1 \le j \le m \end{cases}$$

$$\delta_3(x_{2,i}) = \begin{cases} 1 & \text{if } \ i = 0 \\ 1 + (i + 1)(t + 1) & \text{if } \ 1 \le i \le n. \end{cases}$$

The vertex labeling δ_3 is a bijective function from $K_{1,m} \cup K_{1,n}$ onto the set $\{1, 2, \dots, m + n + 2\}$. The edge-weights of $K_{1,m} \cup K_{1,n}$, under the labeling δ_3, constitute the sets

$$W^1_{\delta_3} = \{w^1_{\delta_3}(x_{1,0}x_{1,j}) : \text{ if } 1 \le j \le m\} = \{2 + t + \lceil \tfrac{j}{t} \rceil + j : \text{ if } 1 \le j \le m\},$$
$$W^2_{\delta_3} = \{w^2_{\delta_3}(x_{2,0}x_{2,i}) : \text{ if } 1 \le i \le n\} = \{2 + (i + 1)(t + 1) : \text{ if } 1 \le i \le n\}.$$

Hence the set $\bigcup_{r=1}^{2} W_{\delta_3}^r = \{t+4, t+5, \ldots, m+n+t+3\}$ consists of consecutive integers. Thus δ_3 is a $(t+4,1)$-edge-antimagic vertex labeling. $\square$

With respect to Lemma 3.0.2, the $(t+4,1)$-edge-antimagic vertex labeling δ_3 extends to a super $(a,0)$-edge-antimagic total labeling, where for $p = m+n+2$ and $q = m+n$, the value $a = 2m+2n+t+6$. Thus we have the following theorem, which was proved by Ivančo and Lučkaničová in [71].

Theorem 4.1.3 [71] *The graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a super $(2m+2n+t+6, 0)$-edge-antimagic total labeling if and only if either m is a multiple of $n+1$ or n is a multiple of $m+1$.*

Furthermore, we obtain the following theorem.

$\diamond$ **Theorem 4.1.4** *If either m is a multiple of $n+1$ or n is a multiple of $m+1$ then the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a super $(m+n+t+7, 2)$-edge-antimagic total labeling.*

Proof. Without loss of generality, we may assume that m is a multiple of $n+1$. Let $m = t(n+1)$. Using the $(t+4,1)$-edge-antimagic vertex labeling δ_3 from Theorem 4.1.2, we define a total labeling $\delta_4 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \rightarrow \{1, 2, \ldots, 2m+2n+2\}$ as follows:

$$\begin{aligned}
\delta_4(v) &= \delta_3(v), \text{ for every vertex } v \in V(K_{1,m} \cup K_{1,n}) \\
\delta_4(x_{1,0}x_{1,j}) &= m+n+1+\left\lceil \frac{j}{t} \right\rceil + j, \quad \text{for } 1 \leq j \leq m \\
\delta_4(x_{2,0}x_{2,i}) &= m+n+2+i(t+1), \quad \text{for } 1 \leq i \leq n.
\end{aligned}$$

The edge-weights of $K_{1,m} \cup K_{1,n}$, under the total labeling δ_4, constitute the sets

$W_{\delta_4}^1 = \{w_{\delta_4}^1(x_{1,0}x_{1,j}) = w_{\delta_3}^1(x_{1,0}x_{1,j}) + \delta_4(x_{1,0}x_{1,j}) : \text{ if } 1 \leq j \leq m\}$

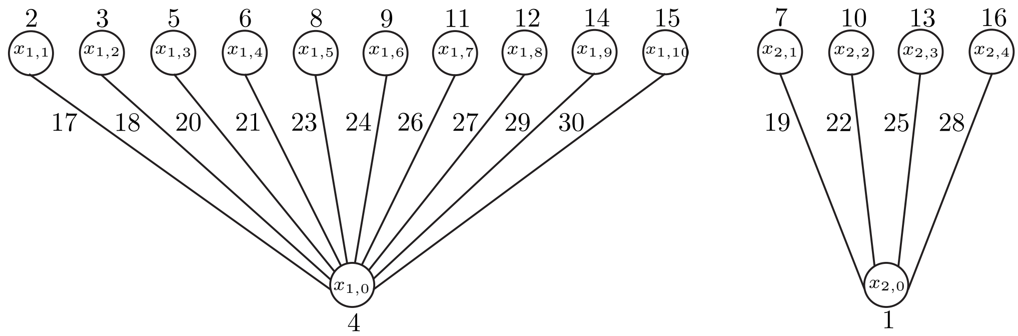


Figure 4.1: Super $(23, 2)$-edge-antimagic total labeling of $K_{1,10} \cup K_{1,4}$.

$$
\begin{aligned}
&= \{m + n + t + 3 + 2\lceil \tfrac{j}{t} \rceil + 2j : \text{ if } 1 \le j \le m\},\\
W^2_{\delta_4} &= \{w^2_{\delta_4}(x_{2,0}x_{2,i}) = w^2_{\delta_3}(x_{2,0}x_{2,i}) + \delta_4(x_{2,0}x_{2,i}) : \text{ if } 1 \le i \le n\}\\
&= \{m + n + 4 + (2i + 1)(t + 1) : \text{ if } 1 \le i \le n\}.
\end{aligned}
$$

Hence the set $\bigcup_{r=1}^{2} W^r_{\delta_4} = \{m + n + t + 7, m + n + t + 9, \ldots, 3m + 3n + t + 5\}$ consists of an arithmetic sequence, with the first term $m + n + t + 7$ and the common difference $d = 2$. Thus δ_4 is a super $(m+n+t+7, 2)$-edge-antimagic total labeling. □

We are not able to give an answer as to whether or not there exists a super $(a, 2)$-edge-antimagic total labeling of $K_{1,m} \cup K_{1,n}$ for other values of m and n. Therefore, we propose the following open problem.

Open Problem 4.1.1 *For the graph $K_{1,m} \cup K_{1,n}$, $m \ge 2$, $n \ge 2$, if m is not a multiple of $n + 1$ and n is not a multiple of $m + 1$, determine whether there is a super $(a, 2)$-edge-antimagic total labeling.*

By using the $(t+4, 1)$-edge-antimagic vertex labeling δ_3, with respect to Lemma 3.0.3, we can claim

◇ **Theorem 4.1.5** *If $m + n$ is odd and either m is a multiple of $n + 1$ or n is a multiple of $m + 1$ then the graph $K_{1,m} \cup K_{1,n}$, $m \ge 2$ and $n \ge 2$, has a super*

$\left(\frac{3(m+n)+2t+13}{2}, 1\right)$*-edge-antimagic total labeling.*

Proof. From Theorem 4.1.2, the graph $K_{1,m} \cup K_{1,n}$ has a $(t+4,1)$-edge-antimagic vertex labeling. Let $\mathfrak{A} = \{c, c+1, c+2, \ldots, c+k\}$ be a set of the edge weights of the vertex labeling δ_3, for $c = t+4$ and $k = m+n-1$. In light of Lemma 3.0.3, there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $\mathfrak{A}+[\Pi(\mathfrak{A}) - c + m + n + 3] = \left\{c + \frac{3m+3n+5}{2}, c + \frac{3m+3n+5}{2} + 1, \ldots, c + \frac{5m+5n+3}{2}\right\}$. If $[\Pi(\mathfrak{A}) - c + m + n + 3]$ is an edge labeling of $K_{1,m} \cup K_{1,n}$ then $\mathfrak{A} + [\Pi(\mathfrak{A}) - c+m+n+3]$ gives the set of the edge weights of $K_{1,m} \cup K_{1,n}$, which implies that the total labeling is super $(a,1)$-edge-antimagic total, where $a = c + \frac{3m+3n+5}{2} = \frac{3(m+n)+2t+13}{2}$. This concludes the proof. □

Figure 4.2 illustrates an example of super $(a,1)$-edge-antimagic total labeling stated in Theorem 4.1.5.

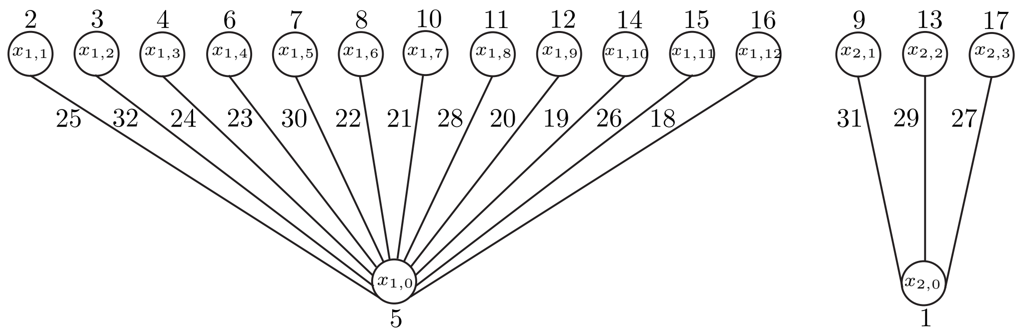


Figure 4.2: Super $(32,1)$-edge-antimagic total labeling of $K_{1,12} \cup K_{1,3}$.

◇ **Theorem 4.1.6** *If* $m = n$ *then the graph* $K_{1,m} \cup K_{1,n}$*,* $m \geq 2$ *and* $n \geq 2$*, has a* $(4,2)$*-edge-antimagic vertex labeling.*

Proof. Let $m = n$ and $m \geq 2$. Consider the bijection $\delta_5 : V(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, m+n+2\}$, where

$$\delta_5(x_{1,j}) = \begin{cases} 1 & \text{if } j = 0 \\ 2j+1 & \text{if } 1 \leq j \leq m \end{cases} \quad \text{and} \quad \delta_5(x_{2,i}) = \begin{cases} m+n+2 & \text{if } i = 0 \\ 2i & \text{if } 1 \leq i \leq n. \end{cases}$$

We observe that the edge-weights of $K_{1,m} \cup K_{1,n}$, under the vertex labeling δ_5, constitute the sets

$W^1_{\delta_5} = \{w^1_{\delta_5}(x_{1,0}x_{1,j}) : \text{if } 1 \leq j \leq m\} = \{2j + 2 : \text{if } 1 \leq j \leq m\}$,
$W^2_{\delta_5} = \{w^2_{\delta_5}(x_{2,0}x_{2,i}) : \text{if } 1 \leq i \leq n\} = \{m + n + 2 + 2i : \text{if } 1 \leq i \leq n\}$.

Hence, the elements of the set $\bigcup_{r=1}^{2} W^r_{\delta_5} = \{4, 6, \ldots, m + 3n + 2\}$ can be arranged to form an arithmetic sequence, with first term 4 and common difference $d = 2$. Thus δ_5 is a $(4, 2)$-edge-antimagic vertex labeling. □

◇ **Theorem 4.1.7** *If $m = n$ then the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$, has super $(2m + 2n + 6, 1)$-edge-antimagic total and super $(m + n + 7, 3)$-edge-antimagic total labelings.*

Proof. Let $m = n$ and $m \geq 2$. From Theorem 4.1.6, it follows that the graph $K_{1,m} \cup K_{1,n}$ has a $(4, 2)$-edge-antimagic vertex labeling. We will distinguish two cases, according to whether $d = 1$ or $d = 3$.

Case 1. $d = 1$

Define $\delta_6 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, 2m + 2n + 2\}$ to be the bijective function

$$\begin{aligned}
\delta_6(v) &= \delta_5(v), \text{ for all vertices } v \in V(K_{1,m} \cup K_{1,n}),\\
\delta_6(x_{1,0}x_{1,j}) &= 2m + 2n + 3 - j, \text{ for } 1 \leq j \leq m,\\
\delta_6(x_{2,0}x_{2,i}) &= m + 2n + 3 - i, \text{ for } 1 \leq i \leq n.
\end{aligned}$$

By direct computation, it is not difficult to verify that the edge-weights constitute the arithmetic progression $2m + 2n + 6, 2m + 2n + 7, \ldots, 3m + 3n + 5$. Thus δ_6 is a super $(2m + 2n + 6, 1)$-edge-antimagic total labeling.

Case 2. $d = 3$

Consider the labeling $\delta_7 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, 2m +$

$2n+2\}$, such that

$$\begin{aligned}\delta_7(v) &= \delta_5(v), \text{ for all vertices } v \in V(K_{1,m} \cup K_{1,n}),\\ \delta_7(x_{1,0}x_{1,j}) &= m+n+2+j, \text{ for } 1 \leq j \leq m,\\ \delta_7(x_{2,0}x_{2,i}) &= 2m+n+2+i, \text{ for } 1 \leq i \leq n.\end{aligned}$$

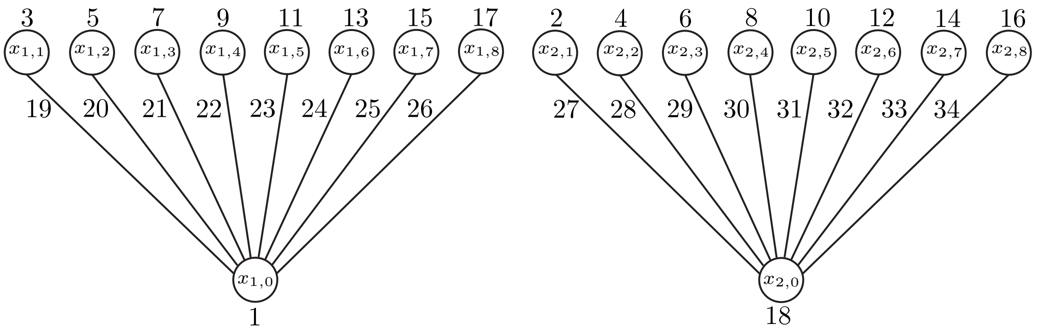


Figure 4.3: Super $(23, 3)$-edge-antimagic total labeling of $K_{1,8} \cup K_{1,8}$.

We can see that the labeling δ_7 uses each integer from the set $\{1, 2, \ldots, 2m+2n+2\}$ exactly once and the set of the edge-weights consists of an arithmetic sequence with the first value $m+n+7$ and common difference $d = 3$. Thus δ_7 is a super $(m+n+7, 3)$-edge-antimagic total labeling. □

In the case when $m+n$ is even and $m \neq n$, we do not know if there exists any super $(a, 1)$-edge-antimagic total labeling for $K_{1,m} \cup K_{1,n}$. Therefore, we propose the following

Open Problem 4.1.2 *For the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$, $n \geq 2$, $m+n$ even and $m \neq n$, determine if there exists a super $(a, 1)$-edge-antimagic total labeling.*

From Theorem 4.1.7, we have that for $m = n$, $m \geq 2$, the graph $K_{1,m} \cup K_{1,n}$ has a super $(m+n+7, 3)$-edge-antimagic total labeling but for $m \neq n$, $m \geq 2$, $n \geq 2$, we do not know if such a labeling exists or not. Therefore, we propose

Open Problem 4.1.3 *For the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$, $n \geq 2$ and $m \neq n$, determine if there exists a super $(a, 3)$-edge-antimagic total labeling.*

To conclude this subsection, we prove the following theorem.

◇ **Theorem 4.1.8** *For the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, there is no $(a, 3)$-edge-antimagic vertex labeling.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a $(a, 3)$-edge-antimagic vertex labeling $\delta : V(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, m+n+1, m+n+2\}$ and $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a+3, a+6, \ldots, a+(m+n-1)3\}$ is the set of edge-weights. The minimum possible edge weight is at least 3. It follows that $a \geq 3$. The maximum possible edge weight is no more than $(p-1)+p = 2m+2n+3$. Consequently, $a+3(m+n-1) \leq 2m+2n+3$ and $3 \leq 2 + \frac{2}{m+n-1}$, which is impossible when $m+n \geq 4$. □

4.2 More than two stars

In Section 4.1, we studied super edge-antimagicness of a disjoint union of two stars. Now we will provide new results on super edge-antimagicness of a disjoint union of more than two copies of stars, denoted by $K_{1,m} \cup 2sK_{1,n}$. The disjoint union of $K_{1,m}$ and $2sK_{1,n}$ is the disconnected graph with vertex set $V(K_{1,m} \cup 2sK_{1,n}) = \{x_{1,j} :\ 0 \leq j \leq m\} \cup \{x_{i,k} :\ 2 \leq i \leq 2s+1, 0 \leq k \leq n\}$ and edge set $E(K_{1,m} \cup 2sK_{1,n}) = \{x_{1,0}x_{1,j} :\ 1 \leq j \leq m\} \cup \{x_{i,0}x_{i,k} :\ 2 \leq i \leq 2s+1, 1 \leq k \leq n\}$. Thus $p = |V(K_{1,m} \cup 2sK_{1,n})| = m + 2s(n+1) + 1$ and $q = |E(K_{1,m} \cup 2sK_{1,n})| = m + 2sn$.

If the graph $K_{1,m} \cup 2sK_{1,n}$ is super (a, d)-edge-antimagic total then, from Lemma 3.0.1, we have that

$$d \leq 3 + \frac{4s}{m + 2sn - 1}. \tag{4.2}$$

By applying Equation (4.2) for values of m, n and s, we obtain the following.

(i) for $K_{1,m} \cup 2sK_{1,n}, m = n = 1, s \geq 1$, there is no super (a, d)-edge-antimagic total labeling with $d > 5$;

(ii) for $K_{1,m} \cup 2sK_{1,n}$, $n+m = 3$, $s \geq 1$, there is no super (a, d)-edge-antimagic total labeling with $d > 4$;

(iii) for $K_{1,m} \cup 2sK_{1,n}, n \geq 2, m \geq 2$ and $s \geq 1$, there is no super (a, d)-edge-antimagic total labeling with $d > 3$.

If $m = n = 1$ then the graph $K_{1,m} \cup 2sK_{1,n}$ is a disjoint union of $2s+1$ copies of P_2, denoted by $(2s+1)P_2$. We have proved in Section 3.2 that for every $s \geq 1$ and $d \in \{0, 1, 2, 3, 4, 5\}$, the graph $(2s+1)P_2$ has a super (a, d)-edge-antimagic total labeling.

◇ **Theorem 4.2.1** *The graph $K_{1,m} \cup 2sK_{1,n}$, $m \geq 1, n \geq 1$ and $s \geq 1$, has an $(3s + 3, 1)$-edge-antimagic vertex labeling.*

Proof. We distinguish two cases.

Case 1. $m \geq n$

Define the vertex labeling $\phi_1 : V(K_{1,m} \cup 2sK_{1,n}) \to \{1, 2, \ldots, m+2s(n+1)+1\}$ in the following way:

$$\phi_1(x_{i,0}) = \begin{cases} s + i & \text{if } 1 \leq i \leq s + 1 \\ i - s - 1 & \text{if } s + 2 \leq i \leq 2s + 1 \end{cases}$$

$$\phi_1(x_{1,j}) = \begin{cases} (2s + 1)j + 1 & \text{if } 1 \leq j \leq n + 1 \\ 2s(n + 1) + j + 1 & \text{if } n + 2 \leq j \leq m \end{cases}$$

$$\phi_1(x_{i,k}) = (2s + 1)k + i, \text{ for } 2 \leq i \leq 2s + 1 \text{ and } 1 \leq k \leq n.$$

Clearly, the values of ϕ_1 are $1, 2, \ldots, m + 2s(n+1) + 1$. The edge-weights of $K_{1,m} \cup 2sK_{1,n}$, under the labeling ϕ_1, constitute the sets

$$\begin{aligned}
W^1_{\phi_1} &= \{w^1_{\phi_1}(x_{1,0}x_{1,j}) = \phi_1(x_{1,0}) + \phi_1(x_{1,j}) :\ 1 \le j \le n+1\} \\
&= \{(2s+1)j + s + 2 :\ 1 \le j \le n+1\}, \\
W^2_{\phi_1} &= \{w^2_{\phi_1}(x_{1,0}x_{1,j}) = \phi_1(x_{1,0}) + \phi_1(x_{1,j}) :\ n+2 \le j \le m\} \\
&= \{(2n+3)s + j + 2 :\ n+2 \le j \le m\}, \\
W^3_{\phi_1} &= \{w^3_{\phi_1}(x_{i,0}x_{i,k}) = \phi_1(x_{i,0}) + \phi_1(x_{i,k}) :\ 2 \le i \le s+1 \text{ and } 1 \le k \le n\} \\
&= \{(2s+1)k + 2i + s :\ 2 \le i \le s+1 \text{ and } 1 \le k \le n\}, \\
W^4_{\phi_1} &= \{w^4_{\phi_1}(x_{i,0}x_{i,k}) = \phi_1(x_{i,0}) + \phi_1(x_{i,k}) :\ s+2 \le i \le 2s+1 \text{ and } 1 \le k \le n\} \\
&= \{(2s+1)k + 2i - s - 1 :\ s+2 \le i \le 2s+1 \text{ and } 1 \le k \le n\}.
\end{aligned}$$

It is not difficult to check that the set $\bigcup_{r=1}^{4} W^r_{\phi_1} = \{3s+3, 3s+4, \ldots, (3+2n)s + m + 2\}$.

Case 2. $m < n$

For $m \ge 1, n \ge 1$ and $s \ge 1$, define the bijection $\phi_2 : V(K_{1,m} \cup 2sK_{1,n}) \to \{1, 2, \ldots, m + 2s(n+1) + 1\}$ as follows:

$$\begin{aligned}
\phi_2(x_{i,0}) &= \phi_1(x_{i,0}) \\
\phi_2(x_{1,j}) &= (2s+1)j + 1, \text{ for } 1 \le j \le m
\end{aligned}$$

$$\phi_2(x_{i,k}) = \begin{cases} (2s+1)k + i & \text{if } 2 \le i \le 2s+1 \text{ and } 1 \le k \le m \\ 2sk + m + i & \text{if } 2 \le i \le 2s+1 \text{ and } m+1 \le k \le n. \end{cases}$$

Then for the edge-weights of $K_{1,m} \cup 2sK_{1,n}$ we have:

$W^1_{\phi_2} = \{w^1_{\phi_2}(x_{1,0}x_{1,j}) = \phi_2(x_{1,0}) + \phi_2(x_{1,j}) :\ 1 \le j \le m\} = \{(2s+1)j + s + 2 :\ 1 \le j \le m\}$,

$W^2_{\phi_2} = \{w^2_{\phi_2}(x_{i,0}x_{i,k}) = \phi_2(x_{i,0}) + \phi_2(x_{i,k}) :\ 2 \le i \le s+1 \text{ and } 1 \le k \le m\} = \{(2s+1)k + 2i + s :\ 2 \le i \le s+1 \text{ and } 1 \le k \le m\}$,

$W^3_{\phi_2} = \{w^3_{\phi_2}(x_{i,0}x_{i,k}) = \phi_2(x_{i,0}) + \phi_2(x_{i,k}) :\ s+2 \le i \le 2s+1 \text{ and } 1 \le k \le m\} = \{(2s+1)k + 2i - s - 1 :\ s+2 \le i \le 2s+1 \text{ and } 1 \le k \le m\}$,

$W^4_{\phi_2} = \{w^4_{\phi_2}(x_{i,0}x_{i,k}) = \phi_2(x_{i,0}) + \phi_2(x_{i,k}) : \ 2 \leq i \leq s+1 \text{ and } m+1 \leq k \leq n\} = \{(2k+1)s + 2i + m : \ 2 \leq i \leq s+1 \text{ and } m+1 \leq k \leq n\}$,

$W^5_{\phi_2} = \{w^5_{\phi_2}(x_{i,0}x_{i,k}) = \phi_2(x_{i,0}) + \phi_2(x_{i,k}) : \ s+2 \leq i \leq 2s+1 \text{ and } m+1 \leq k \leq n\} = \{(2k-1)s+2i+m-1 : \ s+2 \leq i \leq 2s+1 \text{ and } m+1 \leq k \leq n\}$,

and $\bigcup_{r=1}^{5} W^r_{\phi_2} = \{3s+3, 3s+4, \ldots, (3+2n)s+m+2\}$ consists of consecutive integers.

