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LOCAL IRREGULARITY VERTEX COLORING OF GRAPHS

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ABSTRACT

In this paper we study a new notion of coloring type of graph, namely a local irregularity vertex coloring. We define $l: V(G) \to \{1,2,\dots,k\}$ *is called vertex irregular* k*-labeling and* $w: V(G) \to N$ *where* $w(u) = \Sigma_{v \in N(u)} l(v)$*. By*

a local irregularity vertex coloring, we define a condition for f *if for every* $uv \in E(G), w(u) \neq w(v)$ *and* $max(l) = min\{max\{l_i\};\ l_i$ *vertex irregular labeling}. The chromatic number of local irregularity vertex coloring of* G*, denoted by* $\chi_{lis}(G)$*, is the minimum cardinality of the largest label over all such local irregularity vertex* *coloring. In this article, we study the local irregularity vertex coloring of some graphs and we have found the exact value of their chromatic number*

Key words: Irregularity strength, local irregularity, vertex coloring, path, cycle.

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1. INTRODUCTION

In graph theory, graph coloring is a special case of graph labeling. We traditionally label the graph element as "color" and the coloring process must fulfill certain properties. In simple terms, it can be explained that if we color the vertex elements of a graph such that there are no two adjacent vertices having the same color then it is considered to be a vertex coloring. Likewise, if we color the edge elements of a graph such that no two adjacent edges have the same color, it is called an edge coloring. Furthermore, if we color a face of a graph with the same principle, we have a type of face coloring. Kirchoff and Cayley (1821 - 1895) were founding fathers of graph coloring. In their era, it had been born one important thing in face coloring, namely a four-color theorem. It states that given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color. This is actually generalized to color the face of the graph embedded in the plane. With planar dualism, it becomes the colors of the vertices, and in this form it generalizes to all graphs.

By the four-color theorem, we have learned that the coloring of graph vertices is the actually a starting point of a graph coloring. Since other type of coloring problems can be transformed into a vertex versions. For example, the coloring of the edges of a graph, it is simply a coloring of the vertex from the line graph, and the coloring of the face from the plane graph is just a vertex of dual graph. However, non-vertex coloring problems are often stated and studied as they are, since some problems are best studied in non-vertex form, for examples are edge coloring of graph.

The natural extension of graph coloring gives some lot of derivative problems of coloring such as a list coloring, a multi coloring, a generalized circular coloring, and currently we have a local antimagic coloring, a local super (a, d)- antimagic coloring, a local face antimagic coloring, a local H-decomposition antimagic coloring. The antimagic type of coloring, we deals with vertex, edge as well face, thus we also has a total type of antimagic coloring. We assign a color to both elements of graph either vertex or edge. In this paper we study a new notion of coloring type of graph, namely a local irregularity vertex coloring.

From now on, we consider all studied graphs in this paper are finite, simple and connected graph. Graph $G = (V, E)$, for $v \in V(G)$, $d(v)$ and $N(v)$ denote the degree of v in G and the set of vertices adjacent to v in G, respectively. Let $G = (V, E)$ be a graph of order n and size n having no isolated vertices, a bijection $f: E \to \{1,2,\cdots,m\}$ is called a local antimagic labeling if for all $uv \in E(G)$ it has $w(u) \neq w(v)$ where $w(u) = \Sigma_{e\backslash inE(u)} f(e)$. This concept firstly was introduced by Amurugam, et.al [3]. A graph G is local antimagic if G has a local antimagic labeling, introduced by Amurugam, et.al [3] and written as $\chi_{la}(G)$, is the minimum number of colors taken over all colorings of $\boldsymbol{G}$ induced by local antimagic labeling of G. By this definition, Agustin, et.al [9] then introduced a new notion of a local edge antimagic labeling. They defined a bijection $f: V(G) \to \{1, 2, 3, \dots, |V(G)|\}$ is called a local edge antimagic total labeling if for adjacent edges e_1 and e_2, $w(e_1) \neq w(e_2)$, where for

$e = uv \in G, w(e) = f(u) + f(v)$. It is known that any local edge antimagic labeling induces a proper edge coloring of G if each edge e is assigned the color $w(e)$. The local edge antimagic chromatic number $\gamma_{lea}(G)$ is the minimum number of colors taken over all colorings induced by local edge antimagic labeling of G. Furthermore, Kurniawati, et.al [8] extended the concept by introducing a super local edge antimagic total coloring. A bijection $f: V(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ is called a local edge antimagic labeling if for any two adjacent edges e_1 and e_2, $w(e_1) \neq w(e_2)$, where for $e = uv \in E(G), w(e) = f(u) + f(uv) + f(v)$. Thus, any local edge antimagic labeling induces a proper edge coloring of G if each vertex u and edge uv is assigned the color $w(e)$. It is considered to be a super local edge antimagic total coloring, if the smallest labels appear in the vertices. The super local edge antimagic chromatic number, denoted by $\gamma_{leat}(G)$, is the minimum number of colors taken over all colorings induced by super local edge antimagic total labelings of G.

