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The Construction of $P_2 \rhd H$-antimagic graph using smaller edge-antimagic vertex labeling

Rafiantika M. Prihandini[1,4], **I.H Agustin**[1,3], **Dafik**[1,2]

[1]CGANT University of Jember, Indonesia
[2]Mathematics Edu. Depart. University of Jember
[3]Mathematics Depart. University of Jember, Indonesia
[4]Elementary School Teacher Edu, University of Jember, Indonesia

E-mail: `rafiantikap.fkip@unej.ac.id`, `ikahestiagustin@gmail.com`, `d.dafik@unej.ac.id`

Abstract. In this paper we use simple and non trivial graph. If there exist a bijective function $g : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$, such that for all subgraphs $P_2 \rhd H$ of G isomorphic to H, then graph G is called an (a, d)-$P_2 \rhd H$-antimagic total graph. Furthermore, we can consider the total $P_2 \rhd H$-weights $W(P_2 \rhd H) = \sum_{v \in V(P_2 \rhd H)} f(v) + \sum_{e \in E(P_2 \rhd H)} f(e)$ which should form an arithmetic sequence $\{a, a + d, a + 2d, ..., a + (n - 1)d\}$, where a and d are positive integers and n is the number of all subgraphs isomorphic to H. Our paper describes the existence of super (a, d)-$P_2 \rhd H$ antimagic total labeling for graph operation of comb product namely of $G = L \rhd H$, where L is a (b, d^*)-edge antimagic vertex labeling graph and H is a connected graph.

1. Introduction

In this paper we consider simple and nontrivial graphs. One of the graph operation is a comb product. Saputro, *et.al* in [16], defined a comb product of L and H, denoted by $L \rhd H$. Comb product is a graph obtained by taking one copy of L and $|V(L)|$ copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of L. Thus, we have $V(L \rhd H) = \{(a, v)|a \in V(L), v \in V(H)\}$ and $(a, v)(b, w) \in E(L \rhd H)$ whenever $a = b$ and $vw \in E(H)$, or $ab \in E(L)$ and $v = w = o$. Labeling is one to one mapping which maps the set of graph elements into a set of integer. Furthermore, an (a, d)-edge-antimagic vertex labeling is one to one mapping from $g : V(G) \rightarrow \{1, 2, \ldots, v\}$ which maps the set of vertices into a set of integer such that the set of edge weights of all edges in G is $\{a, a + d, \ldots, a + (e - 1)d\}$, where $a > 0$ and $d \geq 0$ are integer set [2]. In this paper we deal with labelings with domain either the set of all vertices and edges.

Suppose $G = L \rhd H$ and $H \subseteq G$, If there exist a bijective function $g : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$, such that for all subgraphs $P_2 \rhd H$ of G isomorphic to H, then graph G is called an (a,d)-$P_2 \rhd H$-antimagic total graph. Furthermore, we can consider the total $P_2 \rhd H$-weights $W(P_2 \rhd H) = \sum_{v\in V(P_2\rhd H)} f(v) + \sum_{e\in E(P_2\rhd H)} f(e)$ which should form an arithmetic sequence $\{a, a+d, a+2d, ..., a+(n-1)d\}$, where a and d are positive integers and n is the number of all subgraphs isomorphic to H. Dafik *et al* in [7] proved cycle-super antimagicness of tensor product of two graphs, namely $C_r \otimes P_n$ for odd $r \geq 3$ any $n \geq 3$. They have showed the tensor product and its disjoint union for any $m \geq 2$, admit a super $(a,d)-C_{2r}$-antimagic total labeling for some feasible difference $d \in \{4r, 4r^2+2r, 6r^2\}$. Many paper have published, some other relevant results can be found in [1, 11, 12, 10] and [8, 14, 13, 15, 17, 18].

Recently, Dafik *et. al* in [5] showed the H-super antimagicness of a graph L when each edge of L is replaced by a graph H. They used a special technique, which is called *an integer set partition technique*, firstly introduced in [3]. They considered, for a connected version of graph and $k = 1, 2, \ldots, n-1$, a partition $\mathcal{P}^n_{c,d}(i,k)$ of the set $\{1, 2, \ldots, cn\}$ of n columns with $n \geq 2$, c-rows such that the sum of the numbers in the kth c-rows forms an arithmetic sequence of difference d.