This implies that ϕ_1 and ϕ_2 are $(3s+3, 1)$-edge-antimagic vertex labelings. □

◇ **Theorem 4.2.2** *For $m \geq 1, n \geq 1$ and $s \geq 1$ the graph $K_{1,m} \cup 2sK_{1,n}$ has a super $((4n+5)s+2m+4, 0)$-edge-antimagic total labeling and a super $((2n+5)s+m+5, 2)$-edge-antimagic total labeling.*

Proof. For $d = 0$, from Theorem 4.2.1, we have that for $m \geq 1, n \geq 1$ and $s \geq 1$, the graph $K_{1,m} \cup 2sK_{1,n}$ has a $(3s+3, 1)$-edge-antimagic vertex labeling. According to Lemma 3.0.2 for $p = m + 2s(n+1) + 1$ and $q = m + 2sn$, there is also a super $((4n+5)s+2m+4, 0)$-edge-antimagic total labeling.

For $d = 2$, we distinguish two cases.

Case 1. $m \geq n$

Let us construct a vertex labeling ϕ_3 of $K_{1,m} \cup 2sK_{1,n}$, $m \geq 2, n \geq 2$ and $s \geq 1$, in the following way:

$$\phi_3(x_{1,0}x_{1,j}) = \begin{cases} 2s(j+n) + m + j + 1 & \text{if } 1 \leq j \leq n+1 \\ 2s(2n+1) + m + j + 1 & \text{if } n+2 \leq j \leq m. \end{cases}$$

If $1 \leq k \leq n$ then

$$\phi_3(x_{i,0}x_{i,k}) = \begin{cases} 2s(n+k) + m + 2i + k - 1 & \text{if } 2 \leq i \leq s+1 \\ 2s(n+k-1) + 2i + m + k - 2 & \text{if } s+2 \leq i \leq 2s+1. \end{cases}$$

We can see that the vertex labeling ϕ_3 is a bijective function from $K_{1,m} \cup 2sK_{1,n}$ onto the set $\{1, 2, \ldots, 2m+2s(2n+1)+1\}$. The edge-weights of $K_{1,m} \cup 2sK_{1,n}$, under the labeling ϕ_3, constitute the sets

$W^1_{\phi_3} = \{w^1_{\phi_3}(x_{1,0}x_{1,j}) = w^1_{\phi_1}(x_{1,0}x_{1,j}) + \phi_3(x_{1,0}x_{1,j}) : \text{ if } 1 \leq j \leq n+1\} = \{(4j+2n+1)s+2j+m+3 : \text{ if } 1 \leq j \leq n+1\},$

$W^2_{\phi_3} = \{w^2_{\phi_3}(x_{1,0}x_{1,j}) = w^2_{\phi_1}(x_{1,0}x_{1,j}) + \phi_3(x_{1,0}x_{1,j}) : \text{ if } n+2 \leq j \leq m\} = \{(6n+5)s+2j+m+3 : \text{ if } n+2 \leq j \leq m\},$

$W^3_{\phi_3} = \{w^3_{\phi_3}(x_{i,0}x_{i,k}) = w^3_{\phi_1}(x_{i,0}x_{i,k}) + \phi_3(x_{i,0}x_{i,k}) : \text{ if } 2 \leq i \leq s+1 \text{ and } 1 \leq k \leq n\} = \{(4k+2n+1)s+4i+2k+m-1 : \text{ if } 2 \leq i \leq s+1 \text{ and } 1 \leq k \leq n\},$

$W^4_{\phi_3} = \{w^4_{\phi_3}(x_{i,0}x_{i,k}) = w^4_{\phi_1}(x_{i,0}x_{i,k}) + \phi_3(x_{i,0}x_{i,k}) : \text{ if } s+2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq n\} = \{(4k+2n-3)s+4i+2k+m-3 : \text{ if } s+2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq n\}.$

Hence the set $\bigcup_{r=1}^{4} W^r_{\phi_3} = \{(2n+5)s+m+5, (2n+5)s+m+7, \dots, (6n+5)s+3m+3\}$ consists of an arithmetic sequence, with the first term $(2n+5)s+m+5$ and the common difference $d = 2$. Thus ϕ_3 is a $((2n+5)s+m+5, 2)$-edge-antimagic total labeling.

Case 2. $m < n$

For $m \geq 2, n \geq 2$ and $s \geq 1$, define the bijection $\phi_4 : V(K_{1,m} \cup 2sK_{1,n}) \to \{1, 2, \dots, 2m+2s(2n+1)+1\}$ as follows:

$$\phi_4(x_{1,0}x_{1,j}) = 2s(n+j)+m+j+1 : \text{ if } 1 \leq j \leq m$$

If $1 \leq k \leq m$ then

$$\phi_4(x_{i,0}x_{i,k}) = \begin{cases} 2s(n+k)+m+2i+k-1 & \text{if } 2 \leq i \leq s+1 \\ 2s(n+k-1)+2i+m+k-2 & \text{if } s+2 \leq i \leq 2s+1. \end{cases}$$

If $m+1 \leq k \leq n$ then

$$\phi_4(x_{i,0}x_{i,k}) = \begin{cases} 2s(n+k)+2m+2i-1 & \text{if } 2 \leq i \leq s+1 \\ 2s(n+k-1)+2m+2i-2 & \text{if } s+2 \leq i \leq 2s+1. \end{cases}$$

We can see that the vertex labeling ϕ_4 is a bijective function from $K_{1,m} \cup 2sK_{1,n}$ onto the set $\{1, 2, \dots, 2m+2s(2n+1)+1\}$. The edge-weights of $K_{1,m} \cup 2sK_{1,n}$, under the labeling ϕ_4, constitute the sets

$W^1_{\phi_4} = \{w^1_{\phi_4}(x_{1,0}x_{1,j}) = w^1_{\phi_2}(x_{1,0}x_{1,j}) + \phi_4(x_{1,0}x_{1,j}) :$ if $1 \leq j \leq m\} = \{(4j + 2n + 1)s + 2j + 3 :$ if $1 \leq j \leq m\}$,

$W^2_{\phi_4} = \{w^2_{\phi_4}(x_{i,0}x_{i,k}) = w^2_{\phi_2}(x_{i,0}x_{i,k}) + \phi_4(x_{i,0}x_{i,k}) :$ if $2 \leq i \leq s+1$ and $1 \leq k \leq m\} = \{(4k+2n+1)s+4i+2k+m-1 :$ if $2 \leq i \leq s+1$ and $1 \leq k \leq m\}$,

$W^3_{\phi_4} = \{w^3_{\phi_4}(x_{i,0}x_{i,k}) = w^3_{\phi_2}(x_{i,0}x_{i,k})+\phi_4(x_{i,0}x_{i,k}) :$ if $s+2 \leq i \leq 2s+1$ and $1 \leq k \leq m\} = \{(4k+2n-3)s+4i+2k+m-3 :$ if $s+2 \leq i \leq 2s+1$ and $1 \leq k \leq m\}$,

$W^4_{\phi_4} = \{w^4_{\phi_4}(x_{i,0}x_{i,k}) = w^4_{\phi_2}(x_{i,0}x_{i,k})+\phi_4(x_{i,0}x_{i,k}) :$ if $2 \leq i \leq s+1$ and $m+1 \leq k \leq n\} = \{(4k+2n)s+4i+k+3m-1 :$ if $2 \leq i \leq s+1$ and $m+1 \leq k \leq n\}$,

$W^5_{\phi_4} = \{w^5_{\phi_4}(x_{i,0}x_{i,k}) = w^5_{\phi_2}(x_{i,0}x_{i,k}) + \phi_4(x_{i,0}x_{i,k}) :$ if $s + 2 \leq i \leq 2s + 1$ and $m+1 \leq k \leq n\} = \{(4k+2n-2)s+4i-k+3m-3 :$ if $s+2 \leq i \leq 2s+1$ and $m+1 \leq k \leq n\}$.

Hence the set $\bigcup_{r=1}^{5} W^r_{\phi_4} = \{(2n+5)s+m+5, (2n+5)s+m+7, \ldots, (6n+5)s+3m+3\}$ consists of an arithmetic sequence, with the first term $(2n+5)s+m+5$ and the common difference $d = 2$. Thus ϕ_4 is a $((2n + 5)s + m + 5, 2)$-edge-antimagic total labeling. □

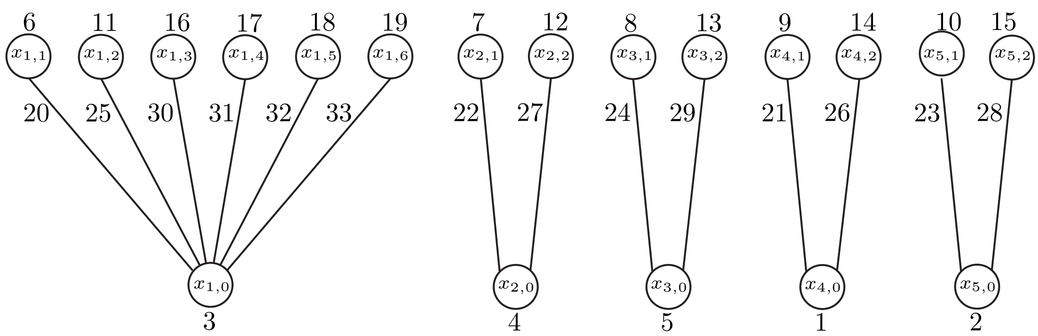


Figure 4.4: Super $(29, 2)$-edge-antimagic total labeling of $K_{1,6} \cup 4K_{1,2}$.

◇ **Theorem 4.2.3** *If m is odd then the graph $K_{1,m} \cup 2sK_{1,n}$, for $m \geq 1, n \geq 1$*

and $s \geq 1$, has a super $(s(3n+5)+\frac{3m+9}{2},1)$-edge-antimagic total labeling.

Proof. Consider the vertex labelings ϕ_1 and ϕ_2 of the graph $K_{1,m} \cup 2sK_{1,n}$ from Theorem 4.2.1, which are $(3s+3,1)$-edge-antimagic vertex labelings. The set of edge-weights gives the sequence $\mathfrak{A} = \{c, c+1, c+2, \dots, c+k\}$, for $c = 3s+3$ an $k = 2ns+m-1$. The value k is even for m odd. According to Lemma 3.0.3, there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$, such that $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + p + 1] = \{s(3n+5)+\frac{3m+9}{2}, s(3n+5)+\frac{3m+11}{2}, s(3n+5)+\frac{3m+13}{2}, \dots, s(5n+5)+\frac{5m+7}{2}\}$. If $[\Pi(\mathfrak{A}) - c + p + 1]$ is an edge labeling of $K_{1,m} \cup 2sK_{1,n}$ for m odd, $m \geq 1, n \geq 1, s \geq 1$, then $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + p + 1]$ determines the set of edge-weights of the graph $K_{1,m} \cup 2sK_{1,n}$ and the resulting total labeling is super $(s(3n+5)+\frac{3m+9}{2},1)$-edge-antimagic total. □

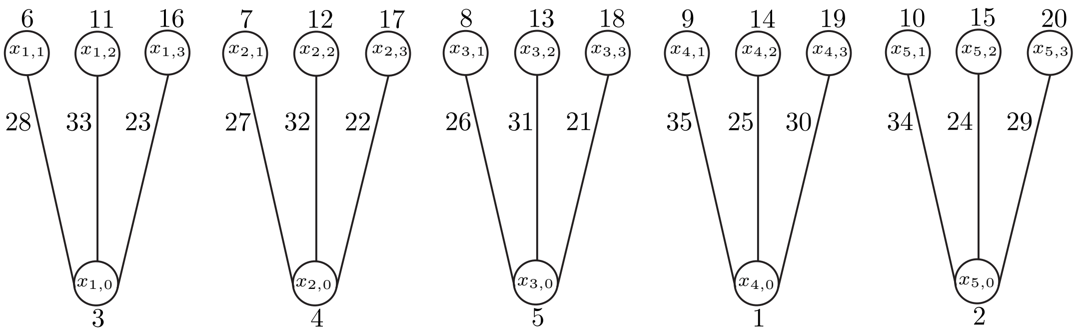


Figure 4.5: Super $(37,1)$-edge-antimagic total labeling of $K_{1,3} \cup 4K_{1,3}$.

For m even we have not yet found any super $(a,1)$-edge-antimagic total labeling. Therefore, we propose the following open problem.

Open Problem 4.2.1 *For m even, $m \geq 2, n \geq 1$ and $s \geq 1$, determine if there is a super $(a,1)$-edge-antimagic total labeling of $K_{1,m} \cup 2sK_{1,n}$.*

◇ **Theorem 4.2.4** *For $s \geq 1$, the graph $K_{1,2} \cup 2sK_{1,1}$ has a super $(5s+7,4)$-edge-antimagic total labeling.*

Proof. For $s \geq 3$ we consider the following function $\phi_5 : V(K_{1,2} \cup 2sK_{1,1}) \rightarrow \{1, 2, \ldots, 4s+3\}$, where

$$\phi_5(x_{i,0}) = \begin{cases} 3s+3 & \text{if } i = 1 \\ s+2i-2 & \text{if } 2 \leq i \leq s+3. \\ 2s+i+1 & \text{if } s+4 \leq i \leq 2s+1 \end{cases}$$

$$\phi_5(x_{1,j}) = \begin{cases} s+3 & \text{if } j = 1 \\ 4s+3 & \text{if } j = 2 \end{cases}$$

$$\phi_5(x_{i,1}) = \begin{cases} i-1 & \text{if } 2 \leq i \leq s+2 \\ 2i-s-1 & \text{if } s+3 \leq i \leq 2s+1. \end{cases}$$

In the case $s = 1$, label $\phi_6(x_{1,0}) = 6, \phi_6(x_{1,1}) = 4, \phi_6(x_{1,2}) = 7, \phi_6(x_{2,0}) = 3, \phi_6(x_{2,1}) = 1, \phi_6(x_{3,0}) = 5$ and $\phi_6(x_{3,1}) = 2$. If $s = 2$ then label $\phi_7(x_{1,0}) = 9$, $\phi_7(x_{1,1}) = 5$, $\phi_7(x_{1,2}) = 11$, $\phi_7(x_{2,0}) = 4$, $\phi_7(x_{2,1}) = 1$, $\phi_7(x_{3,0}) = 6$, $\phi_7(x_{3,1}) = 2$, $\phi_7(x_{4,0}) = 8$, $\phi_7(x_{4,1}) = 3$, $\phi_7(x_{5,0}) = 10$ and $\phi_7(x_{5,1}) = 7$.

It is a matter of routine checking to see that the vertex labelings ϕ_5, ϕ_6 and ϕ_7 are $(s+3, 3)$-edge-antimagic vertex. In the same way as in Theorem 3.3.2, with respect to Theorem 5 from [9], we have that, for $p = 4s+3$ and $s \geq 1$, there is a super $(5s+7, 4)$-edge-antimagic total labeling of $K_{1,2} \cup 2sK_{1,1}$. □

Open Problem 4.2.2 *For $s \geq 1$, determine if there is a super $(a, 4)$-edge-antimagic total labeling of $K_{1,1} \cup 2sK_{1,2}$.*

In the case when $d = 3$, $m \geq 2$, $n \geq 2$ and $s \geq 1$, we do not have any answer for super edge-antimagicness of $K_{1,m} \cup 2sK_{1,n}$. Therefore, we propose

Open Problem 4.2.3 *For the graph $K_{1,m} \cup 2sK_{1,n}$, $m \geq 2$, $n \geq 2$ and $s \geq 1$, determine if there is a super $(a, 3)$-edge-antimagic total labeling.*

Chapter 5

SEATL of Disjoint Union of Complete s-partite Graphs

The study of antimagicness of complete s-partite graphs, for $s = 1$, began by Bača and Barrientos in [3]. They proved that the graph mK_n has a super (a, d)-edge-antimagic total labeling if and only if either (i) $d \in \{0, 2\}$, $n \in \{2, 3\}$ and m is odd, $m \geq 3$; or (ii) $d = 1$, $n \geq 2$ and $m \geq 2$; or (iii) $d \in \{3, 5\}$, $n = 2$ and $m \geq 2$; or (iv) $d = 4$, $n = 2$ and m is odd, $m \geq 3$. In [6] Bača and Brankovic continued the study on antimagicness of complete s-partite graphs when $s = 2$. They proved that the graph $mK_{n,n}$ has a super (a, d)-edge-antimagic total labeling for (i) $d = 1$ if $m \geq 2$ and $n \geq 1$; (ii) for $d = 2$ if $n = 1$ and $m \geq 3$ is odd; (iii) for $d \in \{3, 5\}$ if and only if $n = 1$ and $m \geq 2$; (iv) for $d = 4$ if and only if $n = 1$ and $m \geq 3$ is odd. The question of whether or not the disjoint union of multiple copies of complete s-partite graphs, for general s, admits a super (a, d)-edge-antimagic total labeling, is still open. In this chapter we present new results that partially answer this problem. In the first section we show that disjoint union of complete tripartite graph admits super (a, d)-edge-antimagic total labeling, and we generalise this result for m copies of complete s-partite graph in the second section. However, for some values of parameter s, m and n, the question remains open.

5.1 Complete tripartite graphs

Let $mK_{n,n,n}$ be a disjoint union of m copies of tripartite graph $K_{n,n,n}$ with vertex set $V(mK_{n,n,n}) = \{x_i^l : 1 \leq i \leq n, 1 \leq l \leq m\} \cup \{y_j^l : 1 \leq j \leq n, 1 \leq l \leq m\} \cup \{z_k^l : 1 \leq k \leq n, 1 \leq l \leq m\}$ and with edge set $E(mK_{n,n,n}) = \bigcup_{l=1}^{m}\{x_i^l y_j^l, x_i^l z_k^l, y_j^l z_k^l : 1 \leq i \leq n,\ 1 \leq j \leq n,\ 1 \leq k \leq n\}$. Thus $p = |V(mK_{n,n,n})| = 3mn$ and $q = |E(mK_{n,n,n})| = 3mn^2$. If $mK_{n,n,n}$, $m \geq 2$ and $n \geq 1$, is super (a, d)-edge-antimagic total then, by Lemma 3.0.1, it follows that $d < 3$.

◇ **Theorem 5.1.1** *The graph $mK_{n,n,n}$ has an $(a, 1)$-edge-antimagic vertex labeling if and only if $n = 1$ and m is odd, $m \geq 3$.*

Proof. Assume that $mK_{n,n,n}$ has an $(a, 1)$-edge-antimagic vertex labeling $\lambda_1 : V(mK_{n,n,n}) \rightarrow \{1, 2, \dots, 3mn\}$ and $W = \{w(uv) : uv \in E(mK_{n,n,n})\} = \{a, a + 1, a + 2, \dots, a + (3mn^2 - 1)\}$ is the set of edge-weights. The sum of the edge-weights in the set W is

$$\sum_{uv \in E(mK_{n,n,n})} w(uv) = 3mn^2\left(a + \frac{3mn^2 - 1}{2}\right). \tag{5.1}$$

In the computation of the edge-weights of $mK_{n,n,n}$, the label of each vertex is used $2n$ times. The sum of all vertex labels used to calculate the edge-weights is equal to

$$2n \sum_{u \in V(mK_{n,n,n})} \lambda_1(u) = 3mn^2(1 + 3mn). \tag{5.2}$$

Since Equations (5.1) and (5.2) gives

$$\sum_{uv \in E(mK_{n,n,n})} w(uv) = 2n \sum_{u \in V(mK_{n,n,n})} \lambda_1(u),$$

it immediately follows that

$$a = \frac{3mn(2-n)+3}{2}.$$

The minimum edge weight a is a positive integer if and only if $n = 1$ and m is odd, $m \geq 3$.

The required $\left(\frac{3m+3}{2}, 1\right)$-edge-antimagic vertex labeling λ_1 can be defined in the following way.

$$\lambda_1(x_1^l) = \begin{cases} \frac{l+1}{2} & \text{if } l \text{ is odd} \\ \frac{m+l+1}{2} & \text{if } l \text{ is even} \end{cases}$$

$$\lambda_1(y_1^l) = \begin{cases} \frac{3m+l}{2} & \text{if } l \text{ is odd} \\ m + \frac{l}{2} & \text{if } l \text{ is even} \end{cases}$$

$$\lambda_1(z_1^l) = 3m - l + 1, \quad \text{for all } 1 \leq l \leq m.$$

This completes the proof. □

◇ **Theorem 5.1.2** *For $d \in \{0, 2\}$, the graph $mK_{n,n,n}$ is super (a, d)-edge-antimagic total if and only if $n = 1$ and m is odd, $m \geq 3$.*

Proof.

Case 1. $d = 0$

Figueroa-Centeno, Ichishima and Muntaner-Batle (see Lemma 3.0.2) showed that a (p, q) graph G is super magic (super $(a, 0)$-edge-antimagic total) if and only if there exists an $(a-p-q, 1)$-edge-antimagic vertex labeling. According to Theorem 5.1.1, the graph $mK_{n,n,n}$ has $\left(\frac{3m+3}{2}, 1\right)$-edge-antimagic vertex labeling if and only if $n = 1$ and m is odd. With respect to Lemma 3.0.2, and for $p = 3mn, q = 3mn^2$, we have that the graph $mK_{n,n,n}$ has a super $(\frac{15m+3}{2}, 0)$-edge-antimagic total labeling if and only if $n = 1$ and m is odd.

Case 2. $d = 2$

Assume that $mK_{n,n,n}$, $m \geq 2, n \geq 1$, has a super (a, d)-edge-antimagic total labeling $\lambda_2 : V(mK_{n,n,n}) \cup E(mK_{n,n,n}) \to \{1, 2, \ldots, 3mn + 3mn^2\}$ and

$\{w(uv) = \lambda_2(u)+\lambda_2(uv)+\lambda_2(v) : uv \in E(mK_{n,n,n})\} = \{a, a+d, a+2d, \dots, a+(3mn^2-1)d\}$ is the set of the edge-weights. Then

$$\sum_{uv\in E(mK_{n,n,n})} w(uv) = 3mn^2\big(a + \frac{(3mn^2-1)d}{2}\big) \tag{5.3}$$

is the sum of all the edge-weights. In the computation of the edge-weights of $mK_{n,n,n}$, under the labeling λ_2, the label of each vertex is used $2n$ times and the label of each edge is used once. Thus

$$2n\sum_{l=1}^{m}\Big(\sum_{i=1}^{n}\lambda_2(x_i^l)+\sum_{j=1}^{n}\lambda_2(y_j^l)+\sum_{k=1}^{n}\lambda_2(z_k^l)\Big)+$$
$$\sum_{l=1}^{m}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_2(x_i^l y_j^l)+\sum_{i=1}^{n}\sum_{k=1}^{n}\lambda_2(x_i^l z_k^l)+\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_2(y_j^l z_k^l)\Big)=$$
$$9mn^2\Big(\frac{mn^2+4mn+1}{2}\Big). \tag{5.4}$$

Since we assume that λ_2 is a super (a, d)-edge-antimagic total labeling, the sum of edge-weights is equal to the sum of the vertex and edge labels. Combining (5.3) and (5.4) gives the following equation

$$a = \frac{3mn^2+12mn+3-(3mn^2-1)d}{2}. \tag{5.5}$$

The minimum possible edge weight under the labeling λ_2 is at least $3mn+4$. So, for $d = 2$, Equation (5.5) gives the following inequalities

$$\begin{aligned} 3mn+4 &\leq \frac{12mn-3mn^2+5}{2} \\ mn(n-2) &\leq -1. \end{aligned}$$

The last inequality is true if and only if $n = 1$. Then, from Equation (5.5), it follows that $a = \frac{9m+5}{2}$ and this is an integer if and only if m is odd.

