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$P_2 \rhd H$-super antimagic total labeling of comb product of graphs

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Abstract

Let L and H be two simple, nontrivial and undirected graphs. Let o be a vertex of H, the comb product between L and H, denoted by $L \rhd H$, is a graph obtained by taking one copy of L and $|V(L)|$ copies of H and grafting the ith copy of H at the vertex o to the ith vertex of L. By definition of comb product of two graphs, we can say that $V(L \rhd H) = \{(a, v)|a \in V(L), v \in V(H)\}$ and $(a, v)(b, w) \in E(L \rhd H)$ whenever $a = b$ and $vw \in E(H)$, or $ab \in E(L)$ and $v = w = o$. Let $G = L \rhd H$ and $P_2 \rhd H \subseteq G$, the graph G is said to be an (a, d)-$P_2 \rhd H$-antimagic total graph if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that for all subgraphs isomorphic to $P_2 \rhd H$, the total $P_2 \rhd H$-weights $W(P_2 \rhd H) = \sum_{v \in V(P_2 \rhd H)} f(v) + \sum_{e \in E(P_2 \rhd H)} f(e)$ form an arithmetic sequence $\{a, a + d, a + 2d, \dots, a + (n - 1)d\}$, where a and d are positive integers and n is the number of all subgraphs isomorphic to $P_2 \rhd H$. An (a, d)-$P_2 \rhd H$-antimagic total labeling f is called super if the smallest labels appear in the vertices. In this paper, we study a super (a, d)-$P_2 \rhd H$-antimagic total labeling of $G = L \rhd H$ when $L = C_n$.

Keywords: Super H-antimagic total labeling; Comb product; Cycle graph

1. Introduction

All graphs in this paper are simple, nontrivial and undirected, see [1,2] for more detail definition of graph. A comb product of L and H, denoted by $L \rhd H$, is a graph obtained by taking one copy of L and $|V(L)|$ copies of H and grafting the ith copy of H at the vertex o to the ith vertex of L. Thus, we have $V(L \rhd H) = \{(a, v)|a \in V(L), v \in V(H)\}$ and $(a, v)(b, w) \in E(L \rhd H)$ whenever $a = b$ and $vw \in E(H)$, or $ab \in E(L)$ and $v = w = o$, see Saputro, et al. in [3]. Susilowati in [4] explains in detail about a generalized comb product of graph.

Let $G = L \rhd H$ and let $P_2 \rhd H \subseteq G$, the graph G is said to be an (a, d)-$P_2 \rhd H$-antimagic total graph if there exist a bijective function $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that for all subgraphs isomorphic to

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$P_2 \rhd H$, the total $P_2 \rhd H$-weights $W(P_2 \rhd H) = \sum_{v \in V(P_2 \rhd H)} f(v) + \sum_{e \in E(P_2 \rhd H)} f(e)$ form an arithmetic sequence $\{a, a+d, a+2d, \dots, a+(n-1)d\}$, where a and d are positive integers and n is the number of all subgraphs isomorphic to $P_2 \rhd H$. Inayah et al. in [5] proved that, for H is a non-trivial connected graph and $k \geq 2$ is an integer, $shack(H, k)$ which contains exactly k subgraphs isomorphic to H is H-super antimagic. Some other relevant results can be found in [5–9] and [10–14], but their study only covered a fixed order of the covering H. In this paper, we study a super (a, d)-$P_2 \rhd H$-antimagic total labeling of $G = L \rhd H$ when $L = C_n$, and the covering is the subgraph which is isomorphic to $P_2 \rhd H$ where H is any graph. The resulting graphs of *comb product* $G = L \rhd H$ are not uniques, but for the antimagicness of total labeling study, it will give the same set of weight even we consider different resulting graphs. Thus, we do not consider a certain linkage vertex o of this graph operation.

To show those existence, we will use *an integer set partition technique* introduced by [15,16]. This technique used in determining the feasible difference d. Let n, m and d be positive integers. We consider the partition $\mathcal{P}^n_{m,d}(i, j)$ of the set $\{1, 2, \dots, mn\}$ into n columns, $n \geq 2$, m-rows such that the difference between the sum of the numbers in the $(j+1)$th m-rows and the sum of the numbers in the jth m-rows is always equal to the constant d, where $j = 1, 2, \dots, n-1$. Thus these sums form an arithmetic sequence with the difference d. By the symbol $\mathcal{P}^n_{m,d}(i, j)$ we denote the jth m-rows in the partition with the difference d, where $j = 1, 2, \dots, n$. Let $\sum \mathcal{P}^n_{m,d}(i, j)$ be the sum of the numbers in $\mathcal{P}^n_{m,d}(i, j)$, thus $d = \sum \mathcal{P}^n_{m,d}(j+1) - \sum \mathcal{P}^n_{m,d}(j)$.

