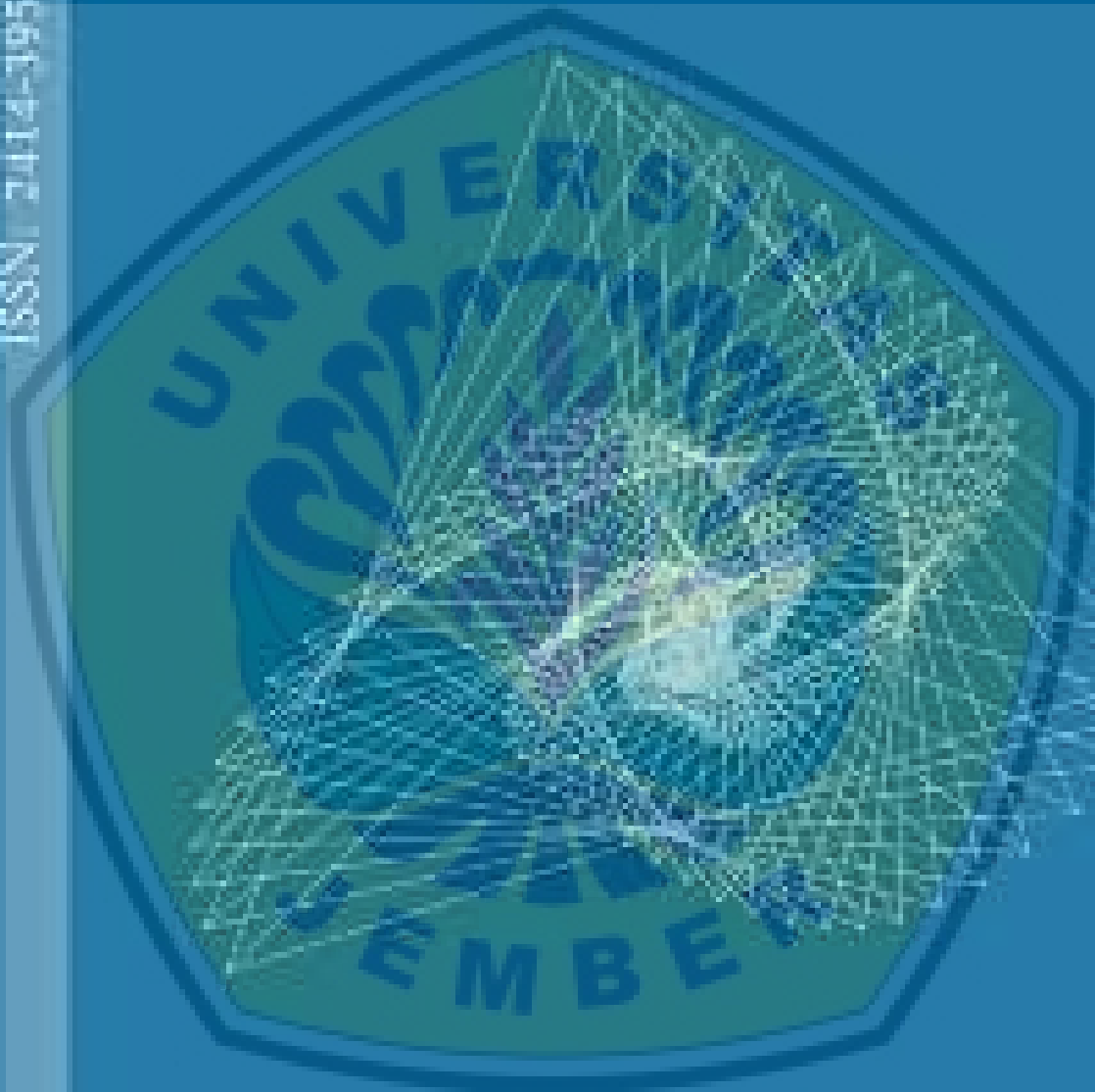


# URAL MATHEMATICAL JOURNAL

M.S. Krasovskii Institute of Mathematics and Mechanics of  
the Ural Branch of Russian Academy of Sciences and  
Ural Federal University named after the first President of Russia B.N. Yeltsin

ISSN 2111-1951





## URAL MATHEMATICAL JOURNAL

### Aims and Scope

The electronic *Ural Mathematical Journal* (UMJ) is a high standard peer-reviewed electronic journal devoted to innovative research which covers the fields of Mathematics, Mechanics and Theoretical Computer Science, Computer Engineering and Optimization, and Control Theory.

We hope that the *Ural Mathematical Journal* will help Russian and foreign experts to exchange ideas, to establish new scientific contacts and to do important joint researches.

### Journal Topics

Pure and Applied Mathematics and Mechanics in general, with a certain emphasis on innovative researches which cover the fields of Modern Mathematical Analysis and Theory of Functions, Algebra, Approximation Theory, Regulation Theory and Applications, Inverse and Ill-Posed Problems, Nonlinear Analysis, Control and Optimization Theory, Game Theory, Fractional Calculus and Applications, Modeling for Optimization and Control, Linear Programming, Nonlinear Programming, Stochastic Programming, Parametric Programming, Discrete Programming, Dynamic Programming, Nonlinear Dynamics, Stochastic Differential Equations, Applications related to Optimization and Engineering.

### Basic information about the Journal

The journal was founded by the Institute of Mathematics and Mechanics named after academician N.N. Krasovskii of the Ural Branch of the Russian Academy of Sciences (<http://imm.uran.ru>) in collaboration with the Ural Federal University named after the first President of Russia B.N.Yeltsin (<http://urfu.ru>).

- The *Ural Mathematical Journal* was founded in 2015 as an electronic scientific publication.
- The *Ural Mathematical Journal* is registered as a scientific periodical publication by the Federal Service for Supervision in the Sphere of Communication, Information Technologies and Mass Communications (*Certificate of Registration Эп № ФС77 - 61719 of 07.05.2015*).
- The *Ural Mathematical Journal* is registered by the International Standard Serial Numbering Center with the assignment of the international standard number **ISSN 2414-3952**.
- Each paper in the journal is assigned a **DOI (Digital Object Identifier)**.
- The journal is periodical, it is published 2 times a year, in July and December.
- All papers submitted to the editors are necessarily reviewed by experts of the highest level.
- Papers are accepted written in English. All articles contain detailed annotations.
- Publication for all authors is free. The full-text version of the journal is in the free access mode on the **journal's website**. Also, the journal's materials are placed on the platform of the Russian Scientific Citation Index of the Russian Universal Scientific Electronic Library **eLIBRARY.ru** and on the all-Russian mathematical portal **Math-Net.Ru**.
- The journal *Ural Mathematical Journal* is included in the database **Scopus**.
- The journal *Ural Mathematical Journal* is included in the database **MathSciNet** (Mathematical Reviews).
- The journal *Ural Mathematical Journal* is included in the database **Zentralblatt MATH (zbMATH)**.
- The journal *Ural Mathematical Journal* is included in **DOAJ (Directory of Open Access Journals)**.
- All the contents of the journal, except where otherwise noted, is licensed under a **Creative Commons Attribution License**.

### EDITORIAL TEAM

#### Editor-in-Chief

[Vitalii I. Berdyshev](#), Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation; Full Member of Russian Academy of Sciences

#### Deputy Editors-in-Chief

[Vitalii V. Arestov](#), Ural Federal University, Ekaterinburg, Russian Federation

[Nikolai Yu. Antonov](#), Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation

[Vladislav V. Kabanov](#), Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation

#### Scientific Editors

[Tatiana F. Filippova](#), Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation

[Vladimir G. Pimenov](#), Ural Federal University, Ekaterinburg, Russian Federation

#### Editorial Council

### USER

Username

Password

Remember me

Login

### NOTIFICATIONS

View  
Subscribe

### LANGUAGE

Select Language

English

Submit

### JOURNAL CONTENT

Search

Search Scope

All

Search

Browse

By Issue  
By Author  
By Title

### FONT SIZE

### INFORMATION

For Readers  
For Authors  
For Librarians

Journal Help



N.N.Krasovskii Institute of  
Mathematics  
and Mechanics of the Ural  
Branch  
of the Russian Academy of  
Sciences



Ural Federal University  
named after  
the first President of Russia  
B.N.Yeltsin

[Alexander G. Chentsov](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Sergei V. Matveev](#), Chelyabinsk State University, Chelyabinsk, Russian Federation  
[Alexander A. Makhnev](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Irina V. Melnikova](#), Ural Federal University, Ekaterinburg, Russian Federation  
[Fernando M.F.L. Pereira](#), Faculdade de Engenharia da Universidade do Porto, Porto, Portugal  
[Stefan W. Pickl](#), University of the Federal Armed Forces, Munich, Germany  
[Szilárd G. Révész](#), Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences, Budapest, Hungary  
[Lev B. Ryashko](#), Ural Federal University, Ekaterinburg, Russian Federation  
[Arseny M. Shur](#), Ural Federal University, Ekaterinburg, Russian Federation  
[Vladimir N. Ushakov](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Vladimir V. Vasin](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Mikhail V. Volkov](#), Ural Federal University, Ekaterinburg, Russian Federation

Copyright © 2015-2022  
 Ural Mathematical Journal  
 Yekaterinburg, Russia

#### Editorial Board

[Elena N. Akimova](#), Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Science, Ekaterinburg, Russia, Russian Federation  
[Alexander G. Babenko](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Vitalii A. Baranskii](#), Ural Federal University, Ekaterinburg, Russian Federation  
[Elena E. Berdysheva](#), University of Cape Town, South Africa  
[Alexey R. Danilin](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Yuri F. Dolgii](#), Ural Federal University, Ekaterinburg, Russian Federation  
[Yakif Ya. Dzhafarov \(Cafer\)](#), Department of Mathematics, Anadolu University, Eskişehir, Turkey  
[Polina Yu. Glazyrina](#), Ural Federal University, Ekaterinburg, Russian Federation  
[Mikhail I. Gusev](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Éva Gyurkovics](#), Department of Differential Equations, Institute of Mathematics, Budapest University of Technology and Economics, Budapest, Hungary  
[Marc Jungers](#), National Center for Scientific Research (CNRS), Nancy and Université de Lorraine, CRAN, Nancy, France  
[Mikhail Yu. Khachay](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Anatolii F. Kleimenov](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Anatoly S. Kondratiev](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Vyacheslav I. Maksimov](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Tapio Palokangas](#), University of Helsinki, Helsinki, Finland  
[Emanuele Rodaro](#), Politecnico di Milano, Department of Mathematics, Italy  
[Dmitrii A. Serkov](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation  
[Alexander N. Sesekin](#), Ural Federal University, Ekaterinburg, Russian Federation  
[Alexander M. Tarashev](#), Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation

#### Managing Editor

[Oxana G. Matviychuk](#), Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation

#### Technical Advisor

[Alexey N. Borbunov](#), Ural Federal University, Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russian Federation

## ANNOUNCEMENTS

### RSCI

The *Ural Mathematical Journal* is included now in RSCI (Russian Science Citation Index).

