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Super edge-antimagicness for a class of disconnected graphs

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Abstract

A graph G of order p and size q is called an (a,d)-*edge-antimagic total* if there exist a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ such that the edge-weights, $w(uv) = f(u) + f(v) + f(uv), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a graph G is called *super* if the smallest possible labels appear on the vertices. In this paper we study properties of super (a,d)-edge-antimagic total labeling of disconnected graphs $K_{1,m} \cup K_{1,n}$.

Key Words: *(a,d)-edge-antimagic total labeling, super (a,d)-edge-antimagic total labeling, disconnected graphs, star graphs.*

1 Introduction

All graphs in this paper are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of G, respectively. A (p,q)-graph G is a graph such that $|V(G)| = p$ and $|E(G)| = q$. We refer the reader to [14] or [15] for all other terms and notation not provided in this paper.

A labeling of graph G is any mapping that sends some set of graph elements to a set of non-negative integers. If the domain is the vertex-set or the edge-set, the labelings are called *vertex labelings* or *edge labelings*, respectively. Moreover, if the domain is $V(G) \cup E(G)$ then the labeling is called a *total labeling.*

Let f be a vertex labeling of a graph G. We define the *edge-weight* of $uv \in E(G)$ to be $w(uv) = f(u) + f(v)$. If f is a total labeling then the edge-weight of uv is $w(uv) = f(u) + f(uv) + f(v)$.

By an (a,d)-*edge-antimagic vertex labeling* of a (p,q)-graph G we mean a bijective function f from $V(G)$ onto the set $\{1, 2, \dots, p\}$ such that the set of all edge-weights, $\{w(uv) : uv \in E(G)\}$, is $\{a, a+d, a+2d, \dots, a+(q-1)d\}$, for two integers $a > 0$ and $d \geq 0$. Note that in his Ph.D thesis, Hegde called this labeling a *strongly (a,d)-indexable* (see Acharya and Hegde [1]).

An (a,d)-*edge-antimagic total labeling* on a (p,q)-graph G is a bijective function $f : V(G) \cup E(G) \to \{1,2,\dots,p+q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v), uv \in E(G)$, form an arithmetic progression $\{a, a+d, a+2d, \dots, a+(q-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists, then G is said to be an (a,d)-*edge-antimagic total graph*. Furthermore, f is a *super* (a,d)-*edge-antimagic total labeling* of G if the vertex labels are the integers $\{1,2,\dots,p\}$. Thus a *super* (a,d)-*edge-antimagic total graph* is a graph that admits a super (a,d)-edge-antimagic total labeling.

These labelings, introduced by Simanjuntak *et al.* in [10], are natural extensions of the concept of magic valuation studied by Kotzig and Rosa [9] (see also [2],[6],[13]) and the concept of super edge-magic labeling defined by Enomoto *et al.* in [5]. Many other researchers investigated different forms of antimagic graphs. For example, see Bodendiek and Walther [3] and [4], and Hartsfield and Ringel [7].

Ivančo and Lučkaničová [8] described some constructions of super edge-magic (super $(a,0)$-edge-antimagic total) labelings for disconnected graphs, namely $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$. The super (a,d)-edge-antimagic labelings for $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana *et al.* in [11].

In this paper we study super (a,d)-edge-antimagic total properties of a disjoint union of two stars $K_{1,m}$ and $K_{1,n}$.

2 Some Useful Lemmas

We start this section by a necessary condition for a graph to be super (a,d)-edge-antimagic total, providing a least upper bound for feasible values of d.

Lemma 2.1. *If a (p,q)-graph is super (a,d)-edge-antimagic total then* $d \leq \frac{2p+q-5}{q-1}$.

Proof. Assume that a (p,q)-graph has a super (a,d)-edge-antimagic total labeling $f : V(G) \cup E(G) \to \{1,2,\dots,p+q\}$. The minimum possible edge-weight in the labeling f is at least $1+2+p+1 = p+4$. Thus, $a \geq p+4$. On the other hand, the maximum possible edge-weight is at most $(p-1)+p+(p+q) = 3p+q-1$. So we obtain $a+(q-1)d \leq 3p+q-1$ which gives the desired upper bound for the difference d. □

The following lemma, proved by Figueroa-Centeno *et al.* in [6], gives a necessary and sufficient condition for a graph to be super edge-magic (super $(a,0)$-edge-antimagic total).

