LAPORAN AKHIR PENELITIAN

HIBAH KERIS-MOBILITY

JUDUL PENELITIAN

***Finishing Manuscript* Untuk Artikel Jurnal Internasional**

PENELITI

Dr. Agustina Pradjaningsih, S.Si., M.Si, 0002087111, Ketua

KELOMPOK RISET

Mathematical Optimization and Computation (MOCo)

FAKULTAS MATEMATIKA DAN ILMU PENGETAHUAN ALAM

UNIVERSITAS JEMBER

KEMENTERIAN RISET,TEKNOLOGI, DAN PENDIDIKAN TINGGI

DESEMBER, 2019

LAPORAN KEGIATAN

KERIS MOBILITY 2019

Nama : Dr. Agustina Pradjaningsih, S.Si., M.Si

KeRis : Mathematical Optimization and Computation (MOCo)

JURUSAN MATEMATIKA

FAKULTAS MATEMATIKA DAN ILMU PENGETAHUAN ALAM

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DESEMBER 2019

KEMENTERIAN RISET, TEKNOLOGI, DAN PENDIDIKAN TINGGI
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SURAT TUGAS

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Memberikan tugas kepada:

No.	Nama	Unit Kerja	Jabatan dalam Kegiatan
1.	Dr. Agustina Pradjaningsih, S.Si., M.Si NIP. 197108022000032009	Jurusan Matematika FMIPA	Ketua Peneliti

Untuk melaksanakan : Kegiatan Hibah *Uninet* atau *Keris Mobility* 2019 yaitu *finishing manuscript* menuju artikel yang siap dipublikasikan pada jurnal internasional

Terhitung mulai : 9 November 2019 sampai dengan 7 Desember 2019
Tempat : PS S3 MIPA, Fakultas Sain dan Teknologi, Universitas Airlangga, Surabaya

Demikian surat tugas ini dibuat untuk dilaksanakan dengan sebaik-baiknya dengan penuh tanggung jawab.

Jember, 30 Oktober 2019
Ketua Lembaga Penelitian dan Pengabdian Kepada Masyarakat,

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DAFTAR ISI

A. DESKRIPSI SINGKAT

Pada berbagai situasi kehidupan terdapat pengambilan keputusan yang menyangkut permasalahan optimasi yang berkaitan dengan pengoptimuman baik maksimum ataupun minimum dalam penggunaan sumber daya yang terbatas. Dari situasi tersebut dapat dikatakan optimasi merupakan masalah yang berhubungan dengan keputusan dan cara penentuan solusi terbaik atau memuaskan. Misalnya permasalahan meminimumkan pembiayaan, jumlah pegawai, bahan baku ataupun memaksimumkan keuntungan. Pengertian optimasi adalah suatu proses untuk menentukan kondisi terbaik yang memberikan nilai maksimum atau minimum dari fungsi tujuan yang memenuhi fungsi kendala yang diberikan (Rao, 2009).

Pemrograman linier adalah suatu pemrograman matematika yang dikembangkan untuk menangani permasalahan optimasi yang melibatkan persamaan linier pada fungsi tujuan dan fungsi kendalanya. **Salah satu asumsi dasar yang ada pada permasalahan pemrograman linier adalah asumsi kepastian**. Asumsi kepastian menunjukkan bahwa semua koefisien atau variabel keputusan pada model, merupakan konstanta yang diketahui dengan pasti (Hillier & Liebermann, 2010). **Namun dalam situasi atau permasalahan riil, dimungkinkan terdapat koefisien atau variabel keputusan yang tidak pasti**. Berdasarkan konsep dan teori analisis interval yang dikembangkan oleh Moore (1966), **permasalahan ketidakpastian ini diantisipasi dengan membuat nilai pendekatan dalam bentuk interval**, sehingga dikembangkan pemrograman linear interval. Perkembangan pemrograman linear interval dimulai dari pemrograman linear dengan koefisien berbentuk interval, baik dalam koefisien fungsi tujuan dan koefisien fungsi kendala. Selanjutnya berkembang menjadi pemrograman linear dengan koefisien dan variabel keputusan berbentuk interval [Shaocheng (1994), Ramadan (1997), Chinnek & Ramadan (2000), dan Kuchta (2008), Suprajitno & Mohd (2008) dan Suprajitno (2010), Suprajitno *et al*. (2009), Suprajitno (2010), Suprajitno & Mohd (2010), Ramesh & Ganesan (2011) serta Ramesh & Ganesan (2012)]. Semua peneliti tersebut menggunakan metode simpleks untuk penyelesaian permasalahannya. **Padahal terdapat metode lain untuk menyelesaikan pemrograman linear yaitu metode titik interior**.