Posted: 2022-07-27

[More Announcements...](#)

VOL 8, NO 2 (2022)

[TABLE OF CONTENTS](#)



РОССИЙСКИЙ ИНДЕКС НАУЧНОГО ЦИТИРОВАНИЯ

Science Index

eLIBRARY.RU

Math-Net.Ru

AMERICAN MATHEMATICAL SOCIETY

MathSciNet  
 Mathematical Reviews

zbMATH

the first resource for mathematics



Scopus

RUSSIAN SCIENCE CITATION INDEX

DOAJ DIRECTORY OF OPEN ACCESS JOURNALS

OPEN ACCESS

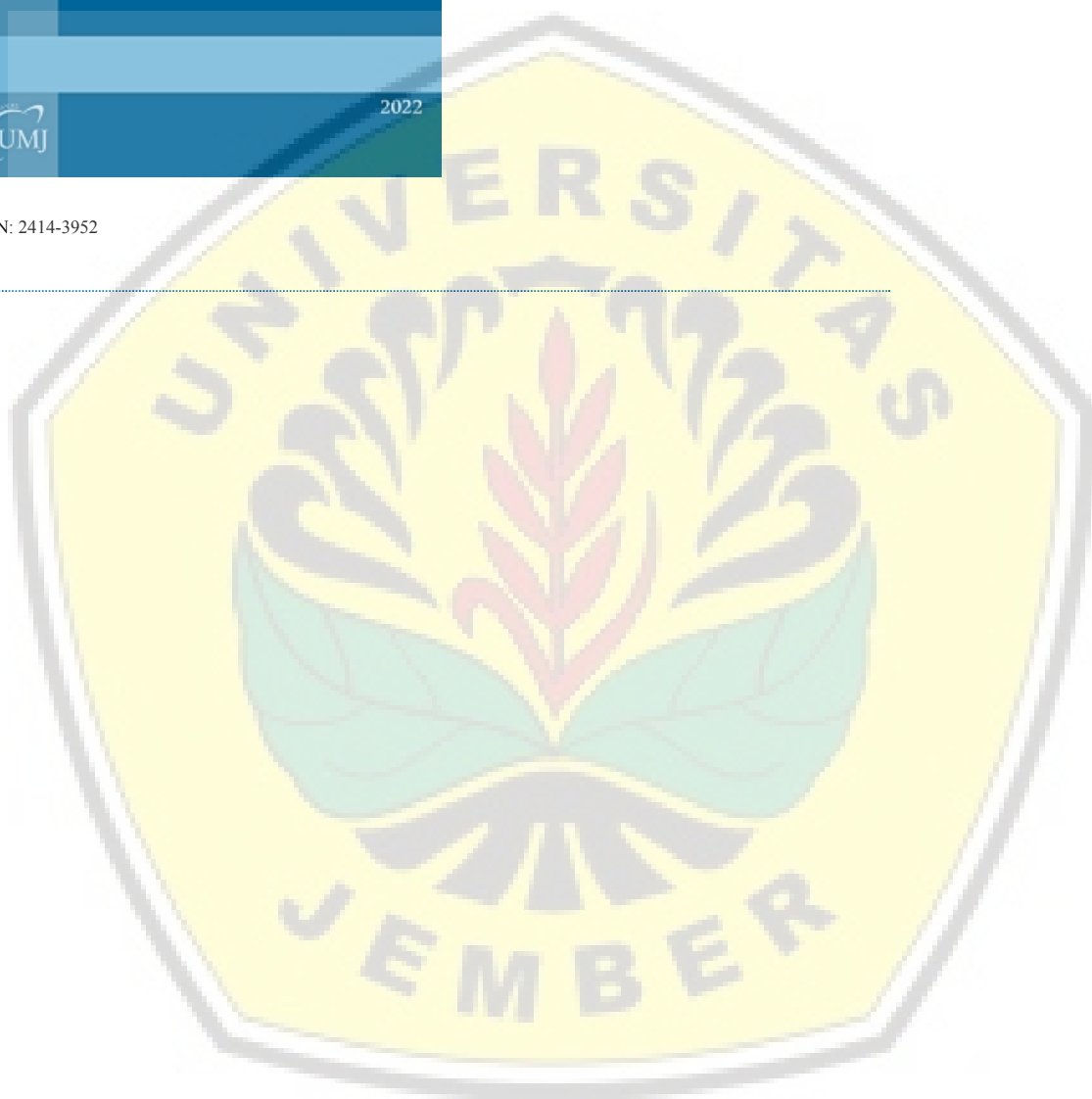
Crossref

Google Scholar



ISSN: 2414-3952

---





- [HOME](#)
[ABOUT](#)
[LOGIN](#)
[REGISTER](#)
[SEARCH](#)
[CURRENT](#)
[ARCHIVES](#)
  
[ANNOUNCEMENTS](#)

*Home > Archives > Vol 8, No 2 (2022)*

## VOL 8, NO 2 (2022)

### FULL ISSUE

[VIEW OR DOWNLOAD THE FULL ISSUE](#)

[PDF](#)

### TABLE OF CONTENTS

#### ARTICLES

**BESSEL POLYNOMIALS AND SOME CONNECTION FORMULAS IN TERMS OF THE ACTION OF LINEAR DIFFERENTIAL OPERATORS**

Baghdadi Aloui, Jihad Souissi

[PDF](#)

4–12

**OUTPUT CONTROLLABILITY OF DELAYED CONTROL SYSTEMS IN A LONG TIME HORIZON**

Boris I. Ananyev

[PDF](#)

13–26

**ON ONE INEQUALITY OF DIFFERENT METRICS FOR TRIGONOMETRIC POLYNOMIALS**

Vitalii V. Arestov, Marina V. Deikalova

[PDF](#)

27–45

**A CHARACTERIZATION OF DERIVATIONS AND AUTOMORPHISMS ON SOME SIMPLE ALGEBRAS**

Farhodjon Arzikulov, Furqatjon Urinboyev, Shahlo Ergasheva

[PDF](#)

46–58

**APPROXIMATE CONTROLLABILITY OF IMPULSIVE STOCHASTIC SYSTEMS DRIVEN BY ROSENBLATT PROCESS AND BROWNIAN MOTION**

Abbes Benchaabane

[PDF](#)

59–70

**PERIODIC SOLUTIONS OF A CLASS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DIFFERENT DELAYS**

Rabah Khemis, Abdelouaheb Ardjouni, Ahlème Bouakkaz

[PDF](#)

71–80

**RESTRAINED ROMAN REINFORCEMENT NUMBER IN GRAPHS**

Saeed Kosari, Seyed Mahmoud Sheikholeslami, Mustapha Chellali, Maryam Hajjari

[PDF](#)

81–93

**ON LOCAL IRREGULARITY OF THE VERTEX COLORING OF THE CORONA PRODUCT OF A TREE GRAPH**

Arika Indah Kristiana, M. Hidayat, Robiatul Adawiyah, D. Dafik, Susi Setiawani, Ridho Alfariis

[PDF](#)

94–114

**COMBINED ALGORITHMS FOR CONSTRUCTING A SOLUTION TO THE TIME–OPTIMAL PROBLEM IN THREE-DIMENSIONAL SPACE BASED ON THE SELECTION OF EXTREME POINTS OF THE SCATTERING SURFACE**

Pavel D. Lebedev, Alexander A. Uspenskii

[PDF](#)

115–126

**ON DISTANCE–REGULAR GRAPHS OF DIAMETER 3 WITH EIGENVALUE  $\theta = 1$**

Alexander A. Makhnev, Ivan N. Belousov, Konstantin S. Efimov

[PDF](#)

127–132

**ANALYSIS OF THE GROWTH RATE OF FEMININE MOSQUITO THROUGH DIFFERENCE EQUATIONS**

Regan Murugesan, Sathish Kumar Kumaravel, Suresh Rasappan, Wardah Abdullah Al Majrafi

[PDF](#)

133–142

**INEQUALITIES PERTAINING TO RATIONAL FUNCTIONS WITH PRESCRIBED POLES**

Nisar Ahmad Rather, Mohammad Shafi Wani, Ishfaq Dar

[PDF](#)

143–152

**A QUADRUPLE INTEGRAL INVOLVING THE EXPONENTIAL LOGARITHM OF QUOTIENT RADICALS IN TERMS OF THE HURWITZ-LERCH ZETA FUNCTION**

Robert Reynolds, Allan Stauffer

[PDF](#)

153–161

**ON SOME VERTEX-TRANSITIVE DISTANCE-REGULAR ANTIPODAL COVERS OF COMPLETE GRAPHS**

Ludmila Yu. Tsiovkina

[PDF](#)

162–176

**BIHARMONIC GREEN FUNCTION AND BISUPERMEDIAN ON INFINITE NETWORKS**

Manivannan Varadha Raj, Venkataraman Madhu

[PDF](#)

