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Super Edge-antimagic Total Labeling of Disjoint Union of Triangular Ladder and Lobster Graphs

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Abstract

A graph G of order p and size q is called an (a, d)-*edge-antimagic total* if there exist a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that the edge-weights, $w(uv) = f(u) + f(v) + f(uv), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a graph G is called *super* if the smallest possible labels appear on the vertices. In this paper we study super (a, d)-edge-antimagic total properties of disconnected graphs triangular ladder and lobster.

Keywords : *(a, d)-edge-antimagic total labeling, super (a, d)-edge-antimagic total labeling, triangular ladder, lobster graph.*

1 Introduction

By a *labeling* we mean any mapping that carries a set of graph elements onto a set of numbers, called *labels*. In this paper, we deal with labelings with domain the set of all vertices and edges. This type of labeling belongs to the class of *total* labelings. We define the *edge-weight* of an edge $uv \in E(G)$ under a total labeling to be the sum of the vertex labels corresponding to vertices u, v and edge label corresponding to edge uv.

An (a, d)-*edge-antimagic total labeling* on a graph G is a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ with the property that the edge-weights $w(uv) = f(u)+f(uv)+f(v), uv \in E(G)$, form an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a, d)-*edge-antimagic total graph*. Such a graph G is called *super* if the smallest possible labels appear on the vertices. Thus, a *super (a, d)-edge-antimagic total graph* is a graph that admits a super (a, d)-edge-antimagic total labeling.

The concept of (a, d)-edge-antimagic total labeling, introduced by Simanjuntak *at al.* in [12], is natural extension of the notion of *edge-magic* labeling defined by Kotzig and Rosa [10] (see also [1], [8], [11] and [15]). The super (a, d)-edge-antimagic total labeling is natural extension of the notion of *super edge-magic* labeling which was defined by Enomoto *et al.* in [7].

In this paper we investigate the existence of super (a,d)-edge-antimagic total labelings for disconnected graphs. Some constructions of super $(a,0)$-edge-antimagic total labelings for $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$ have been shown by Ivančo and Lučkaničová in [9] and super (a,d)-edge-antimagic total labelings for $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana *et al.* in [13]. Dafik *et al* also found some families of graph which admits super (a,d)-edge-antimagic total labelings, namely $mC_n, mP_n, mK\underbrace{n,n,\dots,n}_{s}$ and m caterpilars in [4, 5, 6].

We will now concentrate on the disjoint union of m copies of triangular ladder and lobster, denoted by $m\pounds_n$ and $m\pounds_{i,j,k}$.

2 Some Useful Lemmas

We start this section by a necessary condition for a graph to be super (a,d)-edge-antimagic total, providing a least upper bound for feasible values of d.

Lemma 1 *If a (p,q)-graph is super (a,d)-edge-antimagic total then $d \leq \frac{2p+q-5}{q-1}$.*

Proof. Assume that a (p,q)-graph has a super (a,d)-edge-antimagic total labeling $f : V(G) \cup E(G) \to \{1,2,\dots,p+q\}$. The minimum possible edge-weight in the labeling f is at least $1+2+p+1 = p+4$. Thus, $a \geq p+4$. On the other hand, the maximum possible edge-weight is at most $(p-1)+p+(p+q) = 3p+q-1$. So we obtain $a+(q-1)d \leq 3p+q-1$ which gives the desired upper bound for the difference d. $\square$

The following lemma, proved by Figueroa-Centeno *et al.* in [8], gives a necessary and sufficient condition for a graph to be super edge-magic (super $(a,0)$-edge-antimagic total).

Lemma 2 *A (p,q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1,2,\dots,p\}$ such that the set $S = \{f(u)+f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p+q+s$, where $s = min(S)$ and $S = \{a-(p+1), a-(p+2), \dots, a-(p+q)\}$.*

In our terminology, the previous lemma states that a (p,q)-graph G is super $(a,0)$-edge-antimagic total if and only if there exists an $(a-p-q,1)$-edge-antimagic vertex labeling.

Next, we restate the following lemma that appeared in [14].

Lemma 3 [14] *Let $\mathfrak{A}$ be a sequence $\mathfrak{A} = \{c, c+1, c+2, \dots c+k\}$, k even. Then there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $\mathfrak{A} + \Pi(\mathfrak{A}) = \{2c+\frac{k}{2}, 2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, \dots, 2c+\frac{3k}{2}-1, 2c+\frac{3k}{2}\}$.*

3 Disjoint Union of Triangular Ladder

Disjoint union of m copies of triangular ladder denoted by $m\pounds_n$ is a disconnected graph with vertex set $V(m\pounds_n) = \{u_i^j v_i^j : 1 \le i \le n, 1 \le j \le m\}$ and edge set $E(m\pounds_n) = \{u_i^j u_{i+1}^j, v_i^j v_{i+1}^j, u_i^j v_{i+1}^j : 1 \le i \le n-1, 1 \le j \le m\} \bigcup \{u_i^j v_i^j : 1 \le i \le n, 1 \le j \le m\}$. Thus $|V(m\pounds_n)| = p = 2mn$ and $|E(m\pounds_n)| = q = m(4n-3)$.