Lemma 2.2. *A (p,q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1,2,\dots,p\}$ such that the set $S = \{f(u)+f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p+q+s$, where $s = min(S)$ and $S = \{a-(p+1), a-(p+2), \dots, a-(p+q)\}$.*

In our terminology, the previous lemma states that a (p,q)-graph G is super $(a,0)$-edge-antimagic total if and only if there exists an $(a-p-q,1)$-edge-antimagic vertex labeling.

Next, we restate the following lemma that appeared in [12].

Lemma 2.3. [12] *Let $\mathfrak{A}$ be a sequence $\mathfrak{A} = \{c, c+1, c+2, \dots c+k\}$, k even. Then there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that* $\mathfrak{A} + \Pi(\mathfrak{A}) = \{2c+\frac{k}{2}, 2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, \dots, 2c+\frac{3k}{2}-1, 2c+\frac{3k}{2}\}$.

3 $K_{1,m} \cup K_{1,n}$

In [12] it is proved that the star has a super (a, d)-edge-antimagic total labeling if and only if either (i) $d \in \{0, 1, 2\}$ and $n \geq 1$, or (ii) $d = 3$ and $1 \leq n \leq 2$. Here, we will study super edge-antimagicness of a disjoint union of two stars, denoted by $K_{1,m} \cup K_{1,n}$. The disjoint union of $K_{1,m}$ and $K_{1,n}$ is the disconnected graph with vertex set $V(K_{1,m} \cup K_{1,n}) = \{x_{i,j} :$ either $i = 1$ and $j = 0, 1, \ldots, m$ or $i = 2$ and $j = 0, 1, \ldots, n\}$ and edge set $E(K_{1,m} \cup K_{1,n}) = \{x_{i,0}x_{i,j} : i \in \{1, 2\}, j \geq 1\}$.

We start by finding a least upper bound for the feasible values of d for a super (a, d)-edge-antimagic total labeling of $K_{1,m} \cup K_{1,n}$. If the graph $K_{1,m} \cup K_{1,n}$ is super (a, d)-edge-antimagic total then, by Lemma 2.1, for $p = m + n + 2$ and $q = m + n$, we have $d \leq 3 + \frac{2}{m+n-1}$. If $m \geq 2$ and $n \geq 2$ then $d < 4$. If $m + n = 3$, then $d \leq 4$ and if $m + n = 2$ then $d \leq 5$.

Theorem 3.1. *The graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a $(t + 4, 1)$-edge-antimagic vertex labeling if and only if either m is a multiple of $n + 1$ or n is a multiple of $m + 1$.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a $(a, 1)$-edge-antimagic vertex labeling $f : V(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, m + n + 2\}$ and that $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a + 1, a + 2, \ldots, a + m + n - 1\}$ is the set of edge-weights. The sum of the edge-weights in the set W is

$$\sum_{uv \in E(K_{1,m} \cup K_{1,n})} w(uv) = (m + n)a + \frac{(m + n)(m + n - 1)}{2}.$$

In the computation of the edge-weights of $K_{1,m} \cup K_{1,n}$, the labels of the two central vertices, $f(x_{1,0})$ and $f(x_{2,0})$, are used m and n times, respectively, and the labels of the remaining vertices are used once each. Let $s_1 = f(x_{1,0})$ and $s_2 = f(x_{2,0})$. The sum of all vertex labels used to calculate the edge-weights is equal to

$$\begin{aligned}
&(m - 1)f(x_{1,0}) + (n - 1)f(x_{2,0}) + \sum_{k=1}^{m+n+2} k = \\
&(m - 1)s_1 + (n - 1)s_2 + (1 + 2 + \ldots + m + n + 2) = \\
&(m - 1)s_1 + (n - 1)s_2 + \frac{(m + n + 3)(m + n + 2)}{2}.
\end{aligned}$$

The sum of vertex labels used to obtain the edge-weight is naturally equal to the sum of all the edge-weights. Thus,

$$(m + n)a = 3(m + n + 1) + (m - 1)s_1 + (n - 1)s_2.$$

Clearly, $s_1 + s_2 \notin \{a, a + 1, a + 2, \ldots, a + m + n - 1\}$ because exactly one endpoint of any edge belongs to $\{x_{1,0}, x_{2,0}\}$. Without loss of generality, we may assume that $s_1 + s_2 < a$ (if $s_1 + s_2 > a + m + n - 1$; then we consider $(a', 1)$-edge-antimagic vertex labeling g given by $g(x_{i,j}) = m + n + 3 - f(x_{i,j})$).