177–186

#### USER

Username

Password

Remember me

#### NOTIFICATIONS

[View](#)  
[Subscribe](#)

#### LANGUAGE

Select Language

#### JOURNAL CONTENT

Search

Search Scope

Browse

[By Issue](#)  
[By Author](#)  
[By Title](#)

#### FONT SIZE

#### INFORMATION

[For Readers](#)  
[For Authors](#)  
[For Librarians](#)

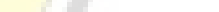
[Journal Help](#)



N.N.Krasovskii Institute of  
Mathematics  
and Mechanics of the Ural  
Branch  
of the Russian Academy of  
Sciences



Ural Federal University  
named after  
the first President of Russia  
B.N.Yeltsin





# ON LOCAL IRREGULARITY OF THE VERTEX COLORING OF THE CORONA PRODUCT OF A TREE GRAPH

Arika Indah Kristiana<sup>1,†</sup>, M. Hidayat<sup>1</sup>, Robiatul Adawiyah<sup>1</sup>, D. Dafik<sup>1</sup>,  
Susni Setiawani<sup>1</sup>, Ridho Alfarisi<sup>2</sup>

<sup>1</sup>Department of Mathematics Education, University of Jember,  
Jalan Kalimantan 37, 68126, Jember, Jawa Timur, Indonesia

<sup>2</sup>Department of Elementary School Education, University of Jember,  
Jalan Kalimantan 37, 68126, Jember, Jawa Timur, Indonesia

<sup>†</sup>arika.fkip@unej.ac.id

**Abstract:** Let  $G = (V, E)$  be a graph with a vertex set  $V$  and an edge set  $E$ . The graph  $G$  is said to be with a local irregular vertex coloring if there is a function  $f$  called a local irregularity vertex coloring with the properties: (i)  $l : (V(G)) \rightarrow \{1, 2, \dots, k\}$  as a vertex irregular  $k$ -labeling and  $w : V(G) \rightarrow N$ , for every  $uv \in E(G)$ ,  $w(u) \neq w(v)$  where  $w(u) = \sum_{v \in N(u)} l(v)$  and (ii)  $\text{opt}(l) = \min\{\max\{l_i : l_i \text{ is a vertex irregular labeling}\}$ . The chromatic number of the local irregularity vertex coloring of  $G$  denoted by  $\chi_{lis}(G)$ , is the minimum cardinality of the largest label over all such local irregularity vertex colorings. In this paper, we study a local irregular vertex coloring of  $P_m \odot G$  when  $G$  is a family of tree graphs, centipede  $C_n$ , double star graph  $(S_{2,n})$ , Weed graph  $(S_{3,n})$ , and  $E$  graph  $(E_{3,n})$ .

**Keywords:** Local irregularity, Corona product, Tree graph family.

## 1. Introduction

Let  $G(V, E)$  be a connected and simple graph with a vertex set  $V$  and an edge set  $E$ . In this paper, we combine two concepts, namely the local antimagic vertex coloring and the distance irregular labelling, with a local irregularity of vertex coloring. This concept firstly was introduced by Kristiana [2, 3], et. al. The latest research was conducted by Azzahra [4], who examined the local irregularity vertex coloring of a grid graph family. In this paper we study the local irregularity of vertex coloring of corona product graph of a tree graph family.

**Definition 1.** Suppose  $l : V(G) \rightarrow \{1, 2, \dots, k\}$  and  $w : V(G) \rightarrow N$ , where

$$w(u) = \sum_{v \in N(u)} l(v),$$

then  $l(v)$  is called the vertex irregular  $k$ -labeling and  $w(u)$  is called the local irregularity of vertex coloring if

- (i)  $\text{opt}(l) = \min\{\max\{l_i : l_i \text{ vertex irregular labeling}\}$ ;
- (ii) for every  $uv \in E(G)$ ,  $w(u) \neq w(v)$ .

**Definition 2.** The chromatic number of local irregular graph  $G$  denoted by  $\chi_{lis}(G)$ , is the minimum of cardinality of the local irregularity of vertex coloring.

In this paper, we will use the following lemma which gives a lower bound on the chromatic number of local irregular vertex coloring:

**Lemma 1** [2]. *Let  $G$  be a simple and connected graph, then  $\chi_{lis}(G) \geq \chi(G)$ .*

**Proposition 1** [2]. *Let  $G$  be a graph each two adjacent vertices of which have a different vertex degree then  $\text{opt}(l) = 1$ .*

**Proposition 2** [2]. *Let  $G$  be a graph each two adjacent vertices have the same vertex degree then  $\text{opt}(l) \geq 2$ .*

**Definition 3** [1]. *Let  $G$  and  $H$  be two connected graphs. Let  $o$  be a vertex of  $H$ . The corona product of the combination of two graphs  $G$  and  $H$  is defined as the graph obtained by taking a duplicate of graph  $G$  and  $|V(G)|$  a duplicate of graph  $H$ , namely  $H_i; i = 1, 2, 3, \dots, |V(G)|$  then connects each vertex  $i$  in  $G$  to each vertex in  $H_i$ . The corona product of the graphs  $G$  and  $H$  is denoted by  $G \odot H$ .*

## 2. Result and discussion

In this paper, we analyze the new result of the chromatic number of local irregular vertex coloring of corona product by family of tree graph ( $P_m \odot G$ ) where  $G$  is centipede graph ( $C_n$ ), double star graph ( $S_{2,n}$ ), and Weed graph ( $S_{3,n}$ ).

**Theorem 1.** *Let  $G = P_m \odot C_{p_n}$ , be a corona product of a path graph of order  $m$  and a centipede graph of order  $n$  for  $n, m \geq 2$ , then*

$$\chi_{lis}(P_m \odot C_{p_n}) = \begin{cases} 5, & \text{for } m = 3 \text{ and } n = 2, 3, \\ 6, & \text{for } m = 2 \text{ and } n = 2, 3 \text{ or for } m = 3 \text{ and } n \geq 4, \\ 7, & \text{for } m = 2 \text{ and } n \geq 4 \text{ or for } m \geq 4 \text{ and } n = 2, 3, \\ 8, & \text{for } m \geq 4 \text{ and } n \geq 4, \end{cases}$$

with  $\text{opt}(l)$  defined as

$$\text{opt}(l)(P_m \odot C_{p_n}) = \begin{cases} 1, & \text{for } m = 3 \text{ and } n = 3, \\ 1, 2, & \text{for } m = 2 \text{ and } n = 2 \text{ or} \\ & \text{for } m = 3 \text{ and } n = 2 \text{ or} \\ & \text{for } m \geq 3 \text{ and } n \geq 4. \end{cases}$$

**P r o o f.** Vertex set is

$$V(P_m \odot C_{p_n}) = \{x_i; 1 \leq i \leq m\} \cup \{x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

and the edge set is

$$E(P_m \odot C_{p_n}) = \{x_i x_{i+1}; 1 \leq i \leq m - 1\} \cup \{x_{ij} x_{i,j+1}; 1 \leq i \leq m, 1 \leq j \leq n - 1\} \\ \cup \{x_{ij} y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\},$$

the order and size respectively are  $2mn + m$  and  $4mn - 1$ .

**Case 1:**  $m \neq p, m \geq 2, p \geq 2, n \geq 3$ .



First step to prove this theorem is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) = 4$ , let  $\chi_{lis}(P_m \odot Cp_n) = 4$ , if  $l(x_1) = l(x_3) = 1$ ,  $l(x_2) = 2$ ,  $l(x_{ij}) = l(y_{ij}) = 1$  then  $w(x_1) = w(x_2)$ , then there are 2 adjacent vertices that have the same color, it contradicts the definition of vertex coloring. If

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq 3, \quad j = 1, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq 3, \quad j = 2, \\ l(x_i) = 1 \rightarrow w(x_i) \neq w(x_{i+1}), \quad w(x_{i1}) \neq w(x_{i2}),$$

then  $\chi_{lis}(P_m \odot Cp_n) \geq 5$ . Based on this, we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 5$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ . Furthermore, the upper bound for the chromatic number of local irregular  $(P_m \odot Cp_n)$ , we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(x_{ij}) = 1, \\ l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq 3 \text{ and } j = 1, \\ 2, & \text{for } 1 \leq i \leq 3 \text{ and } j = 2. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, 3, \\ 8, & \text{for } i = 2, \end{cases} \\ w(x_{ij}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \text{ and } j = 1, \\ 4, & \text{for } 1 \leq i \leq 3 \text{ and } j = 2, \end{cases} \\ w(y_{ij}) = 2, \quad \text{for } 1 \leq i \leq 3 \text{ and } j = 1, 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot Cp_n) \leq 5$ , and we have  $5 \leq \chi_{lis}(P_m \odot Cp_n) \leq 5$ , so  $\chi_{lis}(P_m \odot Cp_n) = 5$  for  $m = 3$  and  $n = 2$ .

**Case 2:**  $m = n = 3$ .

Based on Proposition 1,  $\text{opt}(l) = 1$ . So the lower bound of  $(P_m \odot Cp_n)$  is

$$\chi_{lis}(P_m \odot Cp_n) \geq 5.$$

Hence  $\text{opt}(l) = 1$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, 3, \\ 8, & \text{for } i = 2, \end{cases} \\ w(y_{ij}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \text{ and } j \equiv 1, 3 \pmod{4}, \\ 4, & \text{for } 1 \leq i \leq 3 \text{ and } j = 2, \end{cases} \\ w(x_{ij}) = 2, \quad \text{for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 3.$$

The upper bound is true:  $\chi_{lis}(P_m \odot Cp_n) \leq 5$ . We have  $5 \leq \chi_{lis}(P_m \odot Cp_n) \leq 5$ , so  $\chi_{lis}(P_m \odot Cp_n) = 5$  for  $m = 3$  and  $n = 3$ .