Tahun 2019, Agustina Pradjaningsih (pengusul program ini) bersama-sama dengan Fatmawati dan Herry Suprajitno dalam hal ini berturut-turut adalah promotor dan ko-promotor saat pengusul mengambil Program Doktor di Universitas Airlangga, menulis artikel dengan judul *Interior point method for solving linear programming with interval coefficients using affine scaling* yang dipublikasikan pada Far East Journal of Mathematical Sciences (FJMS) Volume 111, Number 2, 2019, Pages 195-211. Artikel ini merupakan bagian (tahapan pertama) dari hasil disertasi Agustina Pradjaningsih yang ditulis saat mengambil program doktor di Universitas Airlangga. Dalam artikel tersebut **permasalahan pemrograman linear interval dengan koefisien interval diselesaikan dengan metode titik interior**. Sebagai kelanjutan dari artikel tersebut **pengusul dalam proses membuat *draft manuscript* yang merupakan bagian (tahapan kedua) dari hasil disertasinya yang belum pernah dipublikasikan pada jurnal internasional**. Bulan Mei 2019 pengusul kembali aktif sebagai dosen setelah lulus dari program doktor, dan langsung masuk pada **kelompok riset MOCo** (***Mathematical Optimization and Computation***) beranggotakan dosen Jurusan Matematika Fakultas Matematika dan Ilmu Pengetahuan Alam dengan **latar belakang penelitian di bidang komputasi dan optimasi matematika**. Hal ini sesuai dengan **fokus penelitian yang akan dilakukan yaitu pemanfaatan metode dalam komputasi dan optimasi matematika untuk** pembuatan teknologi cerdas dalam bidang pertanian industrial, kesehatan dan **terapan permasalahan nyata**. **Permasalahan-permasalahan nyata yang terjadi dicari penyelesaianya dengan menggunakan teori dan metode matematika, khususnya optimasi dan komputasi matematika**.

Sejalan dengan fokus penelitian pengusul dan Kelompok Riset, maka pengusul mengajukan usulan untuk mengikuti program *Keris Mobility* 2019 dengan kegiatan **melanjutkan *draft manuscript* menjadi *finishing manuscript*** sebagai artikel yang siap dipublikasikan pada jurnal internasional.

B. MEKANISME DAN DESAIN PELAKSANAAN

1. **Kontak Pembimbing**: Kontak Pembimbing dilakukan untuk meminta kesediaan pembimbing untuk memberikan bimbingan dalam rangka *finishing manuscript* yang akan dilaksanakan. Pembimbing terdiri dari dua orang yaitu ibu Dr. Fatmawati dan Bapak Herry Suprajitno, P.hD dari Program Studi S3 MIPA fakultas Sains dan Teknologi Universitas Airlangga Surabaya.
2. **Pembimbingan:** Pembimbingan akan dilakukan selama kurang lebih 4 minggu dengan rincian: (1) **minggu pertama** (9-15 November 2019) dan **minggu kedua** (16-22 November 2019) penulisan/perbaikan draft manuscript yang akan diserahkan kepada pembimbing; terakhir (2) **minggu ketiga** sd **minggu keempat** (23 November sd 7 Desember 2019) draft manuscript didiskusikan, direview,diverifikasi dan dievaluasi bersama pembimbing;
3. **Laporan**: sebagai bagian pertanggung jawaban pengusul kepada pemberi program *Keris Mobility* 2019**, yang akan dibuat setelah selesai akhir program.**