The last, it is very relevant with the one that we will study in this paper is a natural extension of local graph coloring introduced by Slamin, [4]. He introduced a distance irregular labeling of graphs. In this labeling, the weight of a vertices in G, is the sum of the labels of all vertices adjacent to u (distance 1 from u), that is $w(u) = \Sigma_{y \in N(u)} \lambda(y)$. The distance irregularity strength of G, denoted by $dis(G)$, is the minumum cardinality of the largest label k over all such irregular assignments. In this paper, we combine the two concepts, namely combining the local antimagic vertex coloring and the distance irregular labeling. We study the local irregularity vertex coloring. By the local irregularity vertex coloring, we recall a bijection $l: V(G) \to \{1,2,\cdots,k\}$ is called vertex irregular k-labeling and $w: V(G) \to N$ where $w(u) = \Sigma_{v \in N(u)} l(v)$. A condition for f to be local irregularity vertex coloring if for every $uv \in E(G), w(u) \neq w(v)$ and $max(l) = min\{max\{l_i\}; l_i$ vertex irregular labeling$\}$. The chromatic number of local irregularity vertex coloring of G, denoted by $\chi_{lis}(G)$, is the minimum cardinality of the largest label over all such local irregularity vertex coloring. The result of local antimagic vertex coloring graph are as follows.

Proposition 1. [3]

- For any tree T with leaves,$\chi_{la}(T) \geq l + 1$
- For the path P_n, with $n \geq 3$, $\chi_{la}(P_n) = 3$
- For the cycle C_n, $\chi_{la}(C_n) = 3$
- For the friendship graph, Fr_n, $\chi_{la}(Fr_n) = 3$

And the results of distance irregular labeling of graph are as follows.

Proposition 2. [4]

- For the complete graph K_n, with $n \geq 3$, dis$(K_n) = n$
- For the path P_n, with $n \geq 4$, $\boldsymbol{dis(P_n)} = \lceil \frac{n}{2} \rceil$
- For the cycle C_n, with $n \geq 5$, $\boldsymbol{n = 0, 1, 2, 5 (mod\ 8), dis\ (C_n)} = \lceil \frac{n+1}{2} \rceil$
- For the cycle W_n, with $n \geq 5$, $\boldsymbol{n = 0, 1, 2, 5 (mod\ 8), dis\ (W_n)} = \lceil \frac{n+1}{2} \rceil$

2. NEW RESULTS

Now, we are ready to show our result of the local irregularity vertex coloring of a graph and show the chromatic number local irregular of a graph. We present the local irregularity vertex coloring and the chromatic number local irregular of some graph, namely path, cycle, complete, star and friendship graph.

Definition 2.1. Suppose $l: V(G) \to \{1,2,\cdots,k\}$ is called vertex irregular k-labeling and $w: V(G) \to N$ where $w(u) = \Sigma_{v \in N(u)} l(v)$, l is called local irregularity vertex coloring, if

i. $max(l) = min\{max\{l_i\};\ l_i$ vertex irregular labeling$\}$

ii. for every $uv \in E(G), w(u) \neq w(v)$.

Definition 2.2. The chromatic number local irregular denoted by $\chi_{lis}(G)$, is minimum of cardinality local irregularity vertex coloring.

For illustration local irregularity vertex coloring and chromatic number local irregularity is provided in Figure 1.

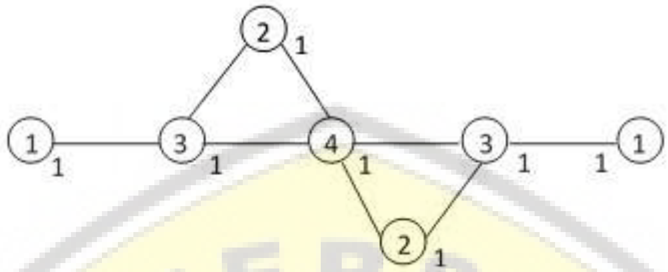

Figure 1. Chromatic number local irregular, $\chi_{lis}(G) = 4$

Lemma 2.1. Let G simple and connected graph, $\chi_{lis}(G) \geq \chi(G)$

Proof. Suppose $l: V(G) \to \{1,2,\cdots,k\}$ is called vertex coloring such that for every $uv \in E(G), l(u) \neq l(v)$ and $\chi(G)$ denoted minimum of cardinality vertex coloring. Based on Definition 1.1, $\chi(G) \leq |w(V(G))|$. Finally, $\chi(G) \leq min\{|w(V(G))|\} = \chi_{lis}(G)$. The proof is complete.

