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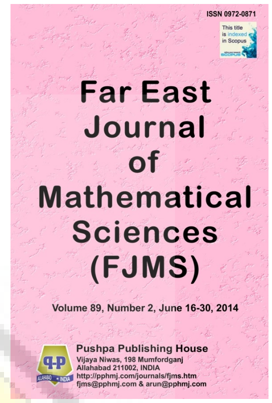

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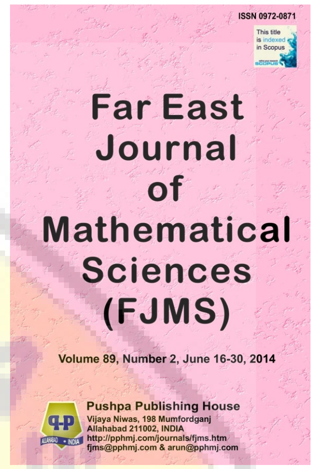
Articles 1 - 9 of 9 [1]



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Content

BACK

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THE 2-DISTANCE CHROMATIC NUMBER OF SOME WHEEL RELATED GRAPHS

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Abstract

Let $G(V, E)$ be a simple and connected graph of vertex set V and edge set E. By the 2-distance chromatic number of a graph G, we mean a map $c : V(G) \to \{1, 2, 3, \ldots, k\}$ such that any two vertices at distance at most two from each other have different colors. The minimum number of colors in 2-distance chromatic number of G is its 2-distance chromatic number, denoted by $\chi^2(G)$. In this paper, we

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study the 2-distance chromatic number of some wheel related graphs, and obtain the 2-distance chromatic number of $\chi^2(W_n)$, $\chi^2(F_n)$, $\chi^2(H_n)$ and $\chi^2(W_{2,n})$ for $n \geq 3$.

1. Introduction

All graphs in this paper are simple and connected graphs. Let $G(V, E)$ be a graph of vertex set V and edge set E. For detail definition of graph, see [10, 4, 1, 5, 6]. A coloring of graph G is a map from $c : V(G) \to \{1, 2, ..., k\}$ having property that $c(u) \neq c(v)$ for every adjacent vertices u, v in G. It is said to be a *k-coloring* of G, if we assign k colors to all vertices in G. We will introduce and study a new concept, namely a 2-distance coloring. By the 2-distance chromatic number of a graph G, we mean a map $c : V(G) \to \{1, 2, 3, ..., k\}$ such that any two vertices at distance at most two from each other have different colors. The minimum number of colors in 2-distance chromatic number of G is its 2-distance chromatic number, denoted by $\chi^2(G)$. This concept was introduced by Borodin and Ivanova [3]. The 2-distance chromatic number has also been studied by several authors, for instance, in [9, 12, 2, 8, 11, 14]. Fertin et al. in [9] found 2-distance chromatic number for the d-dimensional grid graph, $\chi^2(G_d(n_1, n_2, ..., n_d)) = 2d + 1$. Kramer et al. [12] also found the d-distance chromatic number of the hexagonal lattice graph. Borodin and Ivanova [3] proved $\chi^2 = \Delta + 1$ whenever $\Delta \geq 31$ for planar graphs of girth six with the additional assumption that edge is incident with a vertex of degree two. Bonamy et al. [2] proved every graph with maximum degree Δ at least 4 and a maximum average less than $\frac{7}{3}$ admits a 2-distance $(\Delta + 1)$-coloring. Kim et al. [11] found the 2-distance chromatic number of the direct product of two cycles. Some cases of coloring were studied by Kristiana [13] and Dafik et al. [7]. In this paper, we study the 2-distance chromatic number of some wheel related graphs.

2. Main Results

In this paper, we have found the exact value of 2-distance chromatic number of some related wheel graphs, namely wheel, fan, helm and web graphs. All the values of their chromatic numbers attain the best lower bound. The theorems are as follows:

Theorem 2.1. *Let* W_n *be a wheel graph for* $n \geq 3$. *Then the* 2-*distance chromatic number of wheel graph is* $\chi^2(W_n) = n + 1$.

Proof. The graph W_n is a connected graph with vertex set $V(W_n) = \{x, x_i : 1 \leq i \leq n\}$ and edge set

$$E(W_n) = \{xx_i : 1 \leq i \leq n\} \cup \{x_1x_n, x_ix_{i+1} : 1 \leq i \leq n-1\}.$$

The cardinality of vertices $|V(W_n)| = n + 1$ and the cardinality of edges $|E(W_n)| = 2n$.