If the disjoint union of m copies of a triangular ladder $m\pounds_n$, has a super (a,d)-edge-antimagic total labeling then, for $p = 2mn$ and $q = m(4n-3)$, it follows from Lemma 1 that the upper bound of d is $d \le 2 + \frac{3m-3}{4nm-3m}$ or $d \in \{0,1,2\}$.

The following theorem describes an $(a,1)$-edge-antimagic vertex labeling for disjoint union of m copies of a triangular ladder.

Theorem 1 *If $m \ge 3$ is odd and $n \ge 2$, then the graph $m\pounds_n$ has an $(a,1)$-edge-antimagic vertex labeling.*

Proof. Define the vertex labeling $\alpha_1 : V(m\pounds_n) \to \{1, 2, \dots, 2mn\}$ in the following way:

$$\alpha_1(v_i^j) = \begin{cases} \frac{j+1}{2} + (i-1)2m, & \text{for } i \equiv 1(\text{mod}3) \text{ , } j \text{ odd} \\ \frac{m+j+1}{2} + (i-1)2m, & \text{for } i \equiv 1(\text{mod}3) \text{ , } j \text{ even} \\ 3m + 1 - j + (i-2)2m, & \text{for } i \equiv 2(\text{mod}3) \text{ , any } j \\ 4m + \frac{m+j}{2} + (i-3)2m, & \text{for } i \equiv 3(\text{mod}3) \text{ , } j \text{ odd} \\ 4m + \frac{j}{2} + (i-3)2m, & \text{for } i \equiv 1(\text{mod}3) \text{ , } j \text{ even} \end{cases}$$

$$\alpha_1(u_i^j) = \begin{cases} m + \frac{m+j}{2} + (i-1)2m, & \text{for } i \equiv 1(\text{mod}3) \text{ , } j \text{ odd} \\ m + \frac{j}{2} + (i-1)2m, & \text{for } i \equiv 1(\text{mod}3) \text{ , } j \text{ even} \\ 3m + \frac{j+1}{2} + (i-2)2m, & \text{for } i \equiv 2(\text{mod}3) \text{ , } j \text{ odd} \\ 3m + \frac{m+j+1}{2} + (i-2)2m, & \text{for } i \equiv 2(\text{mod}3) \text{ , } j \text{ even} \\ 6m + 1 - j + (i-3)2m, & \text{for } i \equiv 3(\text{mod}3) \text{ , any } j \end{cases}$$

The vertex labeling α_1 is a bijective function. The edge-weights of $m\pounds_n$, under the labeling α_1, constitute the following sets

$$W_{\alpha_1}^1(u_i^j v_i^j) = \begin{cases} \frac{3m+2j+1}{2} + (i-1)4m, & \text{for } i \equiv 1(\text{mod}3) \text{ and any } j \\ \frac{12m-j+3}{2} + (i-2)4m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{13m-j+3}{2} + (i-2)4m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ \frac{21m-j+2}{2} + (i-3)4m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{20m-j+2}{2} + (i-3)4m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

$$W_{\alpha_1}^2(u_i^j u_{i+1}^j) = \begin{cases} \frac{9m+2j+1}{2} + (i-1)4m, & \text{for } i \equiv 1(\text{mod}3) \text{ and any } j \\ \frac{18m-j+3}{2} + (i-2)4m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{19m-j+3}{2} + (i-2)4m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ \frac{27m-j+2}{2} + (i-3)4m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{26m-j+2}{2} + (i-3)4m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

$$W^3_{\alpha_1}(v_i^j v_{i+1}^j) = \begin{cases} \frac{6m-j+3}{2} + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{7m-j+3}{2} + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and } j \text{ even} \\ \frac{15m-j+2}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{14m-j+2}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ \frac{21m+2j+1}{2} + (i-3)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and any } j \end{cases}$$

$$W^4_{\alpha_1}(u_i^j v_{i+1}^j) = \begin{cases} \frac{9m-j+2}{2} + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{8m-j+2}{2} + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and } j \text{ even} \\ \frac{15m+2j+1}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and any } j \\ \frac{24m-j+3}{2} + (i-3)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{25m-j+3}{2} + (i-3)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

It is not difficult to see that the set $\bigcup_{r=1}^4 W^r_{\alpha_1} = \{\frac{3m+3}{2}, \frac{3m+5}{2}, \ldots, \frac{8mn-3m+1}{2}\}$ consists of consecutive integers. Thus α_1 is a $(\frac{3m+3}{2}, 1)$-edge antimagic vertex labeling. $\square$

Theorem 2 *If $m \geq 3$ odd and $n \geq 2$ then the graph $m\pounds_n$ has a super $(\frac{3m(4n-1)+3}{2}, 0)$-edge-antimagic total labeling and a super $(\frac{m(4n+3)+5}{2}, 2)$-edge-antimagic total labeling.*

Proof.