Our paper investigate the existence of super (a,d)-$P_2 \rhd H$-antimagic total labeling of $G = L \rhd H$. We also show connection between $P_2 \rhd H$-antimagic total labeling and edge antimagic vertex labeling. Labels of vertices by following the EAVL pattern, furthermore we also found the general formula of feasible difference d of graph $G = L \rhd H$.

2. A Useful Lemma and Corollary

Let order of graph L, H be respectively $|V(L)|, |V(H)|$ and size $|E(L)|, |E(H)|$ respectively. The graph $G = L \rhd H$ is a connected graph with $|V(G)| = |V(L)||V(H)|$ and $|E(G)| = |V(L)||E(H)| + |E(L)|$.Thus $|V(G)| = np_H$ and $|E(G)| = nq_H + q_L$. The following lemma [4] is to define the upper bound of feasible d for $G = L \rhd H$ to be a super (a,d)-H-antimagic total labeling.

Lemma 1. [4] *Let G be a simple graph of order p and size q. If G is super (a,d)-H-antimagic total labeling then $d \leq \frac{(p_G-p_H)p_H+(q_G-q_H)q_H}{n-1}$, for $p_G = |V(G)|$, $q_G = |E(G)|$, $p_H = |V(H)|$, $q_H = |E(H)|$, and $n = |H|$.*

Corollary 1. *If the graph $G = L \rhd H$ admits super (a,d)-H-antimagic total labeling for integer $n \geq 3$, then $d \leq \frac{(p_H^2+q_H^2)(2p_L-4)+2q_Hq_L}{q_L-1}$*

Lemma 2. [6] *Let n and m be positive integers. The sum of $\mathcal{P}^n_{m,c_1}(i,k) = \{(k-1)n + k,\ \ 1 \leq i \leq c\}$ and $\mathcal{P}^n_{m,c_2}(i,k) = \{(k-1)c + i;\ \ 1 \leq i \leq c\}$ form an aritmatic sequence of difference $d_1 = c, d_2 = c^2\}$, respectively.*

3. The Results

Lemma 3. *Given that $G = L \rhd H$. If L admits an edge antimagic vertex labeling $(EAVL)$, then the sum of the corresponding partition label graph of H form an arithmetic sequence.*

Proof. By an (a,d)-edge-antimagic vertex labeling of a (p,q) graph L we mean a one to one mapping g from $V(L)$ onto $\{1, 2, \ldots, p\}$ such that the set of edge weights of all edges in G, $\{g(u) + g(v) : uv \in E(G)\}$, is $W = \{a, a+d, a+2d, \ldots, a+(q-1)d\}$, where $a > 0$ and $d \geq 0$. Let $\{h_i \in V(L); 1 \leq i \leq p\}$ indicates the order of the label, thus the

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arbitrary pairs, h_i, will form an arithmetic sequence. Let w be a set of an edge weight obtained by arbitrary sum of pairs h_i which $w = \{h_1 + h_2, h_2 + h_3, h_3 + h_4, \ldots, (h_1 + h_2) + (q_L - 1)d\} = \{a, a + d, a + 2d, \ldots, a + (q_L - 1)d\}$, thus we can have the following corresponding partition:

$$\begin{aligned}
\text{(i)} \sum_{\mathrm{i=1}}^{\mathrm{c}} \mathcal{P}^{\mathrm{n}}_{\mathrm{c,d^*}}(\mathrm{h}_1) + \sum_{\mathrm{i=1}}^{\mathrm{c}} \mathcal{P}^{\mathrm{n}}_{\mathrm{c,d^*}}(\mathrm{h}_2) &= \mathcal{C}^n_{c,d^*} + d^*h_1 + \mathcal{C}^n_{c,d^*} + d^*h_2 \\
&= \mathcal{C}^n_{c,d^*} + d^*h_1 + \mathcal{C}^n_{c,d^*} + d^*h_2 \\
&= 2\mathcal{C}^n_{c,d^*} + d^*(h_1 + h_2) \\
\text{(ii)} \sum_{\mathrm{i=1}}^{\mathrm{c}} \mathcal{P}^{\mathrm{n}}_{\mathrm{c,d^*}}(\mathrm{h}_2) + \sum_{\mathrm{i=1}}^{\mathrm{c}} \mathcal{P}^{\mathrm{n}}_{\mathrm{c,d^*}}(\mathrm{h}_3) &= \mathcal{C}^n_{c,d^*} + d^*h_2 + \mathcal{C}^n_{c,d^*} + d^*h_3 \\
&= \mathcal{C}^n_{c,d^*} + d^*h_2 + \mathcal{C}^n_{c,d^*} + d^*h_3 \\
&= 2\mathcal{C}^n_{c,d^*} + d^*(h_2 + h_3) \\
\text{(iii)} \sum_{\mathrm{i=1}}^{\mathrm{c}} \mathcal{P}^{\mathrm{n}}_{\mathrm{c,d^*}}(\mathrm{h}_3) + \sum_{\mathrm{i=1}}^{\mathrm{c}} \mathcal{P}^{\mathrm{n}}_{\mathrm{c,d^*}}(\mathrm{h}_4) &= \mathcal{C}^n_{c,d^*} + d^*h_3 + \mathcal{C}^n_{c,d^*} + d^*h_4 \\
&= \mathcal{C}^n_{c,d^*} + d^*h_3 + \mathcal{C}^n_{c,d^*} + d^*h_4 \\
&= 2\mathcal{C}^n_{c,d^*} + d^*(h_3 + h_4)
\end{aligned}$$