In the same way as in Theorem 3.3.2, since labeling λ_1 from the proof of Theorem 5.1.1 is a $\big(\frac{3m+3}{2}, 1\big)$-edge-antimagic vertex labeling of $mK_{1,1,1}$ when m is odd, with respect to Theorem 5 from [9], we have that $mK_{1,1,1}$, for m odd, $m \geq 3$, has a super $\big(\frac{9m+5}{2}, 2\big)$-edge-antimagic total labeling. □

◇ **Theorem 5.1.3** *The graph $mK_{n,n,n}$ has a super $(6mn+2, 1)$-edge-antimagic total labeling for every $m \geq 2$ and $n \geq 1$.*

Proof. If $d = 1$ then, from Equation (5.5), it follows that $a = 6mn+2$. Define the bijective function $\lambda_3 : V(mK_{n,n,n}) \cup E(mK_{n,n,n}) \to \{1, 2, \ldots, 3mn+3mn^2\}$, for $m \geq 2$ and $n \geq 1$, in the following way:

$$\begin{aligned}
\lambda_3(x_i^l) &= (3i-3)m + l, \text{ for } 1 \leq i \leq n \text{ and } 1 \leq l \leq m \\
\lambda_3(y_j^l) &= (3j-2)m + l, \text{ for } 1 \leq j \leq n \text{ and } 1 \leq l \leq m \\
\lambda_3(z_k^l) &= (3k-1)m + l, \text{ for } 1 \leq k \leq n \text{ and } 1 \leq l \leq m.
\end{aligned}$$

If $1 \leq l \leq m$ then

$\lambda_3(x_i^l y_j^l) = 3mn(n+1-2j+2i) + 3m\sum_{t=0}^{j-i-1}(1+2t) + 1 - l - 3m(i-1)$,

for $1 \leq i \leq n$ and $i \leq j \leq n$,

$\lambda_3(x_i^l y_j^l) = 3mn(n+2-2i+2j) + 6m\sum_{t=0}^{i-j-1} t + 1 - l - 3m(j-1)$,

for $1 \leq j \leq n-1$ and $j+1 \leq i \leq n$,

$\lambda_3(y_j^l z_k^l) = 3mn(n+1-2k+2j) + 3m\sum_{t=0}^{k-j-1}(1+2t) + 1 - l - m(3j-2)$,

for $1 \leq j \leq n$ and $j \leq k \leq n$,

$\lambda_3(y_j^l z_k^l) = 3mn(n+2-2j+2k) + 6m\sum_{t=0}^{j-k-1} t + 1 - l - m(3k-2)$,

for $1 \leq k \leq n-1$ and $k+1 \leq j \leq n$,

$\lambda_3(z_k^l x_i^l) = 3mn(n+3-2i+2k) + 3m\sum_{t=0}^{i-k-2}(1+2t) + 1 - l - m(3k-1)$,

for $1 \leq k \leq n-1$ and $k+1 \leq i \leq n$,

$\lambda_3(z_k^l x_i^l) = 3mn(n-2k+2i) + 6m\sum_{t=0}^{k-i} t + 1 - l - m(3i-4)$,

for $1 \leq i \leq n$ and $i \leq k \leq n$.

Let $A^l = (a_{ij}^l)$ be a system of square matrices, for all $l = 1, 2, \ldots, m$, where $a_{ij}^l = \lambda_3(x_i^l) + \lambda_3(y_j^l)$ for $1 \leq i \leq n, 1 \leq j \leq n$ and $\eta = 3mn+2l, \theta = 6mn+2l$.

$$A^l = \begin{bmatrix} m+2l & 4m+2l & 7m+2l & \dots & \eta-5m & \eta-2m \\ 4m+2l & 7m+2l & 10m+2l & \dots & \eta-2m & \eta+m \\ 7m+2l & 10m+2l & 13m+2l & \dots & \eta+m & \eta+4m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta-5m & \eta-2m & \eta+m & \dots & \theta-11m & \theta-8m \\ \eta-2m & \eta+m & \eta+4m & \dots & \theta-8m & \theta-5m \end{bmatrix}$$

Let $B^l = (b^l_{jk})$ be a system of square matrices for all $l = 1, 2, \dots, m$, where $b^l_{jk} = \lambda_3(y^l_j) + \lambda_3(z^l_k)$ for $1 \le j \le n, 1 \le k \le n$ and $\eta = 3mn + 2l, \theta = 6mn + 2l$.

$$B^l = \begin{bmatrix} 3m+2l & 6m+2l & 9m+2l & \dots & \eta-3m & \eta \\ 6m+2l & 9m+2l & 12m+2l & \dots & \eta & \eta+3m \\ 9m+2l & 12m+2l & 15m+2l & \dots & \eta+3m & \eta+6m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta-3m & \eta & \eta+3m & \dots & \theta-9m & \theta-6m \\ \eta & \eta+3m & \eta+6m & \dots & \theta-6m & \theta-3m \end{bmatrix}$$

Let $C^l = (c^l_{ki})$ be a system of square matrices for all $l = 1, 2, \dots, m$, where $c^l_{ki} = \lambda_3(z^l_k) + \lambda_3(x^l_i)$ for $1 \le k \le n, 1 \le i \le n$ and $\eta = 3mn + 2l, \theta = 6mn + 2l$.

$$C^l = \begin{bmatrix} 2m+2l & 5m+2l & 8m+2l & \dots & \eta-4m & \eta-m \\ 5m+2l & 8m+2l & 11m+2l & \dots & \eta-m & \eta+2m \\ 8m+2l & 11m+2l & 14m+2l & \dots & \eta+2m & \eta+5m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta-4m & \eta-m & \eta+2m & \dots & \theta-10m & \theta-7m \\ \eta-m & \eta+2m & \eta+5m & \dots & \theta-7m & \theta-4m \end{bmatrix}$$

The systems of square matrices A^l, B^l and C^l, for $l = 1, 2, \dots, m$, describe the edge-weights of $mK_{n,n,n}$ under the vertex labeling. The labels of the edges of $mK_{n,n,n}$, described by labeling λ_3, can be exhibited by the systems of square matrices $H^l = (h^l_{ij}), P^l = (p^l_{jk})$ and $R^l = (r^l_{ki})$ for $l = 1, 2, \ldots, m$, where

$$\begin{aligned} h^l_{ij} &= \lambda_3(x^l_i y^l_j), \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq n \\ p^l_{jk} &= \lambda_3(y^l_j z^l_k), \text{ for } 1 \leq j \leq n \text{ and } 1 \leq k \leq n \\ r^l_{ki} &= \lambda_3(z^l_k x^l_i), \text{ for } 1 \leq k \leq n \text{ and } 1 \leq i \leq n, \text{ respectively.} \end{aligned}$$

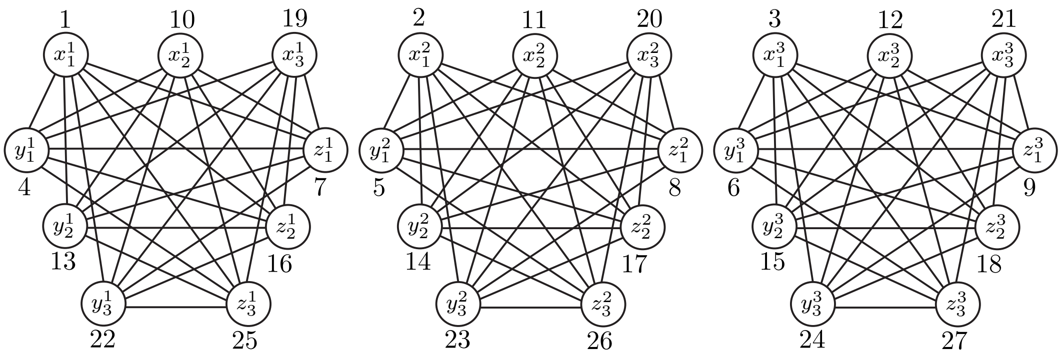
For $\gamma = 3mn^2 + 1 - l$, $\xi = 3mn$ and $\vartheta = \xi + 1 - l$, the systems of square matrices are as follows:

$$H^l = \begin{bmatrix} \gamma+\xi & \gamma-\xi+3m & \gamma-3\xi+12m & \ldots & \vartheta+12m & \vartheta+3m \\ \gamma & \gamma+\xi-3m & \gamma-\xi & \ldots & \vartheta+24m & \vartheta+9m \\ \gamma-2\xi+6m & \gamma-3m & \gamma+\xi-6m & \ldots & \vartheta+42m & \vartheta+21m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vartheta+18m & \vartheta+33m & \vartheta+54m & \ldots & \gamma+6m & \gamma-\xi+6m \\ \vartheta+6m & \vartheta+15m & \vartheta+30m & \ldots & \gamma-\xi+6m & \gamma+3m \end{bmatrix},$$

$$P^l = \begin{bmatrix} \gamma+\xi-m & \gamma-\xi+2m & \gamma-3\xi+11m & \ldots & \vartheta+11m & \vartheta+2m \\ \gamma-m & \gamma+\xi-4m & \gamma-\xi-m & \ldots & \vartheta+23m & \vartheta+8m \\ \gamma-2\xi+5m & \gamma-4m & \gamma+\xi-7m & \ldots & \vartheta+41m & \vartheta+20m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vartheta+17m & \vartheta+32m & \vartheta+53m & \ldots & \gamma+5m & \gamma-2\xi+8m \\ \vartheta+5m & \vartheta+14m & \vartheta+29m & \ldots & \gamma-\xi+5m & \gamma+2m \end{bmatrix},$$

$$R^l = \begin{bmatrix} \gamma+m & \gamma+\xi-2m & \gamma-\xi+m & \ldots & \vartheta+25m & \vartheta+10m \\ \gamma-2\xi+7m & \gamma-2m & \gamma+\xi-5m & \ldots & \vartheta+43m & \vartheta+22m \\ \gamma-4\xi+19m & \gamma-2\xi+4m & \gamma-5m & \ldots & \vartheta+67m & \vartheta+40m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vartheta+7m & \vartheta+16m & \vartheta+31m & \ldots & \gamma-\xi+7m & \gamma+4m \\ \vartheta+m & \vartheta+4m & \vartheta+13m & \ldots & \gamma-3\xi+13m & \gamma-\xi+4m \end{bmatrix}.$$

All edge-weights of $mK_{n,n,n}$, under the total labeling λ_3, can be presented as the systems of square matrices $S^l = A^l + H^l, T^l = B^l + P^l$, and $U^l = C^l + R^l$, for $l = 1, 2, \ldots, m$. It is not difficult to verify that the entries of square matrices S^l, T^l and U^l, for $l = 1, 2, \ldots, m$, are formed from consecutive integers $6mn + 2, 6mn + 3, 6mn + 4, \ldots, 3mn^2 + 6mn, 3mn^2 + 6mn + 1$. This implies that the total labeling λ_3 is super $(6mn + 2, 1)$-edge-antimagic total, for every $m \geq 2$ and $n \geq 1$. □



The edge labels can be exhibited by the following systems of square matrices.

$$H^1 = \begin{bmatrix} 108 & 63 & 36 \\ 81 & 99 & 54 \\ 45 & 72 & 90 \end{bmatrix} \quad H^2 = \begin{bmatrix} 107 & 62 & 35 \\ 80 & 98 & 53 \\ 44 & 71 & 89 \end{bmatrix} \quad H^3 = \begin{bmatrix} 106 & 61 & 34 \\ 79 & 97 & 52 \\ 43 & 70 & 88 \end{bmatrix}$$

$$P^1 = \begin{bmatrix} 105 & 60 & 33 \\ 78 & 96 & 51 \\ 42 & 69 & 87 \end{bmatrix} \quad P^2 = \begin{bmatrix} 104 & 59 & 32 \\ 77 & 95 & 50 \\ 41 & 68 & 86 \end{bmatrix} \quad P^3 = \begin{bmatrix} 103 & 58 & 31 \\ 76 & 94 & 49 \\ 40 & 67 & 85 \end{bmatrix}$$

$$R^1 = \begin{bmatrix} 84 & 48 & 30 \\ 102 & 75 & 39 \\ 57 & 93 & 66 \end{bmatrix} \quad R^2 = \begin{bmatrix} 83 & 47 & 29 \\ 101 & 74 & 38 \\ 56 & 92 & 65 \end{bmatrix} \quad R^3 = \begin{bmatrix} 82 & 46 & 28 \\ 100 & 73 & 37 \\ 55 & 91 & 64 \end{bmatrix}$$

Figure 5.1: Super $(56, 1)$-edge-antimagic total labeling of $3K_{3,3,3}$.

5.2 Complete s-partite graphs

In Section 5.1, we proved that the disjoint union of multiple copies of complete tripartite graph has a super (a, d)-edge-antimagic total labeling for (i) $d \in \{0, 2\}$ if and only if $n = 1$ and $m \geq 3$ is odd; and (ii) $d = 1$ if $m \geq 2$ and $n \geq 1$. Now, we will study super edge-antimagicness of a disjoint union of m copies of complete s-partite graphs, denoted by $mK_{\underbrace{n,n,\ldots,n}_{s}}$. This is a disconnected graph with vertex set $V(mK_{\underbrace{n,n,\ldots,n}_{s}}) = \bigcup_{j=1}^{m}\bigcup_{t=1}^{s}\left\{x_{t,i}^{j} : 1 \leq i \leq n\right\}$ and edge set $E(mK_{\underbrace{n,n,\ldots,n}_{s}}) = \bigcup_{j=1}^{m}\bigcup_{t=1}^{s-1}\bigcup_{i=1}^{n}\left\{x_{t,i}^{j}x_{t+k,r}^{j} : 1 \leq k \leq s-t, 1 \leq r \leq n\right\}$, for $m \geq 2$, $n \geq 1$ and $s \geq 2$. Thus, let $p = |V(mK_{\underbrace{n,n,\ldots,n}_{s}})| = mns$ and $q = |E(mK_{\underbrace{n,n,\ldots,n}_{s}})| = \frac{mn^2s(s-1)}{2}$.

If the graph $mK_{\underbrace{n,n,\ldots,n}_{s}}$ admits a super (a, d)-edge-antimagic total labeling $\sigma : V(mK_{\underbrace{n,n,\ldots,n}_{s}}) \cup E(mK_{\underbrace{n,n,\ldots,n}_{s}}) \rightarrow \left\{1, 2, \ldots, \frac{mns}{2}(n(s-1)+2)\right\}$, then $W = \left\{w(uv) = \sigma(u) + \sigma(uv) + \sigma(v) : uv \in E(mK_{\underbrace{n,n,\ldots,n}_{s}})\right\} = \left\{a, a+d, a+2d, \ldots, a+\left(\frac{mn^2s(s-1)}{2} - 1\right)d\right\}$ is the set of the edge-weights and the sum of all the edge-weights in W is

$$\sum_{uv \in E(mK_{\underbrace{n,n,\ldots,n}_{s}})} w(uv) = \frac{mn^2s(s-1)}{8}\Big[4a + \big(mn^2s(s-1) - 2\big)d\Big]. \quad (5.6)$$

In the computation of the edge-weights of $mK_{\underbrace{n,n,\ldots,n}_{s}}$, each edge label is used once and the label of each vertex is used $(s-1)n$ times. The sum of all the vertex labels and the edge labels used to calculate the edge-weights is thus equal to

$$(s-1)n \sum_{u \in V(mK_{\underbrace{n,n,\ldots,n}_{s}})} \sigma(u) + \sum_{uv \in E(mK_{\underbrace{n,n,\ldots,n}_{s}})} \sigma(uv) =$$

$$\frac{mns+1}{2} mn^2 s(s-1) + \frac{mn^2 s(s-1)}{8} \Big[4mns + mn^2 s(s-1) + 2\Big]. \qquad (5.7)$$

The sum of all the vertex labels and the edge labels used to calculate the edge-weights is equal to the sum of the edge-weights in the set W, under the labeling σ. Thus combining Equations (5.6) and (5.7) gives

$$4a + (mn^2 s(s-1) - 2)d = 8mns + mn^2 s(s-1) + 6. \qquad (5.8)$$

At this point, we are ready to establish an upper bound on the parameter d.

◇ **Lemma 5.2.1** *For the graph* $mK_{\underbrace{n,n,\ldots,n}_{s}}$*,* $m \geq 2$*,* $n = 1$ *and* $s = 4$*, there is no super* (a, d)*-edge-antimagic total labeling with* $d \geq 3$*.*

Proof. Since the minimum possible edge weight, under the labeling σ, is at least $mns + 4$, from Equation (5.8) it follows that

$$d \leq 1 + \frac{4mns - 8}{mn^2 s(s-1) - 2}. \qquad (5.9)$$

It is easy to verify that $1 < \frac{4mns-8}{mn^2 s(s-1)-2} < 2$ only when $m \geq 2$, $n = 1$ and $s = 4$, which completes the proof. □

Since $\frac{4mns-8}{mn^2 s(s-1)-2} < 1$, for $m \geq 2$, $n \geq 2$ and $s \geq 4$, Inequality (5.9) gives $d < 2$ and we have the following lemma.

◇ **Lemma 5.2.2** *For the graph* $mK_{\underbrace{n,n,\ldots,n}_{s}}$*,* $m \geq 2$*,* $n \geq 2$ *and* $s \geq 4$*, there is no super* (a, d)*-edge-antimagic total labeling with* $d \geq 2$*.*

First, we deal with super $(a, 0)$-edge-antimagic total labeling for the disjoint union of m copies of complete s-partite graph.

◇ **Theorem 5.2.1** *If either $s \equiv 0, 1 \pmod 4$, $s \geq 4$, $m \geq 2$, $n \geq 1$, or mn is even, $m \geq 2$, $n \geq 1$, $s \geq 4$, then there is no super $(a, 0)$-edge-antimagic total labeling for $mK_{\underbrace{n, n, \ldots, n}_{s}}$.*

Proof. Assume that $mK_{\underbrace{n, n, \ldots, n}_{s}}$ admits a super $(a, 0)$-edge-antimagic total labeling $\sigma : V(mK_{\underbrace{n, \ldots, n}_{s}}) \cup E(mK_{\underbrace{n, \ldots, n}_{s}}) \to \{1, 2, \ldots, \frac{mns}{2}(n(s-1)+2)\}$. From Equation (5.8) we have

$$a = 2mns + \frac{mn^2 s(s-1)}{4} + \frac{3}{2}. \tag{5.10}$$

If either $s \equiv 0, 1 \pmod 4$, $s \geq 4$, $m \geq 2$, $n \geq 1$, or mn is even, $m \geq 2$, $n \geq 1$ and $s \geq 4$, then from Equation (5.10) it is easy to see that the value a is not an integer, which is a contradiction. □

The minimum edge weight in Equation (5.10) is an integer if and only if mn is odd and $s \equiv 2, 3 \pmod 4$. In this case we do not have any answer concerning the super $(a, 0)$-edge-antimagicness of $mK_{\underbrace{n, n, \ldots, n}_{s}}$. Therefore, we propose

Open Problem 5.2.1 *For the graph $mK_{\underbrace{n, n, \ldots, n}_{s}}$, mn odd, $m \geq 3$, $n \geq 1$ and $s \equiv 2, 3 \pmod 4$, $s \geq 6$, determine if there is a super $\left(2mns + \frac{mn^2 s(s-1)+6}{4}, 0\right)$-edge-antimagic total labeling.*

From Lemma 5.2.1, it follows that the graph $mK_{\underbrace{n, n, \ldots, n}_{s}}$ may possibly be super $(a, 2)$-edge-antimagic total only when $m \geq 2$, $n = 1$ and $s = 4$. However, our next result gives a negative answer.

◇ **Theorem 5.2.2** *If $m \geq 2$, $n = 1$ and $s = 4$, then there is no super $(a, 2)$-edge-antimagic total labeling for the graph $mK_{\underbrace{n, n, \ldots, n}_{s}}$.*

Proof. Assume to the contrary that for $m \geq 2$, $n = 1$ and $s = 4$, the graph $mK_{\underbrace{n, n, \ldots, n}_{s}}$ has a super $(a, 2)$-edge-antimagic total labeling $\sigma : V(mK_{n,n,n,n}) \cup E(mK_{n,n,n,n}) \rightarrow \{1, 2, \ldots, 10m\}$. From Equation (5.8), we get that $2a = 10m + 5$. This contradicts the fact that a is an integer. □

Now, we will concentrate on the existence of super $(a, 1)$-edge-antimagic total labeling of disjoint union of m copies of complete 4-partite graph.

◇ **Theorem 5.2.3** *The graph $mK_{n,n,n,n}$ has a super $(8mn+2, 1)$-edge-antimagic total labeling for every $m \geq 2$ and $n \geq 1$.*

Proof. If $s = 4$ and $d = 1$, then from Equation (5.8), it follows that $a = 8mn + 2$. Consider the bijective function

$\sigma_1 : V(mK_{n,n,n,n}) \cup E(mK_{n,n,n,n}) \rightarrow \{1, 2, \ldots, 2mn(3n + 2)\}$, where

$\sigma_1(x^j_{t,i}) = m(4i + t - 5) + j \quad \text{for} \quad 1 \leq t \leq 4,\ 1 \leq i \leq n,\ 1 \leq j \leq m.$

$$\sigma_1(x^j_{1,i}x^j_{2,r}) = \begin{cases} 2mn(3n + 8 - 6i) + 6m\sum\limits_{k=0}^{i-2}(1 + 2k) + 2m(1 - r) - j + 1, \\ \text{for} \;\; 1 \leq i \leq n, \;\; 1 \leq r \leq n - i + 1 \;\; \text{and} \;\; 1 \leq j \leq m \\ 2mn(6i - 3n - 2) + 4m\sum\limits_{k=0}^{n-i}(1 + 3k) - 2m(r - 2 + i) - j + 1, \\ \text{for} \;\; 2 \leq i \leq n, \;\; n + 2 - i \leq r \leq n \;\; \text{and} \;\; 1 \leq j \leq m \end{cases}$$

$$\sigma_1(x^j_{1,i}x^j_{3,r}) = \begin{cases} 6mn(n + 2 - 2i) + 2m\sum\limits_{k=1}^{i-1}(6k - 1) + m(3 - 2r) - j + 1, \\ \text{for} \;\; 1 \leq i \leq n, \;\; 1 \leq r \leq n - i + 1 \;\; \text{and} \;\; 1 \leq j \leq m \\ 6mn(2i - n) + 12m\sum\limits_{k=0}^{n-1-i}(1 + k) - m(2r - 5 + 2i) - j + 1, \\ \text{for} \;\; 2 \leq i \leq n, \;\; n + 2 - i \leq r \leq n \;\; \text{and} \;\; 1 \leq j \leq m \end{cases}$$

$$\sigma_1(x^j_{1,i}x^j_{4,r}) = \begin{cases} \sigma_1(x^j_{1,i}x^j_{2,r}) - m, \\ \text{for } \ 1 \le i \le n, 1 \le r \le n-i+1 \quad \text{and} \quad 1 \le j \le m \\ \\ \sigma_1(x^j_{1,i}x^j_{2,r}) - m, \\ \text{for } \ 2 \le i \le n, n+2-i \le r \le n \quad \text{and} \quad 1 \le j \le m \end{cases}$$

$$\sigma_1(x^j_{2,i}x^j_{3,r}) = \begin{cases} 2mn(3n+4-6i) + 2m\sum\limits_{k=1}^{i-1}(6k+1) + m(5-2r) - j + 1, \\ \text{for } \ 1 \le i \le n, \ \ 1 \le r \le n-i+1 \quad \text{and} \quad 1 \le j \le m \\ \\ 2mn(6i-3n+2) + 4m\sum\limits_{k=0}^{n-1-i}(2+3k) - m(2r-3+2i) - j + 1, \\ \text{for } \ 2 \le i \le n, \ \ n+2-i \le r \le n \quad \text{and} \quad 1 \le j \le m \end{cases}$$

$$\sigma_1(x^j_{2,i}x^j_{4,r}) = \begin{cases} \sigma_1(x^j_{1,i}x^j_{3,r}) - m, \\ \text{for } \ 1 \le i \le n, 1 \le r \le n-i+1 \quad \text{and} \quad 1 \le j \le m \\ \\ \sigma_1(x^j_{1,i}x^j_{3,r}) - m, \\ \text{for } \ 2 \le i \le n, n+2-i \le r \le n \quad \text{and} \quad 1 \le j \le m \end{cases}$$

$$\sigma_1(x^j_{3,i}x^j_{4,r}) = \begin{cases} 2mn(3n+4-6i) + 2m\sum\limits_{k=1}^{i-1}(6k+1) + m(4-2r) - j + 1, \\ \text{for } \ 1 \le i \le n-1, \ \ 1 \le r \le n-i \quad \text{and} \quad 1 \le j \le m, \\ \\ 2mn(6i-3n+2) + 4m\sum\limits_{k=0}^{n-1-i}(2+3k) - 2m(r-1+i) - j + 1, \\ \text{for } \ 1 \le i \le n, \ \ n+1-i \le r \le n \quad \text{and} \quad 1 \le j \le m. \end{cases}$$

It is not difficult to verify that the system of sets $\bigcup\limits_{j=1}^{m}\bigcup\limits_{t=1}^{3}\bigcup\limits_{i=1}^{n}\left\{\sigma_1(x^j_{t,i}) + \sigma_1(x^j_{t,i}x^j_{t+k,r}) + \sigma_1(x^j_{t+k,r}) : 1 \le k \le 4-t, 1 \le r \le n\right\}$ consists of consecutive integers of the form $8mn+2, 8mn+3, 8mn+4, \ldots, 6n^2m+8mn, 6n^2m+8mn+1$. Thus σ_1 is a super $(8mn+2,1)$-edge-antimagic total labeling. □

A natural question is whether we can say anything about super $(a,1)$-edge-antimagic total labeling for disjoint union of complete s-partite graphs for

$s \geq 5$. Although we have not yet found general formulas for vertex and edge labelings of $mK_{\underbrace{n,n,\ldots,n}_{s}}$ that will produce a required super $(a,1)$-edge-antimagic total labeling, the observed antimagic properties of $mK_{\underbrace{n,n,\ldots,n}_{s}}$ lead us to suggest

Conjecture 5.2.1 *There is a super $(a,1)$-edge-antimagic total labeling for the graph $mK_{\underbrace{n,n,\ldots,n}_{s}}$, for $s \geq 5$ and for every $m \geq 2$ and $n \geq 1$.*

PART II

Chapter 6

Graphs of Large Order

6.1 Motivation

In this part of the thesis, we will deal with another problem in graph theory, namely, the degree/diameter problem. Interestingly, the problem is very different for undirected and directed graphs. Therefore, we consider separately the undirected and directed version of this problem.

