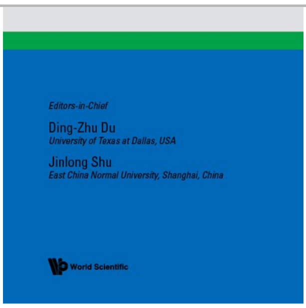
DISCRETE MATHEMATICS, ALGORITHMS AND APPLICATIONS

Editors-in-Chief

Ding-Zhu Du
University of Texas at Dallas, USA

Jinlong Shu
East China Normal University, Shanghai, China

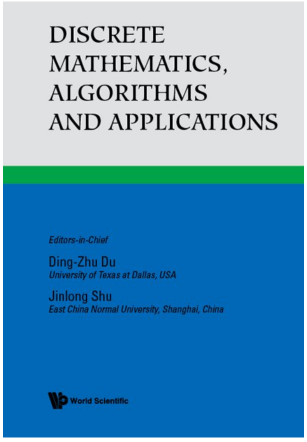
World Scientific



Tools Share

Submit an article Subscribe

Editorial Board



Discrete Mathematics, Algorithms and Applications

ISSN (print): 1793-8309 | ISSN (online): 1793-8317

ONLINE READY

Articles in this section are in the final version. Please cite the article using the **DOI number**.

No Access

Edge-vertex domination in trees

Kijung Kim

2250043

https://doi.org/10.1142/S1793830922500434

Abstract | **PDF/EPUB**

Preview Abstract

No Access

https://doi.org/10.1142/S1793830922500264

Abstract | **PDF/EPUB**

Preview Abstract

No Access

On the local irregularity vertex coloring of volcano, broom, parachute, double broom and complete multipartite graphs

Arika Indah Kristiana, Nafidatun Nikmah, Dafik, Ridho Alfarisi, M. Ali Hasan and Slamin

2250022

https://doi.org/10.1142/S1793830922500227

Abstract | **PDF/EPUB**

Preview Abstract

No Access

Two-ended quasi-transitive graphs

Babak Miraftab and Tim Rühmann

2250023

https://doi.org/10.1142/S1793830922500239

Abstract | **PDF/EPUB**

Preview Abstract

No Access

Edge (m, k)-choosability of graphs

P. Soorya and K. A. Germina

2250025

https://doi.org/10.1142/S1793830922500252

Abstract | **PDF/EPUB**

Preview Abstract

Discrete Mathematics, Algorithms and Applications
(2022) 2250022 (6 pages)

DOI: 10.1142/S1793830922500227

On the local irregularity vertex coloring of volcano, broom, parachute, double broom and complete multipartite graphs

Arika Indah Kristiana*,¶, Nafidatun Nikmah*,‖, Dafik*,**,
Ridho Alfarisi†,††, M. Ali Hasan‡,‡‡ and Slamin§§

*Department of Mathematics Education
University of Jember, Jalan Kalimantan
No. 37 Kampus Tegal Boto, Jember 68121, Indonesia

†Department of Primary School
University of Jember, Jalan Kalimantan
No. 37 Kampus Tegal Boto, Jember 68121, Indonesia

‡Department of Mathematics
Institut Teknologi Bandung
Jalan Ganesa 10 Bandung, Indonesia

§Department of Informatics, University of Jember
Jalan Kalimantan No. 37 Kampus Tegal Boto
Jember 68121, Indonesia
¶arika.fkip@unej.ac.id
‖nafidatun.nikmah@gmail.com
**d.dafik@gmail.com
††alfarisi.fkip@unej.ac.id
‡‡malihasan1996@gmail.com
§§Slamin@unej.ac.id

Received 15 January 2021
Revised 2 September 2021
Accepted 5 September 2021
Published

Communicated by Weifan Wang

Let $G = (V, E)$ be a simple, finite, undirected, and connected graph with vertex set $V(G)$ and edge set $E(G)$. A bijection $l : V(G) \to \{1, 2, \ldots, k\}$ is label function l if $\mathrm{opt}(l) = \min\{\max(l_i) : l_i \textit{ vertex irregular labeling}\}$ and for any two adjacent vertices u and v, $w(u) \neq w(v)$ where $w(u) = \sum_{v \in N(u)} l(v)$ and $N(u)$ is set of vertices adjacent to v. w is called local irregularity vertex coloring. The minimum cardinality of local irregularity vertex coloring of G is called chromatic number local irregular denoted by $\chi_{lis}(G)$. In this paper, we verify the exact values of volcano, broom, parachute, double broom and complete multipartite graphs.