Theorem 2.1. Let P_n be a path graph. For $n \geq 2$, the chromatic number local irregular is

$$\chi_{lis}(P_n) = \begin{cases} 2, & for\ n = 2,3 \\ 3, & for\ n \geq 4 \end{cases}$$

Proof. $\boldsymbol{V(P_n) = \{x_1, x_2, \dots, x_n\}}$ and $E(P_n) = \{x_i x_{i+1};\ 1 \leq i \leq n-1\}$. To prove this theorem, we divide into three cases, namely $n = 2, n = 3$ and $n \geq 4$.

Case 1: For $\boldsymbol{n = 2}$

For $x_1, x_2 \in P_2$, if every vertex is labeled by 1 so $w(x_1) = w(x_2) = 1$. It contradicts to Definition 2.1, it implies that $l(x_1) = 1$ and $l(x_2) = 2$ or $l(x_1) = 2$ and $l(x_2) = 1$. We have $max(l) = 2$. Hence, $|w(V(P\_n))| = 2$, Thus $\chi_{lis}(P_2) = 2$.

Case 2: For $\boldsymbol{n = 3}$

For $x_1, x_2, x_3 \in P_3$, if every vertex labeled by 1 so $w(x_1) = w(x\_3) = 1$; $w(x_2) = 2$. We have $max(l) = 1$. Hence, $|w(V(P\_n))| = 2$, it means $\chi_{lis}(P_3) = 2$.

Case 2: For $\boldsymbol{n \geq 3}$

If every $x_i \in P_n$ labeled by 1, so we have $w(x_i) = l(x_{i-1}) + l(x_{i+1}) = 1 + 1 = 2$ and $w(x_{i+1}) = l(x_i) + l(x_{i+2}) = 1 + 1 = 2$, it means a contradiction by Definition 2.1, since $x_i x_{i+1}\} \in E(P_n), w(x_i) = w(x_{i+1})$. Hence, $max(l) = 2$.

For the proof $\chi_{lis}(P_n) = 3$; $n \geq 4$, based on Lemma 2.1, the lower bound is $\chi_{lis}(P_n) \geq \chi(P_n) = 2$. However, we can not attain the sharpest lower bound. For the upper bound, we define $l: V(P_n) \to \{1,2\}$ and we divide two cases, namely n even and n odd.

i. For n is even

The vertex irregular 2-labeling uses following the formula:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1,2,3(mod\ 4) \\ 2, & for\ i \equiv 0(mod\ 4) \end{cases}$$

Hence, $max(l) = 2$ and the labeling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 1, & for\ i \in \{1, n\} \\ 2, & for\ i \equiv 0(mod\ 2), 2 \le i \le n-1 \\ 3, & for\ i \equiv 1(mod\ 2), 2 \le i \le n-1 \end{cases}$$

For every $uv \in E(P_n), w(x_i) \neq w(x_{i+1})$.

ii. for n is odd

The vertex irregular 2-labeling uses the formula:

$$l(x_i) = \begin{cases} 1, & for\ i\ is\ even, 1 \le i \le n \\ 2, & for\ i\ is\ odd, 1 \le i \le n \end{cases}$$

Hence, $max(l) = 2$ and the labeling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 1, & for\ i \in \{1, n\} \\ 2, & for\ i\ is\ odd, 2 \le i \le n-1 \\ 4, & for\ i\ is\ even, 2 \le i \le n-1 \end{cases}$$

For every $uv \in E(P_n), w(x_i) \neq w(x_{i+1})$.

Clearly, the upper bound is $\chi_{lis}(P_n) \le 3$. Hence, $\chi_{lis}(P_n) = 3$. The proof is completed.

For an example, local irregularity vertex coloring of path graph is provided in Figure 2.