The design of large communication networks has become an issue of growing interest due to recent advances in very large scale integrated technology. In such networks, it is desirable to have connections which achieve the most efficient and reliable communication in view of practical economic constraint.

There are several factors which should be considered in communication network design. Two of the factors which seem to appear most frequently, namely, (i) the number of connections which can be attached to a processing element is limited, and (ii) a short communication route between any two processing elements is required. We would like to end up with a large network subject to these constraints.

Another factor that may be considered when designing a communication network is fault tolerance. The fault tolerance of an interconnection network is the capability of the network to continue working when a number of processing elements and/or network links become faulty. The larger the number of faulty processing elements and/or network links that can be tolerated the better the fault tolerance. Together with this, we may add another constraint, called a diameter vulnerability. In this case, besides requiring the interconnection network to keep working under a faulty condition, the diameter of the resulting network is desired to be the same as or 'close to' the original diameter.

We may require an overall balance of the system when designing a communication network. Given that all the processing elements have the same status, the flow of information and exchange of data between processing elements will be on average faster if there is a similar number of interconnections coming in and going out of each processing element, that is, if there is a balance (or regularity) in the network.

Translating the above required conditions in terms of the underlying graphs, the problem is to find large graph with given maximum degree and diameter. This naturally leads to the well-known fundamental problem, called the $N(\Delta, D)$*-problem*: For given numbers Δ and D, construct graphs of maximum degree Δ and diameter $\leq D$ with the largest possible number of vertices $n_{\Delta,D}$. The $N(\Delta, D)$-problem is also known as the *degree/diameter problem.*

In the degree/diameter problem, the value of $N(\Delta, D)$ is not known for most values of Δ and D. Therefore, it is useful to investigate the lower and upper bounds on $N(\Delta, D)$. A natural number $n_{l_{\Delta,D}}$ is a *lower bound* of $N(\Delta, D)$ if we can prove the existence of a graph of maximum degree at most Δ, diameter D and exactly $n_{l_{\Delta,D}}$ vertices. A natural number $n_{u_{\Delta,D}}$ is an *upper bound* of $N(\Delta, D)$ if we can show that there is no graph of maximum degree at most Δ, diameter D and with the number of vertices more than $n_{u_{\Delta,D}}$.

A natural general upper bound on the order $n_{\Delta,D}$ of a graph is the *Moore bound.* However, there are only a few graphs of order equal to the Moore bound. This gives rise to two directions of research connected with the $N(\Delta, D)$-problem:

(i) Proving the non-existence of graphs of order 'close' to the Moore bound and so lowering the upper bound $n_{u_{\Delta,D}}$;

(ii) Constructing large graphs and so incidentally obtaining better lower bounds $n_{l_{\Delta,D}}$.

When working in either of these directions, it is useful to establish some general structural properties of the graphs in question. For example, we may first wish to establish the regularity of such graphs.

In the next sections we present known results concerning the existence of Moore graphs and graphs of order close to the Moore bound. This section is largely based on the survey by Miller and Širáň [89].

6.2 Moore graphs

For a given maximum degree Δ and diameter D, consider a standard spanning tree of the graph from an arbitrary vertex up to D levels. By counting the vertices at each level, it is easy to derive a natural upper bound for the order $n_{\Delta,D}$ of a graph of maximum degree d and diameter at most k, (see Figure 6.1). Let v be a vertex of the graph G and let n_i, for $0 \leq i \leq D$, be the number of vertices at distance i from v. Then $n_i \leq \Delta(\Delta-1)^{i-1}$, for $1 \leq i \leq D$, and so

$$n_{\Delta,D} = \sum_{i=0}^{D} n_i \leq 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1} \qquad (6.1)$$

The right-hand side of (6.1) is called the *Moore bound* and is denoted by $\mathcal{M}_{\Delta,D}$. The bound was named after E. F. Moore who first proposed the problem, as mentioned in [66]. A graph whose order is equal to the Moore bound $\mathcal{M}_{\Delta,D}$ is called a *Moore graph*; such a graph is necessarily regular of degree Δ.

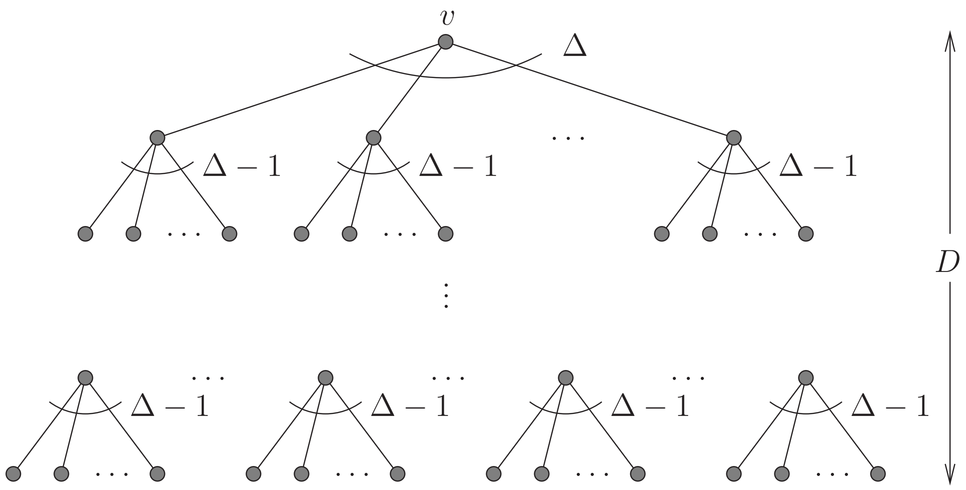


Figure 6.1: A spanning tree of a Moore graph.

Moore graphs exist only for a few cases, namely, if diameter $D = 1$ and degree $\Delta \geq 1$, or if diameter $D = 2$ and degree $\Delta = 2, 3, 7$ and possibly $\Delta = 57$ (proved by Hoffman and Singleton [66]), or if diameter $D \geq 3$ and degree $\Delta = 2$ (proved by Damerell [39], and independently by Bannai and Ito [12]).

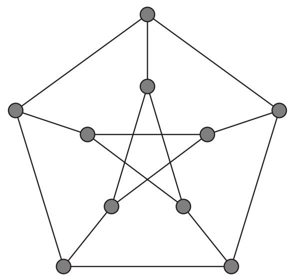


Figure 6.2: Petersen graph $\mathcal{M}_{3,2}$.

Moore graphs for diameter $D = 1$ and degree $\Delta \geq 1$ are the complete graphs $K_{\Delta+1}$. For diameter $D = 2$, Moore graphs are the cycle C_5 for degree $\Delta = 2$, the Petersen graph for degree $\Delta = 3$ (Figure 6.2) and the Hoffman-Singleton graph for degree $\Delta = 7$. The Hoffman-Singleton graph was first constructed by analysing the eigenvalues of the graph which admits the graph's adjacency matrix. Since then, a number of authors have presented different ways of constructing the Hoffman-Singleton graph. In [25], Robertson constructed the Hoffman-Singleton graph by grouping the 50 vertices of the graph into 5 pentagrams $P_0, \ldots, P_4$ and pentagons $Q_0, \ldots, Q_4$ and joining the edges between vertices of the pentagrams and pentagons as follows: vertex i of pentagram P_j is adjacent to vertex $i + jk \pmod 5$ of pentagon Q_k, see Figure 6.3. Finally, for diameter $D \geq 3$ and $\Delta = 2$, the Moore graphs are the cycles on $2D+1$ vertices C_{2D+1}.

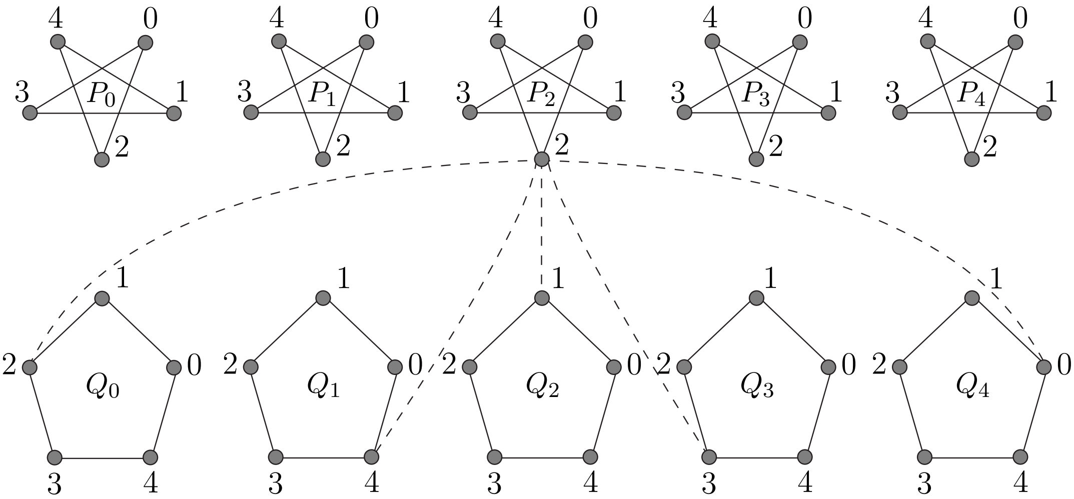


Figure 6.3: Robertson's construction of Hoffman-Singleton graph.

6.3 Graphs of order close to the Moore bound

Since Moore bound is attainable in only a few cases, the study of the existence of large graphs of given diameter and maximum degree focuses on graphs whose order is 'close' to the Moore bound. More precisely, researchers have been considering the question of the existence of graphs of degree at most Δ, diameter $D \geq 2$ and order $\mathcal{M}_{\Delta,D} - \delta$ (δ is the defect), where $1 \leq \delta < \Delta$. For convenience, we denote such graphs as (Δ, D, δ)-*graphs*.

It is easy to see that a (Δ, D, δ)-graph is regular of degree Δ. This can be shown since if there were a vertex v in a (Δ, D, δ)-graph with degree $\Delta_1 < \Delta$, then the order $n_{\Delta,D}$ of G,

$$\begin{aligned} n_{\Delta,D} &\leq 1 + \Delta_1 + \Delta_1(\Delta-1) + \Delta_1(\Delta-1)^2 + \cdots + \Delta_1(\Delta-1)^{D-1} \\ &= 1 + \Delta_1\Delta + \Delta_1(\Delta-1)^2 \cdots + \Delta_1(\Delta-1)^{D-1} \\ &\leq 1 + \Delta(\Delta-1) + \Delta(\Delta-1)^2 \cdots + \Delta(\Delta-1)^{D-1} \\ &= \mathcal{M}_{\Delta,D} - \Delta \\ &< \mathcal{M}_{\Delta,D} - (\Delta-1) \end{aligned}$$

Consequently, if $\delta \leq \Delta - 1$ then $n_{\Delta,D} < \mathcal{M}_{\Delta,D} - (\Delta - 1) \leq \mathcal{M}_{\Delta,D} - \delta$. This implies that all vertices must have the same degree Δ, that is, G must be regular.

Concerning graphs with defect $\delta = 1$, Erdős, Fajtlowitcz and Hoffman [47] proved that, apart from the cycle C_4, there are no graphs of defect 1, degree Δ and diameter 2; for a related result, see Fajtlowicz [49]. This result was generalised by Bannai and Ito [13], and also by Kurosawa and Tsujii [78], to all diameters. Hence, for all $\Delta \geq 3$, there are no $(\Delta, D, 1)$-graphs, and for $\Delta = 2$ the only such graphs are the cycles C_{2D}. It follows that, for $\Delta \geq 3$, we have $n_{\Delta,D} \leq M_{\Delta,D} - 2$.

Graphs with defect $\delta = 2$ represent a wide unexplored area. The $(2, D, 2)$-graphs are the cycles C_{2D-1}. For $\Delta \geq 3$, only five $(\Delta, D, 2)$-graphs are known at present: Two $(3, 2, 2)$-graphs of order 8, a $(4, 2, 2)$-graph of order 15, a $(5, 2, 2)$-graph of order 24 and a $(3, 3, 2)$-graph of order 20. The last three graphs were found by Elspas [45] and are known to be unique; in Bermond, Delorme and Farhi [24], the $(3, 3, 2)$-graph was constructed as a certain product of a 5-cycle with the field of order four. Furthermore, Nguyen and Miller [100, 101] proved some structural properties of $(\Delta, 2, 2)$-graphs and showed that $(\Delta, 2, 2)$-graphs do not exist for many values of Δ.

Little is known about graphs with defect $\delta \geq 3$. Jorgensen [73] proved that a graph with maximum degree 3 and diameter $D \geq 4$ cannot have defect two, which shows that $n_{3,D} \leq M_{3,D} - 3$ if $D \geq 4$; for D equal to 4 this was previously proved by Stanton, Seah and Cowan [111]. Additionally, Miller and Simanjuntak [94] proved that a graph with $\Delta = 4$ and $D \geq 3$ cannot have defect two which shows that $n_{4,D} \leq M_{4,D} - 3$ if $D \geq 3$. We summarise our current knowledge of the upper bound on the order of graphs in Table 6.1.

The current best lower bound on the maximum possible order of graphs of given D and Δ can be found by constructing large graphs. De Bruijn graphs give the lower bound $n_{\Delta,D} \geq (\frac{\Delta}{2})^D$ for any Δ and D. There are improvements on this bound for many small values of Δ and D. For diameter $D = 2$, Brown [36] gave the lower bound on the order of graphs as $n_{\Delta,2} \geq \Delta^2 - \Delta + 1$ for each Δ such that $\Delta - 1$ is a prime power. As shown in [47] and [41], this bound can be improved to $n_{\Delta,2} \geq \Delta^2 - \Delta + 2$ if $\Delta - 1$ is a power of 2.

Diameter D	*Maximum Degree* Δ	*Upper Bound for Order* $n_{\Delta,D}$
$D = 1$	$\Delta \geq 1$	$\mathcal{M}_{\Delta,1}$
$D = 2$	$\Delta = 2, 3, 7,\ 57(?)$	$\mathcal{M}_{\Delta,2}$
	other $\Delta \geq 2$	$\mathcal{M}_{\Delta,2} - 2$
$D = 3$	$\Delta = 2$	$\mathcal{M}_{2,3}$
	$\Delta = 3$	$\mathcal{M}_{3,3} - 2$
	$\Delta = 4$	$\mathcal{M}_{4,D} - 3$
	$\Delta \geq 5$	$\mathcal{M}_{\Delta,3} - 2$
$D \geq 4$	$\Delta = 2$	$\mathcal{M}_{2,4}$
	$\Delta = 3$	$\mathcal{M}_{3,D} - 3$
	$\Delta = 4$	$\mathcal{M}_{4,D} - 3$
	$\Delta \geq 5$	$\mathcal{M}_{\Delta,D} - 2$

Table 6.1: Current upper bounds of $n_{\Delta,D}$.

For the remaining values of Δ, we may use the following fact [69] about the distribution of prime numbers: For an arbitrary $\varepsilon > 0$, there is a constant b_ε such that for any natural m there is a prime between m and $b_\varepsilon m^{7/12+\varepsilon}$. This, in combination with vertex duplication (insertion of new vertices adjacent to all neighbours of certain old vertices) in the graphs of [36], implies that for any $\varepsilon > 0$ there is a constant c_ε such that, for any Δ, we have

$$n_{\Delta,2} \geq \Delta^2 - c_\varepsilon \Delta^{19/12+\varepsilon}. \tag{6.2}$$

For larger diameter, it seems more reasonable to focus on asymptotic behaviour of $n_{\Delta,D}$ for fixed D while $\Delta \to \infty$. Delorme [42] introduced the parameter $\mu_D = \lim\ \inf_{\Delta\to\infty} \frac{n_{\Delta,D}}{\Delta^D}$. Trivially, $\mu_D \leq 1$ for all D, and $\mu_1 = 1$; the bound (6.2) shows that $\mu_2 = 1$ as well. Further results of Delorme [43] imply that μ_D is also equal to 1 for $D = 3$ and $D = 5$. The values of μ_D for other diameters D are unknown. For example, for diameter 4 we only know that $\mu_4 \geq 1/4$; see Delorme [41] for more information.

The above facts can be seen as an evidence in favour of an earlier conjecture of Bollobás [33] that, for each $\varepsilon > 0$, it should be the case that $n_{\Delta,D} > (1-\varepsilon)\Delta^D$ if Δ and D are sufficiently large.

We have included an overview of the degree/diameter problem for undirected graphs for the sake of completeness and because the research on the degree/diameter problem started with undirected case. However, the remaining chapters of this part of the thesis will be dealing only with the directed case. This thesis makes a contribution concerning one such property of a digraph, namely, the diregularity of digraphs of order close to the Moore bound.

Chapter 7

Directed Graphs of Large Order

In Chapter 6, we have described the degree/diameter problem. The directed version of the problem differs only in that 'degree' is replaced by 'out-degree' in the statement of the problems, namely, $N(d, k)$-*problem*: For given numbers d and k, construct digraphs of maximum out-degree d and diameter $\leq k$ with the largest possible number of vertices $n_{d,k}$.

In succeeding sections we present known results concerning the existence of Moore digraphs, and digraphs of order close to the Moore bound. This section is largely based on the survey by Miller and Širáň [89].

7.1 Moore digraphs

For any given vertex v, maximum out-degree d and diameter k of a digraph G, we can count the number of vertices at a particular distance from that vertex. Let n_i, for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i \leq d^i$, for $0 \leq i \leq k$ (see Figure 7.1) and, consequently,

$$n_{d,k} = \sum_{i=0}^{k} n_i \quad \leq \quad 1 + d + d^2 + \cdots + d^k \tag{7.1}$$

The right-hand side of (7.1), denoted by $M_{d,k}$, is called the *Moore bound*. If

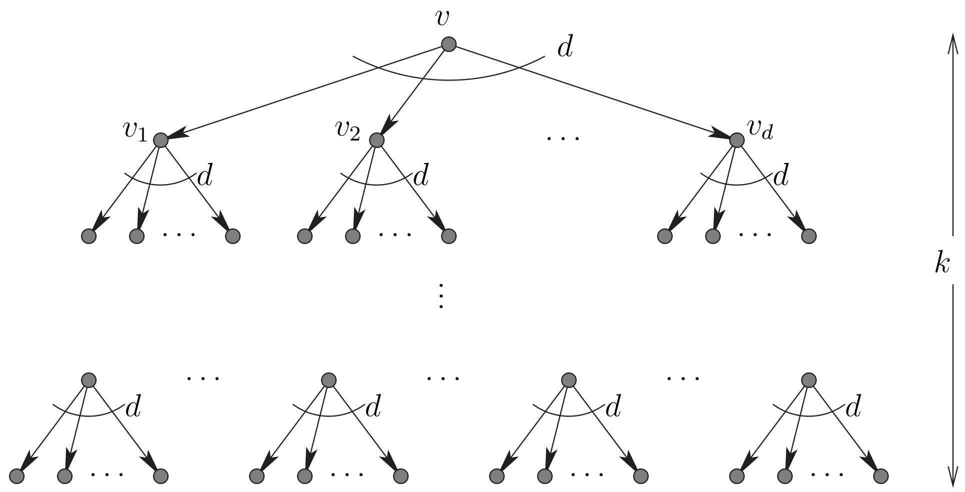


Figure 7.1: Illustration of the layered diagram of a Moore digraph.

the equality sign holds in (7.1) then the digraph is called a *Moore digraph*. It is well known that Moore digraphs exist only in the trivial cases when $d = 1$ (directed cycles of length $k + 1$, C_{k+1}, for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d + 1$, K_{d+1}, for any $d \geq 1$). This was proved by Plesník and Znám in 1974 [103] and independently by Bridges and Toueg in 1980 [35]. The proof due to Bridges and Toueg is very interesting and elegant, and so we include it here.

Let G be a Moore digraph of degree d and diameter k. Then we state the following observations which are used implicitly in the proof.

Observation 7.1.1 *For any pair of vertices u and v in G, $u \neq v$, there exists a unique path from u to v of length $\leq k$.*

Observation 7.1.2 *There are no cycles of length $\leq k$, and every vertex of G lies on exactly d cycles of length $k + 1$. That is, $k + 1$ must divide $d(1 + d + d^2 + \cdots + d^k)$.*

Let A be the adjacency matrix of a digraph G. The number of walks of length k in G from v_i to v_j is the entry in the position (i, j) of the matrix A^k.