We will prove that the lower bound of the 2-distance chromatic number of W_n is $\chi^2(W_n) \geq n + 1$. We assume that $\chi^2(W_n) > n + 1$ by taking $\chi^2(W_n) = n$. Thus, we have n-coloring of W_n and the graph W_n consists of one central vertex and n vertices in its rim (outer cycle). Since the central vertex is adjacent to all vertices in rim, it implies $c_1(x) \neq c_1(v)$ with v is a vertex in its rim. Hence, all vertices in rim have $(n-1)$-coloring. If we assign all vertices in rim with $(n-1)$ colors, then there are at least two vertices which have same colors. It contradicts with the definition of 2-distance chromatic number which states that every two vertices u, v at distance at most two from each other should have different colors. We know that diameter of W_n is 2, since the shortest path between two vertices in rim is $u - x - v$. If two vertices u, v in rim have the same color, then two vertices at distance at most two from $u - x - v$ have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of W_n is $\chi^2(W_n) \geq n + 1$.

Furthermore, we define a map $c_1 : V(W_n) \to \{1, 2, 3, ..., k\}$ with k-color by the following function:

$$c_1(v) = \begin{cases} 1, & \text{if } v = x, \\ i+1, & \text{if } v = x_i, 1 \le i \le n. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n+1$. In other words, the upper bound of the 2-distance chromatic number of W_n is $\chi^2(W_n) \le n+1$. It concludes that the 2-distance chromatic number of W_n is $\chi^2(W_n) = n+1$. □

Theorem 2.2. *Let* F_n *be a fan graph for* $n \ge 3$. *Then the* 2-*distance chromatic number of fan graph is* $\chi^2(F_n) = n+1$.

Proof. The graph F_n is a connected graph with vertex set $V(F_n) = \{x, x_i : 1 \le i \le n\}$ and edge set

$$E(F_n) = \{xx_i : 1 \le i \le n\} \cup \{x_i x_{i+1} : 1 \le i \le n-1\}.$$

The cardinality of vertices $|V(F_n)| = n+1$ and the cardinality of edges $|E(F_n)| = 2n-1$.

We will prove that the lower bound of the 2-distance chromatic number of W_n is $\chi^2(F_n) \ge n+1$. We assume that $\chi^2(F_n) > n+1$ by taking $\chi^2(F_n) = n$. Thus, we have n-coloring of F_n. The graph F_n consists of one central vertex and n vertices in its rim (path). Since it is adjacent to all vertices in its rim, $c_1(x) \ne c_1(v)$ with v is the vertex in rim. Hence, all vertices in rim have $(n-1)$-coloring. If we assign the vertex in rim with $(n-1)$ colors, then there are at least two vertices which have the same color. Based on definition of 2-distance chromatic number, every two vertices u, v at distance at most two from each other must have different colors. We know that diameter of F_n is 2, since the shortest path between two vertices in rim is $u - x - v$. If two vertices u, v in rim have the same color, then two vertices at distance at most two from $u - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of F_n is $\chi^2(F_n) \geq n+1$.

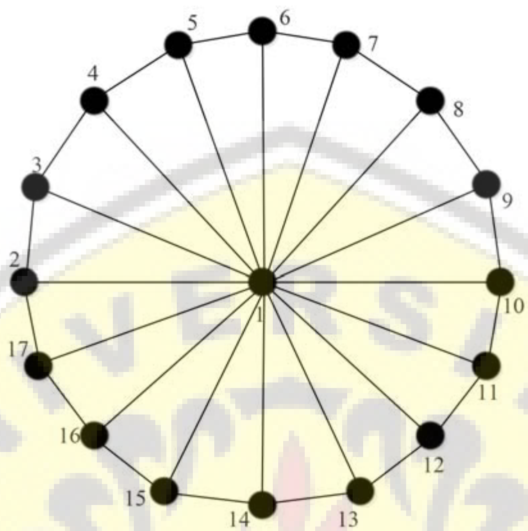

Figure 1. Example of 2-distance coloring of W_{16}.

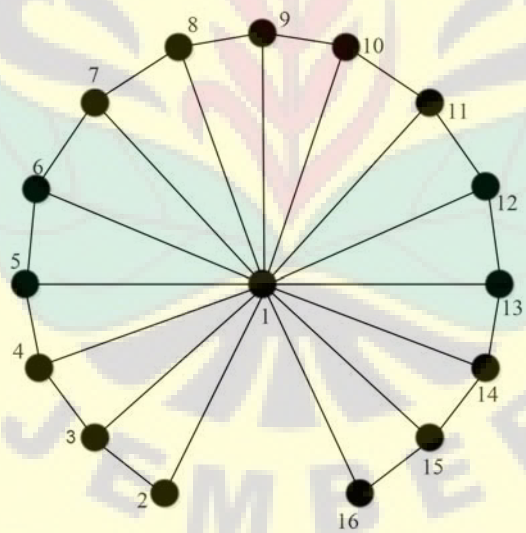

Figure 2. Example of 2-distance coloring of F_{15}.