**Case 3:**  $m = n = 2$ .

First step here is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) = 5$ , if  $l(x_1) = 1, l(x_2) = 2, l(x_{ij}) = l(y_{ij}) = 1$ , then  $w(x_{11}) = w(x_{12})$  and there are 2 adjacent vertices, that have the same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{i1}) = 1, \quad i = 1, 2, \quad l(y_{i2}) = 2, \quad i = 1, 2,$$

then  $w(x_1) \neq w(x_2), w(x_{i1}) \neq w(x_{i2})$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 6$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases} \quad l(x_{ij}) = 1, \quad l(y_{ij}) = \begin{cases} 1, & \text{for } i = 1, 2 \text{ and } j = 1, \\ 2, & \text{for } i = 1, 2 \text{ and } j = 2. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 2, \\ 7, & \text{for } i = 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i = 1 \text{ and } j = 1 \\ 4, & \text{for } i = 1 \text{ and } j = 2, \text{ or for } i = 2 \text{ and } j = 1, \\ 5, & \text{for } i = 2 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i = 1 \text{ and } j = 1, 2, \\ 3, & \text{for } i = 2 \text{ and } j = 1, 2. \end{cases}$$

We have the following upper bound  $\chi_{lis}(P_m \odot Cp_n) \leq 6$ . We have  $6 \leq \chi_{lis}(P_m \odot Cp_n) \leq 6$ , so  $\chi_{lis}(P_m \odot Cp_n) = 6$  for  $m = 2$  and  $n = 2$ .

**Case 4:**  $m = 2$  and  $n = 3$ .

First step here is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) = 5$ , if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{1j}) = 1, \quad l(y_{2j}) = 1, \quad j = 1, 2, \quad l(y_{i3}) = 2,$$

then  $w(x_{22}) = w(x_{23})$ , so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1,$$

then  $w(x_1) \neq w(x_2), w(x_{i,1}) \neq w(x_{i,2})$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 6$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases} \quad l(x_{ij}) = 1, \quad w(y_{ij}) = 1.$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 2, \\ 8, & \text{for } i = 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i = 1 \text{ and } j = 1, 3, \\ 4, & \text{for } i = 1 \text{ and } j = 2, \text{ or for } i = 2 \text{ and } j = 1, 3, \\ 5, & \text{for } i = 2 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i = 1 \text{ and } 1 \leq j \leq 3, \\ 3, & \text{for } i = 2 \text{ and } 1 \leq j \leq 3. \end{cases}$$

The upper bound is true:  $\chi_{lis}(P_m \odot Cp_n) \leq 6$ . So we have  $\chi_{lis}(P_m \odot Cp_n) = 6$  for  $m = 2$  and  $n = 3$ .

**Case 5:**  $m = 3$  and  $n \geq 4$ .

First step to prove this theorem in this case is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) = 5$ , if  $l(x_1) = l(x_3) = 1$ ,  $l(x_2) = 2$ ,  $l(x_{ij}) = l(y_{ij}) = 1$ , then  $w(x_1) = w(x_2)$  so there are 2 adjacent vertices with the have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq 3, \quad j = 1, n, \quad j \equiv 0 \pmod{2},$$

$$l(y_{ij}) = 2, \quad 1 \leq i \leq 3, \quad j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n,$$

with the  $w(x_i) \neq w(x_{i+1})$ ,  $w(x_{ij}) = w(x_{ij+1})$ . Therefore we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 6$ .

After that, we will find the upper bound for  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq 3 \text{ and } j = 1, n \quad \text{or} \quad \text{for } 1 \leq i \leq 3 \text{ and } j \equiv 0 \pmod{2}, \\ 2, & \text{for } 1 \leq i \leq 3 \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1, 3 \text{ and } n \equiv 0 \pmod{2}, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, 3 \text{ and } n \equiv 1, 3 \pmod{4}, \\ 3n + 1 - n/2, & \text{for } i = 2 \text{ and } n \equiv 0 \pmod{2}, \\ 3n + 1 - \lceil n/2 \rceil, & \text{for } i = 2 \text{ and } n \equiv 1, 3 \pmod{4}, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \text{ and } j = 1, n, \\ 4, & \text{for } 1 \leq i \leq 3 \text{ and } j \equiv 0 \pmod{2}, \\ 5, & \text{for } 1 \leq i \leq 3 \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n, \end{cases}$$

$$w(y_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot Cp_n) \leq 6$ . So  $\chi_{lis}(P_m \odot Cp_n) = 6$  for  $m = 3$  and  $n \geq 4$ .

**Case 6:**  $m \equiv 0 \pmod{2}$ ,  $m \geq 4$  and  $n = 2$ .

First step here is to find the lower bound of  $V(P_m \odot CP_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot CP_n) \geq \chi(P_m \odot CP_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot CP_n) = 5$ , let  $\chi_{lis}(P_m \odot CP_n) = 5$ , if

$$l(x_i) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad i \equiv 2 \pmod{4}, \quad l(x_i) = 2, \quad i \equiv 0 \pmod{4}, \quad l(x_{ij}) = l(y_{ij}) = 1$$

then  $w(x_{ij}) = w(x_{ij+1})$ , then there are 2 adjacent vertices that have same color, this contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad i \equiv 2 \pmod{4}, \quad l(x_i) = 2, \quad i \equiv 0 \pmod{4}, \\ l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad j = 2, \quad l(y_{ij}) = 2, \quad j = 1,$$

then  $w(x_{ij}) \neq w(x_{ij+1}); w(x_{i+1}) \neq w(x_{i+2})$ . So we have the lower bound  $\chi_{lis}(P_m \odot CP_n) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot CP_n)$ .

Furthermore, we define  $l : V(P_m \odot CP_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ or for } i \equiv 2 \pmod{4}, \\ 2, & \text{for } i \equiv 0 \pmod{4}, \end{cases} \\ l(x_{ij}) = 1, \quad l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j = 2. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, m, \\ 7, & \text{for } i \equiv 0 \pmod{2}, \quad i \neq m, \\ 8, & \text{for } i \equiv 1, 3 \pmod{4}, \quad i \neq 1, \end{cases} \\ w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 1 \quad \text{or} \quad \text{for } i \equiv 2 \pmod{4} \text{ and } j = 1, \\ 4, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 2 \quad \text{or} \quad \text{for } i \equiv 2 \pmod{4} \text{ and } j = 2 \text{ or} \\ & \text{for } i \equiv 0 \pmod{4} \text{ and } j = 1, \\ 5, & \text{for } i \equiv 0 \pmod{4} \text{ and } j = 2, \end{cases} \\ w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 1, 2 \quad \text{or} \quad \text{for } i \equiv 2 \pmod{4} \text{ and } j = 1, 2, \\ 3, & \text{for } i \equiv 0 \pmod{4} \text{ and } j = 1, 2. \end{cases}$$

We have the upper bound  $\chi_{lis}(P_m \odot CP_n) \leq 7$ . So  $\chi_{lis}(P_m \odot CP_n) = 7$  for  $m \geq 4$  and  $n = 2$ .

**Case 7:**  $m \equiv 0 \pmod{2}$ ,  $m \geq 4$  and  $n = 3$ .

First step to prove this theorem is to find the lower bound of  $V(P_m \odot CP_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot CP_n) \geq \chi(P_m \odot CP_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot CP_n) = 5$ , in this case if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 3, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = 1, 2,$$

then  $w(x_i) = w(x_{i+1})$ , then there are 2 adjacent vertices that have the same color, this contradicts the definition of vertex coloring. If

$$l(x_i) = 1 \quad i \equiv 1, 3 \pmod{4}, \quad i \equiv 2 \pmod{4}, \quad l(x_i) = 2, \quad i \equiv 0 \pmod{2}, \quad l(y_{ij}) = l(x_{ij}) = 1,$$

then  $w(x_{i+1}) \neq w(x_{i+2})$ ,  $w(x_{i1}) \neq w(x_{i2})$ ,  $w(x_{i1}) \neq w(y_{i2})$ . Therefore we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ or for } i \equiv 2 \pmod{4}, \\ 2, & \text{for } i \equiv 0 \pmod{4}, \end{cases}$$

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1.$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, m, \\ 8, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \\ 9, & \text{for } i \equiv 1, 3 \pmod{4}, i \neq 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 1, 3 \text{ or for } i \equiv 2 \pmod{4} \text{ and } j = 1, 3, \\ 4, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 2 \text{ or for } i \equiv 2 \pmod{4} \text{ and } j = 2 \text{ or} \\ & \text{for } i \equiv 0 \pmod{4} \text{ and } j = 1, 3, \\ 5, & \text{for } i \equiv 0 \pmod{4} \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } 1 \leq j \leq 3 \text{ or for } i \equiv 2 \pmod{4} \text{ and } 1 \leq j \leq 3, \\ 3, & \text{for } i \equiv 0 \pmod{4} \text{ and } 1 \leq j \leq 3. \end{cases}$$

We have the upper bound  $\chi_{lis}(P_m \odot Cp_n) \leq 7$ . So  $\chi_{lis}(P_m \odot Cp_n) = 7$  for  $m \geq 4$  and  $n = 3$ .