Theorem 7.1.1 [103] *Moore digraphs exist only for $d = 1$ or $k = 1$.*

Proof. Let G be a Moore digraph of degree d and diameter k. If A is the $n \times n$ adjacency matrix of G, then, by Observation 7.1.1 and 7.1.2, we have the following matrix equation.

$$I + A + A^2 + \cdots + A^k = J, \tag{7.2}$$

where I is the identity matrix and J is the matrix with all its entries equal to one. It is well known (see e.g., [67]) that the eigenvalues of J are n (with multiplicity 1) and 0 (with multiplicity $n-1$). Since G is diregular, A and J commute, and from Equation (7.2), it is clear that the eigenvalues of A are d (this corresponds to n) and some of the roots of

$$1 + x + x^2 + \cdots + x^k = 0 \tag{7.3}$$

The roots of Equation (7.3) are the roots of $x^{k+1} = 1, x \neq 1$. Let us denote the eigenvalues of A, other than d, by $x_1, x_2, \ldots, x_{n-1}$. Since G, by Observation 7.1.2, has no cycle of length less than $k+1$, we have

$$trace(A^p) = 0, \quad \text{for } p = 1, 2, \ldots, k$$

Hence,

$$d^p + \sum_{j=1}^{n-1} x_j^p = 0 \quad (1 \leq p \leq k) \tag{7.4}$$

Since all the eigenvalues lie on a cycle in a complex plane and their sum is an integer (see Equation (7.4), for $p = 1$), we have that for an arbitrary eigenvalue x_i, there exists an eigenvalue x_j such that either $x_i = -x_j$ or $x_i = \bar{x}_j$. Using this fact and the fact that $\bar{x}_i = x_i^k$ we have

$$-d = \sum_{j=1}^{n-1} x_j = \sum_{j=1}^{n-1} \bar{x}_j = \sum_{j=1}^{n-1} x_j^k = -d^k$$

Thus $d = d^k$, which is fulfilled only if $d = 1$ or $k = 1$. □

Therefore, for $d \geq 2$ and $k \geq 2$, the upper bound on the order of a digraph of out-degree at most d and diameter k is less than the Moore bound. Since there are no Moore digraphs with maximum out-degree $d \geq 2$ and diameter $k \geq 2$, the study of the existence of large digraphs next focuses on digraphs whose order is close to the Moore bound, that is, digraphs of order $n = M_{d,k} - \delta$, for δ as small as possible.

In the next section, we present known results concerning the existence of digraphs whose order is close to the Moore bound. We will use the following notation throughout. Let $\mathcal{G}(d, k, \delta)$ be the set of all digraphs of maximum out-degree d and diameter k and defect δ. Then we refer to any digraph $G \in \mathcal{G}(d, k, \delta)$ as a (d, k, δ)-digraph.

7.2 Digraphs of order close to the Moore bound

A digraph of order one less than the Moore bound is called an *almost Moore digraph*, for $d \geq 2$ and $k \geq 2$, if G has maximum out-degree d, diameter at most k, and order $M_{d,k} - 1$. For diameter $k = 2$, line digraphs of complete digraphs are examples of almost Moore digraphs for any $d \geq 2$, showing that $n_{d,2} = M_{d,2} - 1$. Gimbert [60] completely settled the classification problem for diameter 2 and proved that line digraphs of complete digraphs are the only almost Moore digraphs for any out-degree $d \geq 3$.

For out-degree $d = 2$, there are exactly three non-isomorphic diregular digraphs of order $M_{2,2} - 1$, as shown in Figure 7.2.

On the other hand, for $k \geq 3$, focusing on small out-degree instead of diameter, Miller and Fris [87] proved that there are no almost Moore digraphs of maximum out-degree 2. Baskoro, Miller, Širáň and Sutton [22] proved that there are no almost Moore digraphs of maximum out-degree 3 and any diameter greater than or equal to 3. However, the question of whether or not the equality can hold in $n_{d,k} \leq M_{d,k} - 1$, for $d \geq 4$ and $k \geq 3$, is completely open.

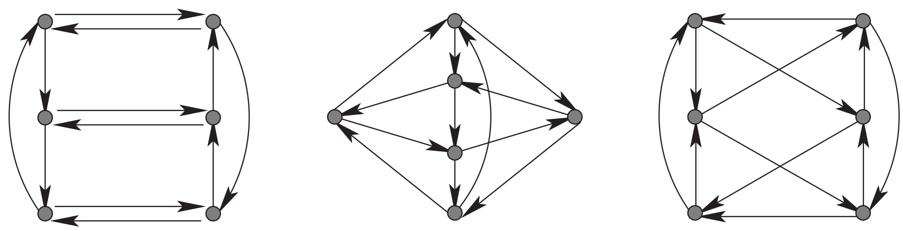


Figure 7.2: Three non-isomorphic diregular digraphs of order $M_{2,2} - 1$.

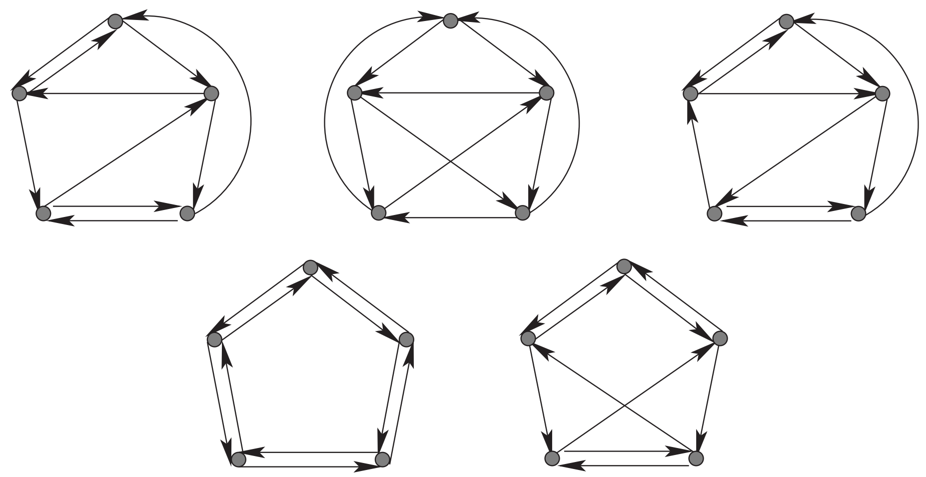


Figure 7.3: Five non-isomorphic diregular digraphs of order $M_{2,2} - 2$.

The study of the existence of large digraphs continued by considering the existence of digraphs of order two less than the Moore bound. We call such digraphs *digraphs of defect two.* Almost Moore digraphs can alternatively be called *digraphs of defect one.* The study of digraphs of defect two so far has concentrated on digraphs of out-degree $d = 2$. In the case of diameter $k = 2$, it was shown in [85] that there are exactly five non-isomorphic diregular digraphs of defect two, as shown in Figure 7.3.

Apart from these five diregular digraphs, there are also four non-isomorphic digraphs of defect two of out-degree 2 and diameter 2, which are not regular with respect to the in-degree, see Figure 7.4. It is interesting to note that there are more diregular digraphs than non-diregular ones for the parameters $n = 5$, $d = 2$, $k = 2$. Miller and Širáň [88] proved that digraphs of defect two do not exist for out-degree $d = 2$ and all $k \geq 3$.

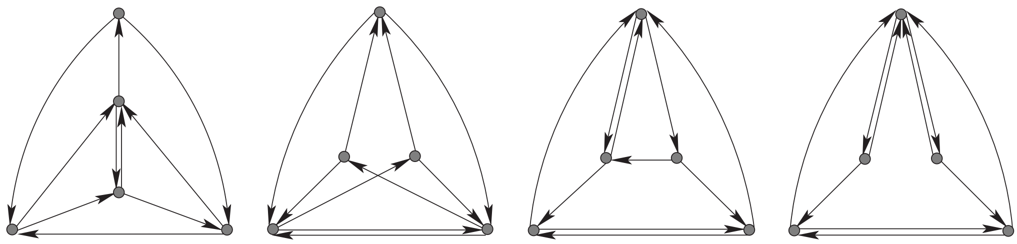


Figure 7.4: Four non-isomorphic non-diregular digraphs of order $M_{2,2} - 2$.

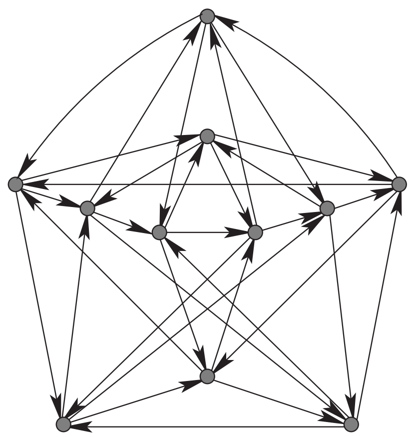


Figure 7.5: The unique diregular digraph of order $M_{3,2} - 2$.

For the case of out-degree $d = 3$, it is not known whether digraphs of defect two exist or not, except that for diameter $k = 2$ there is a unique diregular digraph of order $M_{3,2} - 2$, as proved by Baskoro [15], and shown in Figure 7.5.

Surprisingly, there are more non-diregular digraphs with the same out-degree 3, diameter 2 and order 11. Four such non-isomorphic digraphs were found by Slamin and Miller [108], see Figure 7.6. It is interesting to note that in all these digraphs the number of vertices of in-degree less than 3 is three. For the remaining values of $k \geq 3$ and $d \geq 3$, the question of whether digraphs of defect two exist or not remains completely open.

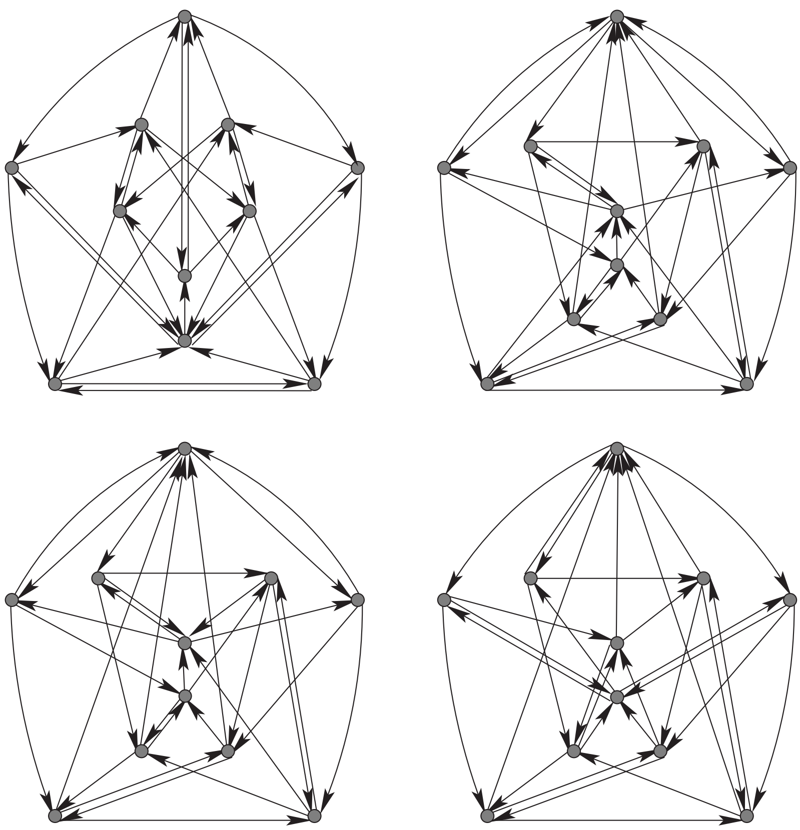


Figure 7.6: Four non-isomorphic non-diregular digraphs of defect two, $d = 3$ and $k = 2$.

In general, we will describe a construction of non-diregular digraphs of out-degree $d \geq 2$ with diameter $k = 2$ and defect 2 in the next chapter.

At present, the best lower bound on the order of digraphs of out-degree d and diameter k is as follows. For out-degree $d = 2$ and diameter $k \geq 4$, $n_{2,k} \geq 25 \times 2^{k-4}$. This lower bound is obtained from *Alegre digraph* which is a digraph of out-degree 2, diameter 4 and order 25 (see Figure 7.7), and from its line digraph iterations. For the remaining values of out-degree and diameter,

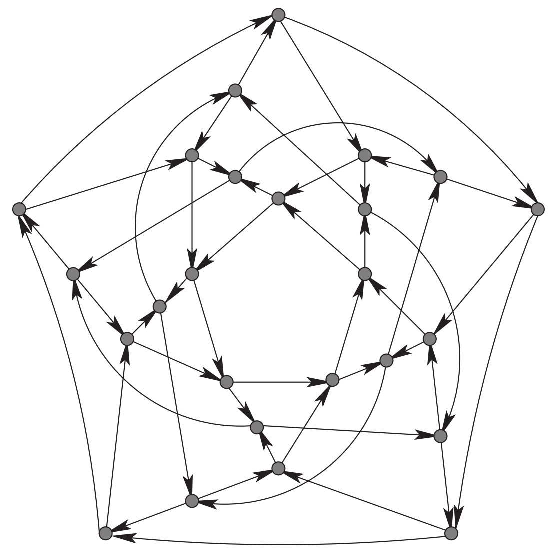


Figure 7.7: Alegre digraph $Al \in \mathcal{G}(2, 4, 6)$.

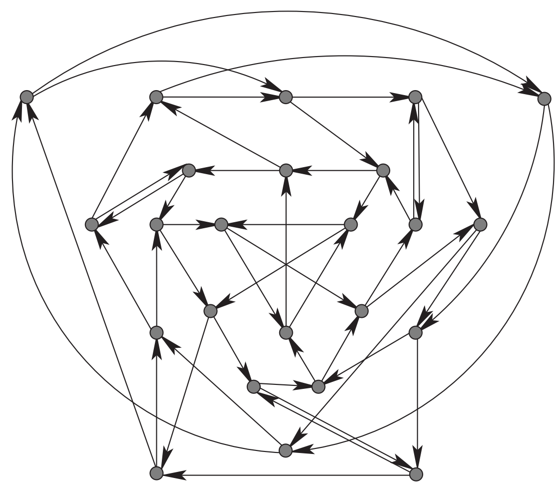


Figure 7.8: Kautz digraph $Ka \in \mathcal{G}(2, 4, 7)$.

a general lower bound is $n_{d,k} \geq d^k + d^{k-1}$. This bound is obtained from Kautz digraphs, the digraphs of out-degree d, diameter k and order $d^k + d^{k-1}$ [75]. For example, Figure 7.8 shows the Kautz digraph of out-degree 2, diameter 4 and order 24.

We summarize our current knowledge of the lower and upper bound on the order of digraphs of out-degree d and diameter k in Table 7.1.

Table 7.1: Lower and upper bounds on the order of digraphs of out-degree d and diameter k.

Degree	*Diameter*	*Lower bound*	*Upper bound*
$d = 1$	$k \geq 1$	$k + 1$	$k + 1$
$d = 2$	$k = 3$	12	12
	$k \geq 4$	$25 \times 2^{k-4}$	$M_{2,k} - 3$
$d \geq 2$	$k = 1$	$d + 1$	$d + 1$
	$k = 2$	$d^2 + d$	$d^2 + d$
$d = 3$	$k \geq 3$	$3^k + 3^{k-1}$	$M_{3,k} - 2$
$d \geq 4$	$k \geq 3$	$d^k + d^{k-1}$	$M_{d,k} - 1$

In view of the huge gap between the best upper bound and the current best lower bound, much effort has been spent in generating large digraphs. There are several techniques for constructing large digraphs. Existing large digraphs provide lower bounds on the order $n_{l_{d,k}}$.

7.3 The notion of a repeat

In this section we present a concept which plays important role in getting an insight into the structure of digraphs of order close to the Moore bound. This notion is called 'repeat', introduced by Miller and Fris [87]. Let G be an almost

Moore digraph of out-degree $d \geq 2$ and diameter $k \geq 2$. A counting argument, presented in [19], shows that for each vertex u of G there exists exactly one vertex $r(u)$ in G with the property that there are two $u \rightarrow r(u)$ walks in G of length not exceeding k. The vertex $r(u)$ is called the *repeat* of u. If digraph G is diregular then it follows that the mapping $v \mapsto r(v)$ is an *automorphism* of the digraph G, see [19].

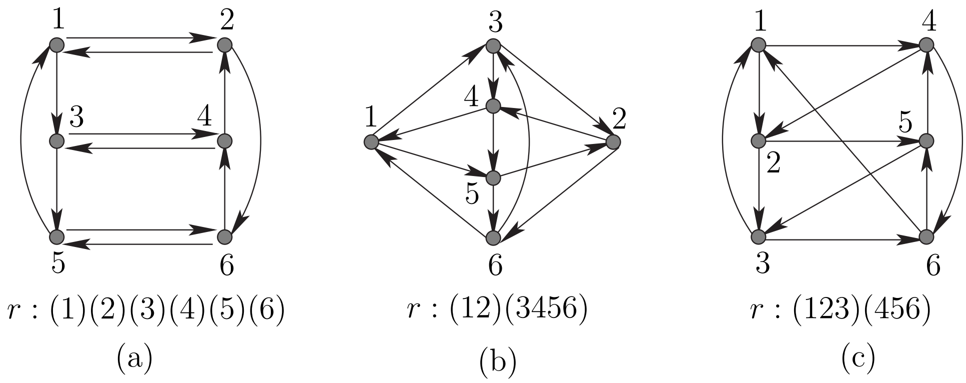


Figure 7.9: The permutation cycles of the three $(2, 2, 1)$-digraphs.

If $r(u) = v$ then $r^{-1}(v) = u$. If $r(u) = u$ (i.e., $v = u$) then u lies on a cycle of length exactly k, in which the two walks have lengths 0 and k, and u is called a *selfrepeat* of G. For $S \subset V(G)$, we define $r(S) = \bigcup_{v \in S} r(v)$ and, similarly, $r^{-1}(S) = \bigcup_{v \in S} r^{-1}(v)$. Hence the function r can be considered as a permutation on the vertex set of G. Figure 7.9 illustrates the notion of repeat for the three $(2, 2, 1)$-digraphs. Each permutation is expressed as a set of permutation cycles. For example, in Figure 7.9(b), we have $r(1) = 2$ (since there are two walks from vertex 1 to vertex 2, namely, $(1, 3, 2)$ and $(1, 5, 2)$, of length at most 2), and $r(2) = 1$, while $r(3) = 4$, $r(4) = 5$, $r(5) = 6$ and $r(6) = 3$.

Next let us consider the idea of repeats in a digraph of defect 2. Let G be a digraph of out-degree $d \geq 2$, diameter $k \geq 3$ and order $M_{d,k} - 2$. Using a counting argument, it is easy to show that, for each vertex u of G, there exist exactly two vertices $r_1(u)$ and $r_2(u)$ (not necessarily distinct) in G with the property that there are two $u \to r_i(u)$ walks, for $i = 1, 2$, in G of length not exceeding k. The vertices $r_i(u)$, $i = 1, 2$, are the repeats of u. If $r_1(x) = r_2(x) = r(x)$ then $r(x)$ is called a *double repeat.*

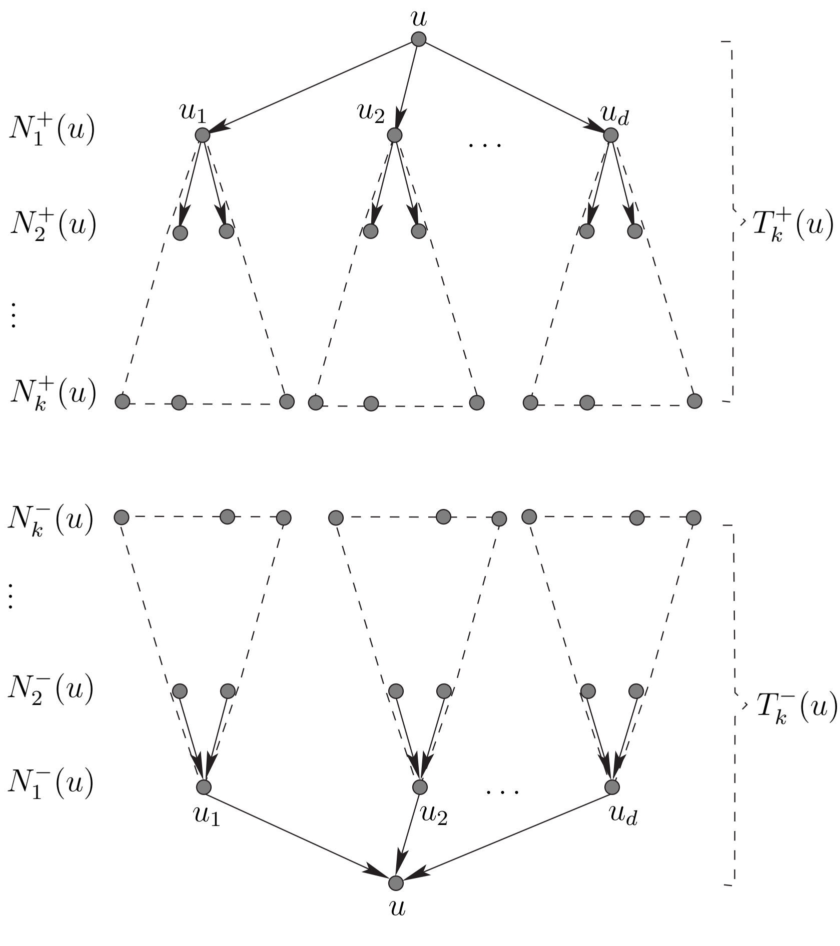


Figure 7.10: Multisets $T_k^+(u)$ and $T_k^-(u)$.

We will also use the following notation throughout. For each vertex u of a digraph G, and for $1 \le s \le k$, let $T_s^+(u)$ be the multiset of all endvertices of directed paths in G of length at most s, which start at u. Similarly, by $T_s^-(u)$ we denote the multiset of all starting vertices of directed paths of length at most s in G, which terminate at u. Observe that the vertex u is in both $T_s^+(u)$ and $T_s^-(u)$, as in both cases it corresponds to a path of zero length. Let $N_s^+(u)$

be the set of all endvertices of directed paths in G, of length exactly s, which start at u. Similarly, by $N_s^-(u)$ we denote the set of all starting vertices of directed paths, of length exactly s in G, which terminate at u. If $s = 1$, the sets $N_1^+(u)$ and $N_1^-(u)$ represent, respectively, the out- and in-neighbourhood of the vertex u in the digraph G; we denote them simply by $N^+(u)$ and $N^-(u)$, respectively. The notations $T_k^+(u)$, $T_k^-(u)$, $N_s^+(u)$ and $N_s^-(u)$, for $1 \leq s \leq k$, are illustrated in Figure 7.10.

One of the reasons why we are interested in establishing diregularity of digraphs is because the *Neighbourhood Theorem* holds whenever the digraph is diregular. The Neighbourhood Theorem first appeared in [19] for digraphs of defect 1. It was later generalised by Miller, Nguyen and Simanjuntak in [93] to all graphs, including undirected, directed and mixed, and for all $\delta \geq 1$.

We denote by $R_m(u)$ the multiset of all repeats of a vertex $u \in G$, containing each repeat v of u exactly $m_v(u)$ times. Here we state the directed version of the Neighbourhood Theorem.

Theorem 7.3.1 *[93] If G is a diregular (d, k, δ)-digraph then for every vertex $u \in G$, $N^+(R_m(u)) = R_m(N^+(u))$.*

Chapter 8

Diregularity of Digraphs

Since Moore digraphs do not exist when both $k \neq 1$ and $d \neq 1$, the problem of finding digraphs of out-degree $d \geq 2$ and diameter $k \geq 2$ and order close to the Moore bound becomes an interesting research problem. To prove the non-existence of such digraphs as well as to assist in finding an algorithm for constructing such digraphs, we may first wish to establish some useful structural properties such digraphs must posses (if they exist). In this thesis we study one such property, the diregularity of potential (d, k, δ)-digraphs, for $\delta \leq 2$.

It is obvious that every Moore digraph must be out-regular. The out-regularity of digraphs of out-degree $d \geq 2$, diameter $k \geq 2$ and order $M_{d,k} - M_{d,k-1} + 1 \leq n \leq M_{d,k} - 1$ considered by Baskoro, Miller and Plesník in [20] follows from a straightforward counting argument.