Case 1. $d = 0$
We have proved that the vertex labeling α_1 is a $(\frac{3m+3}{2}, 1)$-edge antimagic vertex labeling. With respect to Lemma 2, by completing the edge labels $p+1, p+2, \ldots, p+q$, we are able to extend labeling α_1 to a super $(a, 0)$-edge-antimagic total labeling, where, for $p = 2mn$ and $q = m(4n-3)$, the value $a = \frac{3m(4n-1)+3}{2}$.

Case 2. $d = 2$
Label the vertices of $m\pounds_n$ with $\alpha_2(v_i^j) = \alpha_1(v_i^j)$ and $\alpha_2(u_i^j) = \alpha_1(u_i^j)$, for $i = 1, 2, ...n$ and $1 \leq j \leq m$; and label the edges with the following way.

$$\alpha_2(u_i^j v_i^j) = \begin{cases} 2mn + j + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and any } j \\ \frac{m(4n+9)-j+2}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{2m(2n+5)-j+2}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ \frac{2m(2n+9)-j+1}{2} + (i-3)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+17)-j+1}{2} + (i-3)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

$$\alpha_2(u_i^j u_{i+1}^j) = \begin{cases} m(2n+3) + j + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and any } j \\ \frac{m(4n+15)-j+2}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{4m(n+4)-j+2}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ \frac{2m(2n+12)-j+1}{2} + (i-2)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+23)-j+1}{2} + (i-3)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

$$\alpha_2(v_i^j v_{i+1}^j) = \begin{cases} \frac{m(4n+3)-j+2}{2} + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{4m(n+1)-j+2}{2} + (i-1)4m, & \text{for } i \ \equiv 1(\text{mod}3) \text{ and } j \text{ even} \\ \frac{4m(n+3)-j+1}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+11)-j+1}{2} + (i-2)4m, & \text{for } i \ \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ 2mn + 9m + j + (i-3)4m, & \text{for } i \ \equiv 3(\text{mod}3) \text{ and any } j \end{cases}$$

$$\alpha_2(u_i^j v_{i+1}^j) = \begin{cases} \frac{2m(2n+3)-j+1}{2} + (i-1)4m, & \text{for } i \equiv 1(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+5)-j+1}{2} + (i-1)4m, & \text{for } i \equiv 1(\text{mod}3) \text{ and } j \text{ even} \\ 2mn+6m+j+(i-2)4m, & \text{for } i \equiv 2(\text{mod}3) \text{ and any } j \\ \frac{m(4n+21)-j+2}{2} + (i-3)4m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{2m(2n+11)-j+2}{2} + (i-3)4m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

The total labeling α_2 is a bijective function from $V(m\pounds_n) \cup E(m\pounds_n)$ onto the set $\{1, 2, 3, \ldots, 6mn-3m\}$. The edge-weights of $m\pounds_n$, under the labeling α_2, constitute the sets

$$W_{\alpha_2}^1(u_i^j v_i^j) = \begin{cases} \frac{m(4n+3)+4j+1}{2} + (i-1)8m, & \text{for } i \equiv 1(\text{mod}3) \text{ and any } j \\ \frac{m(4n+21)-2j+5}{2} + (i-2)8m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+23)-2j+5}{2} + (i-2)8m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ \frac{m(4n+39)-2j+3}{2} + (i-3)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+37)-2j+3}{2} + (i-3)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

$$W_{\alpha_2}^2(u_i^j u_{i+1}^j) = \begin{cases} \frac{m(4n+15)+4j+1}{2} + (i-1)8m, & \text{for } i \equiv 1(\text{mod}3) \text{ and any } j \\ \frac{m(4n+33)-2j+5}{2} + (i-2)8m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+35)-2j+5}{2} + (i-2)8m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ even} \\ \frac{m(4n+51)-2j+3}{2} + (i-3)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+49)-2j+3}{2} + (i-3)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

$$W_{\alpha_2}^3(v_i^j v_{i+1}^j) = \begin{cases} \frac{m(4n+9)-2j+5}{2} + (i-1)8m, & \text{for } i \equiv 1(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+11)-2j+5}{2} + (i-1)8m, & \text{for } i \equiv 1(\text{mod}3) \text{ and } j \text{ even} \\ \frac{m(4n+27)-2j+3}{2} + (i-2)8m, & \text{for } i \equiv 2(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+25)-2j+3}{2} + (i-2)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ even} \\ \frac{m(4n+39)+4j+1}{2} + (i-3)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and any } j \end{cases}$$

$$W_{\alpha_2}^4(u_i^j v_{i+1}^j) = \begin{cases} \frac{m(4n+15)-2j+3}{2} + (i-1)8m, & \text{for } i \equiv 1(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+13)-2j+3}{2} + (i-1)8m, & \text{for } i \equiv 1(\text{mod}3) \text{ and } j \text{ even} \\ \frac{m(4n+27)+4j+1}{2} + (i-2)8m, & \text{for } i \equiv 2(\text{mod}3) \text{ and any } j \\ \frac{m(4n+45)-2j+5}{2} + (i-3)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ odd} \\ \frac{m(4n+47)-2j+5}{2} + (i-3)8m, & \text{for } i \equiv 3(\text{mod}3) \text{ and } j \text{ even} \end{cases}$$