C. JADWAL KEGIATAN

No	Kegiatan/tahap	Minggu ke...			
		1	2	3	4
1	Kontak Pembimbing				
2	Pembimbingan				
3	Laporan				

D. RINCIAN ANGGARAN

MAK	URAIAN	Biaya (Rp)	Kuantitas	Satuan	Jumlah
524111	Akomodasi (penginapan) 1 bulan Surabaya	400.000	22	H	8.800.000
524111	Transport Jember-Surabaya pp	300.000	8	K	2.400.000
524111	Uang harian 1 bulan Surabaya (Pembina IVa)	410.000	30	H	12.300.000
				TOTAL RAB	**23.500.000**

E. HASIL KEGIATAN

Telah diselesaikan **satu draft artikel** dengan judul **"A Solution of Linear Programming with Interval Variables Using Interior Point Method Based on Calculate The Interval Limit**" yang telah direvisi oleh Dosen Pembimbing 2 (terlampir draft ARTIKEL)

F. LAMPIRAN

1. Surat Ijin Ketua Jurusan Dan Rekomendasi Koordinator Keris
2. Surat Kesediaan Pembimbing
3. Daftar Riwayat Hidup
4. Draft Artikel

1. Surat Ijin Ketua Jurusan Dan Rekomendasi Koordinator Keris

SURAT IZIN

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Demikian surat izin ini saya berikan dengan sepengetahuan/rekomendasi ketua KeRis, untuk selanjutnya agar dipergunakan sebagaimana mestinya.

Jember, 25 September 2019
Hormat saya,
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Dalam rangka kegiatan *Keris Mobility* 2019 yang akan dilaksanakan oleh Agustina Pradjaningsih selama 6 minggu di Program Studi S3 MIPA Fakultas Sains dan Teknologi, Universitas Airlangga.

Demikian surat keterangan ini dibuat untuk dipergunakan sebagaimana mestinya.

Surabaya, 26 September 2019

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3. Daftar Riwayat Hidup

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DAFTAR PENELITIAN, PEMAKALAH dan PUBLIKASI 5 TAHUN TERAKHIR

1. Konstruksi Metode Titik Interior pada Pemrograman Linear Interval dengan Menggunakan Nilai Batas Bawah Terbesar dan Nilai Batas Atas Terkecil, 2016. Penelitian Disertasi Doktor.
2. Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2015. *Construction Interior Point Method to Linear Programming with Interval Coeficients*, Internasonal Seminar ICOWOBAS-5, 16-17 September 2015, Surabaya.
3. Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2016. *Construction Interior Point Method to Linear Programming with Interval Variable*, Internasonal Seminar AMC, 25-29 Juli 2016, Bali.
4. Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2017. Konstruksi Metode Titik Interior untuk Penyelesaian Pemrograman Linear dengan Koefisien Interval, Seminar Nasional SNMA, 21 Oktober 2017, Surabaya.
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6. Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2019. *Interior Point Method for Solving Linear Programming with Interval Coefficients Using Affine Scaling*, *Far East Journal of Mathematical Sciences,* Vol. 111 No 2, 2019.
7. Solution of Linear Programming with Interval Variable Using Interior Point Method, Draft Manuscript, 2019.

SURAT KETERANGAN SELESAI KEGIATAN

Yang bertandatangan di bawah ini :

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Unit Kerja : Dosen Fakultas Sains dan Teknologi Universitas Airlangga

Menyatakan bahwa :

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Unit Kerja : Jurusan Matematika Fakultas MIPA Universitas Jember

Telah selesai melaksanakan kegiatan *Keris Mobility* 2019 di Program Studi S3 MIPA Fakultas Sains dan Teknologi, Universitas Airlangga.

Demikian surat keterangan ini dibuat untuk dipergunakan sebagaimana mestinya.