Furthermore, define a coloring function $c_2 : V(F_n) \to \{1, 2, 3, ..., k\}$ with k-colors by the following:

$$c_2(v) = \begin{cases} 1, & \text{if } v = x, \\ i+1, & \text{if } v = x_i, 1 \leq i \leq n. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n+1$. In other words, the upper bound of the 2-distance chromatic number of F_n is $\chi^2(F_n) \leq n+1$. It concludes that the 2-distance chromatic number of F_n is $\chi^2(F_n) = n+1$. □

Theorem 2.3. *Let H_n be a helm graph for $n \geq 3$. Then the 2-distance chromatic number of helm graph is*

$$\chi^2(H_n) = \begin{cases} 5, & \text{if } n = 3, \\ n+1, & \text{if } n \geq 4. \end{cases}$$

Proof. The helm graph H_n is a graph obtained from the wheel W_n by attaching a pendant edge at each vertex of the cycle C_n. Thus, the vertex set is $V(H_n) = \{x, x_i, y_i : 1 \leq i \leq n\}$ and the edge set is

$$E(H_n) = \{xx_i, x_i y_i : 1 \leq i \leq n\} \cup \{x_n x_1, x_i x_{i+1} : 1 \leq i \leq n-1\}.$$

The cardinality of vertices $|V(H_n)| = 2n+1$ and cardinality of edges $|E(H_n)| = 3n$. To prove the least 2-distance chromatic number, we will describe the proof in two cases as follows:

Case 1. For $n = 3$, we will prove that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \geq 5$. We assume that $\chi^2(H_n) > 5$ by taking $\chi^2(H_n) = 4$. Thus, we have 4-colorings of H_n and the graph H_n consists of one central vertex, 3 vertices in rim (inner cycle) and 3 vertices in pendant edges. Since the central vertex is adjacent to all vertices in rim, $c_4(x) \neq c_4(v) = c_4(u)$ with v, u as the vertices in rim and pendant edges, respectively. Hence, all vertices in rim have 3-coloring. If we assign the vertex in rim and pendant edges with 3 colors, then there are at least two vertices which have same color. Based on definition 2-distance chromatic number, it states that every two vertices *v*, *w* at distance at most two from each other have different colors (the vertex *v* in rim and *w* in pendant edges). We know that distance between each vertex in rim is 2, since the shortest path between two vertices in rim and pendant edges is $w - x - v$. If two vertices *w*, *v* in rim and pendant edges have same color, then two vertices at distance at most two from $w - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \geq 5$.

Furthermore, define a map $c_3 : V(H_n) \to \{1, 2, 3, ..., 5\}$ with *k*-colors by the following coloring function:

$$c_3(v) = \begin{cases} 1, & \text{if } v = x, \\ i + 1, & \text{if } v = x_i, 1 \leq i \leq 3, \\ 5, & \text{if } v = y_i, 1 \leq i \leq 3. \end{cases}$$

It is clear to see that the maximum value of the set of colors is 5. In other words, the upper bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \leq 5$. It concludes that the 2-distance chromatic number of H_n is $\chi^2(H_n) = 5$ for $n = 3$.

Case 2. For $n \geq 4$, we will prove that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \geq n + 1$. We assume that $\chi^2(H_n) > n + 1$ by taking $\chi^2(H_n) = n$. Thus, we have *n*-coloring of H_n and the graph H_n consists of one central vertex, *n* vertices in rim (inner cycle) and *n* vertices in pendant edges with condition that central vertex which is adjacent to all vertices in rim should satisfy $c_4(x) \neq c_4(v) = c_4(u)$ with *v*, *u* as the vertices in rim and pendant edges, respectively. Hence, all vertices in rim have $(n - 1)$-coloring. If we assign the vertex in rim with $(n - 1)$ colors, then there are at least two vertices which have same color. Based on definition of 2-distance chromatic number, it states that every two vertices *v*, *w* at distance at most two from each other have different colors. We know that distance between each vertex in rim is 2, since the shortest path between two vertices in rim is $w - x - v$. If two vertices w, v in rim have same color, then two vertices at distance at most two from $w - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \geq n + 1$.

Furthermore, define a coloring function $c_4 : V(H_n) \to \{1, 2, 3, ..., k\}$ with k-colors by

$$c_4(v) = \begin{cases} 1, & \text{if } v = x, \\ i + 1, & \text{if } v = x_i, 1 \leq i \leq n, \\ i - 1, & \text{if } v = y_i, 3 \leq i \leq n, \\ n - 1, & \text{if } v = y_1, \\ n, & \text{if } v = y_2. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n + 1$. In other words, the upper bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \leq n + 1$. It concludes that the 2-distance chromatic number of H_n is $\chi^2(H_n) = n + 1$ for $n \geq 4$.

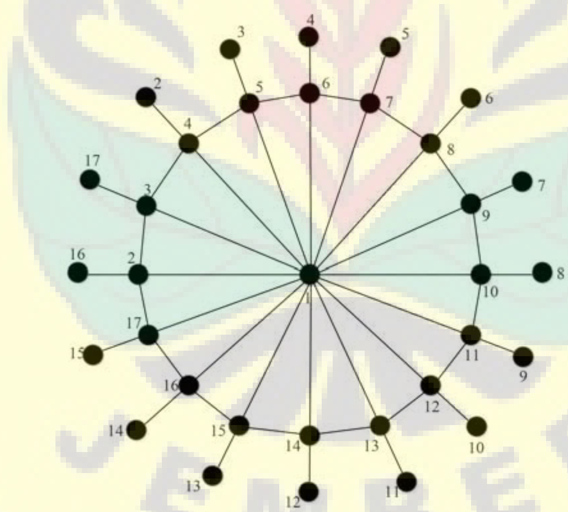

Figure 3. Example of 2-distance coloring of H_{16}.