From the above, we can easily see that the sum of corresponding partition, $\mathcal{P} = \{2\mathcal{C}^n_{c,d^*} + d^*(a), 2\mathcal{C}^n_{c,d^*} + d^*(a + d), 2\mathcal{C}^n_{c,d^*} + d^*(a + 2d) \ldots, 2\mathcal{C}^n_{c,d^*} + d^*(a + (q_L - 1)d)\}$, form an arithmetic sequence with $b = dd^*$. □

Theorem 1. *Given the any graph H. If L admits $(b, d^*)-EAVL$, then the comb product of the connected graph $G = L \rhd H$ admits super $(a, d) - P_2 \rhd H$ antimagic total labeling with $d = d^* + d^*(d_v + d_e) + 1$.*

Proof. Let l be a $(b, d^*) - EAVL$ of graph L. The set of all edge weights of the edges of L under the labeling l is:
$\{w^l(e) : e \in E(L)\} = \{b, b + d^*, b + 2d^*, \ldots, b + (q_L - 1)d^*\}$ Denote the edges of graph L by the symbols $e_1, e_2, \ldots, e_{q_L}$ such that:
$\{w^l(e) = b + (k - 1)d^*$ with $1 \le k \le q_L\}$ Let H be a connected graph, and $G = L \rhd H$ contains p_L subgraphs isomorphic to H, say $H_1, H_2, \ldots, H_{p_L}$ where the subgraphs H_i replaces the vertex v_i in graph L, $i = 1, 2, \ldots, p_L$. Construct a total labeling g,$g : V(L \rhd H) \cup E(L \rhd H) \to \{1, 2, \ldots, p_Lp_H + q_L + p_Lq_H\}$ constitute the following set:

$$\begin{aligned}
g(V_{p_H}) &= \{\mathcal{P}^{p_L}_{p_H-1,d_v}(i, k) \oplus p_L\} \\
g(E_{q_L}) &= p_Hp_L + j; 1 \le k \le q_L \\
g(E_{q_H}) &= \{\mathcal{P}^{p_L}_{q_H,d_e}(i, k) \oplus [p_Lp_H + q_L]\}
\end{aligned}$$

where d_v depends on $p_H - 1$ and d_e depends on q_H. Furthermore the weight of the subgraphs H_i, $i = 1, 2, \ldots, p_L$ in the following way:

$$W = \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e)$$

$$= (b + (k-1)d^*) + (\sum_{i=1}^{p_H-1} (\mathcal{P}^{p_L}_{p_H-1,d_v}(k) \oplus p_L) + (p_H p_L + k) + (\sum_{i=1}^{q_H} (\mathcal{P}^{p_L}_{q_H,d_e}(k) \oplus p_H p_L + q_L)$$

Based on Lemma 3 we are obtained

$$\begin{aligned} &= [b + (k-1)d^*] + [2C^{p_L}_{p_H-1,d_v} + d_v(b + (k-1)d^*)] + [p_H p_L + k] + \\ &\quad [2C^{p_L}_{q_H,d_e} + d_e(b + (k-1)d^*)] \\ &= b - d^* + 2C^{p_L}_{p_H-1,d_v} + bd_v - d_v d_1 + p_H p_L + 2C^{p_L}_{q_H,d_e} + bd_e - d_e d^* + \\ &\quad (d^* + d^*(d_v + d_e) + 1)k \end{aligned}$$