**Case 8:**  $m = 2$  and  $n \geq 4$ .

First step here is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) < 7$ , let  $\chi_{lis}(P_m \odot Cp_n) = 6$ , if

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then  $w(x_{ij+1}) = w(x_{ij+2})$ , then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad j \equiv 0 \pmod{2}, \quad j = 1, n,$$

$$l(y_{ij}) = 2, \quad j \equiv 1, 3 \pmod{4}, \quad j \neq 1, \quad n \rightarrow w(x_1) \neq w(x_2), \quad w(x_{ij+1}) \neq w(x_{ij+2}),$$

then we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases}$$

$$l(x_{ij}) = 1,$$

$$l(y_{ij}) = \begin{cases} 1, & \text{for } i = 1, 2 \text{ and } j = 1, n \text{ or for } i = 1, 2 \text{ and } j \equiv 0 \pmod{2}, \\ 2, & \text{for } i = 1, 2 \text{ and } j \equiv 1, 3 \pmod{4}, j \neq 1, n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 1 - n/2, & \text{for } i = 1 \text{ and } n \equiv 0 \pmod{2}, \\ 3n + 1 - \lceil n/2 \rceil, & \text{for } i = 1 \text{ and } n \equiv 1, 3 \pmod{4}, \\ 2n + n/2, & \text{for } i = 2 \text{ and } n \equiv 0 \pmod{2}, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 2 \text{ and } n \equiv 1, 3 \pmod{4}, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i = 1 \text{ and } j = 1, n, \\ 4, & \text{for } i = 1 \text{ and } j \equiv 0 \pmod{2}, j \neq n \text{ or for } i = 2 \text{ and } j = 1, n, \\ 5, & \text{for } i = 1 \text{ and } j \equiv 1, 3 \pmod{4}, j \neq 1, n \text{ or for } i = 2 \text{ and } j \equiv 0 \pmod{2}, j \neq n, \\ 6, & \text{for } i = 2 \text{ and } j \equiv 1, 3 \pmod{4}, j \neq 1, n, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i = 1 \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i = 2 \text{ and } 1 \leq j \leq n. \end{cases}$$

The upper bound  $\chi_{lis}(P_m \odot Cp_n) \leq 7$  is true. So  $\chi_{lis}(P_m \odot Cp_n) = 7$  for  $m = 2$  and  $n \geq 4$ .

**Case 9:**  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n = 2$ .

First step to prove this theorem in this case is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) < 7$ , and let  $\chi_{lis}(P_m \odot Cp_n) = 6$ , if

$$l(x_i) = 1, \quad i \equiv 1 \pmod{4}, \quad i \equiv 0 \pmod{2}, \quad l(x_i) = 2, \quad i \equiv 3 \pmod{4}, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then  $w(x_{i1}) = w(x_{i2})$ ,  $w(x_{i+1}) = w(x_{i+2})$ , then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1 \pmod{4}, \quad i \equiv 0 \pmod{2}, \quad l(x_i) = 2, \quad i \equiv 3 \pmod{4},$$

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 1, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = 2,$$

then  $w(x_{i+1}) \neq w(x_{i+2})$ ,  $w(x_{ij+1}) \neq w(x_{ij+2})$ . Therefore we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{4} \text{ or for } i \equiv 0 \pmod{2}, \\ 2, & \text{for } i \equiv 3 \pmod{4}, \end{cases}$$

$$w(x_{ij}) = 1; \quad l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j = 2. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, m, \\ 7, & \text{for } i \equiv 1, 3 \pmod{4}, i \neq 1, \\ 8, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \end{cases}$$



$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod{4} \text{ and } j = 1 \text{ or for } i \equiv 0 \pmod{2} \text{ and } j = 1, \\ 4, & \text{for } i \equiv 1 \pmod{4} \text{ and } j = 2 \text{ or for } i \equiv 0 \pmod{2} \text{ and } j = 2 \text{ or} \\ & \text{for } i \equiv 3 \pmod{4} \text{ and } j = 1, \\ 5, & \text{for } i \equiv 3 \pmod{4} \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod{4} \text{ and } j = 1, 2 \text{ or for } i \equiv 0 \pmod{2} \text{ and } j = 1, 2, \\ 3, & \text{for } i \equiv 3 \pmod{4} \text{ and } j = 1, 2. \end{cases}$$

The upper bound  $\chi_{lis}(P_m \odot Cp_n) \leq 7$  is true. So  $\chi_{lis}(P_m \odot Cp_n) = 7$  for  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n = 2$ .

**Case 10:**  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n = 3$ .

First step here is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) < 7$ , let  $\chi_{lis}(P_m \odot Cp_n) = 6$ , if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{i1}) = 1, \quad l(y_{ij}) = 2, \quad j = 2, 3,$$

then  $w(x_{i+1}) = w(x_{i+2})$ , then we have that there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1 \pmod{4}, \quad i \equiv 0 \pmod{2}, \quad l(x_i) = 2, \quad i \equiv 3 \pmod{4}, \quad l(x_{ij}) = 1; l(y_{ij}) = 1,$$

then  $w(x_{i+1}) \neq w(x_{i+2})$ ,  $w(x_{ij+1}) \neq w(x_{ij+2})$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{4} \text{ or for } i \equiv 0 \pmod{2}, \\ 2, & \text{for } i \equiv 3 \pmod{4}, \end{cases}$$

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1.$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, m, \\ 8, & \text{for } i \equiv 1, 3 \pmod{4}, \quad i \neq 1, m, \\ 9, & \text{for } i \equiv 0 \pmod{2}, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod{4} \text{ and } j = 1, 3 \text{ or for } i \equiv 0 \pmod{2} \text{ and } j = 1, 3, \\ 4, & \text{for } i \equiv 1 \pmod{4} \text{ and } j = 2 \text{ or for } i \equiv 0 \pmod{2} \text{ and } j = 2 \text{ or} \\ & \text{for } i \equiv 3 \pmod{4} \text{ and } j = 1, 3, \\ 5, & \text{for } i \equiv 3 \pmod{4} \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod{4} \text{ and } 1 \leq j \leq 3 \text{ or for } i \equiv 0 \pmod{2} \text{ and } 1 \leq j \leq 3, \\ 3, & \text{for } i \equiv 3 \pmod{4} \text{ and } 1 \leq j \leq 3. \end{cases}$$

The upper bound is true:  $\chi_{lis}(P_m \odot Cp_n) \leq 7$ . So  $\chi_{lis}(P_m \odot Cp_n) = 7$  for  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n = 3$ .

**Case 11:**  $m \equiv 0 \pmod{2}$   $m \geq 4$  and  $n \geq 4$ .

First step to prove this theorem is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) < 8$ , let  $\chi_{lis}(P_m \odot Cp_n) = 7$ , if

$$l(x_i) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad i \equiv 2 \pmod{4}, \quad l(x_i) = 2, \quad i \equiv 0 \pmod{4}, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then  $w(x_{ij+1}) = w(x_{ij+2})$ , so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$\begin{aligned} l(x_i) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad i \equiv 2 \pmod{4}, \quad l(x_i) = 2, \quad i \equiv 0 \pmod{4}, \quad l(x_{ij}) = 1, \\ l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 1, n, \quad j \equiv 0 \pmod{2}, \quad l(y_{ij}) = 2, \\ 1 \leq i \leq m, \quad j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n, \end{aligned}$$

then  $w(x_{i+1}) \neq w(x_{i+2})$ ,  $w(x_{ij+1}) \neq w(x_{ij+2})$ ,  $w(x_{ij}) \neq w(y_{ij})$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 8$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$\begin{aligned} l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 2 \pmod{4} \text{ and } 1 \leq j \leq n, \\ 2, & \text{for } i \equiv 0 \pmod{4} \text{ and } 1 \leq j \leq n, \end{cases} \\ l(x_{ij}) = 1, \\ l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, n \text{ or for } 1 \leq i \leq m \text{ and } j \equiv 0 \pmod{2}, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, m. \end{cases} \end{aligned}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$\begin{aligned} w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1, m \text{ and } n \equiv 0 \pmod{2}, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, m \text{ and } n \equiv 1, 3 \pmod{4}, \\ 3n + 1 - n/2, & \text{for } i \equiv 1, 3 \pmod{4}, \quad i \neq 1 \text{ and } n \equiv 0 \pmod{2}, \\ 3n + 1 - \lfloor n/2 \rfloor, & \text{for } i \equiv 0 \pmod{2}, \quad i \neq m \text{ and } n \equiv 1, 3 \pmod{4}, \\ 3n + 2 - n/2, & \text{for } i \equiv 1, 3 \pmod{4}, \quad i \neq 1 \text{ and } n \equiv 0 \pmod{2}, \\ 3n + 1 - \lfloor n/2 \rfloor, & \text{for } i \equiv 0 \pmod{2}, \quad i \neq m \text{ and } n \equiv 1, 3 \pmod{4}, \end{cases} \\ w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 1, n \text{ or for } i \equiv 2 \pmod{4} \text{ and } j = 1, n, \\ 4, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j \equiv 0 \pmod{2}, \quad j \neq n \text{ or} \\ & \text{for } i \equiv 2 \pmod{4} \text{ and } j \equiv 0 \pmod{2}, \quad j \neq n \text{ or for } i \equiv 0 \pmod{4} \text{ and } j = 1, n, \\ 5, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n \text{ or} \\ & \text{for } i \equiv 2 \pmod{4} \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n \text{ or} \\ & \text{for } i \equiv 0 \pmod{4} \text{ and } j \equiv 0 \pmod{2}, \quad j \neq n, \\ 6, & \text{for } i \equiv 0 \pmod{4} \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n, \end{cases} \\ w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 2 \pmod{4} \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i \equiv 0 \pmod{4} \text{ and } 1 \leq j \leq n. \end{cases} \end{aligned}$$

The upper bound is true:  $\chi_{lis}(P_m \odot Cp_n) \leq 8$ . So  $\chi_{lis}(P_m \odot Cp_n) = 8$  for  $m \equiv 0 \pmod{4}$ ,  $m \geq 4$  and  $n \geq 4$ .

**Case 12:**  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n \geq 4$ .

First step to prove this theorem in this case is to find the lower bound of  $V(P_m \odot Cp_n)$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$ .

Assume  $\chi_{lis}(P_m \odot Cp_n) < 8$ , let  $\chi_{lis}(P_m \odot Cp_n) = 7$ , if

$$l(x_i) = 1, \quad i \equiv 1 \pmod{4}, \quad i \equiv 0 \pmod{2}, \quad l(x_i) = 2, \quad i \equiv 3 \pmod{4}, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then  $w(x_{ij+1}) = w(x_{ij+2})$ , so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$\begin{aligned} l(x_i) = 1, \quad i \equiv 1 \pmod{4}, \quad i \equiv 0 \pmod{2}, \quad l(x_i) = 2, \quad i \equiv 3 \pmod{4}, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \\ 1 \leq i \leq m, \quad j = 1, n, \quad j \equiv 0 \pmod{2}, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j \equiv 1, 3 \pmod{4}, \\ j \neq 1, n \rightarrow w(x_{i+1}) \neq w(x_{i+2}), \quad w(x_{ij+1}) \neq w(x_{ij+2}), \quad w(x_{ij}) \neq w(y_{ij}), \end{aligned}$$

therefore we have the lower bound  $\chi_{lis}(P_m \odot Cp_n) \geq 8$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot Cp_n)$ .