□

Theorem 2.4. *Let* $W_{2,n}$ *be a web graph with* $n \geq 3.$ *Then the* 2*-distance chromatic number of web graph is*

$$\chi^2(W_{2,n}) = \begin{cases} 7, & \text{if } n = 3, \\ n+1, & \text{if } n \geq 4. \end{cases}$$

Proof. The web graph $W_{2,n}$ is a graph obtained by joining the pendant vertices of helm graph to cycle (outer cycle) and adding one pendant edge to each vertex in outer cycle with vertex set $V(W_{2,n}) = \{x, x_i, y_i, z_i : 1 \leq i \leq n\}$ and edge set

$$E(W_{2,n}) = \{xx_i, x_iy_i, y_iz_i : 1 \leq i \leq n\}$$
$$\cup \{x_nx_1, y_ny_1, x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n-1\}.$$

The cardinality of vertices $|V(W_{2,n})| = 3n+1$ and cardinality of edges $|E(W_{2,n})| = 5n.$ To prove the least 2-distance chromatic number, we will describe the proof in two cases as follows:

Case 1. For $n = 3,$ we will prove that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq 7.$ We assume that $\chi^2(W_{2,n}) > 7$ by taking $\chi^2(W_{2,n}) = 6.$ Thus, we have 6-colorings of $W_{2,n}$ and the graph $W_{2,n}$ consists of one central vertex, 3 vertices in inner cycle, 3 vertices in outer cycle and 3 vertices in pendant edges. Since the central vertex is adjacent to all vertices in rim, $c_5(x) \neq c_5(v) = c_4(z) \neq c_5(u)$ with v, u, z as the vertices in inner cycle, outer cycle and pendant edges, respectively. Hence, all vertices in outer have 3-colorings. If we assign the vertex in inner cycle and outer cycle with 5 colors, then there are at least two vertices which have same color. Based on definition 2-distance chromatic number, every two vertices v, u at distance at most two from each other have different colors (the vertex v in inner cycle and u in outer cycle). We know that distance between each vertex in inner cycle and outer cycle is 2 such that the shortest path between two vertices in rim and pendant edges is $u - x - v$ If two vertices u, v in rim and pendant edges have same color, then two vertices at distance at most two from $u - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq 7$.

Furthermore, define a coloring function $c_5 : V(W_{2,n}) \to \{1, 2, 3, ..., 7\}$ with k-colors by

$$c_5(v) = \begin{cases} 1, & \text{if } v = x, \\ i + 1, & \text{if } v = x_i, 1 \leq i \leq 3, \\ n + i + 1, & \text{if } v = y_i, 1 \leq i \leq 3, \\ 2, & \text{if } v = z_3, \\ n + i - 1, & \text{if } v = z_i, 1 \leq i \leq 2. \end{cases}$$

It is clear to see that the maximum value of the set of colors is 7. In other words, the upper bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \leq 7$.

It concludes that the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) = 7$ for $n = 3$.

Case 2. For $n \geq 4$, we will prove that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq n + 1$. We assume that $\chi^2(W_{2,n}) > n + 1$ by taking $\chi^2(H_n) = n$. Thus, we have n-colorings of $W_{2,n}$ and the graph $W_{2,n}$ consists of one central vertex, n vertices in inner cycle, n vertices in outer cycle and n vertices in pendant edges with condition that central vertex adjacent to all vertices in inner cycle, then $c_6(x) = c_6(z) \neq c_6(v) = c_6(u)$ with v, u, z as the vertices in inner cycle, outer cycle and pendant edges, respectively. Hence, all vertices in inner cycle have $(n - 1)$-

colorings. If we assign the vertex in inner cycle with $(n-1)$ colors, then there are at least two vertices which have same color. Based on definition 2-distance chromatic number, every two vertices *v*, *w* at distance at most two from each other have different colors. We know that distance between each vertex in inner cycle is 2 such that the shortest path between two vertices in inner cycle is $w-x-v$. If two vertices *w*, *v* in rim have same color, then two vertices at distance at most two from $w-x-v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq n+1$.