From the above, we can easily see that $W = \{[b - d^* + 2C^{p_L}_{p_H-1,d_v} + bd_v - d_v d_1 + p_H p_L + 2C^{p_L}_{q_H,d_e} + bd_e - d_e d^*] + d^* + d^*(d_v + d_e) + 1, [b - d^* + 2C^{p_L}_{p_H-1,d_v} + bd_v - d_v d_1 + p_H p_L + 2C^{p_L}_{q_H,d_e} + bd_e - d_e d^*] + 2(d^* + d^*(d_v + d_e) + 1), \dots, [b - d^* + 2C^{p_L}_{p_H-1,d_v} + bd_v - d_v d_1 + p_H p_L + 2C^{p_L}_{q_H,d_e} + bd_e - d_e d^*] + k(d^* + d^*(d_v + d_e) + 1)\}$ form an arithmetic sequence. It completes the proof □

3.1. Special Families of Connected Graph

We have found a general formula for any graph. Now, in this section we describe the existence of super $(a, d) - P_2 \rhd H$ antimagicness of some special families namely $G = P_n \rhd H$ and $G = S_n \rhd H$.

Theorem 2. *For $n \geq 2$, the graph $G = P_n \rhd H$ admits a super (a, d)-$P_2 \rhd H$ antimagic total labeling with $a = n(c_1^2 - c_1 + {t_1}^2 - t_1) + 3(c_1 + {c_2}^2 + t_1 + {t_2}^2) + 2nc_1 + (c_2 - {c_2}^2) + 2c_2 n(c_1 + 1) + 2t_1(cn + 2n - 1) + (t_2 - {t_2}^2) + 2t_2(cn + nt_1 + 2n - 1) + 4$ and feasible $d = 2(c_1 + {c_2}^2 + t_1 + {t_2}^2) + 3$.*

Proof. Graph $G = P_n \rhd H$ has vertex set $V(G) = \{z_k; 1 \leq k \leq n\} \cup \{z_{i,k}; 1 \leq i \leq c; 1 \leq k \leq n\}$ and edge set $V(G) = \{e_k; 1 \leq k \leq n\} \cup \{e_{i,k}; 1 \leq i \leq c; 1 \leq k \leq n\}$. Suppose c and t are two fix positive integers, with $c = p_H - 1$ and $t = q_H$. By Lemma 2 for $i = 1, 2, \dots, c$ and $k = 1, 2, \dots, n$, we define the vertex and the edge labels as a linear combination of $\mathcal{P}^n_{c_1,c_1}(i, k); \mathcal{P}^n_{c_2,c_2^2}(i, k)$, written as follows:

$$\begin{aligned} g_1(z_k) &= \{k; 1 \leq k \leq n\} \\ g_1(z_{i,k}) &= \{\mathcal{P}^n_{c_1,c_1} \oplus n\} \cup \{\mathcal{P}^n_{c_2,c_2^2} \oplus n(c_1 + 1)\} \\ g_1(e_k) &= \{n(c+1) + k; 1 \leq k \leq n-1\} \\ g_1(e_{l,k}) &= \{\mathcal{P}^n_{t_1,t_1} \oplus [n(c+1) + (n-1)]\} \cup \{\mathcal{P}^n_{t_2,t_2^2} \oplus [n(c+1+t_1) + (n-1)]\} \end{aligned}$$

from the vertex and the edge labels, then it can be determined a function of the total vertex-weight and edge-weight, written as follows:

$$w^1_{g_1} = k + k + 1 = 2k + 1$$

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$$\begin{aligned}
w^2_{g_1} &= [\sum_{i=1}^{c}\mathcal{P}^n_{c_1,c_1}(i,k)\oplus nc_1+\sum_{i=1}^{c}\mathcal{P}^n_{c_1,c_1}(i,k+1)\oplus nc_1]+[\sum_{i=1}^{c}\mathcal{P}^n_{c_2,c_2^2}(i,k)\oplus\\
&\quad nc_2(c_1+1)+\sum_{i=1}^{c}\mathcal{P}^n_{c_2,c_2^2}(i,k+1)\oplus nc_2(c_1+1)]\\
&= [\mathcal{P}^n_{c_1,c_1}(k)\oplus nc_1+\mathcal{P}^n_{c_1,c_1}(k+1)\oplus nc_1]+[\mathcal{P}^n_{c_2,c_2^2}(k)\oplus nc_2(c_1+1)+\\
&\quad \mathcal{P}^n_{c_2,c_2^2}(k+1)\oplus nc_2(c_1+1)]\\
&= \{[\frac{n}{2}(c_1^2-c_1)+c_1k+nc_1]+[\frac{n}{2}(c_1^2-c_1)+c_1k+c_1+nc_1]\}+\\
&\quad \{[\frac{c_2-c_2^2}{2}+c_2^2k+c_2n(c_1+1)]+[\frac{c_2-c_2^2}{2}+c_2^2k+c_2^2+c_2n(c_1+1)]\}\\
&= \{n(c_1^2-c_1)+2c_1k+c_1+2nc_1\}+\{(c_2-{c_2}^2)+2{c_2}^2k+{c_2}^2+\\
&\quad 2c_2n(c_1+1)\}\\
w^3_{g_1} &= [\sum_{l=1}^{t}\mathcal{P}^n_{t_1,t_1}(i,k)\oplus r_1(cn+2n-1)+\sum_{l=1}^{t}\mathcal{P}^n_{t_1,t_1}(i,k+1)\oplus t_1(cn+2n-1)]+\\
&\quad [\sum_{l=1}^{t}\mathcal{P}^n_{t_2,t_2^2}(i,k)\oplus t_2(cn+nt_1+2n-1)+\sum_{l=1}^{t}\mathcal{P}^n_{t_2,t_2^2}(i,k+1)\oplus t_2(cn+nt_1+\\
&\quad 2n-1)]\\
&= [\mathcal{P}^n_{t_1,t_1}(k)\oplus t_1(cn+2n-1)+\mathcal{P}^n_{t_1,t_1}(k+1)\oplus t_1(cn+2n-1)]+[\mathcal{P}^n_{t_2,t_2^2}(k)\\
&\quad \oplus t_2(cn+nt_1+2n-1)+\mathcal{P}^n_{t_2,t_2^2}(k+1)\oplus t_2(cn+nt_1+2n-1)]\\
&= \{[\frac{n}{2}(t_1^2-t_1)+t_1k+t_1(cn+2n-1)]+[\frac{n}{2}(t_1^2-t_1)+t_1k+t_1+t_1(cn+\\
&\quad 2n-1)]\}+[\frac{t_2-t_2^2}{2}+t_2^2k+t_2^2+t_2(cn+nt_1+2n-1)]\}\{[\frac{t_2-t_2^2}{2}+t_2^2k+\\
&\quad t_2(cn+nt_1+2n-1)]\\
&= \{\ n({t_1}^2-t_1)+2t_1k+t_1+2t_1(cn+2n-1)\}+\{(t_2-{t_2}^2)+2{t_2}^2k+{t_2}^2+\\
&\quad 2t_2(cn+nt_1+2n-1)\}
\end{aligned}$$

The vertex and edge label under the labeling g_1 is a bijective function $g_1 : V(G)\cup E(G) \to \{1, 2, \ldots, p_G + q_G\}$. The total edge-weights of $G = P_n \rhd H$ under the labeling g_1, for $k = 1, 2, \ldots, n$, constitute the following sets:

$$\begin{aligned}
W^1_{g_1} &= w^1_{g_1}+w^2_{g_1}+n(c+1)+k+w^3_{g_1}\\
&= 2k+1+w^2_{g_1}+n(c+1)+k+w^3_{g_1}\\
&= C+[2(c_1+c_2^2+t_1+t_2^2)+3]k; 1\le k\le n-1
\end{aligned}$$

with $C = n(c_1^2-c_1)+c_1+2nc_1+(c_2-{c_2}^2)+{c_2}^2+2c_2n(c_1+1)+1+n({t_1}^2-t_1) +t_1+2t_1(cn+2n-1)+(t_2-{t_2}^2)+{t_2}^2+2t_2(cn+nt_1+2n-1)$. It is easy that the set of total edge-weights $W^1_{g_1}$ consists of an arithmetic sequence of the smallest value a when the total edge weights at $k = 1$ and the feasible difference $d = 2[c_1 + c_2^2 + t_1 + t_2^2] + 3$.