If $1 \notin \{s_1, s_2\}$ then $a > s_1 + s_2 > \min\limits_{1 \leq j \leq m} f(x_{1,j}) + s_2 \geq 1 + s_2 \geq a$ or $a > s_1 + s_2 > s_1 + \min\limits_{1 \leq j \leq n} f(x_{2,j}) \geq s_1 + 1 \geq a$, a contradiction.

- Suppose $s_1 = 2$ and $s_2 = 1$ then
$$\begin{aligned}(m+n)a &= 3(m+n+1) + 2(m-1) + (n-1)\\ (m+n)(a-4) &= m,\end{aligned}$$
which implies that m is multiple of $m+n$, a contradiction.

- Suppose $s_1 > 2$ and $s_2 = 1$. We can say that $a = s_1 + 2$ because if $\min_{1 \le j \le n} f(x_{2,j}) = 2$ then $\min_{1 \le j \le n} f(x_{2,j}) + s_2 < s_1 + s_2 < a$, thus the vertex labeled by 2 must belongs to $K_{1,m}$. It follows that
$$\begin{aligned}(m+n)a &= 3(m+n+1) + (m-1)s_1 + (n-1)s_2\\ (m+n)(s_1+2) &= 3(m+n+1) + (m-1)s_1 + (n-1)\\ (s_1-2)(n+1) &= m,\end{aligned}$$
which means that $m > n$ and m is a multiple of $n+1$.

For the sake of completeness, we assume that $m = t(n+1)$ and consider the vertex labeling f_1 described by Ivančo and Lučkaničová in [8].

$$f_1(x_{i,j}) = \begin{cases} 2+t, & \text{if } i = 1 \text{ and } j = 0\\ \lceil \frac{j}{t} \rceil + j, & \text{if } i = 1 \text{ and } j = 1, 2, \dots, m\\ 1 & \text{if } i = 2 \text{ and } j = 0\\ 1 + (j+1)(t+1), & \text{if } i = 2 \text{ and } j = 1, 2, \dots, n. \end{cases}$$

The vertex labeling f_1 is a bijective function from $K_{1,m} \cup K_{1,n}$ onto the set $\{1, 2, \dots, m+n+2\}$. The edge-weights of $K_{1,m} \cup K_{1,n}$, under the labeling f_1, constitute the sets

$$\begin{aligned} W^1_{f_1} &= \{w^1_{f_1}(x_{1,0}x_{1,j}) : \text{ if } 1 \le j \le m\}\\ &= \{2 + t + \lceil \frac{j}{t} \rceil + j : \text{ if } 1 \le j \le m\},\\ W^2_{f_1} &= \{w^2_{f_1}(x_{2,0}x_{2,j}) : \text{ if } 1 \le j \le n\}\\ &= \{2 + (j+1)(t+1) : \text{ if } 1 \le j \le n\}. \end{aligned}$$

Hence the set $\bigcup_{k=1}^{2} W^k_{f_1} = \{t + \lceil \frac{1}{t} \rceil + 3, t + \lceil \frac{2}{t} \rceil + 4, \dots, m + n + t + 2 + \lceil \frac{1}{t} \rceil\}$ consists of consecutive integers. Thus f_1 is a $(t+4, 1)$-edge-antimagic vertex labeling. □

According to Lemma 2.2, the $(t+4, 1)$-edge-antimagic vertex labeling f_1 extends to a super $(a, 0)$-edge-antimagic total labeling, where for $p = m + n + 2$ and $q = m + n$, the value $a = 2m + 2n + t + 6$. Thus we have the following theorem which was proved by Ivančo and Lučkaničová in [8].

Theorem 3.2. [8] *The graph $K_{1,m} \cup K_{1,n}$, $m \ge 2$ and $n \ge 2$, has a super $(2m+2n+t+6, 0)$-edge-antimagic total labeling if and only if either m is a multiple of $n+1$ or n is a multiple of $m+1$.*

Furthermore, we have the following theorem.

