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On The Packing k – Coloring of Edge Corona Product

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***Abstract**: All graph in this paper is connected and simple graph. Let $d(u,v)$ be a distance between any vertex u and v in graph $G=(V,E)$. A function $c:V(G)\rightarrow\{1,2,\ldots,k\}$ is called a packing k-coloring if every two vertices of color i are at least distance $i+1$. $\chi_p(G)$ or packing chromatic number of graph G is the smallest integer of k which has packing coloring. In this paper, we will study about packing k-coloring of graphs and determine their packing chromatic number. We have found the exact values of the packing coloring of edge corona product.*

Keywords: Packing k-Coloring, Packing Chromatic Number, Edge Corona Product.

1. INTRODUCTION

Let G be a connected graph and k be an integer, $k\geq 1$. A packing k−coloring of a graph G is a mapping $c:V(G)\rightarrow\{1,2,\ldots,k\}$ such that any two vertices of color i are at distance at least $i+1$ [1]. The packing chromatic number χ_p of a graph G is the smallest integer k for which G has packing k-coloring [2]. This concept firstly was introduced by Goddard, et.al, under the name broadcast coloring. Goddard, et.al [3] obtain the formula of NP-complete for general graphs in packing coloring problem and it is NP-complete even for trees is by Fiala and Golovach [4]. The packing colorings of distance graphs has found by O. Togni [5]. The study of packing coloring some graphs in [6], [7], [8].

For illustration of packing coloring and packing chromatic number is provided in Figure 1.

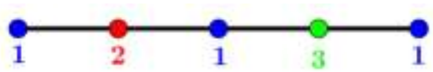

Figure 1: Packing chromatic number of path, $\chi_p(P_5)=3$

Proposition. Let H be a subgraph of graph G. Than $\chi_p(H)\leq\chi_p(G)$. [9]

Definition. Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The edge corona $G_1\Diamond G_2$ is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end-vertices of the *i*-th edge of G_1 to every vertex in the *i*-th copy of G_2. [10]

The last, Goddard, et.al [3] have discovered the exact value of the packing chromatic number of some graph, likely path, cycle, star, bipartite graph, etc.

2. RESULT

Theorem 2.1 The packing chromatic number of $P_2\Diamond P_n$ graph, for $n\geq 2$ is $\chi_p(P_2\Diamond P_n)=\lfloor\frac{n}{2}\rfloor+3$.

Proof. To prove that the packing chromatic number of $P_2\Diamond P_n$ graph, for $n\geq 2$ is $\chi_p(P_2\Diamond P_n)=\lfloor\frac{n}{2}\rfloor+3$, needs to be proven using the lower bound: $\chi_p(P_2\Diamond P_n)\geq\lfloor\frac{n}{2}\rfloor+3$ and upper bound: $\chi_p(P_2\Diamond P_n)\leq\lfloor\frac{n}{2}\rfloor+3$.

First, we prove that the lower bound of the packing chromatic number for $P_2\Diamond P_n$ graph is $\chi_p(P_2\Diamond P_n)\geq\lfloor\frac{n}{2}\rfloor+3$. Assume that $\chi_p(P_2\Diamond P_n)<\lfloor\frac{n}{2}\rfloor+3$. We take $\chi_p(P_2\Diamond P_n)=\lfloor\frac{n}{2}\rfloor+2$, than we give $P_2\Diamond P_n$ graph with $\lfloor\frac{n}{2}\rfloor+2$ color, so that:

- The distance of x_k to x_l, for $1\leq k,l\leq n$ is 2.
- Vertex x_1 neightboring with vertex x_2 and x_i, and vice versa, than color of vertex x_1 and x_2 not equal to x_i, so $c(x_1)\neq c(x_2)$.
- In $P_2\Diamond P_n$ graph, we have $2+\alpha$ color, which α is the color on subgraph P_n.
- In subgraph P_n there is color 1 for n is ood and for n is even have the different color because the distance is 2, so we get that subgraph P_n have $\lfloor\frac{n}{2}\rfloor$ color.
- We known that $\alpha=\lfloor\frac{n}{2}\rfloor$. We construct that the color of vertex with n is ood have color 1 because the distance of x_k to x_l is 2.
- $c(x_k)\neq 1$ for k is even, so $c(x_k)\geq 2$.
- Due to distance of x_k to x_l is 2, than $c(x_k)\neq c(x_l)$, so that we need $\lfloor\frac{n}{2}\rfloor+1$ color.
- If we color the subgraph P_n with $\lfloor\frac{n}{2}\rfloor$ color, than there two vertice with the same color, *contradiction*. So, we got that the lower bound of the packing chromatic number for $P_2\Diamond P_n$ graph is $\chi_p(P_2\Diamond P_n)\geq\lfloor\frac{n}{2}\rfloor+3$.