Lemma 8.0.1 *Any digraph of out-degree $d \geq 2$, diameter $k \geq 2$ and order n, $M_{d,k} - M_{d,k-1} + 1 \leq n \leq M_{d,k} - 1$, must be out-regular of out-degree d.*

Proof. We suppose that the digraph contains a vertex u with out-degree $d_1 < d$ (i.e., $d_1 \leq d - 1$). Then considering the number of vertices in the out-bound spanning tree starting from vertex u, the order of the digraph,

$$\begin{aligned} n &\leq 1 + d_1 + d_1 d + \cdots + d_1 d^{k-1} \\ &= 1 + d_1(1 + d + \cdots + d^{k-1}) \\ &\leq 1 + (d-1)(1 + d + \cdots + d^{k-1}) \\ &= (1 + d + \cdots + d^k) - (1 + d + \cdots + d^{k-1}) \\ &= M_{d,k} - M_{d,k-1}, \end{aligned}$$

which is a contradiction. Hence the out-degree of any vertex in a digraph of order n, $M_{d,k} - M_{d,k-1} + 1 \leq n \leq M_{d,k} - 1$, must be equal to d, that is, the digraph must be out-regular. □

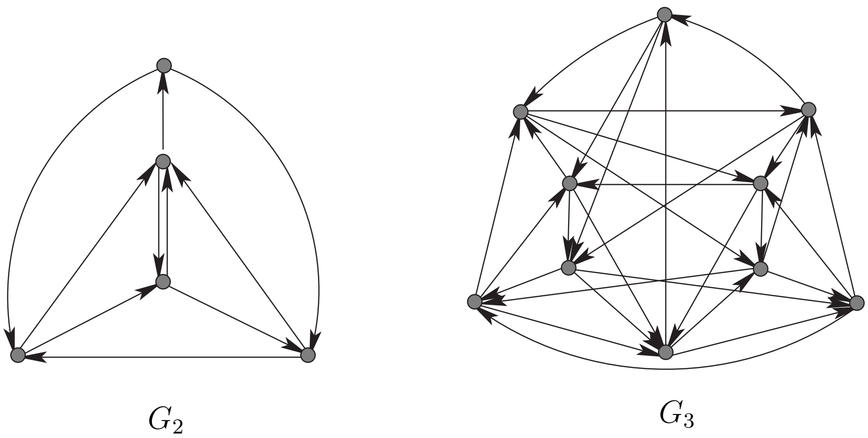


Figure 8.1: Non-diregular digraphs G_2 and G_3.

It is obvious that all Moore digraphs are in- as well as out-regular since the only Moore digraphs are directed cycles C_{k+1} (which are 1-regular) and complete digraphs K_{d+1} (which are d-regular). Alternatively, we can show in-regularity of Moore digraphs as follows. Let v be an arbitrary vertex of a Moore digraph. By the diameter assumption, all $N^+(v)$ must reach v within distance k, otherwise v cannot reach all the other vertices in at most k steps. Since all vertices in the set $N_k^+(v)$ are distinct, then v must be in every multiset $T_k^+(v_i)$, for

$i = 1, \ldots, d$, that is, the in-degree of v is d. Since v is an arbitrary vertex, it follows that every Moore digraph is in-regular as well as out-regular, that is, diregular.

Unlike for Moore digraphs, establishing the regularity of in-degree for an *almost* Moore digraph was not so easy. It is well known that there exist digraphs of out-degree d and diameter k whose order is just two or three less than the Moore bound and in which *not all* vertices have the same in-degree. In Figure 8.1 we give two examples of digraphs of diameter 2, out-degree $d = 2, 3$ and order $M_{d,2} - d$, respectively, with vertices not all of the same in-degree.

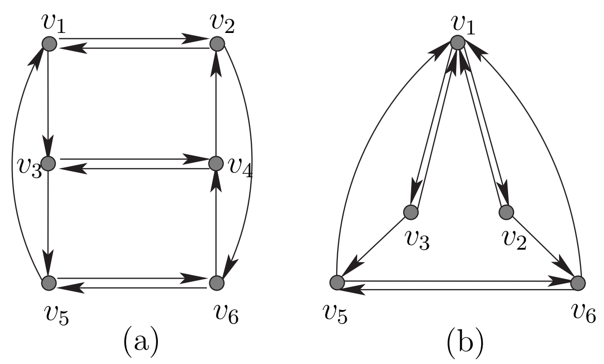


Figure 8.2: Digraph $G \in \mathcal{G}(2, 2, 1)$ and $G_1 \in \mathcal{G}(2, 2, 2)$ obtained from G.

Miller, Gimbert, Širáň and Slamin [90] prove that digraphs of defect one are diregular. For defect two, diameter $k = 2$ and any out-degree $d \geq 2$, non-diregular digraphs always exist. One example can be constructed from Kautz digraphs. Kautz digraph has the property that there exist vertices with identical out-neighbourhoods.

Next we present vertex deletion technique which will be useful later in this thesis. In [95], Miller and Slamin introduced the vertex deletion scheme. They constructed new digraphs from existing digraphs using this technique. Let $G \in \mathcal{G}(d, k, \delta)$. Suppose that $N^+(u) = N^+(v)$ for some vertices $u, v \in G$. Let G_1 be a digraph obtained from G by deleting vertex u together with its outgoing arcs and reconnecting the incoming arcs of u to the vertex v. The new digraph G_1 has maximum out-degree the same as the maximum out-degree of G and the diameter is at most k.

Figure 8.2(a) shows an example of digraph $G \in \mathcal{G}(2,2,1)$ with the property that $N^+(v_1) = N^+(v_4)$. Deleting vertex v_4, together with its outgoing arcs, and then reconnecting its incoming arcs to vertex v_1, we obtain a new digraph $G_1 \in \mathcal{G}(2,2,2)$ as shown in Figure 8.2(b).

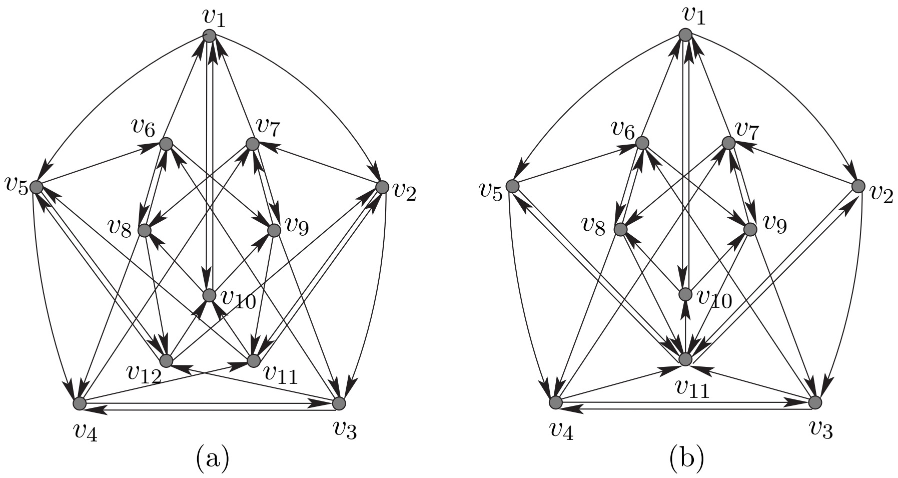


Figure 8.3: Digraph $G \in \mathcal{G}(3,2,1)$ and $G_1 \in \mathcal{G}(3,2,2)$ obtained from G.

By applying vertex deletion scheme, we can obtain non-diregular digraph of defect two, diameter $k = 2$ and out-degree $d \geq 2$. Figure 8.2(b) shows an example of non-diregular digraph G of order $n = M_{2,2} - 2$ generated from Kautz digraph G of order $n = M_{2,2} - 1$ by deleting vertex v_4, together with its outgoing arcs, and then reconnecting its incoming arcs to vertex v_1. Figure 8.3(b) shows an example of non-diregular digraph G of order $n = M_{3,2} - 2$, generated from Kautz digraph G of order $n = M_{3,2} - 1$ by deleting vertex v_{12}, together with its outgoing arcs, and then reconnecting its incoming arcs to vertex v_{11}. For diameter $k \geq 3$, Slamin and Miller [107] proved that digraphs of out-degree $d = 2$ are diregular. For diameter $k \geq 3$ and maximum out-degree $d = 3$, they proposed the following open problem.

Open Problem 8.0.1 *Is every digraph of defect two of maximum out-degree $d = 3$ and diameter $k \geq 3$ diregular?*

Table 8.1 gives a summary of our knowledge of diregular and non-diregular digraphs of maximum out-degree d, diameter k and order equal to $M_{d,k} - \delta$, for $\delta \leq 2$.

Table 8.1: Diregularity of digraphs of defect at most 2

d	k	n	Diregularity	Reference
1	≥ 1	$M_{1,k}$	Only diregular	Plesník and Znám (1974)
≥ 1	1	$M_{d,1}$	Only diregular	
≥ 2	≥ 2	$M_{d,k} - 1$	Only diregular	Miller, Gimbert, Širáň and Slamin (2000)
2	≥ 3	$M_{2,k} - 2$	Only diregular	Miller and Slamin (2000)
$2, 3$	2	$M_{d,2} - 2$	Diregular and non-diregular	Miller and Slamin (2000)
≥ 3	≥ 3	$M_{d,k} - 2$	Unknown	
≥ 4	2	$M_{d,2} - 2$	Unknown	

In this chapter we provide new results concerning diregularity of digraphs of order two less than Moore bound. In the case of defect two with out-degree 2 and diameter $k \geq 3$, we present an alternative proof that a digraph of defect two must be diregular. Furthermore, for any out-degree $d \geq 3$ and diameter $k \geq 2$, we prove that all digraphs of defect two are either diregular or 'almost diregular'.

We now introduce the notion of almost diregularity. Throughout this chapter, let S be the set of all vertices of G whose in-degree is less than d. Let S' be the set of all vertices of G whose in-degree is greater than d; and let σ^- be the *in-excess*, $\sigma^- = \sigma^-(G) = \sum_{w \in S'}(d^-(w) - d) = \sum_{v \in S}(d - d^-(v))$. Similarly, let R be the set of all vertices of G whose out-degree is less than d. Let R' be the set of all vertices of G whose out-degree is greater than d. We define the *out-excess*, $\sigma^+ = \sigma^+(G) = \sum_{w \in R'}(d^+(w) - d) = \sum_{v \in R}(d - d^+(v))$. A digraph of average in-degree d is called *almost in-regular* if the in-excess is at most equal to d. Similarly, a digraph of average out-degree d is called *almost out-regular* if the out-excess is at most equal to d. A digraph is *almost diregular* if it is both almost in-regular and almost out-regular. Note that if $\sigma^- = 0$ (respectively, $\sigma^+ = 0$) then G is in-regular (respectively, out-regular).

We will present our new results concerning the diregularity of digraphs of order close to Moore bound in the following sections.

8.1 Diregularity of $(2, k, 2)$-digraphs

In this section we consider the diregularity of digraphs of defect two for the case of out-degree $d = 2$ and any diameter $k \geq 3$. In the case of diameter $k = 2$, there are four non-isomorphic digraphs of defect two of out-degree 2, with vertices not all of the same in-degree, as shown in Figure 7.4. Recall that S is the set of all vertices of G whose in-degree is 1; and S' is the set of all vertices of G whose in-degree is greater than 2. We first present the following lemma.

◇ **Lemma 8.1.1** *Let $G \in \mathcal{G}(2, k, 2)$. Let S be the set of all vertices of G whose in-degree is* 1*. Let $v \in S$. Then $r(u) \in N^-(v)$, for any vertex $u \in V(G)$.*

Proof. Let $N^+(u) = \{u_1, u_2\}$. Since the diameter of G is equal to k, the vertex v must occur in each of the sets $T_k^+(u_1)$ and $T_k^+(u_2)$. It follows that there exist vertices $x_1, x_2 \in \{u\} \cup T_{k-1}^+(u_1) \cup T_{k-1}^+(u_2)$ such that x_1v and x_2v is an arc of G. However, since the in-degree of v is 1 then $x_1 = x_2$. This means that x_1 is a repeat of u. Therefore, $r(u) \in N^-(v)$. $\square$

Combining Lemma 8.1.1 with the fact that every vertex in G has out-degree 2 gives

$\diamond$ **Corollary 8.1.1** $|S| \leq 4$.

$\diamond$ **Lemma 8.1.2** *For $k \geq 3$ and $d = 2$, every $(2, k, 2)$-digraph is either diregular or almost diregular.*

Proof. First we wish to prove that $|S| \leq 2$. Suppose $|S| \geq 3$. Then there exist $v_1, v_2, v_3 \in S$ such that $d^-(v_i) = 1$, for $i = 1, 2, 3$. The in-excess $\sigma^- = \sum_{v \in S}(d - d^-(v)) \geq 3$. This implies that $|S'| \geq 1$. However, we cannot have $|S'| = 1$. Suppose, for contradiction, $S' = \{x\}$. To reach v_1 (and v_i, $i = 2, 3$) from all the other vertices in G we must have $x \in \bigcap_{i=1}^{3} N^-(v_i)$, which is impossible as the out-degree of x is 2. Hence $|S'| \geq 2$. Let $u \in V(G)$ and $u \neq v_1, v_2, v_3$. To reach v_1, v_2, v_3 from u we must have $N^-(v_1) \cup N^-(v_2) \cup N^-(v_3) \subseteq \{r_1(u), r_2(u)\}$. Since $d = 2$ then $|N^-(v_1) \cup N^-(v_2) \cup N^-(v_3)| = 2$. Without loss of generality, we suppose $N^-(v_1) \cup N^-(v_2) = \{x_1\}$ and $|N^-(v_3)| = \{x_2\}$, where $x_1, x_2 \in S'$. Now consider the multiset $T_k^+(x_1)$. Since v_1 and v_2, respectively, must reach v_2, v_3 and v_1, v_3, within distance at most k then x_1 occurs three times in $T_k^+(x_1)$. This implies that x_1 is a double selfrepeat. Since both v_1 and v_2 occur in the walk joining two selfrepeats then v_1 and v_2 are selfrepeats, see Figure 8.4. Then it is not possible for the two out-neighbours of x_1 to reach v_3. Therefore $|S| \leq 2$.

Now we shall prove that $|S| = 2$. If $|S| = \emptyset$ then $(2, k, 2)$-digraph is diregular. We now suppose $|S| = 1$. Let $v \in S$ and $d^-(v) = 1$. The in-excess $\sigma^- = \sum_{v \in S}(d - d^-(v)) = 1$. This implies that $|S'| = 1$, say $S' = \{x\}$ and $d^-(x) = 3$. By Lemma 8.1.1, $x \in N^-(v)$. Then it would not be possible to reach v from

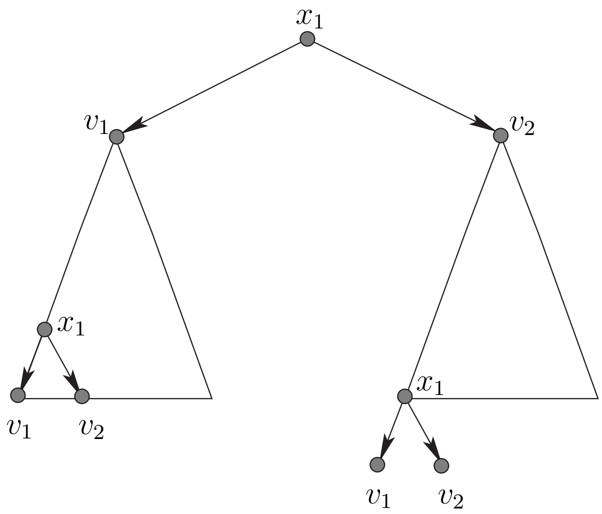


Figure 8.4: Illustration of multiset $T_k^+(x_1)$.

all the other vertices in G, since $|T_k^-(v)| < M_{2,k} - 2$. This implies $|S| = 2$. It follows that $(2, k, 2)$-digraph is almost diregular. □

With Lemma 8.1.2 in hand, we are now in a position to prove that every $(2, k, 2)$-digraph is diregular.

◇ **Theorem 8.1.1** *Every $(2, k, 2)$-digraph is diregular for $k \geq 3$.*

Proof. Let $G \in \mathcal{G}(2, k, 2)$, $k \geq 3$. By Lemma 8.1.2, if G is an almost diregular digraph which is not diregular then $|S| = 2$. Let $S = \{v_1, v_2\}$. Suppose $N^-(v_1) = \{x_1\}$ and $N^-(v_2) = \{x_2\}$. Then the in-excess $\sigma^- = \sum_{v \in S}(d - d^-(v)) = 2$. This implies that $1 \leq |S'| \leq 2$. Suppose $|S'| = 2$. Then $S' = \{x_1, x_2\}$. If $d^-(x_1) = 3$ then it is not possible to reach v_1 from all the other vertices in G.

Therefore, $|S'| = 1$, $x_1 = x_2 (= x)$ and $d^-(x) = 4$. We first consider the multisets $T_k^+(v_1)$ and $T_k^+(v_2)$. Since v_1 must reach v_2 within distance at most k and at the same time v_2 also must reach v_1 within distance at most k, vertex x must occur at distance exactly $k - 1$ from both v_1 and v_2. It follows that x occurs three times in the multiset $T_k^+(x)$, which means that x is a double selfrepeat. Vertex x is also a repeat for every other vertex in G. Let $y_i \in N^-(x)$, for all $i = 1, 2, 3, 4$. Then two of y_i occur at distance $k-2$ from v_1 (respectively, v_2). Without loss of generality, we suppose that $y_1 \in N_{k-2}^+(v_2)$ and $y_2 \in N_{k-2}^+(v_1)$. It follows that y_1 and y_2 are each a selfrepeat exactly once.

Let S_1 and S_2 be multisets. We denote $S = S_1 \uplus S_2$ the multiset defined as follows. If x occurs n_1 times in S_1 and n_2 times in S_2 then x occurs $n_1 + n_2$ times in S. Consider the multiset $T_k^+(y_1) = V(G) \uplus \{x\} \uplus \{y_1\}$. Alternatively, we can express $T_k^+(y_1) = T_{k-1}^+(c_1) \uplus T_{k-1}^+(x) \uplus \{y_1\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T_{k-1}^+(c_1) \uplus T_{k-1}^+(x) \tag{8.1}$$

Consider the multiset $T_k^+(y_2) = V(G) \uplus \{x\} \uplus \{y_2\}$. Similarly, we can express $T_k^+(y_2) = T_{k-1}^+(c_2) \uplus T_{k-1}^+(x) \uplus \{y_2\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T_{k-1}^+(c_2) \uplus T_{k-1}^+(x) \tag{8.2}$$

From Equations (8.1) and (8.2), it follows that $T_{k-1}^+(c_1) = T_{k-1}^+(c_2)$. Since $N_{k-l-1}^+(c_2) \in T_{k-1}^+(x)$, we get $c_1 = c_2$, otherwise y_1 has at least three repeats, namely, $\{y_1\} \uplus \{x\} \uplus \{u | u \in N_{k-l-1}^+(c_2) \cap T_{k-1}^+(c_2)\}$, which is impossible.

We now consider the multiset $T_k^+(y_3) = V(G) \uplus \{x\} \uplus \{r(y_3)\}$. We have also $T_k^+(y_3) = T_{k-1}^+(c_3) \uplus T_{k-1}^+(x) \uplus \{y_3\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T_{k-1}^+(c_3) \uplus T_{k-1}^+(x) \uplus \{y_3\} - \{r(y_3)\} \tag{8.3}$$

We need to show that $r(y_3) = y_3$. We consider the multiset $T_{k-1}^+(c_3)$. Since y_1 and y_2 are each repeat exactly once, namely, $r(y_1) = y_1$ and $r(y_2) = y_2$, it follows that $y_1, y_2 \notin T_{k-1}^+(c_3)$. Vertex y_q must not be y_3, otherwise there exists a cycle of length $k-1$ in G, which is impossible. This implies that $y_p = y_3$. It follows that y_3 occurs twice in the multiset $T_k^+(y_3)$, which means that y_3 is a

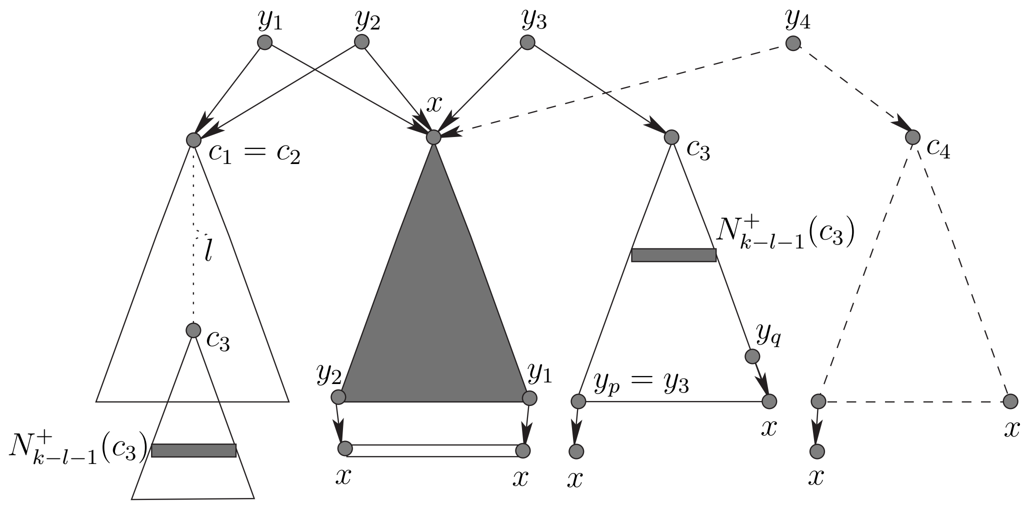


Figure 8.5: Illustration for the case $|S| = 2$.

selfrepeat. Then Equation (8.3) gives

$$V(G) \uplus \{x\} \quad = \quad T^+_{k-1}(c_3) \uplus T^+_{k-1}(x) \tag{8.4}$$

By combining Equations (8.1) and (8.4), we get $T^+_{k-1}(c_1) = T^+_{k-1}(c_3)$. Since $N^+_{k-l-1}(c_3) \in T^+_{k-1}(x)$, see Figure 8.5, we have $c_1 = c_3$, otherwise y_1 has at least three repeats, namely, $\{y_1\} \uplus \{x\} \uplus \{u|u \in N^+_{k-l-1}(c_3) \cap T^+_{k-1}(c_3)\}$, which is impossible. Therefore, $c_1 = c_2 = c_3 (= c)$. Since $c_1 \in N^+(y_1)$, $c_2 \in N^+(y_2)$ and $c_3 \in N^+(y_3)$, it follows that $c \in N^+(y_1) \cap N^+(y_2) \cap N^+(y_3)$. This implies that $S' = \{x, c\}$, which is a contradiction. □

8.2 Diregularity of $(d, k, 2)$-digraphs

In this section we present a new result concerning the diregularity of digraphs of defect two, for maximum out-degree $d \geq 3$ and diameter $k \geq 2$. As before, let S be the set of all vertices of G whose in-degree is less than d; let S' be the set of all vertices of G whose in-degree is greater than d. We first present the following lemma.

◇ **Lemma 8.2.1** *Let $G \in \mathcal{G}(d,k,2)$. Let S be the set of all vertices of G whose in-degree is less than d. Then $S \subseteq N^+(r_1(u)) \cup N^+(r_2(u))$, for any vertex u.*

Proof. Let $v \in S$. Consider an arbitrary vertex $u \in V(G)$, $u \neq v$, and let $N^+(u) = \{u_1, u_2, ..., u_d\}$. Since the diameter of G is equal to k, the vertex v must occur in each of the sets $T_k^+(u_i)$, $i = 1, 2, ..., d$. It follows that, for each i, there exists a vertex $x_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $x_i v$ is an arc of G. Since the in-degree of v is less than d, the in-neighbours x_i of v are not all distinct. This implies that there exists some vertex which occurs at least twice in $T_k^+(u)$. Such a vertex must be a repeat of u. As G has defect 2, there are at most two vertices of G which are repeats of u, namely, $r_1(u)$ and $r_2(u)$. Therefore, $S \subseteq N^+(r_1(u)) \cup N^+(r_2(u))$. □

Combining Lemma 8.2.1 with the fact that every vertex in G has out-degree d gives

◇ **Corollary 8.2.1** *$|S| \leq 2d$.*

In principle, we might expect that the in-degree of $v \in S$ could attain any value between 1 and $d-1$. However, the next lemma asserts that the in-degree cannot be less than $d-1$.