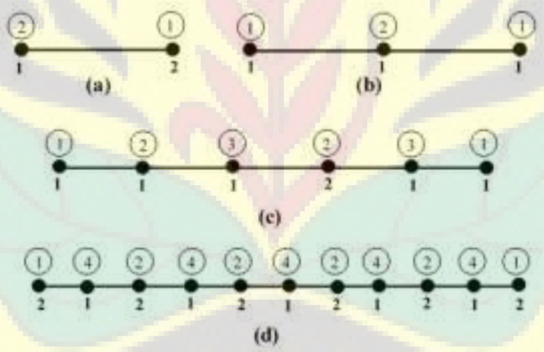

Figure 2. Chromatic number local irregular, (a) $\chi_{lis}(P_2) = 2$; (b) $\chi_{lis}(P_3) = 2$; (c) $\chi_{lis}(P_6) = 3$; (d) $\chi_{lis}(P_{11}) = 3$

Theorem 2.2. Let $\boldsymbol{C_n}$ be a cycle graph. For $n \ge 3$, the chromatic number local irregular is

$$\chi_{lis}(C_n) = \begin{cases} 2, & for\ n\ is\ even \\ 3, & for\ n\ \ is\ odd \end{cases}$$

Proof. The graph C_n is a connected graph with vertex set $V(C_n) = \{x_1, x_2, \dots, x_n\}$ and edge set $E(C_n) = \{x_i x_{i+1};\ 1 \le\ i \le\ n-1\} \cup \{x_n x_1\}$. To prove this theorem, we divide into three cases

Case 1: For $\boldsymbol{n}$ is even

If every $x_i \in C_n$ labeled by 1, so we have $w(x_i) = l(x_{i-1}) + l(x_{i+1}) = 1 + 1 = 2$ and $w(x_{i+1}) = l(x_i) + l(x_{i+2}) = 1 + 1 = 2$, it means a contradiction by Definition 2.1, since $x_i x_{i+1} \in\ E(C_n), w(x_i) = w(x_{i+1})$. Hence, $max(l) = 2$.

For the proof $\chi_{lis}(C_n) = 2$, based on Lemma 2.1, the lower bound is $\chi_{lis}(C_n) \geq \chi(C_n) = 2$. For the upper bound for the chromatic number local irregular, we define $l: V(C_n) \rightarrow \{1,2\}$. The vertex irregular 2-labeling uses the formula:

$$l(x_i) = \begin{cases} 1, & for\ i\ is\ even, 1 \leq i \leq n \\ 2, & for\ i\ is\ odd, 1 \leq i \leq n \end{cases}$$

Hence, $max(l) = 2$ and the labeling provides vertex-weight as follows

$$w(x_i) = \begin{cases} 2, & for\ i\ is\ even, 1 \leq i \leq n \\ 4, & for\ i\ is\ odd, 1 \leq i \leq n \end{cases}$$

For every $uv \in E(C_n)$, take any $u = x_i, v = x_{i+1}, \backslash 1 \leq i \leq n-1, w(x_i) \neq w(x_{i+1})$ and $u = x_1, v = x_n$, $w(x_1) \neq w(x_n)$, we have $|w(V(C_n))| = 2$. Clearly, the upper bound is $\chi_{lis}(C_n) \leq 2$. We have $2 = \chi(C_n) \leq \chi_{lis}(C_n) \leq |w(V(C_n))| = 2$ for n is even. Hence, $\chi_{lis}(C_n) = 2$.

Case 2: For $\boldsymbol{n}$ is odd

If every $x_i \in C_n$ labeled by 1, so we have $w(x_i) = l(x_{i-1}) + l(x_{i+1}) = 1 + 1 = 2$ and $w(x_{i+1}) = l(x_i) + l(x_{i+2}) = 1 + 1 = 2$, it means a contradiction by Definition 2.1, since $x_i x_{i+1} \in E(C_n), w(x_i) = w(x_{i+1})$. We have $max(l) = 3$.

For the proof $\chi_{lis}(C_n) = 3$, based on Lemma 2.1, the lower bound is $\chi_{lis}(C_n) \geq \chi(C_n) = 3$. For the upper bound, we define $l: V(C_n) \rightarrow \{1,2,3\}$ and we divide two case, namely $n = 3$ and $n \geq 4$.

i. for $n = 3$, the vertex irregular 3-labeling uses the formula:

$$l(x_i) = i, 1 \leq i \leq 3$$

Hence, $max(l) = 2$ and the labeling provides vertex-weight as follows

$$w(x_i) = 6 - i, 1 \leq i \leq 3$$

for every $uv \in E(C_n)$, take any $u = x_i, v = x_{i+1}, 1 \leq i \leq n-1, w(x_i) \neq w(x_i)$ and $u = x_1, v = x_n, w(x_1) \neq w(x_n)$.