It is not difficult to see that the set $\bigcup_{r=1}^{4} W_{\alpha_2}^r = \{\frac{m(4n+3)+5}{2}, \frac{m(4n+3)+9}{2}, \ldots, \frac{20mn-9m+1}{2}\}$ contains an arithmetic sequence with the first term $\frac{m(4n+3)+5}{2}$ and common difference 2. Thus α_2 is a super $(\frac{m(4n+3)+5}{2}, 2)$-edge-antimagic total labeling. This concludes the proof. □

Theorem 3 *The graph* $m\pounds_n$ *has a super* $(4mn+2, 1)$*-edge-antimagic total labeling for* $m \geq 2$ *and* $n \geq 2$*.*

Proof. Construct the bijective function of total labeling $\alpha_3 : V(m\pounds_n) \cup E(m\pounds_n) \longrightarrow \{1, 2, 3, \ldots, 6mn-3m\}$, for $i = 1, 2, 3, \ldots n$ and $1 \leq j \leq m$, as follows:

$$\alpha_3(u_i^j) = m + j + (i-1)2m,$$

$$\begin{aligned}
\alpha_3(v_i^j) &= j+(i-1)2m,\\
\alpha_3(u_i^jv_i^j) &= (6n-3)m+1-j-(i-1)2m,\\
\alpha_3(u_i^ju_{i+1}^j) &= (4n-3)m+1-j-(i-1)2m,\\
\alpha_3(v_i^jv_{i+1}^j) &= (4n-2)m+1-j-(i-1)2m,\\
\alpha_3(u_i^jv_{i+1}^j) &= (6n-4)m+1-j-(i-1)2m.
\end{aligned}$$

The total labeling α_3 is a bijective function from $V(m\pounds_n) \cup E(m\pounds_n)$ onto the set $\{1, 2, 3, \ldots, 6mn-3m\}$. The edge-weights of $m\pounds_n$, under the labeling α_3, constitute the sets

$$\begin{aligned}
W^1_{\alpha_3}(u_i^jv_i^j) &= 2m(3n-1)+j+1+(i-1)2m,\\
W^2_{\alpha_3}(u_i^ju_{i+1}^j) &= m(4n+1)+j+1+(i-1)2m,\\
W^3_{\alpha_3}(v_i^jv_{i+1}^j) &= 4nm+j+1+(i-1)2m,\\
W^4_{\alpha_3}u_i^jv_{i+1}^j) &= m(6n-1)+j+1+(i-1)2m.
\end{aligned}$$

Hence, the set $\bigcup_{r=1}^{4} W^r_{\alpha_3} = \{4nm+2, 4nm+3\ldots, 8mn-3m+1\}$ consists of consecutive integers. Thus α_3 is a super $(4nm+2, 1)$-edge-antimagic total labeling. ☐

Apart from those cases, we do not have the complete answer. Therefore we propose the following open problem.

Open Problem 1 *For the graph $m\pounds_n$, $m \geq 2$ even and $n \geq 2$, determine if there is a super (a, d)-edge-antimagic total labeling with $d \in \{0, 2\}$.*

4 Disjoint Union of Lobster Graph

Lobster graph is a *tree* in which if we omit the leaves then it forms a caterpillar. Now, we will study super edge-antimagicness of a disjoint union of m copies of lobster, denoted by $m\pounds_{i,j,k}$. It is a disconnected graph with vertex set $V(\pounds_{i,j,k}) = \{x_i^s \cup x_{i,j}^s \cup x_{i,j,k}^s, 1 \leq i \leq n, 2 \leq j \leq p, 1 \leq k \leq l, 1 \leq s \leq m\}$ and edge set $E(\pounds_{i,j,k}) = \{x_i^s x_{i+1}^s : 1 \leq i \leq n-1, 1 \leq s \leq m\} \cup \{x_i^s x_{i,j}^s \cup x_{i,j}^s x_{i,j,k}^s : 1 \leq i \leq n, 1 \leq j \leq p, 1 \leq k \leq l, 1 \leq s \leq m\}$. Thus $|V(m\pounds_{i,j,k})| = p = 5mn$ and $|E(m\pounds_{i,j,k})| = q = 5mn - m$.

If the disjoint union of m copies of a lobster $m\pounds_{i,j,k}$, has a super (a, d)-edge-antimagic total labeling then, for $p = 5mn$ and $q = 5mn - m$, it follows from Lemma 1 that the upper bound of d is $d \leq 3 + \frac{2m-2}{5nm-m-1}$ or $d \in \{0, 1, 2, 3\}$. We concentrate on the super edge-antimagicness of $m\pounds_{i,j,k}$ for $1 \leq i \leq n, 1 \leq j \leq 2$ and $k = 1$.

The following theorem describes an $(a, 1)$-edge-antimagic vertex labeling for disjoint union of m copies of the lobsters.