Furthermore, we prove that the upper bound of the packing chromatic number for $P_2\Diamond P_n$ graph is $\chi_p(P_2\Diamond P_n)\leq\lfloor\frac{n}{2}\rfloor+3$. We define $c:V(G)\rightarrow\{1,2,\ldots,k\}$ as follows:

$$c(v)=\begin{cases}1, \text{for } v=x_j;\ j=\text{odd}\\ 2+k, \text{for } v=x_j;\ j=\text{even};\ 0\leq k\leq n\\ c(x_j)+l, \text{for } v=x_i;1\leq l\leq 2\end{cases}$$

Based on the coloring function above, we get:

- Every two vertice with color 1, have minumim distance is two from the previous vertex.
- Every two vertice with color 2, have minumim distance is three from the previous vertex.
- Every two vertice with color 3, have minumim distance is four from the previous vertex.

So that, every two vertice with color *i*, have minimum distance is $i+1$ from the previous vertex. So we get upper bound of the packing chromatic number for $P_2\Diamond P_n$ graph is $\chi_p(P_2\Diamond P_n)\leq\lfloor\frac{n}{2}\rfloor+3$.

We got that the lower bound and upper bound of the packing chromatic number for $P_2\Diamond P_n$ graph is $\lfloor\frac{n}{2}\rfloor+3\leq\chi_p(P_2\Diamond P_n)\leq\lfloor\frac{n}{2}\rfloor+3$. So it can be concluded that the packing chromatic number of $P_2\Diamond P_n$ graph, for $n\geq 2$ is $\chi_p(P_2\Diamond P_n)=\lfloor\frac{n}{2}\rfloor+3$.

Theorem 2.2 The packing chromatic number of $P_2\Diamond S_n$ graph, for $n\geq 2$ is $\chi_p(P_2\Diamond S_n)=4$.

Proof. To prove that the packing chromatic number of $P_2\Diamond S_n$ graph, for $n\geq 2$ is $\chi_p(P_2\Diamond S_n)=4$, needs to be proven using the lower bound: $\chi_p(P_2\Diamond S_n)\geq 4$ and upper bound: $\chi_p(P_2\Diamond S_n)\leq 4$.

First, we prove that the lower bound of the packing chromatic number for $P_2\Diamond S_n$ graph is $\chi_p(P_2\Diamond S_n)\geq 4$. Assume that $\chi_p(P_2\Diamond S_n)<4$. We take $\chi_p(P_2\Diamond S_n)=3$, than we give $P_2\Diamond S_n$ graph with 3 color, so that:

- The distance of y_k to x_l, for $1\leq l\leq 2$ is 1.
- Vertex y_j neightboring with vertex y and x_i, and vice versa, than color of vertex y and x_i not equal to y_j, so $c(y)\neq c(x_i)$.
- In $P_2\Diamond S_n$ graph, we have 3 color, which there is 2 color on subgraph S_n.
- In subgraph S_n there is color 1 for $y_j;1\leq j\leq n$ and for vertex y have the different color because the distance is 1, so we get that subgraph S_n have 2 color.
- We known that in subgraph S_n have 2 color. We construct that the color of vertex $x_i\neq 1$ because the distance of y_j to x_i is 1.
- $c(x_i)$ for $1\leq i\leq 2$ cann't have color 2, so $c(x_i)\geq 3$.
- The distance of x_k to x_l is 1, than $c(x_k)\neq c(x_l)$, so we need 4 color.
- If we color the $P_2\Diamond S_n$ graph with 3 color, than there two vertice with the same color, *contradiction*. So, we got that the lower bound of the packing chromatic number for $P_2\Diamond S_n$ graph is $\chi_p(P_2\Diamond S_n)\geq 4$.

Furthermore, we prove that the upper bound of the packing chromatic number for $P_2\Diamond S_n$ graph is $\chi_p(P_2\Diamond S_n)\leq 4$. We define $c:V(G)\rightarrow\{1,2,\ldots,k\}$ as follows:

$$c(v)=\begin{cases}1, \text{for } v=y_j;1\leq j\leq n\\ 2, \text{for } v=y\\ 2+k, \text{for } v=x_i;1\leq i,k\leq 2\end{cases}$$

Based on the coloring function above, we get:

- Every two vertice with color 1, have minumim distance is two from the previous vertex.
- Every two vertice with color 2, have minumim distance is three from the previous vertex.

- Every two vertice with color 3, have minumim distance is four from the previous vertex.