Furthermore, define a coloring function $c_6 : V(W_{2,n}) \to \{1, 2, 3, ..., 7\}$ with *k*-colors by

$$c_6(v) = \begin{cases} 1, & \text{if } v = x \text{ and } v = z_i, 1 \leq i \leq n, \\ i+1, & \text{if } v = x_i, 1 \leq i \leq n, \\ i-1, & \text{if } v = y_i, 3 \leq i \leq n, \\ n+i-1, & \text{if } v = y_i, 1 \leq i \leq 2. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n+1$. In other words, the upper bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \leq n+1$. It concludes that the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) = n+1$ for $n \geq 4$. □

3. Conclusion

In this paper, we have given some results on the 2-distance chromatic number of some wheel related graphs, namely fan, wheel, helm, and web graph. However, we need to characterize the other wheel related graphs, especially for any graph *G*. Hence the following problems arise naturally.

Open problem 3.1. Determine exact value of *k*-distance chromatic number or sharp lower bound and upper bound $\chi^2(G)$ when *G* is any wheel related graph?

Open problem 3.2. Determine lower and upper bounds of the *k*-distance chromatic number of graph operation including corona, Cartesian, comb product and others?

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Abstract

Let $G(V, E)$ be a simple and connected graph of vertex set V and edge set E. By the 2-distance chromatic number of a graph G, we mean a map $c : V(G) \to \{1, 2, 3, ..., k\}$ such that any two vertices at distance at most two from each other have different colors. The minimum number of colors in 2-distance chromatic number of G is its 2-distance chromatic number, denoted by $\chi^2(G)$. In this paper, we

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study the 2-distance chromatic number of some wheel related graphs, and obtain the 2-distance chromatic number of $\chi^2(W_n)$, $\chi^2(F_n)$, $\chi^2(H_n)$ and $\chi^2(W_{2,n})$ for $n \geq 3$.

1. Introduction

All graphs in this paper are simple and connected graphs. Let $G(V, E)$ be a graph of vertex set V and edge set E. For detail definition of graph, see [10, 4, 1, 5, 6]. A coloring of graph G is a map from $c : V(G) \rightarrow \{1, 2, ..., k\}$ having property that $c(u) \neq c(v)$ for every adjacent vertices u, v in G. It is said to be a *k-coloring* of G, if we assign k colors to all vertices in G. We will introduce and study a new concept, namely a 2-distance coloring. By the 2-distance chromatic number of a graph G, we mean a map $c : V(G) \rightarrow \{1, 2, 3, ..., k\}$ such that any two vertices at distance at most two from each other have different colors. The minimum number of colors in 2-distance chromatic number of G is its 2-distance chromatic number, denoted by $\chi^2(G)$. This concept was introduced by Borodin and Ivanova [3]. The 2-distance chromatic number has also been studied by several authors, for instance, in [9, 12, 2, 8, 11, 14]. Fertin et al. in [9] found 2-distance chromatic number for the d-dimensional grid graph, $\chi^2(G_d(n_1, n_2, ..., n_d)) = 2d + 1$. Kramer et al. [12] also found the d-distance chromatic number of the hexagonal lattice graph. Borodin and Ivanova [3] proved $\chi^2 = \Delta + 1$ whenever $\Delta \geq 31$ for planar graphs of girth six with the additional assumption that edge is incident with a vertex of degree two. Bonamy et al. [2] proved every graph with maximum degree Δ at least 4 and a maximum average less than $\frac{7}{3}$ admits a 2-distance $(\Delta + 1)$-coloring. Kim et al. [11] found the 2-distance chromatic number of the direct product of two cycles. Some cases of coloring were studied by Kristiana [13] and Dafik et al. [7]. In this paper, we study the 2-distance chromatic number of some wheel related graphs.

2. Main Results

In this paper, we have found the exact value of 2-distance chromatic number of some related wheel graphs, namely wheel, fan, helm and web graphs. All the values of their chromatic numbers attain the best lower bound. The theorems are as follows:

Theorem 2.1. *Let W_n be a wheel graph for $n \geq 3$. Then the 2-distance chromatic number of wheel graph is $\chi^2(W_n) = n + 1$.*

Proof. The graph W_n is a connected graph with vertex set $V(W_n) = \{x, x_i : 1 \leq i \leq n\}$ and edge set

$$E(W_n) = \{xx_i : 1 \leq i \leq n\} \cup \{x_1x_n, x_ix_{i+1} : 1 \leq i \leq n-1\}.$$

The cardinality of vertices $|V(W_n)| = n + 1$ and the cardinality of edges $|E(W_n)| = 2n$.

We will prove that the lower bound of the 2-distance chromatic number of W_n is $\chi^2(W_n) \geq n + 1$. We assume that $\chi^2(W_n) > n + 1$ by taking $\chi^2(W_n) = n$. Thus, we have n-coloring of W_n and the graph W_n consists of one central vertex and n vertices in its rim (outer cycle). Since the central vertex is adjacent to all vertices in rim, it implies $c_1(x) \neq c_1(v)$ with v is a vertex in its rim. Hence, all vertices in rim have $(n-1)$-coloring. If we assign all vertices in rim with $(n-1)$ colors, then there are at least two vertices which have same colors. It contradicts with the definition of 2-distance chromatic number which states that every two vertices u, v at distance at most two from each other should have different colors. We know that diameter of W_n is 2, since the shortest path between two vertices in rim is $u - x - v$. If two vertices u, v in rim have the same color, then two vertices at distance at most two from $u - x - v$ have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of W_n is $\chi^2(W_n) \geq n + 1$.