Theorem 3.3. *The graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a super $(m+n+t+7, 2)$-edge-antimagic total labeling if and only if either m is a multiple of $n+1$ or n is a multiple of $m+1$.*

Proof. Without loss of generality, we may assume that m is a multiple of $n+1$. Let $m = t(n+1)$. Using the $(t+4, 1)$-edge-antimagic vertex labeling f_1 from Theorem 3.1, we define a total labeling $f_2 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \to \{1, 2, \dots, 2m+2n+2\}$ as follows

$$f_2(x_{i,j}) = f_1(x_{i,j}), \quad \text{for every feasible } i \text{ and } j$$

$$f_2(x_{i,0}x_{i,j}) = \begin{cases} m+n+1+\lceil\frac{j}{t}\rceil+j, & \text{if } i=1 \text{ and } j=1,2,\dots,m \\ m+n+2+j(t+1), & \text{if } i=2 \text{ and } j=1,2,\dots,n. \end{cases}$$

The edge-weights of $K_{1,m} \cup K_{1,n}$, under the total labeling f_2, constitute the sets

$$\begin{aligned} W_{f_2}^1 &= \{w_{f_2}^1(x_{1,0}x_{1,j}) = w_{f_1}^1(x_{1,0}x_{1,j}) + f_2(x_{1,0}x_{1,j}) : \text{ if } 1 \leq j \leq m\} \\ &= \{m+n+t+3+2\lceil\frac{j}{t}\rceil+2j : \text{ if } 1 \leq j \leq m\}, \\ W_{f_2}^2 &= \{w_{f_2}^2(x_{2,0}x_{2,j}) = w_{f_1}^2(x_{2,0}x_{2,j}) + f_2(x_{2,0}x_{2,j}) : \text{ if } 1 \leq j \leq n\} \\ &= \{m+n+4+(2j+1)(t+1) : \text{ if } 1 \leq j \leq n\}. \end{aligned}$$

Hence the set $\bigcup_{k=1}^{2} W_{f_2}^k = \{m+n+t+2\lceil\frac{1}{t}\rceil+5, m+n+t+2\lceil\frac{2}{t}\rceil+7, \dots, 3m+3n+t+2\lceil\frac{1}{t}\rceil+3\}$ consists of arithmetic sequence with first term $m+n+t+2\lceil\frac{1}{t}\rceil+5$ and common difference 2. Thus f_2 is a super $(m+n+t+7, 2)$-edge-antimagic total labeling. $\square$

Theorem 3.4. *For the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, there is no $(a, 3)$-edge-antimagic vertex labeling.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a $(a, 3)$-edge-antimagic vertex labeling $f : V(K_{1,m} \cup K_{1,n}) \to \{1, 2, \dots, m+n+1, m+n+2\}$ and $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a+3, a+6, \dots, a+(m+n-1)3\}$ is the set of edge-weights. The minimum possible edge weight is at least $1+2=3$. It follows that $a \geq 3$. The maximum possible edge weight is no more than $(p-1)+p = 2m+2n+3$.

Consequently, $a+3(m+n-1) \leq 2m+2n+3$ and $3 \leq 2+\frac{2}{m+n-1}$, which is impossible when $m+n \geq 4$. $\square$

By using $(t+4)$-edge-antimagic vertex labeling f_1, with respect to Lemma 2.3, we have the following theorem.

Theorem 3.5. *If $m+n$ is odd and either m is a multiple of $n+1$ or n is a multiple of $m+1$, then the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a super $\left(\frac{3(m+n)+2t+13}{2}, 1\right)$-edge-antimagic total labeling.*

Proof. From Theorem 3.1, the graph $K_{1,m} \cup K_{1,n}$ has $(t+4, 1)$-edge-antimagic vertex labeling. Let a set $\mathfrak{A} = \{c, c+1, c+2, \dots, c+k\}$ be the set of edge weights of the vertex labeling f_1 for $c = t+4$ and $k = m+n-1$. In light of Lemma 2.3, there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + m + n + 3] =$

$\{c + \frac{3m+3n+5}{2}, c + \frac{3m+3n+5}{2} + 1, \ldots, c + \frac{5m+5n+3}{2}\}$. If $[\Pi(\mathfrak{A}) - c + m + n + 3]$ is an edge labeling of $K_{1,m} \cup K_{1,n}$ then $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + m + n + 3]$ gives the set of the edge weights of $K_{1,m} \cup K_{1,n}$, which implies that the total labeling is super $(a, 1)$-edge-antimagic total, where $a = c + \frac{3m+3n+5}{2} = \frac{3(m+n)+2t+13}{2}$. This concludes the proof. □

Theorem 3.6. *If $m = n$ then the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$ and $n \geq 2$, has a $(4, 2)$-edge-antimagic vertex labeling.*