So that, every two vertice with color *i*, have minimum distance is $i+1$ from the previous vertex. So we get upper bound of the packing chromatic number for $P_2 \lozenge S_n$ graph is $\chi_p(P_2 \lozenge S_n) \le 4$.

We got that the lower bound and upper bound of the packing chromatic number for $P_2 \lozenge S_n$ graph is $4 \le \chi_p(P_2 \lozenge S_n) \le 4$. So it can be concluded that the packing chromatic number of $P_2 \lozenge S_n$ graph, for $n \ge 2$ is $\chi_p(P_2 \lozenge S_n) = 4$.

Theorem 2.3 The packing chromatic number of $P_2 \lozenge C_n$ graph, for $n \ge 3$ is $\chi_p(P_2 \lozenge C_n) = \begin{cases} 5, \text{ for } n = 0 (\text{mod} 4) \\ 6, \text{ for } n = 1,2,3 (\text{mod} 4) \end{cases}$.

Proof. There are two cases in the packing chromatic number of $P_2 \lozenge C_n$ graph, for $n \ge 3$ namely for $n = 0(\text{mod})4$ and for $n = 1,2,3(\text{mod})4$. The explanation of the two cases as follows.

Case 1: for $n = 0(\text{mod})4$

To prove that the packing chromatic number of $P_2 \lozenge C_n$ graph, for $n \ge 3$ is $\chi_p(P_2 \lozenge C_n) = 5$, needs to be proven using the lower bound: $\chi_p(P_2 \lozenge C_n) \ge 5$ and upper bound: $\chi_p(P_2 \lozenge C_n) \le 5$.

First, we prove that the lower bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \ge 5$. Assume that $\chi_p(P_2 \lozenge C_n) < 5$. We take $\chi_p(P_2 \lozenge C_n) = 4$, than we give $P_2 \lozenge P_n$ graph with 4 color, so that:

- The distance of y_k to y_l, for $1 \le k,l \le n$ is 2.
- Vertex y_1 neightboring with vertex y_2 and x_i, and vice versa, than color of vertex y_1 and y_2 not equal to x_i, so $c(y_1) \ne c(y_2)$.
- In $P_2 \lozenge P_n$ graph, we have 4 color, which there is 3 color on subgraph C_n.
- In subgraph C_n there is color 1 for $n = 1(\text{mod})2$, color 2 for $n = 2(\text{mod})4$ because the distance is 3, and color 3 for $n = 4(\text{mod})4$ because the distance is 4, so we get that subgraph C_n have 3 color.
- We known that in subgraph C_n have 3 color. We construct that the color of vertex $x_i \ne 1$ because the distance of y_j to x_i is 1.
- $c(x_i)$ for $1 \le i \le 2$ cann't have color 2 because the distance of y_j to x_i is 1, so $c(x_i) \ge 3$.
- $c(x_i)$ for $1 \le i \le 2$ cann't have color 3 too, because the distance of y_j to x_i is 1, so $c(x_i) \ge 4$.
- The distance of x_k to x_l is 1, than $c(x_k) \ne c(x_l)$, so we need 5 color.
- If we color the $P_2 \lozenge C_n$ graph with 4 color, than there two vertice with the same color, *contradiction*. So, we got that the lower bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \ge 5$.

Furthermore, we prove that the upper bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \le 5$. We define $c : V(G) \to \{1, 2, \ldots, k\}$ as follows:

$$c(v) = \begin{cases} 1, \text{ for } v = y_j;\ j = 1(\text{mod})2; 1 \le j \le n \\ 2, \text{ for } v = y_j;\ j = 2(\text{mod})4; 1 \le j \le n \\ 3, \text{ for } v = y_j;\ j = 4(\text{mod})4; 1 \le j \le n \\ c(y_j) + k, \text{ for } v = x_i; 1 \le i,k \le 2 \end{cases}$$

Based on the coloring function above, we get:

- Every two vertice with color 1, have minumim distance is two from the previous vertex.
- Every two vertice with color 2, have minumim distance is three from the previous vertex.
- Every two vertice with color 3, have minumim distance is four from the previous vertex.

So that, every two vertice with color *i*, have minimum distance is $i+1$ from the previous vertex. So we get upper bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \le 5$.

We got that the lower bound and upper bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $5 \le \chi_p(P_2 \lozenge C_n) \le 5$. So it can be concluded that the packing chromatic number of $P_2 \lozenge C_n$ graph, for $n \ge 3$ is $\chi_p(P_2 \lozenge C_n) = 5$, for $n = 0(\text{mod})4$.