Furthermore, we define  $l : V(P_m \odot Cp_n) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$\begin{aligned} l(x_i) &= \begin{cases} 1, & \text{for } i \equiv 1 \pmod{4} \text{ or for } i \equiv 0 \pmod{2}, \\ 2, & \text{for } i \equiv 3 \pmod{4}, \end{cases} \\ l(x_{ij}) &= 1, \\ l(y_{ij}) &= \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, n \text{ or for } 1 \leq i \leq m \text{ and } j \equiv 0 \pmod{2}, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n. \end{cases} \end{aligned}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$\begin{aligned} w(x_i) &= \begin{cases} 2n + n/2, & \text{for } i = 1, m \text{ and } n \equiv 0 \pmod{2}, \\ 3n + 2 - n/2, & \text{for } i = 0 \pmod{2} \text{ and } n \equiv 0 \pmod{2}, \\ 3n + 1 - n/2, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } n \equiv 0 \pmod{2}, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, m \text{ and } n \equiv 1, 3 \pmod{4}, \\ 3n - \lfloor n/2 \rfloor, & \text{for } i \equiv 1, 3 \pmod{4}, \quad i \neq 1 \text{ and } n \equiv 1, 3 \pmod{4}, \\ 3n + 1 - \lfloor n/2 \rfloor, & \text{for } i = 0 \pmod{4} \text{ and } n \equiv 1, 3 \pmod{4}, \end{cases} \\ w(x_{ij}) &= \begin{cases} 3, & \text{for } i \equiv 1 \pmod{4} \text{ and } j = 1, n \text{ or for } i \equiv 0 \pmod{2} \text{ and } j = 1, n, \\ 4, & \text{for } i \equiv 1 \pmod{4} \text{ and } j \equiv 0 \pmod{2}, \quad j \neq n \text{ or} \\ & \text{for } i \equiv 0 \pmod{2} \text{ and } j \equiv 0 \pmod{2}, \quad j \neq n \text{ or} \\ & \text{for } i \equiv 3 \pmod{4} \text{ and } j = 1, n, \\ 5, & \text{for } i \equiv 1 \pmod{4} \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n \text{ or} \\ & \text{for } i \equiv 0 \pmod{2} \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n \text{ or} \\ & \text{for } i \equiv 3 \pmod{4} \text{ and } j \equiv 0 \pmod{2}, \quad j \neq n, \\ 6, & \text{for } i \equiv 3 \pmod{4} \text{ and } j \equiv 1, 3 \pmod{4}, \quad j \neq 1, n, \end{cases} \\ w(y_{ij}) &= \begin{cases} 2, & \text{for } i \equiv 1 \pmod{4} \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 0 \pmod{2} \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i \equiv 3 \pmod{4} \text{ and } 1 \leq j \leq n. \end{cases} \end{aligned}$$

The upper bound is true:  $\chi_{lis}(P_m \odot Cp_n) \leq 8$ . So  $\chi_{lis}(P_m \odot Cp_n) = 8$  for  $m \geq 5$  and  $n \geq 4$ .  $\square$

**Theorem 2.** *Let  $G = P_m \odot S_{2,n}$  for  $n, m \geq 2$ , then the chromatic number of local irregular  $G$  is*

$$\chi_{lis}(P_m \odot S_{2,n}) = \begin{cases} 5, & \text{for } m = 3 \text{ and } n \geq 2, \\ 6, & \text{for } m = 2 \text{ and } n \geq 2, \\ 7, & \text{for } m \geq 4 \text{ and } n \geq 2, \end{cases}$$

with  $\text{opt}(l)(P_m \odot S_{2,n}) = 1, 2$ , for  $m \geq 2$  and  $n \geq 2$ .

**P r o o f.** Vertex set is

$$V(P_m \odot S_{2,n}) = \{x_i; 1 \leq i \leq m\} \cup \{a_i; 1 \leq i \leq m\} \cup \{b_i; 1 \leq i \leq m\} \\ \cup \{a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

and the edge set is

$$E(P_m \odot S_{2,n}) = \{x_i x_{i+1}, 1 \leq i \leq m-1\} \cup \{a_i b_i; 1 \leq i \leq m\} \cup \{x_i a_i; 1 \leq i \leq m\} \\ \cup \{x_i b_i; 1 \leq i \leq m\} \cup \{x_i a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \\ \cup \{a_i a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{b_i b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The order and the size respectively are  $2mn + 3m$  and  $4mn + 4m - 1$ . This proof is divided into 4 cases as follows.

**Case 1:**  $m = 3$  and  $n \geq 2$ .

First step to prove this theorem is to find the lower bound of  $V(P_m \odot S_{2,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{2,n}) \geq \chi(P_m \odot S_{2,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{2,n}) = 4$ , if  $l(a_i) = l(b_i) = 1$ ,  $l(x_i) = l(a_{ij}) = l(b_{ij}) = 1$  then  $w(a_i) = w(b_i)$ , then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad 1 \leq j \leq n-1, \quad l(b_{in}) = 2,$$

then

$$w(a_i) \neq w(b_i), \quad w(x_1) = w(x_3) \neq w(x_2),$$

therefore we have the lower bound  $\chi_{lis}(P_m \odot S_{2,n}) \geq 5$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{2,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{2,n}) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1, \\ l(b_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } 1 \leq i \leq 3 \text{ and } j = n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + 4, & \text{for } i = 1, 3, \\ 2n + 5, & \text{for } i = 2, \end{cases} \\ w(a_i) = n + 2, \quad \text{for } 1 \leq i \leq 3, \\ w(b_i) = n + 3, \quad \text{for } 1 \leq i \leq 3, \\ w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound  $\chi_{lis}(P_m \odot S_{2,n}) \leq 5$  is true. So  $\chi_{lis}(P_m \odot S_{2,n}) = 5$  for  $m = 3$  and  $n \geq 2$ .

**Case 2:**  $m = 2$  and  $n \geq 2$ .

First step here is to find the lower bound of  $V(P_m \odot S_{2,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{2,n}) \geq \chi(P_m \odot S_{2,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{2,n}) = 5$ , if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = l(b_{2j}) = 1, \quad l(b_{1j}) = 1, \quad 1 \leq j \leq n-1, \quad l(b_{1n}) = 2,$$

and then  $w(a_2) = w(b_2)$ , and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \\ l(b_{1,j}) = 1, \quad l(b_{1,n}) = 2, \quad l(b_{2,j}) = 2, \quad j = 1, n, \quad l(b_{2j}) = 1, \quad 2 \leq j \leq n-1,$$

then  $w(a_i) \neq w(b_i)$ ,  $w(x_1) \neq w(x_2)$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot S_{2,n}) \geq 6$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{2,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{2,n}) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1, \\ l(b_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } 1 \leq j \leq n-1 \text{ or for } i = 2 \text{ and } 2 \leq j \leq n-1, \\ 2, & \text{for } i = 1 \text{ and } j = n \text{ or for } i = 2 \text{ and } j = 1, n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + 4, & \text{for } i = 1, \\ 2n + 5, & \text{for } i = 2, \end{cases} \\ w(a_i) = n + 2, \quad \text{for } i = 1, 2, \\ w(b_i) = \begin{cases} n + 3, & \text{for } i = 1, \\ n + 4, & \text{for } i = 2, \end{cases} \\ w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{2,n}) \leq 6$ . So  $\chi_{lis}(P_m \odot S_{2,n}) = 6$  for  $m = 2$  and  $n \geq 2$ .

**Case 3:**  $m \equiv 0 \pmod{4}$ ,  $m \geq 4$  and  $n \geq 2$ .

First step to prove this theorem in this case is to find the lower bound of  $V(P_m \odot S_{2,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{2,n}) \geq \chi(P_m \odot S_{2,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{2,n}) = 6$ , if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad i \equiv 2 \pmod{4}, \\ l(b_{ij}) = 1, \quad i \equiv 0 \pmod{4}, \quad j \neq 1, n, \quad l(b_{ij}) = 2, \quad i \equiv 0 \pmod{4}, \quad j = 1, n,$$

then  $w(a_i) = w(b_i)$ , so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad i = m, \quad 1 \leq j \leq n-1, \\ i \equiv 0 \pmod{2}, \quad i \neq m, \quad 2 \leq j \leq n-1, \quad l(b_{ij}) = 2, \quad i \equiv 1, 3 \pmod{4}, \\ i = m, \quad j = n, \quad i \equiv 0 \pmod{2}, \quad i \neq m, \quad j = 1, n,$$

then  $w(a_i) \neq w(b_i)$ ,  $w(x_i) \neq w(x_{i+1})$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot S_{2,n}) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{2,n})$

Furthermore, we define  $l : V(P_m \odot S_{2,n}) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$

$$l(b_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } 1 \leq j \leq n-1 \text{ or} \\ & \text{for } i \equiv 0 \pmod{2}, i \neq m \text{ and } 2 \leq j \leq n-1 \text{ or} \\ & \text{for } i = m, \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = n \text{ or} \\ & \text{for } i \equiv 0 \pmod{2}, i \neq m \text{ and } j = 1, n \text{ or} \\ & \text{for } i = m, \text{ and } j = n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, m, \\ 2n+5, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \\ 2n+6, & \text{for } i \equiv 1, 3 \pmod{4}, i \neq 1, \end{cases}$$

$$w(a_i) = n+2, \quad \text{for } 1 \leq i \leq m,$$

$$w(b_i) = \begin{cases} n+3, & \text{for } i \equiv 1, 3 \pmod{4}, i = m, \\ n+4, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{2,n}) \leq 7$ . So  $\chi_{lis}(P_m \odot S_{2,n}) = 7$  for  $m \equiv 0 \pmod{2}$ ,  $m \geq 4$  and  $n \geq 2$ .