Surabaya, 8 Desember 2019

Dr. Herry Suprajitno, M.Si.

SURAT KETERANGAN SELESAI KEGIATAN

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A SOLUTION OF LINEAR PROGRAMMING WITH INTERVAL VARIABLES USING INTERIOR POINT METHOD BASED ON CALCULATE THE INTERVAL LIMIT.

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Abstract

Linear programming with interval variables is developed from linear programming with interval coefficients to obtain optimum solutions in the form of interval. In this paper, the interior point method based on calculate the interval limit is used to complete linear programming with interval variables. The interior-point method procedure is to change the linear programming model with interval variables into a pair of a classic linear programming model. It would be (the best optimum and the worst optimum problems). The procedure for solving (the best and the worst optimum) problems using the interior point method was also built to get the optimum solution in interval form.

Keywords: linear programming, interval linear programming, interval variables, interior point method, the best optimum problem, the worst optimum problem.

1. Introduction

An optimization problem could be categorized as a linear programming model if it satisfies assumptions of proportionality, additivity, divisibility, and certainty [1]. The certainty assumption is that all coefficients and variables in the model are known. However, in real situations, sometimes the values of coefficients and variables are not certainty known. The uncertainty problem could be overcome using the interval approach to the coefficients and variables. This approach supported the concept and theory of interval analysis developed by [2]. Classical linear programming, which constructed by transforming the coefficients in objective functions and constraints into interval form, is called linear programming with interval coefficients. If the coefficients and variables in the objective function and constraints are both of interval form, it is called linear programming with interval variables or interval linear programming.

Researches on linear programming with interval coefficients have been discussed by [3-6]. The research on linear programming with interval variables was inspired by [7-11]. The researchers [7-11] used the simplex method to solve it. The interior-point method has been used to solve linear programming, or fuzzy number linear programming has been discussing by several researchers [12-16]. The interior-point method has several variants. One of them is a gradient projection algorithm was popular since the 1990s [17].

This paper presents the linear programming solution with interval variables using an interior point method, notably the gradient projection algorithm. Stages to obtain an optimum solution in the form of interval: the first stage is to define coefficients and variables in the linear programming interval model as an interval form. The second step is to change linear programming with an interval variables model into a pair of the classical linear programming model. The last stage is a pair of classical linear programming models that must be solved using an interior point method. This last stage begins by constructing an interval solution procedure in a classic linear programming model by adding new constraints to the model that has an unbounded solution into the feasible region. The purpose of adding new constraints is to determine whether the model has an unbounded solution or bounded solution.

Organized from this paper as follows: We demonstrate some preliminaries of interval arithmetic and formula of the linear programming with interval variables in Section 2. Section 3 would to solving linear programming with interval variables using an interior point method. Numerical examples will show in Section 4. Finally, we will allocate the Section 5 to conclusions.

2. Preliminaries

In this section, we review some of the concepts needed, such as interval arithmetic and formula of linear programming with interval variables. For more details, we refer to [1, 2, 9, 18].

2.1 Interval Arithmetic

The basic concepts, definition and properties of interval number, interval arithmetic, and comparison of two intervals can be found in [2, 9, 18]. Let R denote the set of all real numbers.

Definition 1. A closed real interval $\underline{x} = [x_I, x_S]$ is a real interval number which can be defined by

$$\underline{x} = [x_I, x_S] = \{x \in R | x_I \leq x \leq x_S;\ x_I, x_S \in R\}, \tag{1}$$

where x_I and x_S are called infimum and supremum of $\underline{x}$, respectively.

Definition 2. A real interval number $\underline{x} = [x_I, x_S]$ is called a degenerate, if $x_I = x_S$.

Definition 3. Let $I(R)$ is the set of all interval on R. A real interval vector $\underline{\mathbf{x}} \in I(R^n)$, is a vector in the form $\underline{\mathbf{x}} = (\underline{x}_i)_{nx1}$, where $\underline{x}_i = [x_{iI}, x_{iS}] \in I(R)$,$i$=1,2,…, n.