◇ **Lemma 8.2.2** *Let $G \in \mathcal{G}(d,k,2)$. If $v_1 \in S$ then $d^-(v_1) = d-1$.*

Proof. Let $v_1 \in S$. Consider an arbitrary vertex $u \in V(G)$, $u \neq v_1$, and let $N^+(u) = \{u_1, u_2, ..., u_d\}$. Since the diameter of G is equal to k, the vertex v_1 must occur in each of the sets $T_k^+(u_i)$, $i = 1, 2, ..., d$. It follows that, for each i, there exists a vertex $x_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $x_i v_1$ is an arc of G. If $d^-(v_1) \leq d-3$ then there are at least three repeats of u, which is impossible. Suppose that $d^-(v_1) \leq d-2$. By Lemma 8.2.1, the in-excess must satisfy

$$\sigma^- = \sum_{x \in S'} (d^-(x) - d) = \sum_{v_1 \in S} (d - d^-(v_1)) = |S| \leq 2d.$$

We now consider the number of vertices in the multiset $T_k^-(v_1)$. To reach v_1 from all the other vertices in G, the number of distinct vertices in $T_k^-(v_1)$ must be

$$|T_k^-(v_1)| \leq \sum_{t=0}^{k} |N_t^-(v_1)|. \tag{8.5}$$

To estimate the above sum we can observe the following inequality

$$|N_t^-(v_1)| \leq \sum_{u \in N_{t-1}^-(v_1)} d^-(u) = d|N_{t-1}^-(v_1)| + \varepsilon_t, \tag{8.6}$$

where $2 \leq t \leq k$ and $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma$. If $d^-(v_1) = d-2$ then $|N^-(v_1)| = |N_1^-(v_1)| = d-2$. It is not difficult to see that a safe upper bound on the sum of $|T_k^-(v_1)|$ is obtained from Inequality (8.6) by setting $\varepsilon_2 = 2d$, and $\varepsilon_t = 0$ for $3 \leq t \leq k$. This gives

$$\begin{aligned}
|T_k^-(v_1)| &\leq 1 + |N_1^-(v_1)| + |N_2^-(v_1)| + |N_3^-(v_1)| + \ldots + |N_k^-(v_1)| \\
&= 1 + (d-2) + (d(d-2) + \varepsilon_2) + (d(d(d-2) + \varepsilon_2) + \varepsilon_3) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&= 1 + (d-2) + (d(d-2) + 2d) + (d(d(d-2) + 2d) + 0) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&= 1 + d - 2 + d^2 + d^3(1 + d + \cdots + d^{k-3}) \\
&= M_{d,k} - 2.
\end{aligned}$$

Since $\varepsilon_2 = 2d$, $\varepsilon_t = 0$ for $3 \leq t \leq k$, and G contains a vertex of in-degree $d-2$, we have $|S| = d$. Let $S = \{v_1, v_2, \ldots, v_d\}$. Every v_i, for $i = 2, 3, \ldots, d$, has to reach v_1 at distance at most k. Since v_1 and every v_i have exactly the same in-neighbourhood, vertex v_1 is forced to be selfrepeat. This implies that v_1 occurs twice in the multiset $T_k^-(v_1)$. Hence $|T^-(v_1)| < M_{d,k} - 2$, which is a contradiction. Therefore, $d^-(v_1) = d-1$ for any $v_1 \in S$. □

◇ **Lemma 8.2.3** *If S is the set of all vertices of G whose in-degree is $d-1$ then $|S| \leq d$.*

Proof. Suppose $|S| \geq d+1$. Then there exist $v_i \in S$ such that $d^-(v_i) = d-1$, for $i = 1, 2, \ldots, d+1$. The in-excess $\sigma^- = \sum_{v \in S}(d - d^-(v)) \geq d+1$. This implies that $|S'| \geq 1$. However, we cannot have $|S'| = 1$. Suppose, for a contradiction, $S' = \{x\}$. To reach v_1 (and v_i, $i = 2, 3, \ldots, d+1$) from all the other vertices in G, we must have $x \in \bigcap_{i=1}^{d+1} N^-(v_i)$, which is impossible as the out-degree of x is d. Hence $|S'| \geq 2$.

Let $u \in V(G)$ and $u \neq v_i$. To reach v_i from u, we must have $\bigcup_{i=1}^{d+1} N^-(v_i) \subseteq \{r_1(u), r_2(u)\}$. Since G has out-degree d, it follows that $|\bigcup_{i=1}^{d+1} N^-(v_i)| = d$. Let $r_1(u) = x_1$ and $r_2(u) = x_2$. Without loss of generality, we suppose $x_1 \in \bigcup_{i=1}^{d} N^-(v_i)$ and $x_2 \in N^-(v_{d+1})$. Now consider the multiset $T_k^+(x_1)$. Since every v_i, for $i = 1, 2, \ldots, d$, respectively, must reach $\{v_{j \neq i}\}$, for $j = 1, 2, \ldots, d+1$, within distance at most k, then x_1 occurs three times in $T_k^+(x_1)$, otherwise x_1 will have at least three repeats, which is impossible. This implies that x_1 is a double selfrepeat. Since two of v_i, say v_k and v_l, for $k, l \in \{1, 2, \ldots, d+1\}$, occur in the walk joining two selfrepeats then v_k and v_l are selfrepeats. Then it is not possible for the d out-neighbours of x_1 to reach v_{d+1}. □

◇ **Theorem 8.2.1** *For $d \geq 3$ and $k \geq 2$, every $(d, k, 2)$-digraph is out-regular and almost in-regular. Moreover, if $k = 2$ then $d - 1 \leq |S| \leq d$ and if $k \geq 3$ then $|S| = d$.*

Proof. Out-regularity of $(d, k, 2)$-digraphs was established in the introduction. Hence we only need to prove that every $(d, k, 2)$-digraph is almost in-regular. If $S = \emptyset$ then $(d, k, 2)$-digraph is diregular. By Lemma 8.2.2, if $S \neq \emptyset$ then all vertices in S have in-degree $d - 1$. This gives

$$\sigma = \sum_{x \in S'}(d^-(x) - d) = \sum_{v \in S}(d - d^-(v)) = |S| \leq 2d.$$

Take an arbitrary vertex $v \in S$; then $|N^-(v)| = |N_1^-(v)| = d - 1$. By the diameter assumption, the union of all the sets $N_t^-(v)$ for $0 \leq t \leq k$ is the entire vertex set $V(G)$ of G, which implies that

$$|V(G)| \leq \sum_{t=0}^{k} |N_t^-(v)|. \tag{8.7}$$

To estimate the above sum we can observe that

$$|N_t^-(v)| \leq \sum_{u \in N_{t-1}^-(v)} d^-(u) = d|N_{t-1}^-(v)| + \varepsilon_t, \tag{8.8}$$

where $2 \leq t \leq k$ and $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma$.

It is not difficult to see that a safe upper bound on the sum of $|V(G)|$ is obtained from Inequality (8.8) by setting $\varepsilon_2 = \sigma = |S|$, and $\varepsilon_t = 0$, for $3 \leq t \leq k$; note that the latter is equivalent to assuming that *all* vertices from $S \setminus \{v\}$ are contained in $N_k^-(v)$ and that all vertices of S' belong to $N_1^-(v)$. This way we successively obtain:

$$\begin{aligned}
|V(G)| &\leq 1 + |N_1^-(v)| + |N_2^-(v)| + |N_3^-(v)| + \ldots + |N_k^-(v)| \\
&\leq 1 + (d-1) + (d(d-1) + |S|)(1 + d + \cdots + d^{k-2}) \\
&= d + d^2 + \cdots + d^k + (|S| - d)(1 + d + \cdots + d^{k-2}) \\
&= M_{d,k} - 2 + (|S| - d)(1 + d + \cdots + d^{k-2}) + 1.
\end{aligned}$$

But G is a digraph of order $M_{d,k} - 2$; this implies that

$$\begin{aligned}
(|S| - d)(1 + d + \cdots + d^{k-2}) + 1 &\geq 0 \\
(|S| - d)\frac{d^{k-1} - 1}{d - 1} + 1 &\geq 0 \\
|S| &\geq d - \frac{d-1}{d^{k-1} - 1}.
\end{aligned}$$

If $k = 2$ and $d \geq 3$ then $|S| \geq d - 1$. Since $1 \leq |S| \leq d$. This implies $d - 1 \leq |S| \leq d$. If $k \geq 3$ and $d \geq 3$ then $|S| \geq d$ as $0 < \frac{d-1}{d^{k-1}-1} < 1$. This implies $|S| = d$. In both cases $d - 1 \leq |S| \leq d$, for $k = 2$, and $|S| = d$, for $k \geq 3$, we obtain an almost in-regular digraph. □

Chapter 9

On the Diregularity of $(3, k, 2)$-digraphs

In this chapter we present a new result concerning the diregularity of a digraph of defect two, for the case of out-degree $d = 3$ and diameter $k \geq 3$. When $k = 2$ and out-degree 3, there are four non-isomorphic digraphs of defect two with vertices not all of the same in-degree, see Figure 7.6.

As in the previous chapter, let S be the set of all vertices of G whose in-degree is less than 3; let S' be the set of all vertices of G whose in-degree is greater than 3. By the notion of almost dregularity, a digraph G of average in-degree 3 is called *almost in-regular* if the in-excess is at most equal to 3.

From now on, let $G \in \mathcal{G}(3, k, 2), k \geq 3$. In this chapter, we present a possible approach towards proving the in-regularity of $(3, k, 2)$-digraphs. Although we did not manage to completely prove the diregularity, our method is novel and we hope it will lead to the full proof of the diregularity of $(3, k, 2)$-digraphs.

Applying Theorem 8.2.1 from the previous chapter asserts that $|S| = 3$. This means that the in-excess δ^- is 3 and so $|S'| \leq 3$. We will next show that $|S'| \neq 3$.

$\diamond$ **Lemma 9.0.4** $|S'| \neq 3$.

Proof. Suppose $|S'| = 3$. Then G has in-degree sequence $(2, 2, 2, 3, \ldots, 4, 4, 4)$. Let $S = \{v_1, v_2, v_3\}$ and $S' = \{x, y, z\}$, where $d^-(x) = d^-(y) = d^-(z) = 4$. Since the only possible in-neighbours of v_1 are the vertices x, y which form a pair of vertices whose sum of in-degree is less than nine, it would not be possible to reach v_1 from all other vertices in G. $\square$

If $|S'| \neq 3$ then the only possible in-degree sequences are $(2, 2, 2, 3, \ldots, 3, 4, 5)$ and $(2, 2, 2, 3, \ldots, 3, 6)$. Next we shall outline our method for proving the diregularity of $(3, k, 2)$-digraphs. When trying to prove the nonexistence of the two remaining cases, we shall combine the two cases into one case by transforming G with either of those in-degree sequences to a particular digraph G^* of defect three with in-degree sequence $(1, 1, 1, 3, 3, \ldots, 3, 3, 3, 3, 9)$. We will utilize Theorem 1 from [95] to achieve the transformations.

Theorem 9.0.2 [95] *If $G \in \mathcal{G}(n, d, k)$ and $N^+(u) = N^+(v)$, for any vertex $u, v \in G$, then there exists $G_1 \in \mathcal{G}(n-1, d, k')$, $k' \leq k$.*

To be able to utilise the above theorem, first we need to establish that in $G \in \mathcal{G}(3, k, 2)$, there exist two vertices with the same out-neighbourhoods.

Let G be a digraph of in-degree sequence $(2, 2, 2, 3, \ldots, 3, 4, 5)$. To reach each vertex of in-degree 2 from other vertices in G, clearly, only the vertices of in-degree 4 and 5 can be chosen as the in-neighbours of three vertices of in-degree 2. This implies that the out-neighbourhoods of the vertices of in-degree 4 and 5 are the same. Therefore, Theorem 9.0.2 can be applied to this case.

Let G be a digraph of in-degree sequence $(2, 2, 2, 3, \ldots, 3, 6)$. Let $S = \{v_1, v_2, v_3\}$ and $S' = \{x\}$, where $d^-(x) = 6$. To reach v_1 (and v_2, v_3) from other vertices in G, clearly, x and some vertex of in-degree 3 can be chosen as the in-neighbours of a vertex of in-degree 2. Since there are three vertices of in-degree 2, there will be three pairs of vertices, namely, $\{x, y_1\}$, $\{x, y_2\}$ and $\{x, y_3\}$. Since the in-degree of all vertices, apart from v_1, v_2, v_3, x, is three, there are many possibilities to chose y_1, y_2, y_3 of in-degree 3. To be able to apply Theorem 9.0.2, we establish that $y_1 = y_2 = y_3$. The following observation and lemmas are very useful prior to establishing the equality.

◇ **Observation 9.0.1** *Let $v_i, v_j \in S$. The distance $dist(v_i, v_j) = k$, for $i \neq j$.*

◇ **Lemma 9.0.5** *Let $v_1, v_2, v_3 \in S$. The vertices v_1, v_2, v_3 are all selfrepeats.*

Proof. First we consider the number of distinct vertices in the multiset $T_k^-(v_1)$, denoted by $|T_k^-(v_1)|$. By diameter assumption, both vertices v_2 and v_3 have to occur at distance at most k to v_1. Both vertices v_2 and v_3 cannot be in the multiset $T_{k-1}^-(y_1)$ at the same time, otherwise we will have

$$\begin{aligned} |T_k^-(y_1)| &\leq 1 + 3 + 9 + 27 + ... + 3^k - 2 \\ &= M_{3,k} - 2. \end{aligned}$$

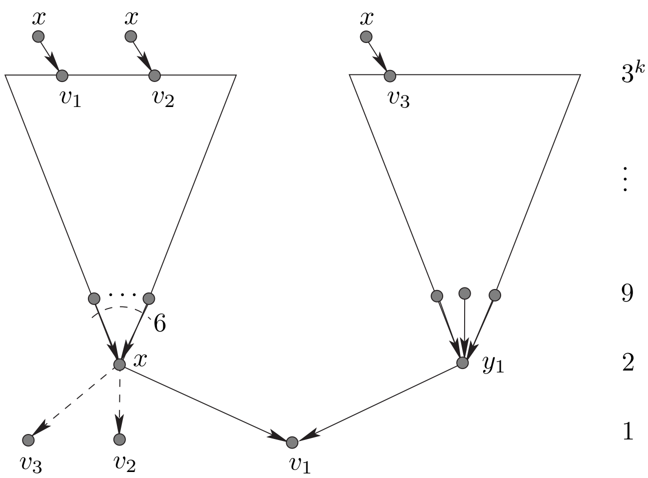


Figure 9.1: Illustration of the multiset $T_k^-(v_1)$.

However, since x occurs twice in the multiset $T_k^-(y_1)$, it follows that $|T_k^-(y_1)| \leq M_{3,k} - 3$, which is impossible. Therefore, without loss of generality, we suppose that v_2 is in the multiset $T^-_{k-1}(x)$, see Figure 9.1. We now consider $T^-_k(v_2)$. By Observation 9.0.1, all other vertices of in-degree 2 must occur at distance k to v_2. On the other hand, v_2 is in the multiset $T^-_{k-1}(x)$ and x is an in-neighbour of v_2. Therefore v_2 is a selfrepeat.

We now consider the multiset $T^-_k(v_2)$. By diameter assumption, both vertices v_1 and v_3 have to occur at distance at most k to v_2. But both v_1 and v_3 cannot be in the multiset $T^-_{k-1}(y_2)$, otherwise $|T^-_k(y_2)| \leq M_{3,k} - 3$, which is impossible. Without loss of generality, we suppose that v_1 is in the multiset $T^-_{k-1}(x)$. We now consider the number of distinct vertices in the multiset $T^-_k(v_1)$. By Observation 9.0.1, all other vertices of in-degree 2 must occur at distance k to v_1. But v_1 is in the multiset $T^-_{k-1}(x)$ which implies that v_1 is a selfrepeat.

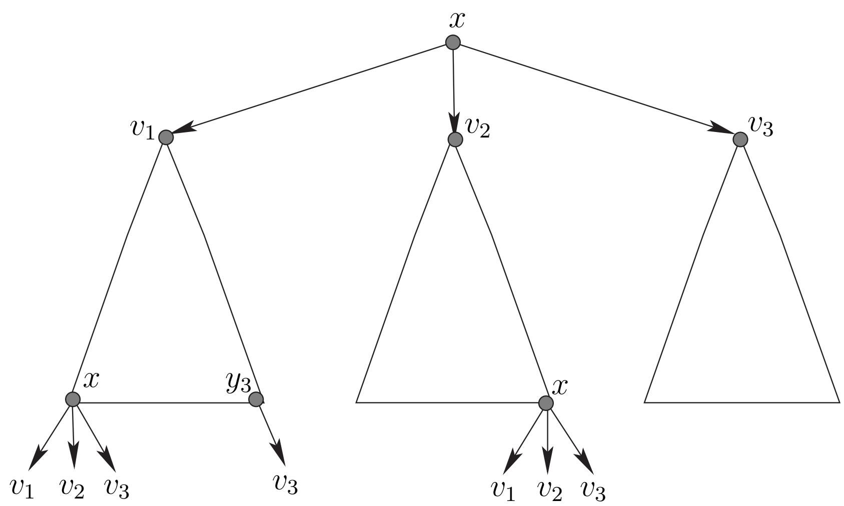


Figure 9.2: Illustration of the multiset $T^+_k(x)$.

Finally, we consider v_3. Since both vertices v_1, v_2 are selfrepeats, x must be a double selfrepeat. We now consider $T^+_k(x)$, see Figure 9.2. Vertex y_3 must occur within distance at most k from x. But y_3 cannot be in either $T^+_{k-1}(v_1)$ or $T^+_{k-1}(v_2)$. Suppose $y_3 \in T^+_{k-1}(v_1)$. Then v_3 is a repeat of v_1. Let $N^+(v_1) = \{w_1, w_2, w_3\}$. To reach v_3 from w_1, w_2, w_3, we must have either x or y occur twice in $T_k^+(v_1)$ which would give too many repeats for v_1. This implies that v_3 is a selfrepeat. □

◇ **Lemma 9.0.6** $N^-(v_1) = N^-(v_2) = N^-(v_3)$.

Proof. Let $N^-(v_1) = \{x, y_1\}, N^-(v_2) = \{x, y_2\}$ and $N^-(v_3) = \{x, y_3\}$. We shall prove that $y_1 = y_2 = y_3$ $(= y)$. Let $N^+(v_1) = \{w_1, w_2, w_3\}$. To reach v_1 from w_1, w_2, w_3, either x or y_1 must occur twice in $T_k^+(v)$. To prove this lemma, we consider two cases.

Case 1. $\exists\, v \in \{v_1, v_2, v_3\}$ such that x is not a repeat of v. Suppose $v = v_1$ and x is not a repeat of v. Then y_1 is a repeat of v, and also y_2 and y_3 must occur twice in $T_k^+(v)$ in order to reach v_2 and v_3. Since the vertex v is a selfrepeat, we can only have one other repeat of v. This is possible only if $y_1 = y_2 = y_3\,(= y)$.

Case 2. Vertex x is a repeat of $v \in \{v_1, v_2, v_3\}$. Consider the multiset $T_k^-(v_1)$. By using Lemma 9.0.5, we have $T_k^-(v_1) = V(G) \uplus \{x\} \uplus \{v_1\}$ and $T_k^-(v_1) = T_{k-1}^-(y_1) \uplus T_{k-1}^-(x) \uplus \{v_1\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T_{k-1}^-(y_1) \uplus T_{k-1}^-(x) \tag{9.1}$$

Similarly, consider the multisets $T_k^-(v_2)$, we have $T_k^-(v_2) = V(G) \uplus \{x\} \uplus \{v_2\}$ and $T_k^-(v_2) = T_{k-1}^-(y_2) \uplus T_{k-1}^-(x) \uplus \{v_2\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T_{k-1}^-(y_2) \uplus T_{k-1}^-(x) \tag{9.2}$$

From Equations 9.1 and 9.2, it follows that $T_{k-1}^-(y_1) = T_{k-1}^-(y_2)$, see Figure 9.3. Since $N_{k-l-1}^-(y_2) \in T_{k-1}^-(x)$, it follows that $y_1 = y_2$, otherwise v_1 has at least three repeats, namely, $\{v_1\} \uplus \{x\} \uplus \{u | u \in N_{k-l-1}^-(y_2) \cap T_{k-1}^-(y_2)\}$ which is impossible.

Finally, we consider the multisets $T_k^-(v_3)$, we have $T_k^-(v_3) = V(G) \uplus \{x\} \uplus \{v_3\}$ and $T_k^-(v_3) = T_{k-1}^-(y_3) \uplus T_{k-1}^-(x) \uplus \{v_3\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T_{k-1}^-(y_3) \uplus T_{k-1}^-(x) \tag{9.3}$$

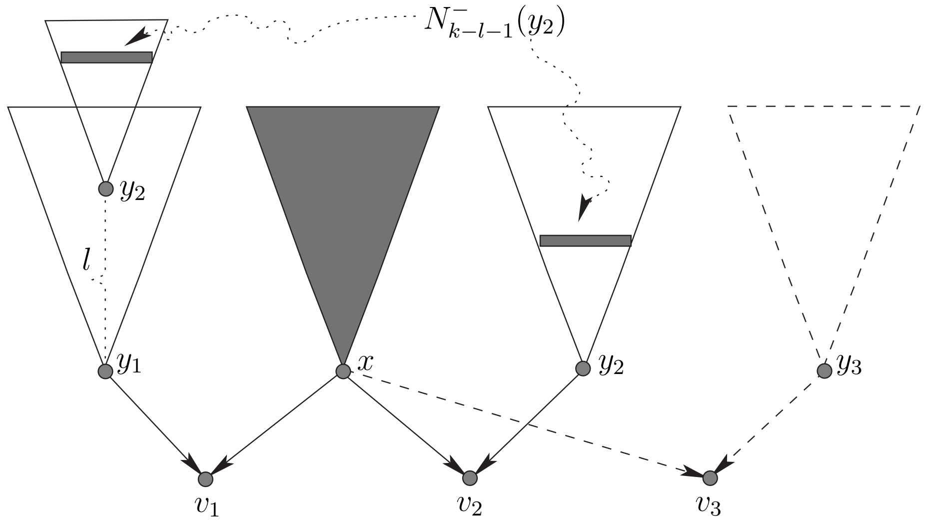


Figure 9.3: Illustration of the multisets $T_k^-(v_1)$ and $T_k^-(v_2)$.

From Equations 9.2 and 9.3, it follows that $T^-_{k-1}(y_2) = T^-_{k-1}(y_3)$. Similarly, since $N^-_{k-l-1}(y_3) \in T^-_{k-1}(x)$, it follows that $y_2 = y_3$, otherwise v_2 has at least three repeats, namely, $\{v_2\} \uplus \{x\} \uplus \{u|u \in N^-_{k-l-1}(y_3) \cap T^-_{k-1}(y_3)\}$ which is impossible. This implies that $y_1 = y_2 = y_3 (= y)$. □

◇ **Lemma 9.0.7** *If G has in-degree sequence* $(2, 2, 2, 3, \ldots, 3, 4, 5)$ *or* $(2, 2, 2, 3, \ldots, 3, 6)$ *then G can be transformed into $G^* \in \mathcal{G}(3, k, 3)$ of in-degree sequence* $(1, 1, 1, 3, \ldots, 3, 9)$.