**Case 4:**  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n \geq 2$ .

First step here is to find the lower bound of  $V(P_m \odot S_{2,n})$ . Based on Lemma 1, we have

$$\chi_{lis}(P_m \odot S_{2,n}) \geq \chi(P_m \odot S_{2,n}) = 3.$$

Assume  $\chi_{lis}(P_m \odot S_{2,n}) = 6$ , if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1 \pmod{4}, \quad i \equiv 0 \pmod{2},$$

$$l(b_{ij}) = 1, \quad i \equiv 3 \pmod{4}, \quad j \neq n, \quad l(b_{ij}) = 2, \quad i \equiv 3 \pmod{4}, \quad j = n,$$

then  $w(a_i) = w(b_i)$ , and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad 1 \leq j \leq n-1,$$

$$i \equiv 0 \pmod{2}, \quad 2 \leq j \leq n-1, \quad l(b_{ij}) = 2, \quad i \equiv 1, 3 \pmod{4}, \quad j = n, \quad i \equiv 0 \pmod{2}, \quad j = 1, n,$$

then  $w(a_i) \neq w(b_i)$ ,  $w(x_i) \neq w(x_{i+1})$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot S_{2,n}) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{2,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{2,n}) \rightarrow \{1, 2\}$  with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$

$$l(b_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } 1 \leq j \leq n-1 \text{ or for } i \equiv 0 \pmod{2}, \text{ and } 2 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = n \text{ or for } i \equiv 0 \pmod{2} \text{ and } j = 1, n. \end{cases}$$



Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + 4, & \text{for } i = 1, m, \\ 2n + 5, & \text{for } i \equiv 1, 3 \pmod{4}, \\ 2n + 6, & \text{for } i \equiv 0 \pmod{2}, \end{cases}$$

$$w(a_i) = n + 2, \quad \text{for } 1 \leq i \leq m,$$

$$w(b_i) = \begin{cases} n + 3, & \text{for } i \equiv 1, 3 \pmod{4}, \\ n + 4, & \text{for } i \equiv 0 \pmod{2}, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{2,n}) \leq 7$ . So  $\chi_{lis}(P_m \odot S_{2,n}) = 7$  for  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n \geq 2$ .  $\square$

**Theorem 3.** *Let  $G = P_m \odot S_{3,n}$  for  $n, m \geq 2$ , then the chromatic number of local irregular  $G$  is*

$$\chi_{lis}(P_m \odot S_{3,n}) = \begin{cases} 5, & \text{for } m = 3 \text{ and } n \geq 2, \\ 6, & \text{for } m = 2 \text{ and } n \geq 2, \\ 7, & \text{for } m \geq 4 \text{ and } n \geq 3, \end{cases}$$

with

$$\text{opt}(l)(P_m \odot S_{3,n}) = \begin{cases} 1, & \text{for } m = 3 \text{ and } n = 3, \\ 1, 2, & \text{for } m = 2 \text{ and } n = 2 \text{ or for } m = 3 \text{ and } n = 2 \text{ or} \\ & \text{for } m \geq 4 \text{ and } n \geq 2. \end{cases}$$

**P r o o f.** The vertex set is

$$V(P_m \odot S_{3,n}) = \{x_i; 1 \leq i \leq m\} \cup \{a_i; 1 \leq i \leq m\} \cup \{b_i; 1 \leq i \leq m\} \cup \{c_i; 1 \leq i \leq m\} \\ \cup \{a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{c_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

and the edge set is

$$V(P_m \odot S_{3,n}) = \{x_i x_{i+1}; 1 \leq i \leq m - 1\} \cup \{x_i y_i; 1 \leq i \leq m\} \cup \{x_i a_i; 1 \leq i \leq m\} \\ \cup \{x_i b_i; 1 \leq i \leq m\} \cup \{x_i c_i; 1 \leq i \leq m\} \cup \{y_i a_i; 1 \leq i \leq m\} \cup \{y_i b_i; 1 \leq i \leq m\} \\ \cup \{y_i c_i; 1 \leq i \leq m\} \cup \{x_i a_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{x_i b_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \\ \cup \{x_i c_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{a_i a_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \\ \cup \{b_i b_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{c_i c_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\}.$$

The order and size respectively are  $3mn + 5m$  and  $6mn + 8n - 1$ . This proof can be divided into 8 following cases.

**Case 1:**  $m = 3$  and  $n = 2$ .

First step to prove this theorem is to find the lower bound of  $V(P_m \odot S_{3,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{3,n}) = 4$ , if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1,$$

then  $w(a_i) = w(b_i) = w(c_i) = w(y_i)$ , and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(a_i) = l(b_i) = l(c_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(y_i) = 2,$$

then  $(w(a_i) = w(b_i) = w(c_i)) \neq w(y_i)$ ,  $w(x_1) \neq w(x_2)$ . Therefore we have the lower bound  $\chi_{lis}(P_m \odot S_{3,n}) \geq 5$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{3,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{3,n}) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad l(c_{ij}) = 1.$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, 3, \\ 13, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5, \quad w(c_i) = 5, \quad w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{3,n}) \leq 5$ . So  $\chi_{lis}(P_m \odot S_{3,n}) = 5$  for  $m = 3$  and  $n = 2$ .

**Case 2:**  $m = 3$  and  $n = 3$ .

Based on Proposition 1, we have  $\text{opt}(l) = 1$ . So the lower bound  $(P_m \odot S_{3,n})$  is  $\chi_{lis}(P_m \odot S_{3,n}) \geq 5$

Since  $\text{opt}(l) = 1$ , the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5, & \text{for } i = 1, 3, \\ 3n + 6, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4,$$

$$w(a_i) = n + 1 \quad \text{for } 1 \leq i \leq 3,$$

$$w(b_i) = n + 1 \quad \text{for } 1 \leq i \leq 3,$$

$$w(c_i) = n + 1 \quad \text{for } 1 \leq i \leq 3,$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{3,n}) \leq 5$ . So  $\chi_{lis}(P_m \odot S_{3,n}) = 5$  for  $m = 3$  and  $n \geq 2$ .

**Case 3:**  $m = 2$  and  $n = 2$ .

First step to prove this theorem is to find the lower bound of  $V(P_m \odot S_{3,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{3,n}) = 5$ , if

$$l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(y_1) = 1, \quad l(y_2) = 2,$$

then  $w(a_2) = w(b_2) = w(c_2) = w(y_2)$  and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(c_i) = l(b_{ij}) = 1, \quad l(y_i) = 2, \quad l(c_{1j}) = 1, \quad l(c_{2,1}) = 1, \quad l(c_{2,2}) = 2,$$

then  $w(x_1) \neq w(x_2)$ ,  $w(y_i) \neq ((w(a_i) = w(b_i) = w(c_i)))$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot S_{3,n}) \geq 6$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{3,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{3,n}) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } j = 1, 2 \text{ or for } i = 2 \text{ and } j = 1, \\ 2, & \text{for } i = 2 \text{ and } j = 2. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, \\ 13, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5,$$

$$w(c_i) = \begin{cases} 5, & \text{for } i = 1, \\ 6, & \text{for } i = 2, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{2,n}) \leq 6$ . So  $\chi_{lis}(P_m \odot S_{2,n}) = 6$  for  $m = 2$  and  $n = 2$ .

**Case 4:**  $m = 2$  and  $n \geq 3$ .

First step here is to find the lower bound of  $V(P_m \odot S_{3,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{3,n}) = 5$ , if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i = 1, 2, \quad 1 \leq j \leq n - 1$$

$$l(c_{ij}) = 2, \quad i = 1, 2, \quad j = n,$$

then  $w(x_1) = w(x_2)$ , then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i = 1, \quad 1 \leq j \leq n, \quad i = 2, \quad 1 \leq j \leq n - 1,$$

$$l(c_{ij}) = 2, \quad i = 2, \quad j = n,$$

then  $w(x_1) \neq w(x_2), w(y_i) \neq ((w(a_i) = w(b_i) = w(c_i)))$ . Therefore we have the lower bound  $\chi_{lis}(P_m \odot S_{3,n}) \geq 6$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{3,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{3,n}) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } 1 \leq j \leq n \text{ or for } i = 2 \text{ and } 1 \leq j \leq n - 1, \\ 2, & \text{for } i = 2 \text{ and } j = n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5, & \text{for } i = 1, \\ 3n + 6, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4,$$

$$w(a_i) = n + 1, \quad \text{for } i = 1, 2,$$

$$w(b_i) = n + 1, \quad \text{for } i = 1, 2,$$

$$w(c_i) = \begin{cases} n + 1, & \text{for } i = 1, \\ n + 2, & \text{for } i = 2, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{3,n}) \leq 6$ . So  $\chi_{lis}(P_m \odot S_{3,n}) = 6$  for  $m = 2$  and  $n \geq 3$ .