ii. for $n \neq 3$, the vertex irregular 3-labeling uses the formula:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1,2,3 (mod\ 4), 1 \leq i \leq n-1 \\ 2, & for\ i = n \\ 3, & for\ i \equiv 0 (mod\ 4), 1 \leq i \leq n-1 \end{cases}$$

Hence, max(l) = 2 and the labeling provides vertex-weight as follows

$$w(x_i) = \begin{cases} 2, & for\ i = n\ or\ i \equiv 0 (mod\ 2), 2 \leq i \leq n-2 \\ 3, & for\ i = 1, n-1 \\ 4, & for\ i \equiv 1 (mod\ 2), 2 \leq i \leq n-2 \end{cases}$$

for every $uv \in E(C_n)$, take any $u = x_i, v = x_{i+1}, 1 \leq i \leq n-1, w(x_i) \neq w(x_{i+1})$ and $u = x_1, v = x_n$, $w(x_1) \neq w(x_n)$.

Clearly, $|w(V(C_n)| = 3$. It means $\chi_{lis}(C_n) \leq 3$. We have $3 = \chi(C_n) \leq \chi_{lis}(C_n) \leq |w(V(C_n)| = 3$ for n odd. Hence, $\chi_{lis}(C_n) = 3$. The proof is complete.

For an example, local irregularity vertex coloring of cycle graph is provided in Figure 3.

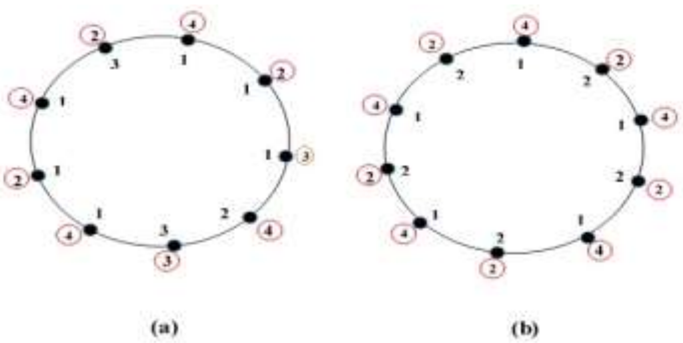

Figure 3. Chromatic number local irregular, (a) $\chi_{lis}(C_9) = 3$; (b) $\chi_{lis}(C_{10}) = 2$

Observation 1 [4]

Let u and w be any two adjacent vertices in a connected graph G. If $N(u) - \{w\} = N(w) - \{u\}$, then the labels of u and w must be distinct, that is, $l(u) \neq l(w)$.

Theorem 2.3

Let K_n be a complete graph. For $n \geq 4$, the chromatic number local irregular is $\chi_{lis}(K_n) = n$.

Proof:

$V(K_n) = \{x_i, 1 \leq i \leq n\}$ and $E(K_n) = \{x_i x_{i+k}, 1 \leq i \leq n, 1 \leq k \leq n - i\}$. Suppose $u, v \in V(K_n)$ such that $N(u) - \{v\} = N(v) - \{u\}$, based on Observation 1 that $l(u) \neq l(v)$ then the labels of all vertices in K_n are distinct, consequently that max(l) = n. Based on Lemma 2.1, the lower bound is $\chi_{lis}(K_n) \geq \chi(K_n) = n$. For the upper bound for the chromatic number local irregular, we define $l: V(K_n) \longrightarrow \{1,2, \dots, n\}$ where $l(x_i) = i$, $1 \leq i \leq n$ and we have $w(x_i) = \frac{n(n+1)}{2} - i, 1 \leq i \leq n$, such that $|w(V(K_n))| = n$. Finally, $n = \chi(K_n) \leq \chi_{lis}(K_n) \leq |w(V(K_n)| = n$, it means $\chi_{lis}(K_n = n$. The proof is complete.

Theorem 2.4

Let Fr_n be a Friendship graph. For $n \geq 3$, the chromatic number local irregular is $\chi_{lis}(Fr_n) = 3$.

Proof:

$V(Fr_n) = \{x_i, 0 \leq i \leq 2n\}$ and $E(Fr_n) = \{x_i x_{i+1}, i \text{ odd}, 1 \leq i \leq 2n\} \cup \{x_0 x_i, 1 \leq i \leq 2n\}$.
Similar with the proof of cycle graph, if every vertex in Fr_n labeled 1, so $w(x_i) = l(x_{i-1}) + l(x_{i+1}) = 1 + 1 = 2$ and $w(x_{i+1}) = l(x_i) + l(x_{i+2}) = 1 + 1 = 2$. Hence max($l$) = 2.