Furthermore, we define a map $c_1 : V(W_n) \to \{1, 2, 3, ..., k\}$ with k-color by the following function:

$$c_1(v) = \begin{cases} 1, & \text{if } v = x, \\ i+1, & \text{if } v = x_i, \ 1 \le i \le n. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n+1$. In other words, the upper bound of the 2-distance chromatic number of W_n is $\chi^2(W_n) \le n+1$. It concludes that the 2-distance chromatic number of W_n is $\chi^2(W_n) = n+1$. □

Theorem 2.2. *Let F_n be a fan graph for $n \ge 3$. Then the 2-distance chromatic number of fan graph is $\chi^2(F_n) = n+1$.*

Proof. The graph F_n is a connected graph with vertex set $V(F_n) = \{x, x_i : 1 \le i \le n\}$ and edge set

$$E(F_n) = \{xx_i : 1 \le i \le n\} \cup \{x_i x_{i+1} : 1 \le i \le n-1\}.$$

The cardinality of vertices $|V(F_n)| = n+1$ and the cardinality of edges $|E(F_n)| = 2n-1$.

We will prove that the lower bound of the 2-distance chromatic number of W_n is $\chi^2(F_n) \ge n+1$. We assume that $\chi^2(F_n) > n+1$ by taking $\chi^2(F_n) = n$. Thus, we have n-coloring of F_n. The graph F_n consists of one central vertex and n vertices in its rim (path). Since it is adjacent to all vertices in its rim, $c_1(x) \ne c_1(v)$ with v is the vertex in rim. Hence, all vertices in rim have $(n-1)$-coloring. If we assign the vertex in rim with $(n-1)$ colors, then there are at least two vertices which have the same color. Based on definition of 2-distance chromatic number, every two vertices u, v at distance at most two from each other must have different colors. We know that diameter of F_n is 2, since the shortest path between two vertices in rim is $u - x - v$. If two vertices u, v in rim have the same color, then two vertices at distance at most two from $u - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of F_n is $\chi^2(F_n) \geq n + 1$.

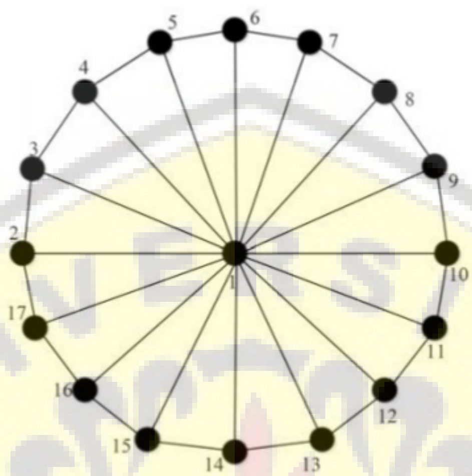

Figure 1. Example of 2-distance coloring of W_{16}.

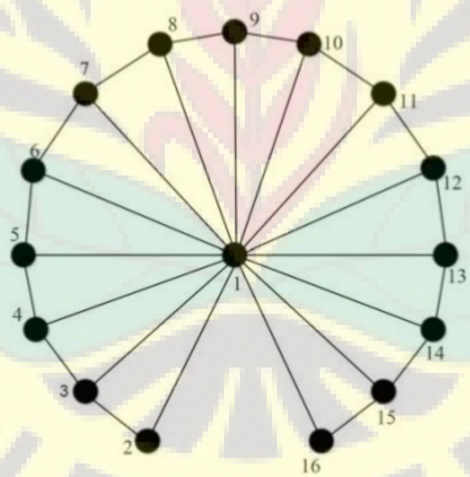

Figure 2. Example of 2-distance coloring of F_{15}.

Furthermore, define a coloring function $c_2 : V(F_n) \to \{1, 2, 3, ..., k\}$ with k-colors by the following:

$$c_2(v) = \begin{cases} 1, & \text{if } v = x, \\ i+1, & \text{if } v = x_i, 1 \le i \le n. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n+1$. In other words, the upper bound of the 2-distance chromatic number of F_n is $\chi^2(F_n) \le n+1$. It concludes that the 2-distance chromatic number of F_n is $\chi^2(F_n) = n+1$. □

Theorem 2.3. *Let H_n be a helm graph for $n \ge 3$. Then the 2-distance chromatic number of helm graph is*

$$\chi^2(H_n) = \begin{cases} 5, & \text{if } n = 3, \\ n+1, & \text{if } n \ge 4. \end{cases}$$