Keywords: Local irregular; chromatic number; chromatic number local irregular.

Mathematics Subject Classification 2020: 05C78

1. Introduction

Let $G = (V, E)$ be a simple, finite, undirected, and connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance irregular vertex k-labeling l is an assignment $l : V(G) \to \{1, 2, \ldots, k\}$ such that for two distinct vertices u and v in G, $w(u) \neq w(v)$, where $w(u)$, a weight of u, is the sum of the labels of all vertices distance 1 from u (or neighbor of vertex u). Thus, this labeling gives a condition that all weights of the vertices are not the same. This concept is introduced by Slamin [6]. For the development of this concept, Kristiana *et al.* [3] considered another condition for any two adjacent vertices having to different weights. This concept is named as local irregularity vertex coloring of graph. In future discussion, we will explain about the local irregularity vertex coloring of graphs. Formally, they state in the following definition.

Definition 1.1 ([3]). By the local irregularity vertex coloring, we recall a bijection $l : V(G) \to \{1, 2, \ldots, k\}$ is called vertex k-labeling and $w : V(G) \to N$, where $w(u) = \sum_{v \in N(u)} l(v)$. A condition for w to be local irregularity vertex coloring if

(i) $\mathrm{opt}(l) = \min\{\max\{l_i\}; l_i \text{ vertex irregular labeling}\}$
(ii) for every $uv \in E(G), w(u) \neq w(v)$.

Definition 1.2 ([3]). The chromatic number of local irregular of G, denoted by $\chi_{lis}(G)$, is the minimum cardinality of the largest label over all such local irregularity vertex coloring.

Lemma 1.3 ([3]). *Let G be a simple and connected graph, $\chi_{lis}(G) \geq \chi(G)$.*

There are some results of this coloring. Kristiana *et al.* [2–4] presented the exact value of the local irregular chromatic number of some of graphs as follows: bipartite complete graph $K_{\{n,m\}}$, the chromatic number local irregular is $\chi_{\mathrm{lis}}(K_{\{n,m\}}) = 2$; fan graph F_n, where $n \geq 4$, the chromatic number local irregular is $\chi_{lis}(F_n) = 4$; and square book graph Bs_n where $n \geq 2$, the chromatic number local irregular is $\chi_{lis}(Bs_n) = 4$. Then, Nadia *et al.* [1] presented the exact value of the chromatic number local irregular of H_n graph, where $n \geq 2$, the chromatic number local irregular is $\chi_{lis}(H_n) = 4$.

Umilasari *et al.* [7] presented the exact value of vertex shackle product graphs as follows.

(a) Let Shack (S_n, v, m) be the vertex shackle of m star S_n for $m \geq 2$ and $n \geq 3$, the chromatic number local irregular is $\chi_{lis}(\mathrm{Shack}(S_n, v, m)) = 3$.
(b) Let Shack (K_n, v, m) be the vertex shackle of m complete graphs K_n for $m \geq 2$ and $n \geq 3$, the chromatic number local irregular is $\chi_{lis}(\mathrm{Shack}(K_n, v, m)) = n$.
(c) Let Shack (W_n, v, m) be the vertex shackle of m wheels W_n for $m \geq 2$ and $n \geq 3$, the chromatic number local irregular is $\chi_{lis}(\mathrm{Shack}(W_n, v, m)) = 4$ for n even and $\chi_{lis}(\mathrm{Shack}(W_n, v, m)) = 5$ for n odd.

We refer to [5] for the other terms and notations.

2. Main Result

In this section, we investigate the chromatic number local irregular of some special graphs, i.e., volcano, parachute, broom, double broom and multipartite graph.