Since the biggest d is attained when $d = 2(c_2^2 + t_2^2)$ then, for $c = p_H$ and $t = q_H$, it gives $d \leq \frac{(p_H^2+q_H^2)(2n-4)+2(n-1)q_H}{n-2}$. It concludes the proof. $\square$

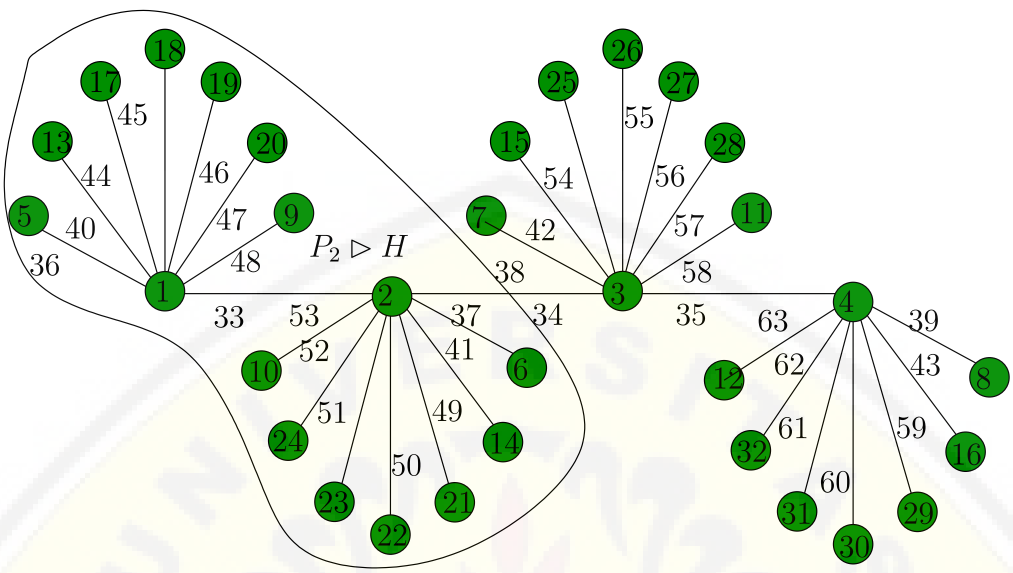

Figure 1. Illustration of graph $G = P_4 \rhd S_7$

Theorem 3. *For $n \geq 1$, the graph $G = S_n \rhd H$ admits a super (a, d)-$P_2 \rhd H$ antimagic total labeling with $a = n + 2 + (n+1)(c_1^2 - c_1) + c_1(n+2) + 2c_1(n+1) + c_2 + {c_2}^2(n+1) + 2c_2((n+1)(c_1+1)) + (n+1)({t_1}^2 - t_1) + t_1(n+2) + 2t_1((n+1)(c+1)+n) + t_2 + {t_2}^2(n+1) + 2{t_2}^2((n+1)(c+t_1+1)+n)$ feasible $d = c_1 + c_2^2 + t_1 + t_2^2 + 2$.*

Proof. Graph $G = S_n \rhd H$ has vertex set $V(G) = \{x\} \cup \{z_k; 1 \leq k \leq n\} \cup \{z_{i,k}; 1 \leq i \leq c; 1 \leq j \leq n\}$ and edge set $V(G) = \{e_k; 1 \leq k \leq n\} \cup \{e_{l,k}; 1 \leq l \leq t; 1 \leq k \leq n\}$. Let c and t be positive integers, with $c = p_H - 1$ and $t = q_H$. For $i = 1, 2, \ldots, c$ and $k = 1, 2, \ldots, n$, by Lemma 2, 3, 4 and 5 we define the vertex and the edge labels as a linear combination of $\mathcal{P}^n_{c_1,c_1}(i,k)$ and $\mathcal{P}^n_{c_2,c_2^2}(i,k)$ as follows:

$$\begin{aligned}
g_2(x) &= \{n+1\} \\
g_2(z_k) &= \{k; 1 \leq k \leq n\} \\
g_2(x_{i,k}) &= \{\mathcal{P}^n_{c_1,c_1} \oplus (n+1)\} \cup \{\mathcal{P}^n_{c_2,c_2^2} \oplus (n+1)(c_1+1)\} \\
g_2(e_k) &= \{(n+1)(c+1) + k; 1 \leq k \leq n\} \\
g_2(e_{l,k}) &= \{\mathcal{P}^n_{t_1,t_1} \oplus [(n+1)(c+1)+n]\} \cup \{\mathcal{P}^n_{t_2,t_2^2} \oplus [(n+1)(c+t_1+1)+n]\}
\end{aligned}$$