Theorem 4 *If $m \geq 3$ is odd and $n \geq i \geq 2$ is even, then the graph $m\pounds_{i,j,k}$ has an $(a, 1)$-edge-antimagic vertex labeling, for $1 \leq j \leq 2$ and $k = 1$.*

Proof. Define the vertex labeling $\alpha_4 : V(m\pounds_{i,j,k}) \rightarrow \{1, 2, \ldots, 5mn\}$ in the following way:

$$\alpha_4(x_i^s) = \begin{cases} \frac{m(5i-3)}{2}+\mathrm{s}, & \text{for } i \text{ odd}, \ \ 1 \leq \mathrm{s} \leq \mathrm{m} \\ \frac{m(5n+5i-3)-s}{2} + 1, & \text{for } i \text{ even}, s \text{ odd} \\ \frac{5m(n+i)-s}{2} + 1-\mathrm{m}, & \text{for } i \text{ even}, s \text{ even} \end{cases}$$

$$\alpha_4(x_{i,j}^s) = \begin{cases} \frac{5mi}{2}+\mathrm{s} \text{ - } 2\mathrm{m}, & \text{for } j = \ 1, \ \ i \text{ even}, \ 1 \leq \mathrm{s} \leq \mathrm{m} \\ \frac{5mi}{2}+\mathrm{s} \text{ - } \mathrm{m}, & \text{for } j = \ 2, \ \ i \text{ even}, \ 1 \leq \mathrm{s} \leq \mathrm{m} \\ \frac{5m(n+i)-s}{2}+1 \text{ - } 2\mathrm{m}, & \text{for } j = \ 1, \ \ i \text{ odd}, s \text{ odd} \\ \frac{5m(n+i)-s}{2}+1 \text{ - } \mathrm{m}, & \text{for } j = \ 2, \ \ i \text{ odd}, s \text{ odd} \\ \frac{m(5n+5i-3)-s}{2} + 1, & \text{for } j = \ 1, \ \ i \text{ odd}, s \text{ even} \\ \frac{m(5n+5i-1)-s}{2} + 1, & \text{for } j = \ 2, \ \ i \text{ odd}, s \text{ even} \end{cases}$$

$$\alpha_4(x_{i,j,k}^s) = \begin{cases} 5\mathrm{m}(\frac{i-1}{2})+\mathrm{s} \ , & \text{for } j = \ 1, \ \ i \text{ odd}, \ 1 \leq \mathrm{s} \leq \mathrm{m} \\ \frac{m(5i-1)}{2}+\mathrm{s}, & \text{for } j = \ 2, \ \ i \text{ odd}, \ \ 1 \leq \mathrm{s} \leq \mathrm{m} \\ \frac{5m(n+i-1)-s}{2}+1 \ , & \text{for } j = \ 1, \ \ i \text{ even}, s \text{ odd} \\ \frac{5m(n+i)-(s+m)}{2}+1 \ , & \text{for } j = \ 2, \ \ i \text{ even}, s \text{ odd} \\ \frac{5m(n+i)-s}{2}+1\text{-}2\mathrm{m} \ , & \text{for } j = \ 1, \ \ i \text{ even}, s \text{ even} \\ \frac{5m(n+i)-s}{2}+1 \ , & \text{for } j = \ 2, \ \ i \text{ even}, s \text{ even} \end{cases}$$

The vertex labeling α_4 is a bijective function. The edge-weights of $m\pounds_{i,j,k}$, under the labeling α_4, constitute the following sets

$$\begin{aligned} W_{\alpha_4}^1 &= \{w_{\alpha_1}^1(x_i^s x_{i+1}^s) : \text{for } 1 \leq i \leq n-1 \ \text{ and } \ s \text{ odd}\} \\ &= \{\tfrac{5mn+10mi-m+s}{2} + 1 : \text{for } 1 \leq i \leq n-1 \ \text{ and } \ s \text{ odd}\}, \\ W_{\alpha_4}^2 &= \{w_{\alpha_1}^1(x_i^s x_{i+1}^s) : \text{for } 1 \leq i \leq n-1 \ \text{ and } \ s \text{ even}\} \\ &= \{\tfrac{5mn+s}{2} + 5mi + 1 : \text{for } 1 \leq i \leq n-1 \ \text{ and } \ s \text{ even}\}, \end{aligned}$$

$$\begin{aligned} W_{\alpha_4}^3 &= \{w_{\alpha_1}^1(x_i^s x_{i,j}^s) : \ \text{for } j = \ 1, 1 \leq i \leq n \ \text{ and } \ s \text{ odd}\} \\ &= \{\tfrac{m(5n+10i-7)+s}{2} + 1 : \ \text{for } j = \ 1, 1 \leq i \leq n \ \text{ and } \ s \text{ odd}\} \\ W_{\alpha_4}^4 &= \{w_{\alpha_1}^1(x_i^s x_{i,j}^s) : \ \text{for } j = \ 2, 1 \leq i \leq n \ \text{ and } \ s \text{ odd}\} \\ &= \{\tfrac{m(5n+10i-5)+s}{2} + 1 : \ \text{for } j = \ 2, 1 \leq i \leq n \ \text{ and } \ s \text{ odd}\} \\ W_{\alpha_4}^5 &= \{w_{\alpha_1}^1(x_i,^s x_{i,j}^s) : \ \text{for } j = \ 1, 1 \leq i \leq n \ \text{ and } \ s \text{ even}\} \\ &= \{\tfrac{5mn+s}{2} + m(5i-3) + 1 : \ \text{for } j = \ 1, 1 \leq i \leq n \ \text{ and } \ s \text{ even}\} \\ W_{\alpha_4}^6 &= \{w_{\alpha_1}^1(x_i^s x_{i,j}^s) : \ \text{for } j = \ 2, 1 \leq i \leq n \ \text{ and } \ s \text{ even}\} \\ &= \{\tfrac{5mn+s}{2} + m(5i-2) + 1 : \ \text{for } j = \ 2, 1 \leq i \leq n \ \text{ and } \ s \text{ even}\} \end{aligned}$$