We will show $\chi_{lis}(Fr_n) = 3$ for n even.

Based on Lemma 2.1, the lower bound for the chromatic number is $\chi_{lis}(Fr_n) \geq \chi(Fr_n) = 3$ so that $\chi_{lis}(Fr_n) \geq 3$.

Furthermore, the upper bound for the chromatic number local irregular, we define $l: V(Fr_n) \rightarrow \{1,2\}$ with vertex irregular 2-labeling as follow:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 0, i \text{ is odd}, 1 \leq i \leq 2n \\ 2, & \text{for } i \text{ is even}, 1 \leq i \leq 2n \end{cases}$$

Hence, max(l) = 2 and the labeling provides vertex-weight as follows

$$w(x_i) = \begin{cases} 3n, & for\ i = 0 \\ 2, & \text{for } i \text{ is even, } 1 \le i \le 2n \\ 3, & \text{for } i \text{ is odd, } 1 \le i \le 2n \end{cases}$$

For every $uv \in E(Fr_n)$, take any $u = x_0$, $v = x_i$, $1 \le i \le 2n$, $w(x_0) \ne w(x_i)$ and $u = x_i$, $v = x_{i+1}$, $1 \le i \le 2n$, i odd, then $w(x_i) \ne w(x_{i+1})$. Thus $|w(V(Fr_n))| = 3$.

Clearly, $\chi_{lis}(Fr_n) \le 3$. We have $3 = \chi(Fr_n) \le \chi_{lis}(Fr_n) \le |w(V(Fr_n))| = 3$.

Hence $\chi_{lis}(Fr_n) = 3$. The proof is complete.

For an example, local irregularity vertex coloring of friendship graph is provided in Figure 4.

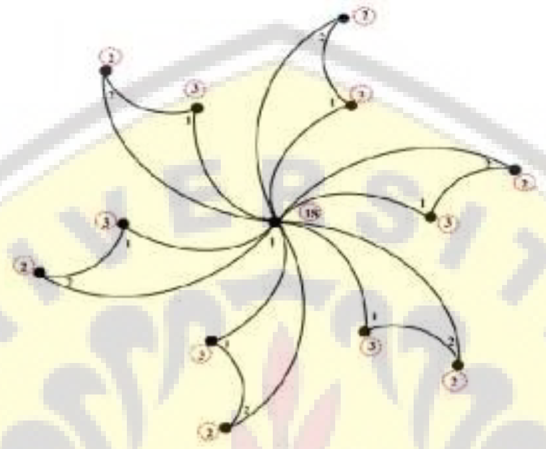

Figure 4. Chromatic number local irregular $\chi_{lis}(Fr_6) = 3$

Theorem 2.5

Let W_n be a wheel graph. For $n \ge 4$, the chromatic number local irregular is

$$\chi_{lis}(W_n) = \begin{cases} 3, & for\ n\ even \\ 4, & for\ n\ odd \end{cases}$$

Proof:

$V(W_n) = \{x, x_i, 1 \le i \le n\}$ dan $E(W_n) = \{xx_i, 1 \le i \le n\} \cup \{x_i x_{i+1}, 1 \le i \le n-1\} \cup \{x_n x_1\}$.

Case 1: For n is even

If every $v \in V(W_n)$ labeled by 1, so we have $w(x_i) = l(x) + l(x_{i-1} + l(x_{i+1}) = 1 + 1 + 1 = 3$ and $w(x_{i+1}) = l(x) + l(x_i) + l(x_{i+2}) = 1 + 1 + 1 = 3$, it means a contradiction by Definition 2.1, since $x_i x_{i+1} \in E(W_n)$, $w(x_i) = w(x_{i+1})$. Hence, $\max(l) = 2$.