Volcano graph, denoted by V_n, is a graph constructed by a cycle C_3 and $\overline{K_n}$ where one vertex of C_3 is adjacent to all vertices of $\overline{K_n}$.

Theorem 2.1. *If V_n be a volcano graph with $n \geq 2$, then $\chi_{lis}(V_n) = 4$.*

Proof. Let $V(V_n) = \{x_i \,|\, 1 \leq i \leq 3\} \cup \{y_i \,|\, 1 \leq i \leq n\}$ and $E(V_n) = \{x_1x_2, x_2x_3, x_3x_1\} \cup \{x_2y_i \,|\, 1 \leq i \leq n\}$ be vertex set and edge set of V_n, respectively. First, we show the lower bound $\chi_{lis}(V_n) \geq 4$. If each vertex receives label 1, we get $w(x_1) = w(x_3) = 2$. It contradicts with Definition 1.1. It must be $\max(l_i) \geq 2$. Choose $\mathrm{opt}(l) = 2$. For any 2-labeling l, we know $w(y_i) = l(x_2) < w(x_j)$ for $j \in \{1, 3\}$. It means $w(y_i) \neq w(x_1) \neq w(x_2) \neq w(x_3)$. So, we get $\chi_{lis}(V_n) \geq 4$. Next, we show the upper bound value of $\chi_{lis}(V_n)$. Define a 2-labeling l by formula $l(x_1, x_2, x_3, y_i) = (1, 1, 2, 1)$. This labeling gives the weights $w(x_1, x_2, x_3, y_i) = (3, n + 3, 2, 1)$. We obtain $\chi_{lis}(V_n) \leq 4$. Combining with the lower bound, we get the exact value $\chi_{lis}(V_n) = 4$. □

Parachute graph, denoted by PC_n, is a graph constructed by a cycle C_{2n} and K_1 where the first n vertices of C_n are adjacent to K_1. The chromatic local irregular of this graph is given by the following theorem.

Theorem 2.2. *If PC_n is a parachute graph for $n \equiv 0, 1 \pmod 4$ then $\chi_{lis}(PC_n) = 3$.*

Proof. Let $V(PC_n) = \{c\} \cup \{x_i, y_i \,|\, 1 \leq i \leq n\}$ and $E(PC_n) = \{cx_i \,|\, 1 \leq i \leq n\} \cup \{x_1y_1, x_ny_n\} \cup \{x_ix_{i+1}, y_iy_{i+1} \,|\, 1 \leq i \leq n-1\}$. If each vertex is labeled by 1, then $w(y_1) = w(y_2) = 2$. We get a contradiction with Definition 1.1. So, it must be $\max(l_i) \geq 2$. Choose $\mathrm{opt}(l) = 2$. Clearly, Lemma 1.3 implies $\chi_{lis}(PC_n) \geq 3$. Then, we prove that $\chi_{lis}(PC_n) \leq 3$. Divide into two cases.

Case 1. For $n \equiv 0 \pmod 4$.

Define a 2-labeling l by formula $l(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \ldots) = (1, 1, 1, 2, 1, 1, 1, 2, \ldots)$, $l(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, \ldots) = (2, 2, 2, 1, 2, 2, 2, 1, \ldots)$ and $l(c) = 1$. By this labeling, we have $w(x_1, x_2, x_3, x_4, \ldots) = (4, 3, 4, 3, \ldots)$, $w(y_1, y_2, y_3, y_4, \ldots) = (3, 4, 3, 4, \ldots)$ and $w(c) = \frac{5n}{4}$. So, we have $\chi_{lis}(PC_n) \leq 3$.

Case 2. For $n \equiv 1 \pmod 4$.

Use the 2-labeling l by formula $l(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \ldots) = (2, 1, 1, 1, 2, 1, 1, 1, \ldots)$, $l(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, \ldots) = (1, 2, 2, 2, 1, 2, 2, 2, \ldots)$ and $l(c) = 1$. This labeling gives the vertices weights $w(x_1, x_2, x_3, x_4, \ldots) = (3, 4, 3, 4, \ldots)$, $w(y_1, y_2, y_3, y_4, \ldots) = (4, 3, 4, 3, \ldots)$ and $w(c) = \frac{5n+3}{4}$. So, we have $\chi_{lis}(PC_n) \leq 3$.