$$\begin{aligned}
W^7_{\alpha_4} &= \{w^1_{\alpha_1}(x^s_{i,j}x^s_{i,j,k}):\ \text{for } j=\ 1,\ 1\le i\le n \text{ and } s \text{ odd}\}\\
&= \{\tfrac{m(5n+10i-9)+s}{2}+1:\ \text{for } j=\ 1,\ 1\le i\le n \text{ and } s \text{ odd}\}\\
W^8_{\alpha_4} &= \{w^1_{\alpha_1}(x^s_{i,j}x^s_{i,j,k}):\ \text{for } j=\ 2,\ 1\le i\le n \text{ and } s \text{ odd}\}\\
&= \{\tfrac{m(5n+10i-3)+s}{2}+1:\ \text{for } j=\ 2,\ 1\le i\le n \text{ and } s \text{ odd}\}\\
W^9_{\alpha_4} &= \{w^1_{\alpha_1}(x^s_{i,j}x^s_{i,j,k}):\ \text{for } j=\ 1,\ 1\le i\le n \text{ and } s \text{ even}\}\\
&= \{\tfrac{m(5n+10i-8)+s}{2}+1:\ \text{for } j=\ 1,\ 1\le i\le n \text{ and } s \text{ even}\}\\
W^{10}_{\alpha_4} &= \{w^1_{\alpha_1}(x^s_{i,j}x^s_{i,j,k}):\ \text{for } j=\ 2,\ 1\le i\le n \text{ and } s \text{ even}\}\\
&= \{\tfrac{m(5n+10i)+s}{2}\text{+1-m:}\ \text{for } j=\ 2,\ 1\le i\le n \text{ and } s \text{ even}\}
\end{aligned}$$

It is not difficult to see that the set $\bigcup_{r=1}^{10} W^r_{\alpha_4} = \{\frac{5mn+m+3}{2}, \frac{5mn+m+5}{2}, ..., \frac{15mn-m+1}{2}\}$ consists of consecutive integers. Thus α_4 is a $(\frac{5mn+m+3}{2}, 1)$-edge antimagic vertex labeling. □

Theorem 5 *If $m \ge 3$ is odd and $n \ge i \ge 2$ is even then the graph $m\pounds_{i,j,k}$ has a super $\left(\frac{25mn-m-3}{2}, 0\right)$-edge-antimagic total labeling and a super $(\frac{15mn+m+5}{2}, 2)$-edge-antimagic total labeling.*

Proof.

Case 1. $d = 0$

We have proved that the vertex labeling α_4 is a $(\frac{5mn+m+3}{2}, 1)$-edge antimagic vertex labeling. With respect to Lemma 2, by completing the edge labels $p+1, p+2, \ldots, p+q$, we are able to extend labeling α_4 to a super $(a, 0)$-edge-antimagic total labeling, where, for $p = 5mn$ and $q = 5mn - m$, the value $a = \frac{25mn-m-3}{2}$.

Case 2. $d = 2$

Label the vertices of $m\pounds_{i,j,k}$ with $\alpha_5(x^s_i) = \alpha_4(x^s_i), \alpha_5(x^s_{i,j}) = \alpha_4(x^s_{i,j})$ and $\alpha_5(x^s_{i,j,k}) = \alpha_4(x^s_{i,j,k})$, for $1 \le i \le n, 1 \le j \le 2, k = 1$ and $1 \le s \le m$; and label the edges with the following way.

$$\alpha_5(x^s_i x^s_{i+1}) = \begin{cases} m(5n+5i-1)+(\frac{1+s}{2}), & \text{for } 1\le \text{i} \le \text{n}-1,\ s \text{ odd}\\ m(5n+5i)+(\frac{1+s-m}{2}), & \text{for } 1\le \text{i} \le \text{n}-1,\ s \text{ even}\end{cases}$$

For $1 \le i \le n$ and $1 \le j \le 2$

$$\alpha_5(x^s_i x^s_{i,j}) = \begin{cases} m(5n+5i-4)+(\frac{1+s}{2}), & \text{for } j=\ 1,\ \ 1\le \text{s}\le \text{n},\ s \text{ odd}\\ m(5n+5i-3)+(\frac{1+s}{2}), & \text{for } j=\ 2,\ \ 1\le \text{s}\le \text{n},\ s \text{ odd}\\ 5m(n+i)+(\frac{s+1-7m}{2}), & \text{for } j=\ 1,\ \ 1\le \text{s}\le \text{n},\ s \text{ even}\\ 5m(n+i)+(\frac{s+1-5m}{2}), & \text{for } j=\ 2,\ \ 1\le \text{s}\le \text{n},\ s \text{ even}\end{cases}$$