Proof. The helm graph H_n is a graph obtained from the wheel W_n by attaching a pendant edge at each vertex of the cycle C_n. Thus, the vertex set is $V(H_n) = \{x, x_i, y_i : 1 \le i \le n\}$ and the edge set is

$$E(H_n) = \{xx_i, x_iy_i : 1 \le i \le n\} \cup \{x_nx_1, x_ix_{i+1} : 1 \le i \le n-1\}.$$

The cardinality of vertices $|V(H_n)| = 2n+1$ and cardinality of edges $|E(H_n)| = 3n$. To prove the least 2-distance chromatic number, we will describe the proof in two cases as follows:

Case 1. For $n = 3$, we will prove that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \ge 5$. We assume that $\chi^2(H_n) > 5$ by taking $\chi^2(H_n) = 4$. Thus, we have 4-colorings of H_n and the graph H_n consists of one central vertex, 3 vertices in rim (inner cycle) and 3 vertices in pendant edges. Since the central vertex is adjacent to all vertices in rim, $c_4(x) \ne c_4(v) = c_4(u)$ with v, u as the vertices in rim and pendant edges, respectively. Hence, all vertices in rim have 3-coloring. If we assign the vertex in rim and pendant edges with 3 colors, then there are at least two vertices which have same color. Based on definition 2-distance chromatic number, it states that every two vertices v, w at distance at most two from each other have different colors (the vertex v in rim and w in pendant edges). We know that distance between each vertex in rim is 2, since the shortest path between two vertices in rim and pendant edges is $w - x - v$. If two vertices w, v in rim and pendant edges have same color, then two vertices at distance at most two from $w - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \geq 5$.

Furthermore, define a map $c_3 : V(H_n) \to \{1, 2, 3, ..., 5\}$ with k-colors by the following coloring function:

$$c_3(v) = \begin{cases} 1, & \text{if } v = x, \\ i+1, & \text{if } v = x_i, 1 \leq i \leq 3, \\ 5, & \text{if } v = y_i, 1 \leq i \leq 3. \end{cases}$$

It is clear to see that the maximum value of the set of colors is 5. In other words, the upper bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \leq 5$. It concludes that the 2-distance chromatic number of H_n is $\chi^2(H_n) = 5$ for $n = 3$.

Case 2. For $n \geq 4$, we will prove that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \geq n+1$. We assume that $\chi^2(H_n) >$ $n+1$ by taking $\chi^2(H_n) = n$. Thus, we have n-coloring of H_n and the graph H_n consists of one central vertex, n vertices in rim (inner cycle) and n vertices in pendant edges with condition that central vertex which is adjacent to all vertices in rim should satisfy $c_4(x) \neq c_4(v) = c_4(u)$ with v, u as the vertices in rim and pendant edges, respectively. Hence, all vertices in rim have $(n-1)$-coloring. If we assign the vertex in rim with $(n-1)$ colors, then there are at least two vertices which have same color. Based on definition of 2-distance chromatic number, it states that every two vertices v, w at distance at most two from each other have different colors. We know that distance between each vertex in rim is 2, since the shortest path between two vertices in rim is $w - x - v$. If two vertices w, v in rim have same color, then two vertices at distance at most two from $w - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \geq n+1$.

Furthermore, define a coloring function $c_4 : V(H_n) \to \{1, 2, 3, ..., k\}$ with k-colors by

$$c_4(v) = \begin{cases} 1, & \text{if } v = x, \\ i+1, & \text{if } v = x_i, 1 \leq i \leq n, \\ i-1, & \text{if } v = y_i, 3 \leq i \leq n, \\ n-1, & \text{if } v = y_1, \\ n, & \text{if } v = y_2. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n+1$. In other words, the upper bound of the 2-distance chromatic number of H_n is $\chi^2(H_n) \leq n+1$. It concludes that the 2-distance chromatic number of H_n is $\chi^2(H_n) = n+1$ for $n \geq 4$.

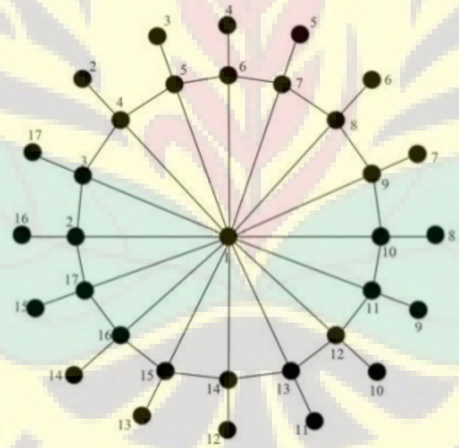

Figure 3. Example of 2-distance coloring of H_{16}.