then can be determined a function of the total vertex-weight and edge-weight

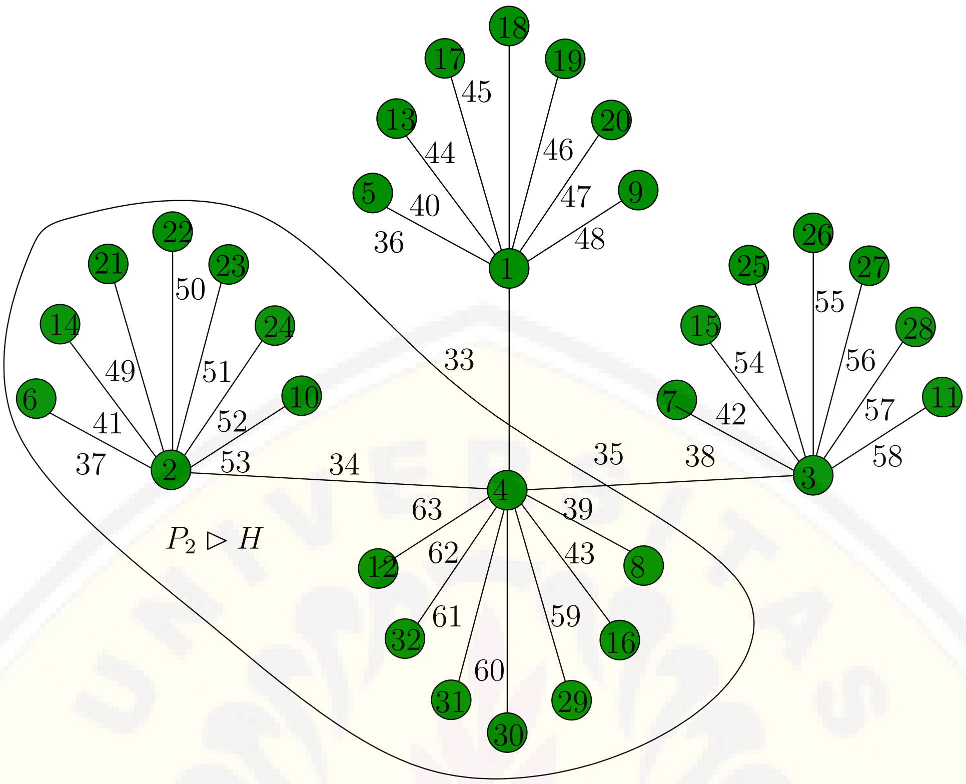

Figure 2. Illustration of graph $G = S_3 \rhd S_7$

$$
\begin{aligned}
w^1_{g_2} &= n + 1 + k \\
w^2_{g_2} &= [\sum_{i=1}^{c} \mathcal{P}^n_{c_1,c_1}(i,k) \oplus c_1(n+1) + \sum_{i=1}^{c} \mathcal{P}^n_{c_1,c_1}(i,n+1) \oplus c_1(n+1)] + \\
&\quad [\sum_{i=1}^{c} \mathcal{P}^n_{c_2,c_2^2}(i,k) \oplus c_2((n+1)(c_1+1)) + \sum_{i=1}^{c} \mathcal{P}^n_{c_2,c_2^2}(i,n+1) \\
&\quad \oplus c_2((n+1)(c_1+1))] \\
&= [\mathcal{P}^n_{c_1,c_1}(k) \oplus c_1(n+1) + \mathcal{P}^n_{c_1,c_1}(n+1) \oplus c_1(n+1)] + [\mathcal{P}^n_{c_2,c_2^2}(k) \\
&\quad \oplus c_2((n+1)(c_1+1)) + \mathcal{P}^n_{c_2,c_2^2}(n+1) \oplus c_2((n+1)(c_1+1))] \\
&= \{[\frac{n+1}{2}(c_1^2 - c_1) + c_1 k + c_1(n+1)] + [\frac{n+1}{2}(c_1^2 - c_1) + 2c_1(n+1)] \\
&\quad +[\frac{c_2 - c_2^2}{2} + c_2^2 k + c_2^2 + c_2((n+1)(c_1+1))] + [\frac{c_2 - c_2^2}{2} + c_2^2(n+1) \\
&\quad +c_2^2 + c_2((n+1)(c_1+1))]\} \\
&= [(n+1)(c_1^2 - c_1) + c_1(n+k+1) + 2c_1(n+1)] + [c_2 + c_2^2(n+k) \\
&\quad +2c_2((n+1)(c_1+1))] \\
w^3_{g_2} &= [\sum_{l=1}^{t} \mathcal{P}^n_{t_1,t_1}(i,k) \oplus t_1((n+1)(c+1)+n) + \sum_{l=1}^{t} \mathcal{P}^n_{t_1,t_1}(i,n+1)
\end{aligned}
$$