For $1 \le i \le n, 1 \le j \le 2$ and $k = 1$

$$\alpha_5(x^s_{i,j} x^s_{i,j,k}) = \begin{cases} 5m(n+i-1)+(\frac{1+s}{2}), & \text{for } j=\ 1,\ \ 1\le \text{s}\le \text{n},\ s \text{ odd}\\ m(5n+5i-2)+(\frac{1+s}{2}), & \text{for } j=\ 2,\ \ 1\le \text{s}\le \text{n},\ s \text{ odd}\\ 5m(n+i)+(\frac{s+1-9m}{2}), & \text{for } j=\ 1,\ \ 1\le \text{s}\le \text{n},\ s \text{ even}\\ 5m(n+i)+(\frac{s+1-3m}{2}), & \text{for } j=\ 2,\ \ 1\le \text{s}\le \text{n},\ s \text{ even}\end{cases}$$

The total labeling α_5 is a bijective function from $V(m\pounds_{i,j,k}) \cup E(m\pounds_{i,j,k})$ onto the set $\{1, 2, 3, \dots, 10mn - m\}$. The edge-weights of $m\pounds_{i,j,k}$, under the labeling α_5, constitute the sets

$$\begin{aligned}
W^1_{\alpha_5} &= W^1_{\alpha_4} + \alpha_5(x^s_i x^s_{i+1});\ \text{for } 1 \le i \le n-1, \text{ and } s \ \text{ odd}\\
&= \{\tfrac{5mn+10mi-m+s}{2} + 1\} + \{5mn + 5mi - m + (\tfrac{1+s}{2})\}\\
W^2_{\alpha_5} &= W^2_{\alpha_4} + \alpha_5(x^s_i x^s_{i+1});\ \text{for } 1 \le i \le n-1, \text{ and } s \ \text{ even}\\
&= \{\tfrac{5mn+s}{2} + 5mi + 1\} + \{5mn + 5mi + (\tfrac{1+s-m}{2})\}\\
W^3_{\alpha_5} &= W^3_{\alpha_4} + \alpha_5(x^s_i x^s_{i,j});\ \text{for } 1 \le i \le n,\ j = 1, \text{ and } s \ \text{ odd}\\
&= \{\tfrac{5mn+10mi-7m+s}{2} + 1\} + \{5mn + 5mi - 4m + (\tfrac{s+1}{2})\}\\
W^4_{\alpha_5} &= W^4_{\alpha_4} + \alpha_5(x^s_i x^s_{i,j});\ \text{for } 1 \le i \le n\ ,\ j = 2, \text{ and } s \ \text{ odd}\\
&= \{\tfrac{5mn+10mi-5m+s}{2} + 1\} + \{5mn + 5mi - 3m + (\tfrac{s+1}{2})\}\\
W^5_{\alpha_5} &= W^5_{\alpha_4} + \alpha_5(x^s_i x^s_{i,j});\ \text{for } 1 \le i \le n\ ,\ j = 1, \text{ and } s \ \text{ even}\\
&= \{\tfrac{5mn+s}{2} + 5mi - 3m + 1\} + \{5mn + 5mi + (\tfrac{s+1-7m}{2})\}\\
W^6_{\alpha_5} &= W^6_{\alpha_4} + \alpha_5(x^s_i x^s_{i,j});\ \text{for } 1 \le i \le n\ ,\ j = 2, \text{ and } s \ \text{ even}\\
&= \{\tfrac{5mn+s}{2} + 5mi - 2m + 1\} + \{5mn + 5mi + (\tfrac{s+1-5m}{2})\}\\
W^7_{\alpha_5} &= W^7_{\alpha_4} + \alpha_5(x^s_{i,j} x^s_{i,j,k});\ \text{for } 1 \le i \le n\ ,\ j = 1, \text{ and } s \ \text{ odd}\\
&= \{\tfrac{m(5n+10i-9)+s}{2} + 1\} + \{5mn + 5mi - 5m + (\tfrac{s+1}{2})\}\\
W^8_{\alpha_5} &= W^8_{\alpha_4} + \alpha_5(x^s_{i,j} x^s_{i,j,k});\ \text{for } 1 \le i \le n\ ,\ j = 2, \text{ and } s \ \text{ odd}\\
&= \{\tfrac{m(5n+10i-3)+s}{2} + 1\} + \{5mn + 5mi - 2m + (\tfrac{s+1}{2})\}\\
W^9_{\alpha_5} &= W^9_{\alpha_4} + \alpha_5(x^s_{i,j} x^s_{i,j,k});\ \text{for } 1 \le i \le n\ ,\ j = 1, \text{ and } s \ \text{ even}\\
&= \{\tfrac{m(5n+10i-8)+s}{2} + 1\} + \{5mn + 5mi + (\tfrac{s+1-9m}{2})\}\\
W^{10}_{\alpha_5} &= W^{10}_{\alpha_4} + \alpha_5(x^s_{i,j} x^s_{i,j,k});\ \text{for } 1 \le i \le n\ ,\ j = 2, \text{ and } s \ \text{ even}\\
&= \{\tfrac{m(5n+10i)+s}{2} + 1 - m\} + \{5mn + 5mi + (\tfrac{s+1-3m}{2})\}
\end{aligned}$$