□

Theorem 2.4. *Let $W_{2,n}$ be a web graph with $n \geq 3$. Then the 2-distance chromatic number of web graph is*

$$\chi^2(W_{2,n}) = \begin{cases} 7, & \text{if } n = 3, \\ n+1, & \text{if } n \geq 4. \end{cases}$$

Proof. The web graph $W_{2,n}$ is a graph obtained by joining the pendant vertices of helm graph to cycle (outer cycle) and adding one pendant edge to each vertex in outer cycle with vertex set $V(W_{2,n}) = \{x, x_i, y_i, z_i : 1 \leq i \leq n\}$ and edge set

$$E(W_{2,n}) = \{xx_i, x_iy_i, y_iz_i : 1 \leq i \leq n\}$$
$$\cup \{x_nx_1, y_ny_1, x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n-1\}.$$

The cardinality of vertices $|V(W_{2,n})| = 3n+1$ and cardinality of edges $|E(W_{2,n})| = 5n$. To prove the least 2-distance chromatic number, we will describe the proof in two cases as follows:

Case 1. For $n = 3$, we will prove that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq 7$. We assume that $\chi^2(W_{2,n}) > 7$ by taking $\chi^2(W_{2,n}) = 6$. Thus, we have 6-colorings of $W_{2,n}$ and the graph $W_{2,n}$ consists of one central vertex, 3 vertices in inner cycle, 3 vertices in outer cycle and 3 vertices in pendant edges. Since the central vertex is adjacent to all vertices in rim, $c_5(x) \neq c_5(v) = c_4(z) \neq c_5(u)$ with v, u, z as the vertices in inner cycle, outer cycle and pendant edges, respectively. Hence, all vertices in outer have 3-colorings. If we assign the vertex in inner cycle and outer cycle with 5 colors, then there are at least two vertices which have same color. Based on definition 2-distance chromatic number, every two vertices v, u at distance at most two from each other have different colors (the vertex v in inner cycle and u in outer cycle). We know that distance between each vertex in inner cycle and outer cycle is 2 such that the shortest path between two vertices in rim and pendant edges is $u - x - v$ If two vertices u, v in rim and pendant edges have same color, then two vertices at distance at most two from $u - x - v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq 7$.

Furthermore, define a coloring function $c_5 : V(W_{2,n}) \to \{1, 2, 3, ..., 7\}$ with k-colors by

$$c_5(v) = \begin{cases} 1, & \text{if } v = x, \\ i+1, & \text{if } v = x_i, 1 \leq i \leq 3, \\ n+i+1, & \text{if } v = y_i, 1 \leq i \leq 3, \\ 2, & \text{if } v = z_3, \\ n+i-1, & \text{if } v = z_i, 1 \leq i \leq 2. \end{cases}$$

It is clear to see that the maximum value of the set of colors is 7. In other words, the upper bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \leq 7$.

It concludes that the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) = 7$ for $n = 3$.

Case 2. For $n \geq 4$, we will prove that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq n+1$. We assume that $\chi^2(W_{2,n}) > n+1$ by taking $\chi^2(H_n) = n$. Thus, we have n-colorings of $W_{2,n}$ and the graph $W_{2,n}$ consists of one central vertex, n vertices in inner cycle, n vertices in outer cycle and n vertices in pendant edges with condition that central vertex adjacent to all vertices in inner cycle, then $c_6(x) = c_6(z) \neq c_6(v) = c_6(u)$ with v, u, z as the vertices in inner cycle, outer cycle and pendant edges, respectively. Hence, all vertices in inner cycle have $(n-1)$-

colorings. If we assign the vertex in inner cycle with $(n-1)$ colors, then there are at least two vertices which have same color. Based on definition 2-distance chromatic number, every two vertices v, w at distance at most two from each other have different colors. We know that distance between each vertex in inner cycle is 2 such that the shortest path between two vertices in inner cycle is $w-x-v$. If two vertices w, v in rim have same color, then two vertices at distance at most two from $w-x-v$ also have some colors, it is a contradiction. Hence, we obtain that the lower bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \geq n+1$.

Furthermore, define a coloring function $c_6 : V(W_{2,n}) \to \{1, 2, 3, \ldots, 7\}$ with k-colors by

$$c_6(v) = \begin{cases} 1, & \text{if } v = x \text{ and } v = z_i, 1 \leq i \leq n, \\ i+1, & \text{if } v = x_i, 1 \leq i \leq n, \\ i-1, & \text{if } v = y_i, 3 \leq i \leq n, \\ n+i-1, & \text{if } v = y_i, 1 \leq i \leq 2. \end{cases}$$

It is clear to see that the maximum value of the set of colors is $n+1$. In other words, the upper bound of the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) \leq n+1$. It concludes that the 2-distance chromatic number of $W_{2,n}$ is $\chi^2(W_{2,n}) = n+1$ for $n \geq 4$. □

3. Conclusion

In this paper, we have given some results on the 2-distance chromatic number of some wheel related graphs, namely fan, wheel, helm, and web graph. However, we need to characterize the other wheel related graphs, especially for any graph G. Hence the following problems arise naturally.

Open problem 3.1. Determine exact value of k-distance chromatic number or sharp lower bound and upper bound $\chi^2(G)$ when G is any wheel related graph?

Open problem 3.2. Determine lower and upper bounds of the k-distance chromatic number of graph operation including corona, Cartesian, comb product and others?

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