Definition 4. Let $\underline{x}, \underline{y} \in I(R)$ where $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$, then

a. $\underline{x} + \underline{y} = [x_I + y_I, x_S + y_S]$ (addition),

b. $\underline{x} - \underline{y} = [x_I - y_S, x_S - y_I]$ (subtraction),

c. $\underline{x} \cdot \underline{y} = [\min\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}, \max\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}]$ (multiplication),

d. $\underline{x}/\underline{y} = [x_I, x_S][1/y_S, 1/y_I]$, $0 \notin \underline{y}$ (division).

Definition 5. Let $\underline{x}, \underline{y} \in I(R)$, $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$. $\underline{x} \leq \underline{y}$ if only if $u x_I + v x_S \leq u y_I + v y_S$ where $u, v \in (0,1]$ dan $u \leq v$.

2.2. Linear Programming with Interval Variables

Linear programming with interval coefficients and interval variables, after this, referred to as Interval Linear Programming. The general form of interval linear programming is defined as follows:

Maximize (objective function)

$$\underline{Z} = \sum_{j=1}^{n} [c_{jI}, c_{jS}][x_{jI}, x_{jS}], \tag{2}$$

subject to

$$\sum_{j=1}^{n} [a_{ijI}, a_{ijS}][x_{jI}, x_{jS}] \leq [b_I, b_S], \tag{3}$$

$$[x_{jI}, x_{jS}] \geq 0, \tag{4}$$

where $i = 1,2, \dots m,\ j = 1,2, \dots, n.$, $[x_{jI}, x_{jS}]$ is decision variable, $[c_{jI}, c_{jS}]$ is coefficient of the objective function, $[a_{ijI}, a_{ijS}]$ is coefficient of the constraint function, $[b_I, b_S]$ is coefficient of resource constraint, $\underline{Z}$ is the objective function, and $[x_{jI}, x_{jS}], [c_{jI}, c_{jS}], [a_{ijI}, a_{ijS}], [b_I, b_S] \in I(R)$.

[9] have proposed Theorem 1 and Theorem 2. Theorem 1 could be used to determine the largest feasible region and the smallest feasible region of an interval linear programming. Theorem 2 described some property of the objective function of an interval linear programming model.

Teorema 1 [9]. Suppose the interval inequality of an interval linear programming

$$\sum_{j=1}^{n} [a_{jI}, a_{jS}][x_{jI}, x_{jS}] \leq [\, b_I, b_S] \tag{5}$$

where $[x_{jI}, x_{jS}] \geq 0$, for every $j = 1,2, \dots, n$ is given. Then

a. The largest feasible region is a region satisfies the following inequality

$$\sum_{j=1}^{n} \min\{a_{jI} x_{jI}\, , a_{jI} x_{jS}\} \leq b_S\, , \tag{6}$$

b. The smallest feasible region is a region satisfies the following inequality

$$\sum_{j=1}^{n} \max\{a_{jS} x_{jI}\, , a_{jS} x_{jS}\} \leq b_I\, . \tag{7}$$

Teorema 2 [9]. Suppose an objective function of an interval linear programming

$$\underline{Z} = \sum_{j=1}^{n} [c_{jI}, c_{jS}][x_{jI}, x_{jS}] \tag{8}$$

where $[x_{jI}, x_{jS}] \geq 0 \; (j = 1,2,\ldots,n)$ is given. Then for every $\underline{\mathbf{x}} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)^{\boldsymbol{T}}$ in the feasible region,

$$\sum_{j=1}^{n} \min\{ c_{JI}x_{JI}, c_{JI}x_{JS}\} \leq \sum_{j=1}^{n} \max\{ c_{JS}x_{JI}, c_{JS}x_{JS}\}. \tag{9}$$

3. Solving Linear Programming with Interval Variables Using Interior Point Method

In this sub-section, the interior point algorithm for interval linear programming will be constructed. The steps of the algorithm are as follows.