From the two cases above, combine the lower bound and the upper bound, which have been found, then $\chi_{lis}(PC_n) = 3$ for $n \equiv 0, 1 \pmod 4$, □

Next, we determine the chromatic number local irregular of broom graph. A broom graph, denoted by $B_{n,m}$, is a graph which is constructed by a path P_m where one of the end vertices is incident to $n-m$ pendant edges.

Theorem 2.3. *If $B_{n,m}$ is a broom graph with $n-m \geq 4$ and $m \geq 4$, then $\chi_{lis}(B_{n,m}) = 4$.*

Proof. Let vertex set of $B_{n,m}$ be given by $V(B_{n,m}) = \{x_i \,|\, 1 \leq i \leq m\} \cup \{y_i \,|\, 1 \leq i \leq n-m\}$ such that the edge set be $E(B_{n,m}) = \{x_i x_{i+1} \,|\, 1 \leq i \leq m-1\} \cup \{x_1 y_i \,|\, 1 \leq i \leq n-m\}$. If each vertex in $B_{n,m}$ receives label 1, then $w(x_2) = w(x_3) = 2$. But, that is not allowed by Definition 1.1. So, it must be $\max(l_i) \geq 2$. Choose $\mathrm{opt}(l) = 2$. For any 2-labeling l, we know that $w(x_1) \geq n-m+1$ and $w(x_m) = l(x_{m-1}) < w(x_{m-2})$. It forces the condition $w(x_m) \neq w(x_{m-1}) \neq w(x_{m-2}) \neq w(x_1)$. Therefore, $\chi_{lis}(B_{n,m}) \geq 4$. Now, we show that $\chi_{lis}(B_{n,m}) \leq 4$. Define 2-labeling l by $l(x_1, x_2, x_3, x_4, \ldots, x_{m-1}) = (2, 1, 2, 1, \ldots)$, $l(x_m) = 2$ and $l(y_i) = 1$. Furthermore, we have the weights $w(x_1) = n-m+1$, $w(x_2, x_3, x_4, x_5, \ldots, x_{m-2}) = (4, 2, 4, 2, \ldots)$, $w(x_{m-1}, x_m) = (3, 2)$ for m is even, $w(x_{m-1}, x_m) = (4, 1)$ for m is odd and $w(y_i) = 2$. Consequently, the upper bound is $\chi_{lis}(B_{n,m}) \leq 4$. Thus, $\chi_{lis}(B_{n,m}) = 4$. □

Theorem 2.4. *If $n-m \geq 2$ and $m = 3$, then $\chi_{lis}(B_{n,m}) = 3$.*

Proof. Use 1-labeling on this graph. □

A double broom graph, denoted by $\mathrm{DB}_{n,m}$, is a graph which is constructed by a path P_m where the first and the last end vertices are incident to $\frac{n-m}{2}$ pendant edges, respectively. This following theorem gives an exact value of chromatic number local irregular of double broom graph.

Theorem 2.5. *If $\mathrm{DB}_{n,m}$ is a double broom graph, where $\frac{n-m}{2} \geq 4$ and m is odd, $m \geq 5$, then $\chi_{lis}(\mathrm{DB}_{n,m}) = 3$.*

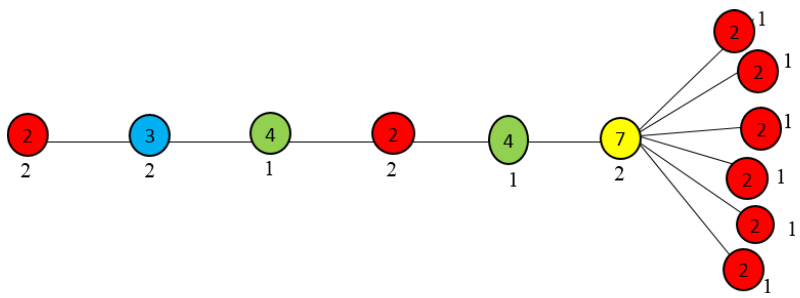


Fig. 1. The chromatic number local irregular $B_{12,6}$ be a broom graph with $n-m=6$ and $m=4$, then $\chi_{lis}(B_{12,6}) = 4$.