Case 2: for $n = 1,2,3(\text{mod})4$

To prove that the packing chromatic number of $P_2 \lozenge C_n$ graph, for $n \ge 3$ is $\chi_p(P_2 \lozenge C_n) = 6$, needs to be proven using the lower bound: $\chi_p(P_2 \lozenge C_n) \ge 6$ and upper bound: $\chi_p(P_2 \lozenge C_n) \le 6$.

First, we prove that the lower bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \ge 6$. Assume that $\chi_p(P_2 \lozenge P_n) < 6$. We take $\chi_p(P_2 \lozenge C_n) = 5$, than we give $P_2 \lozenge C_n$ graph with 5 color, so that:

- The distance of y_k to y_l, for $1 \le k,l \le n$ is 2.
- Vertex y_1 neightboring with vertex y_2 and x_i, and vice versa, than color of vertex y_1 and y_2 not equal to x_i, so $c(y_1) \ne c(y_2)$.
- In $P_2 \lozenge P_n$ graph, we have 5 color, which there is 4 color on subgraph C_n.

- In subgraph C_n there is color 1 for $n=1(\bmod)2$, color 2 for $n=2(\bmod)4$ because the distance is 3, color 3 for $n=4(\bmod)4$ because the distance is 4, and color 4 for *n* is otherwise, so we get that subgraph C_n have 4 color.
- We known that in subgraph C_n have 4 color. We construct that the color of vertex $x_i \neq 1$ because the distance of y_j to x_i is 1.
- $c(x_i)$ for $1 \le i \le 2$ cann't have color 2 because the distance of y_j to x_i is 1, so $c(x_i) \ge 3$.
- $c(x_i)$ for $1 \le i \le 2$ cann't have color 3 too, because the distance of y_j to x_i is 1, so $c(x_i) \ge 4$.
- $c(x_i)$ for $1 \le i \le 2$ cann't have color 4 too, because the distance of y_j to x_i is 1, so $c(x_i) \ge 5$.
- The distance of x_k to x_l is 1, than $c(x_k) \neq c(x_l)$, so we need 6 color.
- If we color the $P_2 \lozenge C_n$ graph with 5 color, than there two vertice with the same color, *contradiction*. So, we got that the lower bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \ge 6$.

Furthermore, we prove that the upper bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \le 6$. We define $c : V(G) \to \{1, 2, \ldots, k\}$ as follows:

$$c(v) = \begin{cases} 1, \text{ for } v = y_j;\ j = 1(\bmod)2; 1 \le j \le n \\ 2, \text{ for } v = y_j;\ j = 2(\bmod)4; 1 \le j \le n \\ 3, \text{ for } v = y_j;\ j = 4(\bmod)4; 1 \le j \le n \\ 4, \text{ for } v = y_j;\ j = n; 1 \le j \le n \\ c(y_j) + k, \text{ for } v = x_i; 1 \le i, k \le 2 \end{cases}$$

Based on the coloring function above, we get:

- Every two vertice with color 1, have minumim distance is two from the previous vertex.
- Every two vertice with color 2, have minumim distance is three from the previous vertex.
- Every two vertice with color 3, have minumim distance is four from the previous vertex.

So that, every two vertice with color *i*, have minimum distance is $i+1$ from the previous vertex. So we get upper bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $\chi_p(P_2 \lozenge C_n) \le 6$.

We got that the lower bound and upper bound of the packing chromatic number for $P_2 \lozenge C_n$ graph is $6 \le \chi_p(P_2 \lozenge C_n) \le 6$. So it can be concluded that the packing chromatic number of $P_2 \lozenge C_n$ graph, for $n \ge 3$ is $\chi_p(P_2 \lozenge C_n) = 6$, for $n = 1,2,3(\bmod)4$.

So we can conclede that in **Theorem 2.3** there are two cases namely for $n = 0(\bmod)4$ and for $n = 1,2,3(\bmod)4$ which have been proven right or true from the expanation above.

3. CONCLUSION

In this paper, we have studied packing coloring of edge corona product. We have concluded the exact value of the packing chromatic number of $P_2 \lozenge P_n$ graph, namely $\chi_p(P_2 \lozenge P_n) = \lfloor \frac{n}{2} \rfloor + 3$, the packing chromatic number of $P_2 \lozenge S_n$ graph, namely $\chi_p(P_2 \lozenge S_n) = 4$, and the packing chromatic number of $P_2 \lozenge C_n$ graph, namely for $n = 0(\bmod)4$ is $\chi_p(P_2 \lozenge C_n) = 5$ and for $n = 1,2,3(\bmod)4$ is $\chi_p(P_2 \lozenge C_n) = 6$. Hence the following problem arises naturally.

4. ACKNOWLEDGEMENT

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