**Case 5:**  $m \equiv 0 \pmod{2}$   $m \geq 4$  and  $n = 2$ .

First step to prove this theorem in this case is to find the lower bound of  $V(P_m \odot S_{3,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{3,n}) = 6$ , if

$$\begin{aligned} l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod{4}, \\ j = 1, 2, \quad i \equiv 0 \pmod{4}, \quad j = 1, 2, \quad l(c_{ij}) = 1, \quad i \equiv 2 \pmod{4}, \\ j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 2 \pmod{4}, \quad j = 2, \end{aligned}$$

then  $w(y_i) = w(a_i)$ . Then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$\begin{aligned} l(x_i) = 1, \quad l(a_i) = l(b_i) = l(c_i) = 1, \quad l(y_i) = 2, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \\ i \equiv 0 \pmod{2}, \quad j = 1, \quad i \neq m, \quad i \equiv 1, 3 \pmod{4}, \quad j = 1, 2, \quad l(c_{ij}) = 2, \\ i \equiv 0 \pmod{2}, \quad i \neq m, \quad j = 2, \end{aligned}$$

then  $w(x_{i+1}) \neq w(x_{i+2}); w(y_i) \neq w(a_i)$ . Therefore we have the lower bound  $\chi_{lis}(P_m \odot S_{3,n}) \geq 7$ . After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{3,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{3,n}) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows:

$$\begin{aligned} l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \\ l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 1, 2 \text{ or for } i = m \text{ and } j = 1, 2 \text{ or} \\ & \text{for } i \equiv 0 \pmod{2}, i \neq m \text{ and } j = 1, \\ 2, & \text{for } i \equiv 0 \pmod{2}, i \neq m \text{ and } j = 2. \end{cases} \end{aligned}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$\begin{aligned} w(x_i) = \begin{cases} 12, & \text{for } i = 1, m, \\ 13, & \text{for } i \equiv 1, 3 \pmod{4}, i \neq 1, \\ 14, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \end{cases} \\ w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5, \\ w(c_i) = \begin{cases} 5, & \text{for } i \equiv 1, 3 \pmod{4}, \\ 6, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \end{cases} \\ w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2, \end{aligned}$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{3,n}) \leq 7$ . So  $\chi_{lis}(P_m \odot S_{3,n}) = 7$  for  $m \equiv 0 \pmod{2}; m \geq 4$  and  $n = 2$ .

**Case 6:**  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n = 2$ .

First step here is to find the lower bound of  $V(P_m \odot S_{3,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{3,n}) = 6$ , if

$$\begin{aligned} l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \\ i \equiv 1 \pmod{4}, \quad j = 1, 2, \quad i \equiv 0 \pmod{2}, \quad j = 1, 2, \quad l(c_{ij}) = 1, \\ i \equiv 3 \pmod{4}, \quad j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 3 \pmod{4}, \quad j = 2, \end{aligned}$$

then  $w(y_i) = w(a_i)$ , then there are 2 adjacent vertices that have same color, it contradicts to definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod{4}, \\ j = 1, 2, \quad i \equiv 0 \pmod{2}, \quad j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod{2}, \quad j = 2, \quad l(y_i) = 2,$$

then

$$w(x_{i+1}) \neq w(x_{i+2}), \quad w(y_i) \neq w(a_i), \quad w(y_i) \neq w(b_i), \quad w(y_i) \neq w(c_i).$$

We have the lower bound  $\chi_{lis}(P_m \odot S_{3,n}) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{3,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{3,n}) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \\ l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } j = 1, 2 \text{ or} \\ & \text{for } i \equiv 0 \pmod{2}, i \neq m \text{ and } j = 1, \\ 2, & \text{for } i \equiv 0 \pmod{2}, i \neq m \text{ and } j = 2. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, m \\ 13, & \text{for } i \equiv 1, 3 \pmod{4}, i \neq 1, m, \\ 14, & \text{for } i \equiv 0 \pmod{2}, \end{cases} \\ w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5, \\ w(c_i) = \begin{cases} 5, & \text{for } i \equiv 1, 3 \pmod{4}, i = 1, m, \\ 6, & \text{for } i \equiv 0 \pmod{2}, \end{cases} \\ w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{3,n}) \leq 7$ . So  $\chi_{lis}(P_m \odot S_{3,n}) = 7$  for  $m \equiv 1, 3 \pmod{4}$ ;  $m \geq 5$  and  $n = 2$ .

**Case 7:**  $m \equiv 0 \pmod{2}$   $m \geq 4$  and  $n \geq 3$ .

First step to prove this theorem is to find the lower bound of  $V(P_m \odot S_{3,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{3,n}) = 6$ , it is true if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad 1 \leq i \leq m, \\ 1 \leq j \leq n - 1, \quad l(c_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = n,$$

then  $w(x_{i+1}) = w(x_{i+2})$ , then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(y_i) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod{4}, \quad 1 \leq j \leq n, \\ i \equiv 0 \pmod{2}, \quad i \neq m, \quad 1 \leq j \leq n - 1, \quad i = m, \quad 1 \leq j \leq n, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod{2}, \\ i \neq m, \quad i \neq m, \quad j = n, \quad w(x_{i+1}) \neq w(x_{i+2}),$$

we have the lower bound of  $\chi_{lis}(P_m \odot S_{3,n}) \geq 7$ . After that, we will find the upper bound  $\chi_{lis}(P_m \odot S_{3,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{3,n}) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4} \text{ and } 1 \leq j \leq n \text{ or} \\ & \text{for } i \equiv 0 \pmod{2}, i \neq m, \text{ and } 1 \leq j \leq n - 1 \text{ or} \\ & \text{for } i = m, \text{ and } 1 \leq j \leq n, \\ 2, & \text{for } i \equiv 0 \pmod{2}, i \neq m \text{ and } j = n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5, & \text{for } i = 1, m, \\ 3n + 6, & \text{for } i \equiv 1, 3 \pmod{4}, i \neq 1, \\ 3n + 7, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \end{cases}$$

$$w(y_i) = 4,$$

$$w(a_i) = n + 2, \quad \text{for } 1 \leq i \leq m,$$

$$w(b_i) = n + 2, \quad \text{for } 1 \leq i \leq m,$$

$$w(c_i) = \begin{cases} n + 2, & \text{for } i = m, \text{ or for } i \equiv 1, 3 \pmod{4}, \\ n + 3, & \text{for } i \equiv 0 \pmod{2}, i \neq m, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2,$$

The upper bound  $\chi_{lis}(P_m \odot S_{3,n}) \leq 7$ . So  $\chi_{lis}(P_m \odot S_{3,n}) = 7$  for  $m \equiv 0 \pmod{2}$ ;  $m \geq 4$  and  $n \geq 3$ .

**Case 8:**  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n \geq 3$ .

First step to prove the theorem in this case is to find the lower bound of  $V(P_m \odot S_{3,n})$ . Based on Lemma 1, we have  $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$ .

Assume  $\chi_{lis}(P_m \odot S_{3,n}) = 6$ , if

$$l(x_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(a_i) = l(b_i) = 1, \quad l(c_i) = 2,$$

then  $w(x_{i+1}) = w(x_{i+2})$ . Then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(c_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(y_i) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod{4},$$

$$1 \leq j \leq n, \quad i \equiv 0 \pmod{2}, \quad 1 \leq j \leq n - 1, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod{2}, \quad j = n,$$

then  $w(x_{i+1}) \neq w(x_{i+2})$ . Based on that we have the lower bound  $\chi_{lis}(P_m \odot S_{3,n}) \geq 7$ .

After that, we will find the upper bound of  $\chi_{lis}(P_m \odot S_{3,n})$ .

Furthermore, we define  $l : V(P_m \odot S_{3,n}) \rightarrow \{1, 2\}$  with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod{4}, \text{ and } 1 \leq j \leq n \text{ or} \\ & \text{for } i \equiv 0 \pmod{2}, \text{ and } 1 \leq j \leq n - 1, \\ 2, & \text{for } i \equiv 0 \pmod{2}, \text{ and } j = n. \end{cases}$$

Hence,  $\text{opt}(l) = 2$  and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5, & \text{for } i = 1, m, \\ 3n + 6, & \text{for } i \equiv 1, 3 \pmod{4}, i \neq 1, m, \\ 3n + 7, & \text{for } i \equiv 0 \pmod{2}, \end{cases}$$



$$\begin{aligned}
 w(y_i) &= 4, \\
 w(a_i) &= n + 2, \text{ for } 1 \leq i \leq m, \\
 w(b_i) &= n + 2, \text{ for } 1 \leq i \leq m, \\
 w(c_i) &= \begin{cases} n + 2, & \text{for } i \equiv 1, 3 \pmod{4}, \\ n + 3, & \text{for } i \equiv 0 \pmod{2}, \end{cases} \\
 w(a_{ij}) &= 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.
 \end{aligned}$$

The upper bound is true:  $\chi_{lis}(P_m \odot S_{3,n}) \leq 7$ . So  $\chi_{lis}(P_m \odot S_{3,n}) = 7$  for  $m \equiv 1, 3 \pmod{4}$ ,  $m \geq 5$  and  $n \geq 3$ .  $\square$

### 3. Conclusion

In this paper, we have studied the coloring of the vertices of the local irregular corona product by the graph of the family tree. We determined the exact value of the local irregular chromatic number of the corona product from the graph of the family tree, namely  $\chi_{lis}(P_m \odot Cp_n)$ ,  $\chi_{lis}(P_m \odot S_{2,n})$  and  $\chi_{lis}(P_m \odot S_{3,n})$ .

### Acknowledgements

We gratefully acknowledge the support from University of Jember of year 2023.

### REFERENCES

1. Frucht R., Harary F. On the corona of two graphs. *Aequationes Math.*, 1970. Vol. 4. P. 322–325. DOI: [10.1007/BF01844162](https://doi.org/10.1007/BF01844162)
2. Kristiana A. I., Dafik, Utoyo M. I., Slamir, Alfarisi R., Agustin I. H., Venkatachalam M. Local irregularity vertex coloring of graphs. *Int. J. Civil Eng. Technol.*, 2019. Vol. 10, No. 3. P. 1606–1616.
3. Kristiana A. I., Utoyo M. I., Dafik, Agustin I. H., Alfarisi R., Waluyo E. On the chromatic number local irregularity of related wheel graph. *J. Phys.: Conf. Ser.*, 2019. Vol. 1211. Art. no. 0120003. P. 1–10. DOI: [10.1088/1742-6596/1211/1/012003](https://doi.org/10.1088/1742-6596/1211/1/012003)
4. Kristiana A. I., Alfarisi R., Dafik, Azahra N. Local irregular vertex coloring of some families of graph. *J. Discrete Math. Sci. Cryptogr.*, 2020. P. 15–30. DOI: [10.1080/09720529.2020.1754541](https://doi.org/10.1080/09720529.2020.1754541)