Algorithm 1

Step 1. Given an interval linear programming in (2)-(4).

Step 2. Determine the largest feasible region and the smallest feasible region of the interval linear programming. The largest feasible region of the model is shown by equation (6), and the smallest feasible region of the model is shown by equation (7).

Step 3. Determine the least upper bound and the greatest lower bound of the objective function. The least upper bound and the greatest lower bound of the model in equation (8) as follows:

a. The least upper bound satisfies the following inequality

$$z_S = \sum_{j=1}^{n} \max\{c_{jS}x_{jI}, c_{jS}x_{jS}\}, \tag{10}$$

b. The greatest lower bound satisfies the following inequality

$$z_I = \sum_{j=1}^{n} \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\}. \tag{11}$$

Step 4. By using interior point algorithm (algorithm 2), solve the calculation step for the least upper bound of the objective function. The least upper bound z_S is determined by using the largest feasible region.

Step 5. By using interior point algorithm (algorithm 2), solve the calculation step for the greatest lower bound of the objective function. The greatest lower bound z_I is determined by using the smallest feasible region.

Step 6. Check the following condition

a. If the interval linear programming does not have the largest feasible region and the smallest feasible region, the interval linear programming does not have any solution.

b. If the least upper bound and the greatest lower bound are finite, then an optimum solution of the interval linear programming is obtained.

c. If the least upper bound is infinite, then add some constraints to the smallest feasible region. So the corresponding linear programming has bounded solution. Set the least upper bound value obtained from the smallest feasible region.

d. If the greatest lower bound is infinite, then add some constraints to the largest feasible region. So the corresponding linear programming has bounded solution. Set the greatest upper bound value obtained from the largest feasible region.

Step 7. An optimum solution of the objective function is $\underline{x} = [x_I, x_S]$ and the optimal objective function

$$\underline{Z} = [z_I, z_S] \tag{12}$$

Algorithm 2 (Interior Point Algorithm)

(1) Choose an initial interior point $\widetilde{\mathrm{X}}^0 = (x_1, x_2, \dots, x_{n+m})$. The interior point $\widetilde{\mathrm{X}}^0$ should satisfy all constraints. Then evaluate the objective function $\mathrm{Z}_0 = \mathrm{c}^{\mathrm{T}}\widetilde{\mathrm{X}}^0$.

Define a diagonal matrix $\mathbf{D}_{i+1} = \mathrm{diag}(\widetilde{\mathrm{X}}^0)$,

$$\mathbf{D}_{i+1} = \begin{bmatrix} x_1 & 0 & 0 & & 0 \\ 0 & x_2 & 0 & & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots \ddots & x_{n+m} \end{bmatrix}.$$

(2) Calculate $\mathbf{A}_{i+1} = \mathbf{A}\mathbf{D}_{i+1}$and $\mathbf{C}_{i+1} = \mathbf{D}_{i+1}\mathbf{c}$.

(3) a. Calculate the projected matrix

$$\mathbf{P}_{i+1} = \mathbf{I} - \mathbf{A}_{i+1}{}^T(\mathbf{A}_{i+1}\mathbf{A}_{i+1}{}^T)^{-1}\mathbf{A}_{i+1},$$

where I is an identity matrix.

b. Calculate the projected gradient

$$\mathbf{C}_{P_{i+1}} = \mathbf{P}_{i+1}\mathbf{C}_{i+1}.$$

(4) Calculate $\mathbf{V}_{i+1} = \left|\min\left(\mathbf{C}_{P_{i+1}}\right)\right|$ and

$$\mathbf{M}_{i+1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\alpha}{\mathbf{V}_{i+1}}\mathbf{C}_{P_{i+1}},$$

where $\mathbf{M}_{i+1}$ is (n+m) × 1 matrix and $\alpha \in (0,1)$

(5) Calculate the next interior point

$$\widetilde{\mathrm{X}}^{i+1} = \mathbf{D}_{i+1}\mathbf{M}_{i+1}.$$

(6) Calculate the objective function

$$\mathrm{Z}_{i+1} = \mathrm{c}^{\mathrm{T}}\widetilde{\mathrm{X}}^{i+1},$$

a. If $\mathrm{Z}_{i+1} > \mathrm{Z}_i$,then the iteration should be continued. Go to the Step 1 where $\widetilde{\mathrm{X}}^{i+1}$ is the next interior point.

b. If $Z_{i+1} \leq Z_i$, then the optimal solution was reached. Here, Z_i is the optimal solution.