$$
\begin{aligned}
& \oplus t_1((n+1)(c+1)+n)] + [\sum_{l=1}^{t} \mathcal{P}^n_{t_2,t_2^2}(i,k) \oplus t_2(n(c+t_1+1)+1+n) \\
& + \sum_{l=1}^{t} \mathcal{P}^n_{t_2,t_2^2}(i,n+1) \oplus t_2(n(c+t_1+1)+1+n)] \\
= \; & [\mathcal{P}^n_{t_1,t_1}(k) \oplus t_1((n+1)(c+1)+n) + \mathcal{P}^n_{t_1,t_1}(n+1) \oplus t_1((n+1)(c+1)+n)] \\
& +[\mathcal{P}^n_{t_2,t_2^2}(k) \oplus t_2((n+1)(c+t_1+1)+n) + \mathcal{P}^n_{t_2,t_2^2}(n+1) \\
& \oplus t_2((n+1)(c+t_1+1)+n)] \\
= \; & \{[\frac{n+1}{2}(t_1^2-t_1) + t_1k + t_1((n+1)(c+1)+n)] + [\frac{n+1}{2}(t_1^2-t_1) \\
& +t_1(n+1) + t_1((n+1)(c+1)+n)] + [\frac{t_2-t_2^2}{2} + t_2^2k + t_2^2 + \\
& t_2((n+1)(c+t_1+1)+n)] + [\frac{t_2-t_2^2}{2} + t_2^2(n+1) + t_2^2 + \\
& t_2((n+1)(c+t_1+1)+n)]\} \\
= \; & [(n+1)(t_1^2-t_1) + t_1(n+k+1) + 2t_1((n+1)(c+1)+n)] + [t_2 + t_2^2(n+k) \\
& +2t_2((n+1)(c+k_1+1)+n)]
\end{aligned}
$$

The vertex labeling g_2 is a bijective function $g_2 : V(G) \cup E(G) \to \{1, 2, \dots, p_G + q_G\}$. The total edge-weights of $G = S_n \rhd H$ under the labeling g_2, for $k = 1, 2, \dots, n$, constitute the following sets:

$$
\begin{aligned}
W_{g_2}^2 &= w_{g_2}^1 + w_{g_2}^2 + (n+1)(c+1) + k + w_{g_2}^3 \\
&= n+1+k+w_{k_2}^2 + (n+1)(c+1) + k + w_{g_2}^3 \\
&= C + [c_1 + c_2^2 + t_1 + t_2^2 + 2]k; 1 \le k \le n
\end{aligned}
$$

with $C = n+1+(n+1)(c_1^2-c_1)+c_1(n+1)+2c_1(n+1)+c_2+{c_2}^2n+2c_2((n+1)(c_1+1)) + (n+1)({t_1}^2 - t_1) + t_1(n+1) + 2t_1((n+1)(c+1)+n) + t_2 + {t_2}^2n + 2{t_2}^2((n+1)(c+t_1+1)+n)$. It is easy that the set of total edge-weights $W_{2_{g_2}}$ consists of an arithmetic sequence of the smallest value a when the edge weights at $k=1$ and the difference $d = c_1 + c_2^2 + t_1 + t_2^2 + 2$. Since the biggest d is attained when $d = c_2^2 + t_2^2$ then, for $c = p_H$ and $t = q_H + n$, it gives $d \le \frac{(p_H^2+q_H^2)(2(n+1)-4)+2nq_H}{n-1}$ It concludes the proof. □

Concluding Remarks

We have shown the existence of super antimagic labeling for graph operation $G = L \rhd H$ where L is a $(b, d^*) - EAV$ labeling. We have found super(a,d)-$P_2 \rhd H$ antimagic labelings for all differences $d = d^* + d^*(d_v + d_e) + 1$ where d^* is the feasible value of difference in super edge antimagic graph L and d_v and d_e respectively are feasible values of differences in the partitions $\mathcal{P}^{p_L}_{p_H-1,d_v}$ and $\mathcal{P}^{p_L}_{q_H,d_e}$. We have not found the result for disconnected of graph G. Thus, we propose the following open problems.

Open Problem 1. *Let L be a subgraph of G and $G = s(L \rhd H)$. Does G admit a super $(a,d) - P_2 \rhd H$ antimagic total labeling for $n \geq 2$ and feasible d?*

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