It is not difficult to see that the set $\bigcup_{r=1}^{10} W^r_{\alpha_5} = \{\frac{15mn+m+5}{2}, \frac{15mn+m+9}{2}, \dots \frac{35mn-3m+1}{2}\}$ contains an arithmetic sequence with the first term $\frac{15mn+m+5}{2}$ and common difference 2. Thus α_5 is a super $(\frac{15mn+m+5}{2}, 2)$-edge-antimagic total labeling. This completes the proof. □

Theorem 6 *The graph $m\pounds_{i,j,k}$ has a super $(10mn+2, 1)$-edge-antimagic total labeling for $m \ge 3$ odd and $n \ge i \ge 2$ even.*

Proof. For $m \ge 3$ odd and $n \ge i \ge 2$ even, consider the vertex labeling α_4 of the graph $m\pounds_{i,j,k}$ from Theorem 4 which is a $(\frac{5mn+m+3}{2}, 1)$-EAV labeling. Let a sequence $\mathfrak{A} = \{c, c+1, c+2, \dots, c+k\}$ be the set of edge-weights of the vertex labeling α_4 for $c = \frac{5mn+m+3}{2}$ and $k = 5mn - m - 1$. In light of Lemma 3, there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + 5mn + 1] = \{c + \frac{15mn-m+1}{2}, c + \frac{15mn-m+1}{2} + 1, \dots, c + \frac{25mn-3m-1}{2}\}$. If $[\Pi(\mathfrak{A}) - c + 5mn + 1]$ is an edge labeling of

$m\pounds_{i,j,k}$ then $\mathfrak{A}+[\Pi(\mathfrak{A})-c+5mn+1]$ gives the set of the edge-weights of $m\pounds_{i,j,k}$, which implies that the resulting total labeling is super $(10mn+2,1)$-EAT. This concludes the proof. $\square$

Apart from those cases, we have not found any super (a,d)-edge-antimagic total labeling. Therefore we propose the following open problems.

Open Problem 2 *For the graph $m\pounds_{i,j,k}$, $m \geq 3$ odd and $n \geq i \geq 2$ even, determine if there is a super (a,d)-edge-antimagic total labeling with $d=3$.*

Open Problem 3 *For the graph $m\pounds_{i,j,k}$, either $m \geq 3$ odd and $n \geq i \geq 2$ odd; or $m \geq 3$ even and $n \geq i \geq 2$, determine if there is a super (a,d)-edge-antimagic total labeling with $d \in \{0,1,2,3\}$.*

References

[1] M. Bača, Y. Lin, M. Miller and R. Simanjuntak, New constructions of magic and antimagic graph labelings, *Utilitas Math.* **60** (2001), 229–239.

[2] M. Bača and L. Brankovic, Edge-antimagicness for a class of disconnected graphs, *Ars Combin.*, in press.

[3] M. Bača, Dafik, M. Miller and J. Ryan, On super (a,d)-edge antimagic total labeling of caterpillars, *J. Combin. Math. Combin. Comput.*, **65** (2008), 61–70.

[4] Dafik, M. Miller, J. Ryan and M. Bača, Super edge-antimagic total labelings of $mK_{n,n,n}$, *Ars Combinatoria* (2006), in press.

[5] Dafik, M. Miller, J. Ryan and M. Bača, Antimagic total labeling of disjoint union of complete s-partite graphs, *J. Combin. Math. Combin. Comput.*, **65** (2008), 41–49.

[6] Dafik, M. Miller, J. Ryan and M. Bača, On super (a,d)-edge antimagic total labeling of disconnected graphs, *Discrete Math.*, **309** (2009), 4909-4915.

[7] H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[8] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (2001), 153–168.

[9] J. Ivančo and I. Lučkaničová, On edge-magic disconnected graphs, *SUT Journal of Math.* **38** (2002), 175–184.

[10] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[11] G. Ringel and A.S. Lladó, Another tree conjecture, *Bull. Inst. Combin. Appl.* **18** (1996), 83–85.

[12] R. Simanjuntak, F. Bertault and M. Miller, Two new (a, d)-antimagic graph labelings, *Proc. of Eleventh Australasian Workshop on Combinatorial Algorithms* (2000), 179–189.

[13] I.W. Sudarsana, D. Ismaimuza, E.T. Baskoro and H. Assiyatun, On super (a, d)-edge-antimagic total labeling of disconnected graphs, *JCMCC* **55** (2005), 149–158.

[14] K.A. Sugeng, M. Miller, Slamin and M. Bača, (a, d)-edge-antimagic total labelings of caterpillars, *Lecture Notes in Computer Science* 3330 (2005), 169–180.

[15] W. D. Wallis, E. T. Baskoro, M. Miller and Slamin , Edge-magic total labelings, *Austral. J. Combin.* **22** (2000), 177–190.

[16] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston - Basel - Berlin, 2001.