4. Numerical Example

In this section, one example of interval linear programming has been solved in [8], and we will compare the results.

According to Zhou et al. (2009), three criteria can be used to determine the best solution for linear interval programming problems. These criteria are:

1) Meet the constraint function

2) Interval width of the optimal if the narrowest value

3) Level uncertainty is the ratio of the interval width from the optimal amount to the midpoint of the optimal interval (the smallest).

Maximize (objective function)

$$Z = [-20,50][x_{1I}, x_{1S}] + [0,10][x_{2I}, x_{2S}],$$

subject to

$$10[x_{1I}, x_{1S}] + 60[x_{2I}, x_{2S}] \leq 1080,$$

$$10[x_{1I}, x_{1S}] + 20[x_{2I}, x_{2S}] \leq 400,$$

$$10[x_{1I}, x_{1S}] + 10[x_{2I}, x_{2S}] \leq 240,$$

$$30[x_{1I}, x_{1S}] + 10[x_{2I}, x_{2S}] \leq 420,$$

$$40[x_{1I}, x_{1S}] + 10[x_{2I}, x_{2S}] \leq 520,$$

$$[x_{1I}, x_{1S}], [x_{2I}, x_{2S}] \geq 0.$$

a. The linear programming could be considered as an interval linear programming as follows.

Maximize (objective function)

$$Z = [-20,50][x_{1I}, x_{1S}] + [0,10][x_{2I}, x_{2S}],$$

subject to

$$[10x_{1I} + 60x_{2I}, 10x_{1S} + 60x_{2S}] \leq [1080,1080],$$

$$[10x_{1I} + 20x_{2I}, 10x_{1S} + 20x_{2S}] \leq [400,400],$$

$$[10x_{1I} + 10x_{2I}, 10x_{1S} + 10x_{2S}] \leq [240,240],$$

$$[30x_{1I} + 10x_{2I}, 30x_{1S} + 10x_{2S}] \leq [420,420],$$

$$[40x_{1I} + 10x_{2I}, 40x_{1S} + 10x_{2S}] \leq [520,520],$$

$$x_{1I}, x_{1S}, x_{2I}, x_{2S} \geq 0.$$

b. Determine the largest feasible region and the smallest feasible region by using equation (6) and (7), respectively. We obtain the largest feasible region and the smallest feasible region given in Table 1.

Table 1. The largest and the smallest feasible region

1. The Largest Feasible Region	**2. The Smallest Feasible Region**
$10x_{1I} + 60x_{2I} \leq 1080,$	$10x_{1S} + 60x_{2S} \leq 1080,$
$10x_{1I} + 20x_{2I} \leq 400,$	$10x_{1S} + 20x_{2S} \leq 400,$

1. The Largest Feasible Region	2. The Smallest Feasible Region
$10x_{1I} + 10x_{2I} \leq 240$,	$10x_{1S} + 10x_{2S} \leq 240$,
$30x_{1I} + 10x_{2I} \leq 420$,	$30x_{1S} + 10x_{2S} \leq 420$,
$40x_{1I} + 10x_{2I} \leq 520$,	$40x_{1S} + 10x_{2S} \leq 520$,
$x_{1I}, x_{2I} \geq 0$.	$x_{1S}, x_{2S} \geq 0$.

c. From the objective function, we obtain the least upper bound and the greatest lower bound that given in Table 2.