Proof. Let $m = n$ and $m \geq 2$. Consider the bijection $f_3 : V(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, m + n + 2\}$, where

$$f_3(x_{i,j}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j = 0 \\ 2j + 1, & \text{if } i = 1 \text{ and } j = 1, 2, \ldots, m \\ m + n + 2, & \text{if } i = 2 \text{ and } j = 0 \\ 2j, & \text{if } i = 2 \text{ and } j = 1, 2, \ldots, n. \end{cases}$$

We observe that the edge-weights of $K_{1,m} \cup K_{1,n}$, under the vertex labeling f_3, constitute the sets

$$\begin{aligned} W^1_{f_3} &= \{w^1_{f_3}(x_{1,0}x_{1,j}) : \text{ if } 1 \leq j \leq m\} \\ &= \{2j + 2 : \text{ if } 1 \leq j \leq m\}, \\ W^2_{f_3} &= \{w^2_{f_3}(x_{2,0}x_{2,j}) : \text{ if } 1 \leq j \leq n\} \\ &= \{m + n + 2 + 2j : \text{ if } 1 \leq j \leq n\}. \end{aligned}$$

Hence the elements of set $\bigcup_{k=1}^{2} W^k_{f_3} = \{4, 6, \ldots, m + 3n + 2\}$ can be arranged to form an arithmetic sequence with first term 4 and common difference 2. Thus f_3 is a $(4, 2)$-edge-antimagic vertex labeling. □

Theorem 3.7. *If $m = n$ then the graph $K_{1,m} \cup K_{1,n}$, $m \geq 2$, has super $(2m + 2n + 6, 1)$-edge-antimagic total and super $(m + n + 7, 3)$-edge-antimagic total labeling.*

Proof. Let $m = n$ and $m \geq 2$. From Theorem 3.6, it follows that the graph $K_{1,m} \cup K_{1,n}$ has a $(4, 2)$-edge-antimagic vertex labeling. We will distinguish two cases, according to whether $d = 1$ or $d = 3$.

Case 1. $d = 1$

Define $f_4 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, 2m + 2n + 2\}$ to be the bijective function such that

$$f_4(x_{i,j}) = f_3(x_{i,j}), \quad \text{for every feasible } i \text{ and } j$$

$$f_4(x_{i,0}x_{i,j}) = \begin{cases} 2m + 2n + 3 - j, & \text{if } i = 1 \text{ and } j = 1, 2, \ldots, m \\ m + 2n + 3 - j, & \text{if } i = 2 \text{ and } j = 1, 2, \ldots, n. \end{cases}$$

The edge-weights of $K_{1,m} \cup K_{1,n}$, under the labeling f_4, constitute the sets

$$\begin{aligned} W^1_{f_4} &= \{w^1_{f_4}(x_{1,0}x_{1,j}) = w^1_{f_3}(x_{1,0}x_{1,j}) + f_4(x_{1,0}x_{1,j}) : \text{ if } 1 \leq j \leq m\} \\ &= \{2m + 2n + 5 + j : \text{ if } 1 \leq j \leq m\}, \\ W^2_{f_4} &= \{w^2_{f_4}(x_{2,0}x_{2,j}) = w^2_{f_3}(x_{2,0}x_{2,j}) + f_4(x_{2,0}x_{2,j}) : \text{ if } 1 \leq j \leq n\} \\ &= \{2m + 3n + 5 + j : \text{ if } 1 \leq j \leq n\}. \end{aligned}$$

Hence the set $\bigcup_{k=1}^{2} W_{f_4}^k = \{2m+2n+6, 2m+2n+7, \ldots, 3m+3n+5\}$ consists of consecutive integers. Thus f_4 is a super $(2m+2n+6, 1)$-edge-antimagic total labeling.

Case 2. $d = 3$

Consider the labeling $f_5 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, 2m+2n+2\}$ such that

$$f_5(x_{i,j}) = f_3(x_{i,j}), \quad \text{for every feasible } i \text{ and } j$$

$$f_5(x_{i,0}x_{i,j}) = \begin{cases} m+n+2+j, & \text{if } i=1 \text{ and} j = 1, 2, \ldots, m \\ 2m+n+2+j, & \text{if } i=2 \text{ and} j = 1, 2, \ldots, n. \end{cases}$$