Proof. Consider the in-degree sequence $(2, 2, 2, 3, \ldots, 3, 4, 5)$. Let $S = \{v_1, v_2, v_3\}$ and $S' = \{x, y\}$ where $d^-(x) = 4$ and $d^-(y) = 5$. Consider the number of vertices in the multiset $T_k^-(v_1)$. To reach v_1 (and v_2, v_3) from other vertices in G, clearly, only x and y can be chosen as the in-neighbours of three vertices of in-degree 2. Let $x, y \in \bigcap_{i=1}^{3} N^-(v_i)$. It follows that $N^+(x) = N^+(y)$. We now consider the in-degree sequence $(2, 2, 2, 3, \ldots, 3, 6)$. Let $S = \{v_1, v_2, v_3\}$ and $S' = \{x\}$, where $d^-(x) = 6$. To reach v_1 (and v_2, v_3) from other vertices in G, clearly x and some vertex of in-degree 3 can be chosen as the in-neighbours of three vertices of in-degree 2. Since there are three vertices of in-degree 2, there will be three pairs of vertices, namely, $\{x, y_1\}$, $\{x, y_2\}$ and $\{x, y_3\}$. By Lemma 9.0.6, we have $y_1 = y_2 = y_3 (= y)$. It follows that $N^+(x) = N^+(y)$. Applying Theorem 9.0.2 in both cases, we can delete vertex y, together with its outgoing arcs, and then reconnect its incoming arcs to vertex x, so that we obtain a new digraph G^* of defect three, with in-degree sequence $(1, 1, 1, 3, \ldots, 3, 9)$. □

From now on, we assume that $G^* \in \mathcal{G}(3, k, 3)$ is a digraph of defect three, out-degree 3 and diameter $k \geq 3$, with in-degree sequence $(1, 1, 1, 3, \ldots, 3, 9)$. Next we prove some structural properties of $(3, k, 3)$-digraphs. Based on this properties, for $k = 3$, we prove that a digraph with out-degree 3 and in-degree sequence $(1, 1, 1, 3, \ldots, 3, 9)$ does not exist.

◇ **Lemma 9.0.8** *Let $G^* \in \mathcal{G}(3, k, 3)$. Let $x \in S'$. If $d^-(x) = 9$ then x is a triple selfrepeat.*

Proof. Let $S = \{v_1, v_2, v_3\}$ and $S' = \{x\}$, where $d^-(x) = 9$. To reach vertex v_1 (and respectively v_2, v_3) from all the other vertices in G, we must have $\{x\} = N^-(v_1) = N^-(v_2) = N^-(v_3)$. This implies that x occurs at distance at most $k - 1$ from v_1, v_2 and v_3, respectively. It then follows that x occurs four times in the multiset $T_k^+(x)$. Therefore, x is a triple selfrepeat. □

◇ **Lemma 9.0.9** *Let $N^-(x) = \{y_1, y_2, \ldots, y_9\}$. Then three of $y_1, y_2, \ldots, y_9$ are selfrepeats.*

Proof. By Lemma 9.0.8, vertex x is a triple selfrepeat and $y_j \in N^-(x)$, for $j = 1, 2, \ldots, 9$. This implies that three of y_j occur at distance $k - 2$ from v_1, v_2 and v_3, respectively. Without loss of generality, we suppose that $y_1 \in N^+_{k-2}(v_1)$, $y_2 \in N^+_{k-2}(v_2)$ and $y_3 \in N^+_{k-2}(v_3)$. We now consider the multiset $T_k^+(y_1), T_k^+(y_2)$ and $T_k^+(y_3)$. Since $x \in N^+(y_1) \cap N^+(y_2) \cap N^+(y_3)$, it follows that $T_k^+(x) \subseteq T_k^+(y_1) \cap T_k^+(y_2) \cap T_k^+(y_3)$. This implies that y_1, y_2, y_3 are selfrepeats. □

◇ **Lemma 9.0.10** *If $y_p, y_q \in \{y_j\}$ then $|N^+(y_p) \cap N^+(y_q)| = 1$ or 3.*

Proof. Since $y_j \in N^-(x)$, for all $j = 1, \ldots, 9$, it is obvious that $|N^+(y_p) \cap N^+(y_q)| = 1$. Now, suppose $|N^+(y_p) \cap N^+(y_q)| = 2$, this situation is depicted in Figure 9.4. We wish to prove that $c_s = c_t$. Suppose $c_s \neq c_t$ and either c_t is in the multiset $T_k^+(c_s)$ or c_s is in the multiset $T_k^+(c_t)$. If c_s occurs at distance at most $k-2$ from c_t or c_t occurs at distance at most $k-2$ from c_s then there will obviously exist more than three repeats, which is impossible. However, if c_s occurs at distance exactly $k-1$ from c_t, or c_t occurs at distance exactly $k-1$ from c_s, then $N^+(c_s) = \{\{r(c_t)\} \cup T_{k-1}^+(d_l) \cup T_{k-1}^+(x)\} = \{\{r(c_t)\} \cup \{r(c_s)\} \cup \{x\}\}$ and $N^+(c_t) = \{\{r(c_s)\} \cup T_{k-1}^+(d_l) \cup T_{k-1}^+(x)\} = \{\{r(c_s)\} \cup \{r(c_t)\} \cup \{x\}\}$, giving $N^+(c_s) = N^+(c_t)$. This forces c_s, c_t to be selfrepeats, which is a contradiction. This completes the proof. □

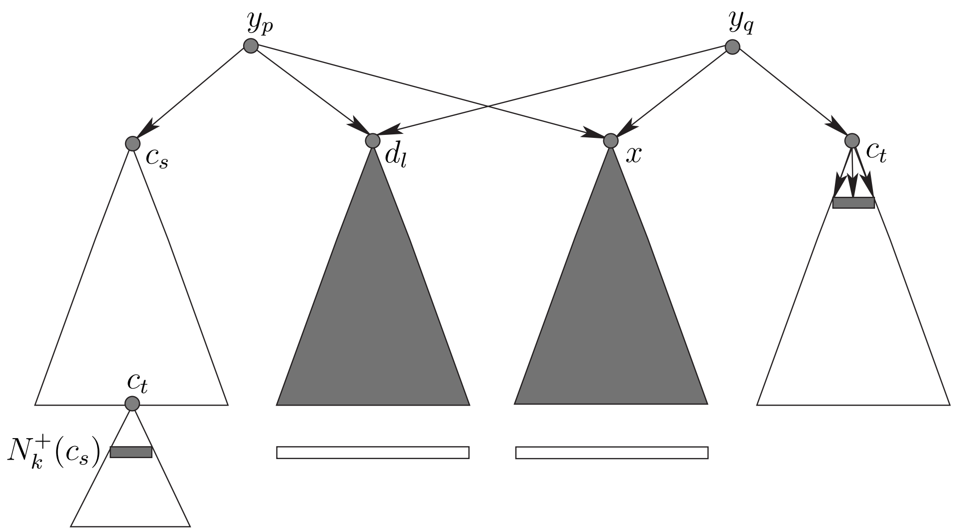


Figure 9.4: Illustration of the multisets $T_k^+(y_p)$ and $T_k^+(y_q)$.

From Lemma 9.0.10, it immediately follows that

◇ **Corollary 9.0.2** *All the vertices $y_1, y_2, \ldots, y_9$ are selfrepeats.*

◇ **Corollary 9.0.3** *The set of vertices* $\{y_1, y_2, \ldots, y_9\}$ *can be partitioned into three sets of three vertices, each triple having the same out-neighbourhoods.*

We will next prove that $(3, k, 2)$-digraphs must be diregular if $k = 3$. To prove the diregularity of such digraphs we will prove that $G^* \in \mathcal{G}(3, k, 3)$ with in-degree sequence $(1, 1, 1, 3, \ldots, 3, 9)$ does not exist if $k = 3$. By applying the previous lemmas and theorems, such a digraph can be partially depicted in Figure 9.5.

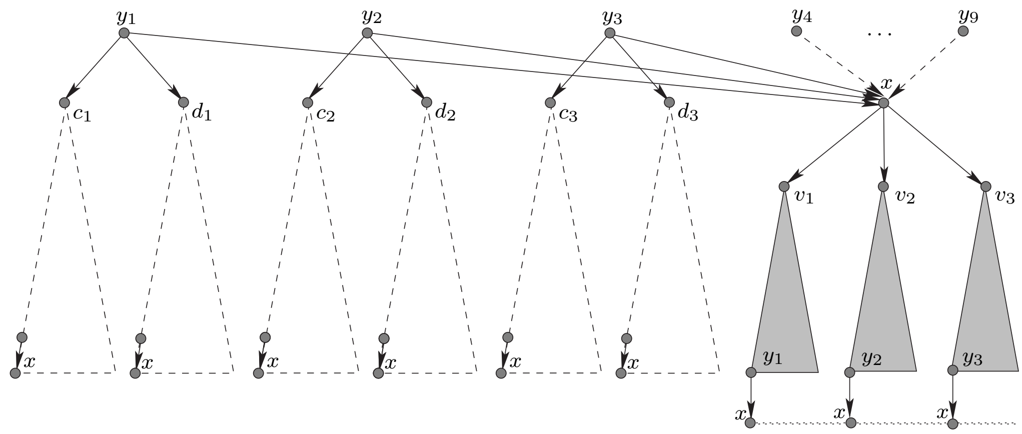


Figure 9.5: Illustration of the multisets $T_k^+(y_i)$, for $i = 1, 2, \ldots, 9$.

◇ **Theorem 9.0.3** *Every* $(3, k, 2)$*-digraph is diregular for* $k = 3$.

Proof. By Corollary 9.0.3, it follows that $N^+(y_1) = N^+(y_6) = N^+(y_8) = \{c_1, d_1, x\}$, $N^+(y_2) = N^+(y_4) = N^+(y_9) = \{c_2, d_2, x\}$ and $N^+(y_3) = N^+(y_5) = N^+(y_7) = \{c_3, d_3, x\}$. Since $k = 3$, we can assume that $N^+(v_1) = \{y_1, v_4, v_5\}$, $N^+(v_2) = \{y_2, v_6, v_7\}$ and $N^+(v_3) = \{y_3, v_8, v_9\}$. The multiset $T_3^+(x)$ is depicted in Figure 9.6.

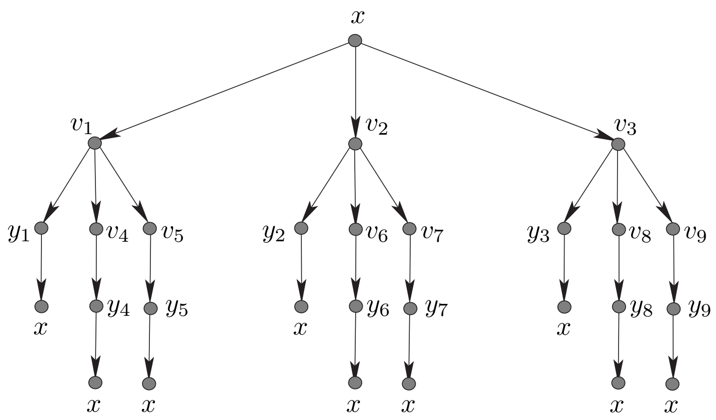


Figure 9.6: Illustration of the multiset $T_3^+(x)$.

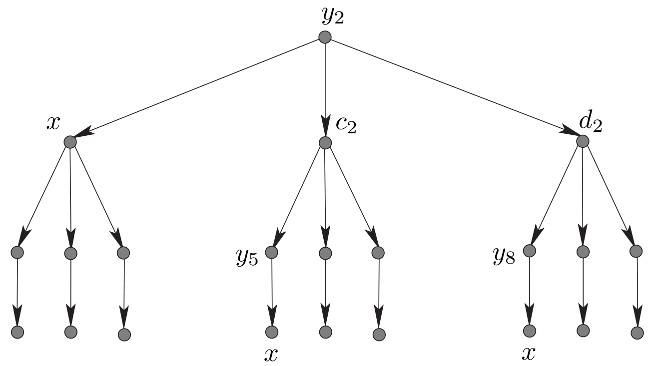


Figure 9.7: Illustration of the multiset $T_3^+(y_2)$.

We now consider the multisets $T_3^+(y_1)$ and $T_3^+(y_2)$. By Lemma 9.0.8, it follows that x is a double repeat of y_1. This implies that x occurs in both $T_3^+(c_1)$ and $T_3^+(d_1)$. To reach x from c_1 and d_1, we must have $N^-(x) \in \{y_7, y_9\}$ otherwise y_1 will have either too many repeats, namely, $\{y_1\} \uplus \{x\} \uplus \{x\}\{u|u \in N_2^+(x) \cap T_2^+(c_1) \cap T_2^+(d_1)\}$, or a cycle of length less than 3, both impossible situations. Similarly, to reach x from c_2 and d_2, we must have $N^-(x) \in \{y_5, y_8\}$, see Figure 9.7. Without loss of generality we suppose that $N^+(c_1) = \{y_7\}$, $N^+(d_1) = \{y_9\}$ and $N^+(c_2) = \{y_5\}$, $N^+(d_2) = \{y_8\}$.

We now consider the multiset $T_3^-(y_8)$. We know that $v_8, d_2 \in N^-(y_8)$, see

Figures 9.6 and 9.7. To reach y_8 from v_1 we have $N^-(y_8) \in \{u|u \in (N^+(v_4) \cup N^+(v_5))\backslash\{y_4, y_5\}\}$ as $x \in \{y_4, y_5\}$. But then it is not possible to reach y_8 from y_1 within distance at most $k = 3$ as $\big(N^+(c_1) \cup N^+(d_1)\big) \bigcap \big((N^+(v_4) \cup N^+(v_5))\backslash\{y_4, y_5\}\big) = \emptyset$. This contradiction completes the proof. □

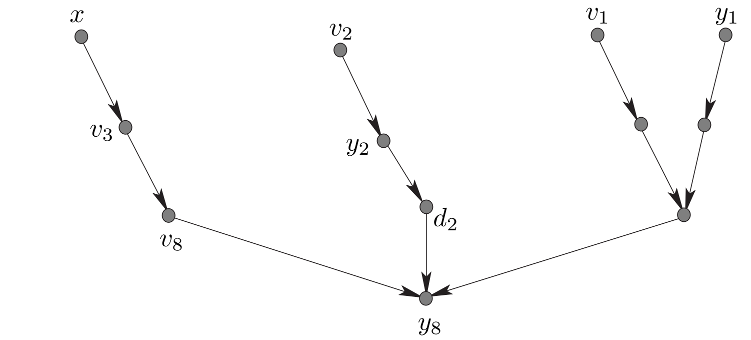


Figure 9.8: Illustration of the multiset $T_3^-(y_8)$.

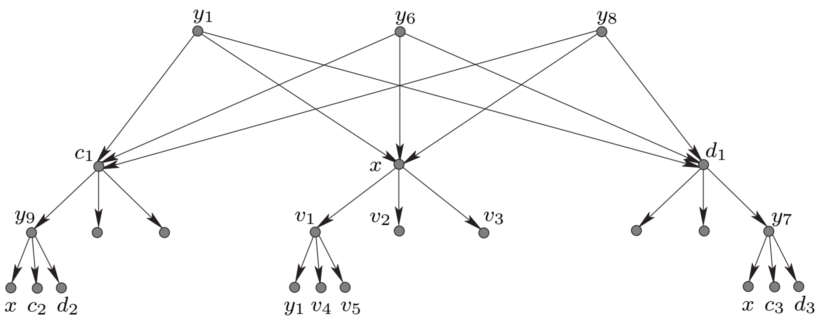


Figure 9.9: Illustration of the multiset $T_3^+(y_1) = T_3^+(y_6) = T_3^+(y_8)$.

Alternatively, we can prove Theorem 9.0.3 as follows.

Alternative Proof of Theorem 9.0.3. Consider the multisets $T_3^+(y_1) = T_3^+(y_6) = T_3^+(y_8)$ in Figure 9.9. By diameter assumption, the vertices c_2, d_2 and c_3, d_3 occur at distance at most k from y_1, y_6 and y_8. But they will not be in the multiset $T_2^+(x)$, otherwise there will be too many repeats. Without loss of generality, we assume $c_2, d_2 \in T_2^+(c_1)$ and $c_3, d_3 \in T_2^+(d_1)$. Then we have $y_9 \in N^+(c_1)$ and $y_7 \in N^+(d_1)$, otherwise v_1 will have too many repeats, namely, either v_1, x, x, y_4 or v_1, x, x, y_5, both impossible.

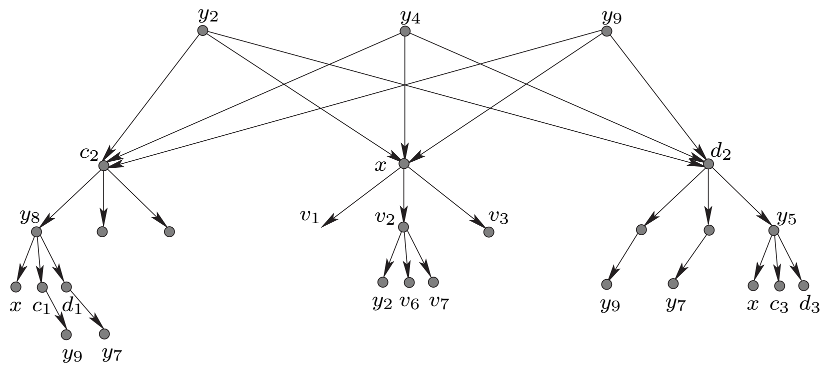


Figure 9.10: Illustration of the multiset $T_3^+(y_2) = T_3^+(y_4) = T_3^+(y_9)$.

We consider the multisets $T_3^+(y_2) = T_3^+(y_4) = T_3^+(y_9)$ in Figure 9.10. By diameter assumption, the vertices c_1, d_1 and c_3, d_3 occur at distance at most k from y_2, y_4 and y_9. Without loss of generality, we assume $c_1, d_1 \in T_2^+(c_2)$ and $c_3, d_3 \in T_2^+(d_2)$. Then we have $y_8 \in N^+(c_2)$ and $y_5 \in N^+(d_2)$, otherwise v_2 will have too many repeats, namely, either v_1, x, x, y_6 or v_1, x, x, y_7, both impossible. We now finally consider the multisets $T_3^+(y_3) = T_3^+(y_5) = T_3^+(y_7)$ in Figure 9.11. By diameter assumption, the vertices c_1, d_1 and c_2, d_2 occur at distance at most k from y_3, y_5 and y_7. Without loss of generality, we assume $c_1, d_1 \in T_2^+(c_3)$ and $c_2, d_2 \in T_2^+(d_3)$. Then we have $y_6 \in N^+(c_3)$ and $y_4 \in N^+(d_3)$, otherwise v_3 will have too many repeats, namely, either v_1, x, x, y_8 or v_1, x, x, y_9, both impossible.

We finally consider the vertices y_9, y_7. The vertices y_9, y_7 must occur in both multisets $T_3^+(y_1) = T_3^+(y_6) = T_3^+(y_8)$ and $T_3^+(y_2) = T_3^+(y_4) = T_3^+(y_9)$. Since $N^-(y_7) = \{d_1, v_7\}$ and $N^-(y_9) = \{c_1, v_9\}$, see Figure 9.9 and Figure 9.6, we have $|N^+(d_2) \cap N^+(d_3)| = 2$, which contradicts Lemma 9.0.10. □

At this stage we have proved that $(3,3,2)$-digraphs are diregular. To settle the question of the diregularity of all $(3,k,2)$-digraphs, it remains to answer the

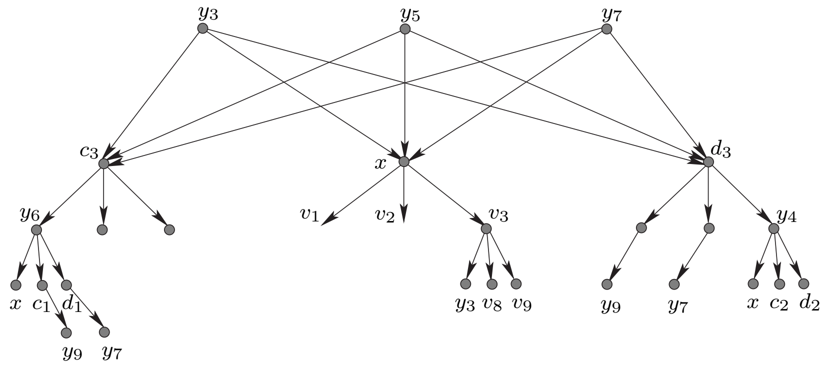


Figure 9.11: Illustration of the multiset $T_3^+(y_3) = T_3^+(y_5) = T_3^+(y_7)$.

following open problem.

Open Problem 9.0.1 *Does there exist a digraph* $G \in \mathcal{G}(3, k, 3)$*, for* $k \geq 4$*, with in-degree sequence* $(1, 1, 1, 3, 3, \ldots, 3, 3, 3, 3, 9)$*?*

Although we are unable to completely settle the question of diregularity of all $(3, k, 2)$-digraphs, we propose the following conjecture.

Conjecture 9.0.1 *Every* $(3, k, 2)$*-digraph is diregular, for* $k \geq 3$.

CONCLUSION

In this thesis, we have considered two different problems of the underlying graph of a network, namely, graph labeling and structural properties of graphs.

In this chapter we summarise our results and list open problems and conjectures arising from this thesis.

Graph labeling

The question of whether a particular family of graphs admits a particular labeling is in general still open, even though there is a large number of results concerning various types of graph labelings, including antimagic labeling. To decide whether a graph G admits a vertex-magic or an edge-magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution. No polynomial time bounded algorithm is known for determining whether G is a vertex-magic or an edge-magic graph.

We have presented a new result on super (a, d)-edge antimagic total labeling for disjoint union of multiple copies of special families of graphs. Our main problem was the following: if a graph G is super (a, d)-edge-antimagic total, is the disjoint union of multiple copies of the graph G super (a, d)-edge-antimagic total as well? We gave some new results when the graph G was either cycle, path, star, m-crowns, caterpillar or complete s-partite graph.

Based on our results, we have proposed the following conjecture.

Conjecture 1 (5.2.1) *There is a super $(a, 1)$-edge-antimagic total labeling for the graph $mK_{\underbrace{n, n, \ldots, n}_{s}}$, for $s \geq 5$ and for every $m \geq 2$ and $n \geq 1$.*

The outstanding open problems in this area can be found in Chapters 3,4 and 5. In addition there are many other classes of graphs, such as multiple copies of friendship graphs, fans, wheels, generalised prisms and antiprisms, ladders and generalised Petersen graphs, on which almost no work has been done so far.

It is not true that if a graph G admits super (a, d)-edge-antimagic total labeling then the disjoint union of multiple copies of the graph G admits super (a, d)-edge-antimagic total labeling as well. There is scope to explore further properties of this type of labelings for future research.

Open Problem 1 *Find relationships between labeling of connected graph and labeling of disconnected graph.*

Open Problem 2 *Find new methods of generating antimagic labeling schemes for disconnected graphs from known antimagic labeling schemes for connected graphs.*

Structural properties of graphs

In this area, we considered the diregularity of digraphs of order close to the Moore bound, that is, digraphs of defect two. We gave an alternative proof for the diregularity of digraphs, defect two of out-degree $d = 2$ and diameter $k \geq 3$. We proved that digraphs of order two less than Moore bound, with maximum out-degree $d \geq 3$ and diameter $k \geq 2$, are out-regular and almost in-regular. Additionally, concerning the direguarity of $(3, k, 2)$-digraphs, for $k \geq 3$, we partially solved an open problem. We have proved that $(3,3,2)$-digraphs are diregular. To settle the question of the diregularity of all $(3,k,2)$-digraphs, it remains to answer the following open problem.

Open Problem 3 (9.0.1) *Does there exist a digraph $G \in \mathcal{G}(3,k,3)$, for $k \geq 4$, with in-degree sequence $(1,1,1,3,3,\ldots,3,3,3,3,9)$?*

As stated in Chapter 8, we believe that the answer is "no". This has led us to propose the following conjecture.

Conjecture 2 (9.0.1) *Every $(3,k,2)$-digraph is diregular, for $k \geq 3$.*

Finally, we believe that the following stronger conjecture also holds.

Conjecture 3 *Every digraph of defect two of out-degree $d \geq 3$ and diameter $k \geq 3$ is diregular.*

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