For the proof $\chi_{lis}(W_n) = 3$, based on Lemma 2.1, the lower bound is $\chi_{lis}(W_n) \ge \chi(W_n) = 3$ n even. For the upper bound for the chromatic number local irregular, we define $l: V(W_n) \to \{1,2\}$

The vertex irregular 2-labeling uses the formula:

$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ is odd, } 1 \le i \le n \\ 2, & \text{for } i \text{ is even, } 1 \le i \le n \end{cases}$$

$$l(x) = 1$$

Hence, $\max(l) = 2$ and the labeling provides vertex-weight as follows

$$w(x_i) = \begin{cases} 3, & \text{for } i = 0, i \text{ is even, } 1 \le i \le 2n \\ 5, & \text{for } i \text{ is odd, } 1 \le i \le 2n \end{cases}$$

$$w(x) = \frac{3n}{2}$$

For every $uv \in E(W_n)$, take any $u = x_i$, $v = x_{i+1}$, $1 \le i \le n-1$, $w(x_i) \ne w(x_{i+1})$ and $u = x_1$, $v = x_n$, $w(x_1) \ne w(x_n)$ we have $|w(V(W_n))| = 3$.

Clearly, it means $\chi_{lis}(W_n) \le 3$. We have $3 = \chi(W_n) \le \chi_{lis}(W_n) \le \left|w\left(V(W_n)\right)\right| = 3$ for n is even. Hence, $\chi_{lis}(W_n) = 3$.

For an example, local irregularity vertex coloring of wheel graph is provided in Figure 4.

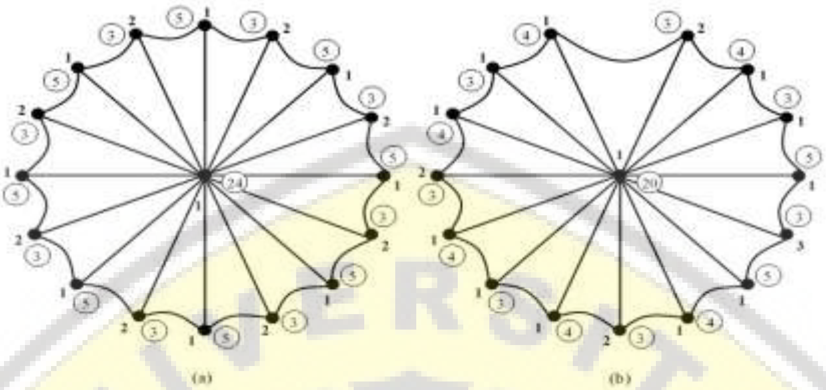

Figure 5. Chromatic number local irregular (a) $\chi_{lis}(W_{16}) = 3$; (a) $\chi_{lis}(W_{15}) = 4$

Case 2: For n is odd

If every $v \in V(W_n)$ labeled by 1, so we have $w(x_i) = l(x) + l(x_{i-1}) + l(x_{i+1}) = 1 + 1 + 1 = 3$ and $w(x_{i+1}) = l(x) + l(x_i) + l(x_{i+2}) = 1 + 1 + 1 = 3$, it means a contradiction by Definition 2.1, since $x_i x_{i+1} \in E(W_n)$, $w(x_i) = w(x_{i+1})$. Hence, $\max(l) = 3$.

For the proof $\chi_{lis}(W_n) = 4$, based on Lemma 2.1, the lower bound is $\chi_{lis}(W_n) \ge \chi(W_n) = 3$. For the upper bound for the chromatic number local irregular, we define $l: V(W_n) \to \{1,2,3\}$ the vertex irregular 3-labeling uses the formula:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1,2,3 (mod\ 4), 1 \le i \le n-1 \\ 2, & for\ i \equiv 0 (mod\ 4), 1 \le i \le n-1 \\ 3, & for\ i = n \end{cases}$$

$$l(x) = 1$$

Hence, $\max(l) = 3$ and the labeling provides vertex-weight as follows

$$w(x_i) = \begin{cases} 3, & for\ i\ even, 2 \le i \le n-2\ or\ i = n\ or\ n = 7 \\ 4, & for\ i\ odd, 2 \le i \le n-2\ or\ i = n\ or\ n \ne 7 \\ 5, & i = 1, n-1 \end{cases}$$

$$w(x) = \begin{cases} \frac{5(n-1)}{4} + 1, & for\ n \equiv 1 (mod\ 4) \\ \frac{5(n-3)}{4} + 5, & for\ n \equiv 3 (mod\ 4) \end{cases}$$

for every $uv \in E(W_n)$, take any $u = x_i$, $v = x_{i+1}$, $1 \le i \le n-1$, $w(x_i) \ne w(x_{i+1})$ and $u = x_1$, $v = x_n$, $w(x_1) \ne w(x_n)$.

Clearly, $|w(V(W_n))| = 4$. It means $\chi_{lis}(W_n) \le 4$. We have $4 = \chi(W_n) \le \chi_{lis}(W_n) \le \left|w\left(V(W_n)\right)\right| = 4$ for n odd. Hence, $\chi_{lis}(Fr_n) = 4$. The proof is complete.