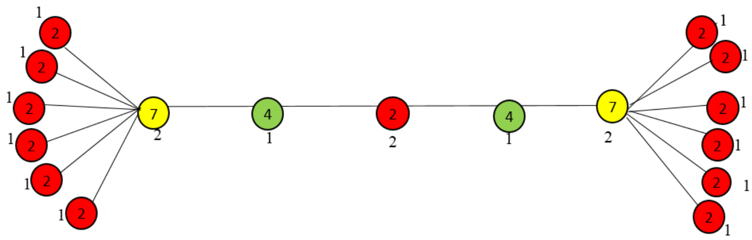


Fig. 2. The chromatic number local irregular $\mathrm{DB}_{17,5}$ be a broom graph with $n-m=6$ and $m=5$, then $\chi_{lis}(\mathrm{DB}_{17,5})=3$.

Proof. We define the vertex set of $\mathrm{DB}_{n,m}$ as $V(\mathrm{DB}_{n,m}) = \{x_i \mid 1 \le i \le m\} \cup \{y_i \mid 1 \le i \le \frac{n-m}{2}\} \cup \{z_i \mid 1 \le i \le \frac{n-m}{2}\}$ and the edge set as $E(\mathrm{DB}_{n,m}) = \{x_i x_{i+1} \mid 1 \le i \le m-1\} \cup x_1 y_i \mid 1 \le i \le \frac{n-m}{2}\} \cup \{x_m z_i \mid 1 \le i \le \frac{n-m}{2}\}$. If we use 1-labeling, we get $w(x_2) = w(x_3) = 2$. Contradiction with Definition 1.1. It must be $\max(l_i) \ge 2$. Choose $\mathrm{opt}(l) = 2$. We know that for any 2-labeling, it is clearly that the weight of x_1 and x_m is greater than the other vertices. Also, we avoid two adjacent vertices to receive the same color. This condition requires $w(x_1) \ne w(x_2) \ne w(x_3)$. So, we obtain the lower bound $\chi_{lis}(\mathrm{DB}_{n,m}) \ge 3$. Now, put 2-labeling l on $\mathrm{DB}_{m,n}$ by rules $l(x_1, x_2, x_3, x_4, \ldots, x_m) = (2, 1, 2, 1, \ldots, 2)$ and $l(y_i) = l(z_i) = 1$. By this labeling, the vertices weights are $w(x_1) = w(x_m) = \frac{n-m}{2} + 1$, $w(x_2, x_3, x_4, x_5, \ldots, x_{m-1}) = (4, 2, 4, 2, \ldots, 4)$ and $w(y_i) = w(z_i) = 2$. It shows $\chi_{lis}(\mathrm{DB}_{n,m}) \le 3$. Combining with the upper bound above, we then have $\chi_{lis}(\mathrm{DB}_{n,m}) = 3$. □

Theorem 2.6. *If $m = 3$ and $\frac{n-m}{2} \ge 2$, then $\chi_{lis}(\mathrm{DB}_{n,m}) = 3$.*

Proof. Use 1-labeling on this graph. □

A complete multipartite graph, denoted by $K_{n_1,n_2,\ldots,n_m}$, is a graph which has a set of graph vertices decomposed into m disjoint sets such that no two vertices within the same set are adjacent and every pair of vertices in the m sets is adjacent.