The total labeling f_5 is a bijective function. The edge-weights of $K_{1,m} \cup K_{1,n}$, under the labeling f_5, constitute the sets

$$\begin{aligned} W_{f_5}^1 &= \{w_{f_5}^1(x_{1,0}x_{1,j}) = w_{f_3}^1(x_{1,0}x_{1,j}) + f_5(x_{1,0}x_{1,j}) : \text{ if } 1 \le j \le m\} \\ &= \{m+n+4+3j : \text{ if } 1 \le j \le m\}, \\ W_{f_5}^2 &= \{w_{f_5}^2(x_{2,0}x_{2,j}) = w_{f_3}^2(x_{2,0}x_{2,j}) + f_5(x_{2,0}x_{2,j}) : \text{ if } 1 \le j \le n\} \\ &= \{3m+2n+4+3j : \text{ if } 1 \le j \le n\}. \end{aligned}$$

Hence, the set $\bigcup_{k=1}^{2} W_{f_5}^k = \{m+n+7, m+n+10, \ldots, 4(m+n)+4\}$ consists of arithmetic sequence with first value $m+n+7$ and common difference 3. Thus f_5 is a super $(m+n+7, 3)$-edge-antimagic total labeling. ☐

Theorem 3.8. *For the graph $K_{1,m} \cup K_{1,n}$, $m+n=3$, there is no super $(a,4)$-edge-antimagic total labeling.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, for $m+n=3$, has a super $(a,4)$-edge-antimagic total labeling $f : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \to \{1, 2, \ldots, 8\}$, and $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a+4, a+8\}$ is the set of edge-weights. In the computation of the edge-weights of $K_{1,m} \cup K_{1,n}$, a label of a vertex of degree two is used twice, but the labels of remained vertices are used once each, and also the labels of edges are used once each. The sum of all vertex and edge labels used to calculate the edge-weights is equal to the sum of edge-weights. If s_1 is a label of the vertex of degree two then

$$\begin{aligned} s_1 + \sum_{u \in V} f(u) + \sum_{uv \in E} f(uv) &= \sum_{uv \in E} w(uv) \\ s_1 + (1+2+3+4+5) + (6+7+8) &= a+a+4+a+8 \end{aligned}$$

Thus

$$a = 8 + \frac{s_1}{3}.$$

Since a must be an integer, then for s_1 we have only one possible value $s_1 = 3$, which gives $a = 9$.

The smallest value of edge-weight $a = 9$ can be obtained only from the triple $(1, 2, 6)$, where 1 and 2 are values of adjacent vertices of degree one and 6 is the value of the edge. The remained vertices of degree one must be labeled by values 4 and 5. Thus, we have the triples $(3,4,7)$ and $(3,5,8)$ or $(3,4,8)$ and $(3,5,7)$. This contradicts the fact that $K_{1,m} \cup K_{1,n}$, for $m+n=3$, has super $(a,4)$-edge-antimagic total labeling. □

Remark 3.1. *If $m+n=2$ then the graph $K_{1,m} \cup K_{1,n}$ has a super $(8,5)$-edge-antimagic total labeling.*

The wanted super $(8,5)$-edge-antimagic total labeling f_6 of the graph $K_{1,m} \cup K_{1,n}$, for $m+n=2$, can be defined in the following way $f_6(x_{1,0})=2$, $f_6(x_{2,0})=4$, $f_6(x_{1,1})=1$, $f_6(x_{2,1})=3$, $f_6(x_{1,0}x_{1,1})=5$ and $f_6(x_{2,0}x_{2,1})=6$.

4 Conclusion

We have considered edge-antimagic labelings of disconnected graphs $K_{1,m} \cup K_{1,n}$. We summarize that the graph $K_{1,m} \cup K_{1,n}$ has a super (a,d)-edge-antimagic total labeling for (i) $d \in \{0,2\}$, if either m is a multiple of $n+1$ or n is a multiple of $m+1$, for $m \geq 2$ and $n \geq 2$; (ii) $d=1$, if $m+n$ is odd and either m is a multiple of $n+1$ or n is a multiple of $m+1$, for $m \geq 2$ and $n \geq 2$; (iii) $d \in \{1,3\}$, if $m \geq 2$ and $n \geq 2$ and $m=n$; (iv) $d=5$, when $m+n=2$.

In the case when $m+n$ is even and either m is a multiple of $n+1$ or n is a multiple of $m+1$ we do not have any answer. Therefore we propose the following open problem.

Open Problem 1. *For the graph $K_{1,m} \cup K_{1,n}$, $m+n$ is even and either m is a multiple of $n+1$ or n is a multiple of $m+1$, determine if there is a super $(a,1)$-edge-antimagic total labeling.*

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