Theorem 2.6

Let $K_{\{n,m\}}$ be a bipartite complete graph, the chromatic number local irregular is $\chi_{lis}(K_{\{n,m\}}) = 2$

Proof:

$V(K_{\{n,m\}}) = \{x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(K_{\{n,m\}}) = \{x_i y_j; 1 \leq i \leq n, 1 \leq j \leq m\}$

Case 1: for $n \neq m$

For every $v \in K_{\{n,m\}}$ labeled by 1, and we have max(l) = 1. For the proof $\chi_{lis}(K_{\{n,m\}}) = 2$, based on Lemma 2.1, the lower bound is $\chi_{lis}(K_{\{n,m\}}) \geq \chi(K_{\{n,m\}}) = 2$. For the upper bound, we define $\boldsymbol{l}: \boldsymbol{V}(\boldsymbol{K}_{\{n,m\}}) \rightarrow \{\mathbf{1}\}$, the vertex irregular 1-labeling uses the formula:

$$l(x_i) = 1,\ 1 \leq i \leq n$$

$$l(y_j) = 1,\ 1 \leq i \leq m$$

Hence, max(l) = 1 and the labeling provides vertex-weight as follows

$$w(x_i) = m,\ 1 \leq i \leq n$$

$$w(y_j) = n,\ 1 \leq i \leq m$$

for every $uv \in E(K_{\{n,m\}})$, take any $u = x_i$, $v = y_j$, $w(x_i) \neq w(y_j)$. Clearly, $|w(V(K_{\{n,m\}}))| = 2$. It means $\chi_{lis}(K_{\{n,m\}}) \leq 2$.

For an example, local irregularity vertex coloring of complete bipartite graph is provided in Figure 4.

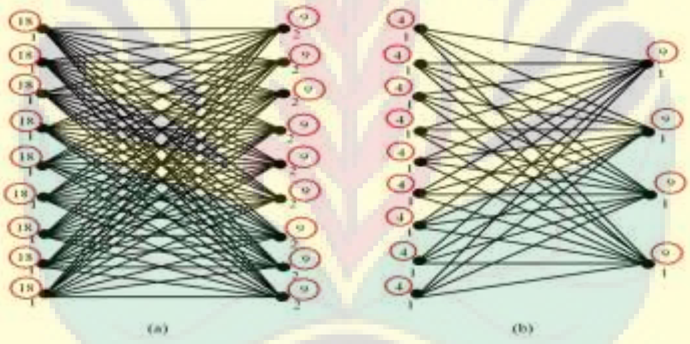

Figure 6. Chromatic number local irregular (a) $\chi_{lis}(K_{9,9}) = 2$; (a) $\chi_{lis}(K_{9,4}) = 2$

Case 2: for $n = m$

If every $v \in K_{\{n,m\}}$ labeled by 1, there $w(x_i) = w(y_j)$ so we have max(l) = 2. For the proof $\chi_{lis}(K_{\{n,m\}}) = 2$, based on Lemma 2.1, the lower bound is $\chi_{lis}(K_{\{n,m\}}) \geq \chi(K_{\{n,m\}}) = 2$. For the upper bound, we define $\boldsymbol{l}: \boldsymbol{V}(\boldsymbol{K}_{\{n,m\}}) \rightarrow \{\mathbf{1}, \mathbf{2}\}$,, the vertex irregular 2-labeling uses the formula:

$$l(x_i) = 1,\ 1 \leq i \leq n$$

$$l(y_j) = 2,\ 1 \leq j \leq m$$

Hence, max(l) = 2 and the labeling provides vertex-weight as follows

$$w(x_i) = m,\ 1 \leq i \leq n$$

$$w(y_j) = 2m,\ 1 \leq j \leq m$$

for every $uv \in E(K_{\{n,m\}})$, take any $u = x_i$, $v = y_j$, $w(x_i) \neq w(y_j)$. Clearly, $|w(V(K_{\{n,m\}}))| = 2$. It means $\chi_{lis}(K_{\{n,m\}}) \leq 2$.

Finally, we have $2=\chi(K_{\{n,m\}}) \leq \chi_{lis}(K_{\{n,m\}}) \leq |w(V(K_{\{n,m\}}))| = 2$ for $n = m$. Hence, $\chi_{lis}(K_{\{n,m\}}) = 2$. The proof is complete.

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