Theorem 2.7. *Let $n_1, n_2, \ldots, n_m$ be positive integer numbers, if $K_{n_1,n_2,\ldots,n_m}$ is a complete multipartite graph for $m \ge 2$, then $\chi_{lis}(K_{n_1,n_2,\ldots,n_m}) = m$.*

Proof. For simplify, we assume $G = K_{n_1,n_2,\ldots,n_m}$. Let the vertex and the edge set of G be $V(G) = \bigcup_{j=1}^{m} V_j$ where $V_j = \{v_i^j \mid 1 \le i \le n_j\}$ and $E(G) = \{v_i^j v_s^t \mid 1 \le i \le n_j, 1 \le s \le n_t, 1 \le j \ne t \le m\}$. Clearly, for each $u, v \in V_j$ with $j \in \{1, 2, \ldots, m\}$, we have $w(u) = w(v)$. It means that if there is a vertex irregular k-labeling of G, then we can conclude $\chi_{lis}(G) \le m$. Also, we have $\chi_{lis}(G) \ge m$ by Lemma 1.3. Therefore, if there exists some k such that $\mathrm{opt}(l) = k$, then $\chi_{lis}(G) = m$. It is important to show the existence of such k number. We show that $\mathrm{opt}(l) \le m$. Without loss generality, suppose that $n_1 \leq n_2 \leq \cdots \leq n_m$. Label the vertices of G with $l(u) = j$ for every $u \in V_j$ and $j \in \{1, 2, \ldots, m\}$, then we obtain $w(u) \neq w(v)$ for each $uv \in E(G)$. So, we get $\chi_{lis}(G) = m$. Suppose that $\text{opt}(l) = m - 1$, then there are two possibilities: (i) for each $uv \in E(G)$, $w(u) \neq w(v)$ that is the graph G has a vertex irregular $(m-1)$-labeling, or (ii) there is an $uv \in E(G)$ satisfying $w(u) = w(v)$ for any $(m-1)-$ labeling that requires $\text{opt}(l) = m$. Similarly, suppose that $\text{opt}(l) = m - 2$ then also we have the condition either: (i) for each $uv \in E(G)$, $w(u) \neq w(v)$, for some vertex irregular $(m-2)$-labeling of G, or (ii) there is an edge uv such that $w(u) = w(v)$ for any $(m-2)$-labeling that forces $m-1 \leq \max(l) \leq m$. By continuing this process, we certainly obtain $\text{opt}(l) = k$ for some $k \in \{1, 2, \ldots, m\}$. The existence of the value k has been shown. Thus, we can conclude $\chi_{lis}(K_{n_1,n_2,\ldots,n_m}) = m$. □

Acknowledgment

We gratefully acknowledge the support from the University of Jember of year 2021.

References

[1] N. Azahra, A. I. Kristiana, Dafik and R. Alfarisi, On the local irregularity vertex coloring of related grid graph, *Int. J. Acad. Appl. Res.* **4**(2) (2020) 1–4.
[2] A. I. Kristiana, R. Alfarisi, Dafik and N. Azahra, Local irregular vertex coloring of some families graph, *J. Discrete Math. Sci. Cryptogr.* (2020) 1–16.
[3] A. I. Kristiana, Dafik, M. I. Utoyo, Slamin, R. Alfarisi, I. H. Agustin and M. Venkatachalam, Local irregularity vertex coloring of graphs, *Int. J. Civil Eng. Technol.* **10** (2019) 1606–1616.
[4] A. I. Kristiana, M. I. Utoyo, I. H. Agustin, R. Alfarisi and E. Waluyo, On the chromatic number local irregularity of related wheel graph, in *2nd Int. Conf. Combinatorics, Graph Theory and Network Topology*, *Journal of Physics*: *Conference Series*, Vol. 1211(1), (IOP Publishing, 2019), pp. 012003.
[5] N. Hartsfield and G. Ringel, *Pearls in Graph Theory* (Academic Press, UK, 1994).
[6] Slamin, On distance irregular labeling of graph, *Far East J. Math. Sci.* **105**(5) (2017) 919–932.
[7] R. Umilasari, L. Susilowati and Slamin, Local irregularity chromatic number of vertex shackle product of graphs, in *Int. Conf. Engineering and Applied Technology* (*ICEAT*), IOP Conference Series, Vol. 821(1), Sorong, Indonesia, 2020, pp. 012038.