Table 2. The least upper bound and the greatest lower bound

1. The Least Upper Bound	2. The Greatest Lower Bound
Maximize	Maximize
$z_S = 50x_{1S} + 10x_{2S}$	$z_I = -20x_{1S}$

d. By using algorithm two and α = 0.95 (cited from [1]), we determined the least upper bound value (Table 2) by using the largest feasible region constraints (Table 1). We found that the least upper bound was infinite. Therefore, we added some constraints on the smallest feasible region (table 1. We obtained the least upper bound that given in Table 3. From the iteration, we obtain $x_{1S} \cong 12.999986$, $x_{2S} \cong 0.0000003$, $x_{1I} = 0$, $x_{2I} = 0$.

e. By using algorithm two and α = 0.95 (cited from [1]), we determined the greatest lower bound (Table 2) by using the smallest feasible region constraints (Table 1). We obtained the greatest lower bound given in Table 4. From the iteration, we obtain $x_{1I} = 0$, $x_{2I} = 0, x_{1S} \cong 0$, $x_{2S} \cong 0.100006$.

f. From the step d and step e, we obtain $x_{1S} \cong 12.999986$, $x_{2S} \cong 0.0000003$, $x_{1I} = 0$, $x_{2I} = 0$. The optimal solution of the interval linear programming is $[x_{1I}, x_{1S}] = [0\,, 12.999986]$, $[x_{2I}, x_{2S}] = [0\,, 0.0000003]$ and $Z = [-259.99972\,, 649.999305]$. This solution gives the same value as obtained by [8].

Table 3. The least upper bound from interior point method

Iteration	x_i	z_i	Optimality Test	Decision
0	(10 , 5)	550.000000	Z_0	
1	(12.010457 , 3.608170)	636.604573	$Z_1 > Z_0$	Continue
2	(12.926704 , 0.180408)	648.139304	$Z_2 > Z_1$	Continue
3	(12.967216 , 0.125496)	649.615776	$Z_3 > Z_2$	Continue
4	(12.997322 , 0.006274)	649.928877	$Z_4 > Z_3$	Continue
5	(12.998797 , 0.004587)	649.985758	$Z_5 > Z_4$	Continue
6	(12.999899 , 0.000229)	649.997292	$Z_6 > Z_5$	Continue
7	(12.999955 , 0.000170)	649.999469	$Z_7 > Z_6$	Continue
8	(12.999995 , 0.000008)	649.999864	$Z > Z_7$	Continue
9	(12.999996 , 0.000006)	649.999893	$Z_9 > Z_8$	Continue
10	(12.999986 , 0.0000003)	649.999305	$Z_{10} \leq Z_9$	Stop

Table 4. The greatest lower bound from interior point method

Iteration	x_i	z_i	Optimality Test	Decision
0	(0.1 , 0.1)	-2.000000	Z_0	
1	(0.004999 , 0.100006)	-0.099999	$Z_1 > Z_0$	Continue
2	(0.000249 , 0.100006)	-0.004999	$Z_2 > Z_1$	Continue
3	(0.000012 , 0.100006)	-0.000249	$Z_3 > Z_2$	Continue
4	($6.2\ 10^{-7}$, 0.100006)	-0.000012	$Z_4 > Z_3$	Continue
5	($3.1\ 10^{-8}$, 0.100006)	$-6.2\ 10^{-7} \cong 0$	$Z_5 > Z_4$	Continue
6	($1.5\ 10^{-9}$, 0.100006)	$-3.1\ 10^{-8} \cong 0$	$Z_6 \leq Z_5$	Stop

5. Conclusions

This paper presents the use of an interior point method that can be used to solve linear programming with interval variables or interval linear programming. The use of an interior point method compared to the simplex method of existing papers. Linear programming with interval variables could be solved with converting the problem into two classic linear programming models. The first model called the least upper bound in objective function corresponding with the largest feasible region constraints. The second model called the greatest lower bound in objective function corresponding with the smallest feasible region constraints. Furthermore, two problems were solved by an interior point method, notably a gradient projection algorithm. The optimum interval value is obtained from a combination of these two problems.

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