In this study, we will focus for the connected version of the graph $G = L \rhd H$. Let L, H be two graphs of order $|V(L)|, |V(H)|$ and size $|E(L)|, |E(H)|$ respectively. The graph $G = L \rhd H$ is a connected graph with $|V(G)| = |V(L)||V(H)|$ and $|E(G)| = |V(L)||E(H)| + |E(L)|$. When $L = C_n$, thus $|V(L)| = |E(L)| = n$. Let $p_H = |V(H)|, q_H = |E(H)|$, the vertex set and edge set of the graph $G = C_n \rhd H$ can be split in the following sets: $V(G) = \{x_j; 1 \leq j \leq n\} \cup \{x_{ij}; 1 \leq i \leq p_H - 1, 1 \leq j \leq n\}$ and $E(G) = \{x_j x_{j+1}, x_1 x_n; 1 \leq j \leq n-1\} \cup \{e_{lj}; 1 \leq l \leq q_H, 1 \leq j \leq n\}$. Thus $|V(G)| = np_H$ and $|E(G)| = nq_H + n$.

The upper bound of feasible d for $G = C_n \rhd H$ to be a super (a, d)-$P_2 \rhd H$-antimagic total labeling follows the following lemma, proved by [7].

Lemma 1 ([7]). *Let G be a simple graph of order p and size q. If G is super (a, d)-$\mathcal{H}$-antimagic total labeling then $d \leq \frac{(p_G - p_H)p_H + (q_G - q_H)q_H}{n-1}$, for $p_G = |V(G)|$, $q_G = |E(G)|$, $p_H = |V(H)|$, $q_H = |E(H)|$, and $n = |H_i|$.*

If $G = C_n \rhd H$, the upper bound of feasible d follows the following corollary.

Corollary 1. *Let $K = P_2 \rhd H$, for odd integer $n \geq 3$, if the graph $G = C_n \rhd H$ admits super (a, d)-K-antimagic total labeling with $p_K = 2p_H$ and $q_K = 2q_H + 1$, then $d \leq ({p_K}^2 + {q_K}^2) - (\frac{n}{n-1})(\frac{1}{2}{p_K}^2 + \frac{1}{2}{q_K}^2 - q_K)$.*

The following theorem will be useful to show the variation of feasible d for $G = C_n \rhd H$ admits super (a, d)-K-antimagic total labeling.

Theorem 1 ([17]). *The number of r-combinations, with repetition allowed (multisets of size r), that can be selected from a set of n elements is $\binom{r+n-1}{r}$. This equals with the number of ways of choosing r objects which can be selected from n categories of objects with allowed repetition.*

Furthermore, a partition theorem has been developed by Dafik et al. in [16]. This theorem is used to have a different permutation of partition technique.

Lemma 2 ([16]). *Let n and m be positive integers. The sum of $\mathcal{P}^n_{m,d_1}(i, j) = \{(i-1)n + j,\ \ 1 \leq i \leq m\}$ and $\mathcal{P}^n_{m,d_2}(i, j) = \{(j-1)m + i;\ \ 1 \leq i \leq m\}$ forms an arithmetic sequence of difference $d_1 = m$, $d_2 = m^2\}$, respectively.*

2. The result

Establishing some lemmas related to the partition $\mathcal{P}^n_{m,d}(i, j)$ is a first important step prior to developing the super (a, d)-$P_2 \rhd H$ antimagic total labeling of $G = C_n \rhd H$ when $K = P_2 \rhd H$. We have $p_G = |V(G)| = n\frac{p_K}{2}$ and $q_G = |E(G)| = n(\frac{q_K - 1}{2} + 1)$.

Based on Lemma 2, we can derive two new lemmas with $d_1 = m$ and $d_2 = m^2$, but it has a different bijective function to Lemma 2.

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Lemma 3. *Let n, m be positive integers. For $1 \le j \le n$, the sum of*

$$\mathcal{P}^n_{m,d_1}(i,j) = \begin{cases} (\frac{j+1}{2}) + (i-1)n;\ 1 \le i \le m;\ j \text{ odd} \\ \lceil \frac{n}{2} \rceil + \frac{j}{2} + (i-1)n;\ 1 \le i \le m;\ j \text{ even} \end{cases}$$

forms an arithmetic sequence of difference $d_1 = m$.

Proof. By simple calculation. It gives $\sum_{i=1}^{m} \mathcal{P}^n_{m,d_1}(i,j) = \mathcal{P}^n_{m,d_1}(j)$, where

$$\mathcal{P}^n_{m,d_1}(j) = \begin{cases} m(\frac{j+1}{2}) + (\frac{m+m^2}{2})n - mn;\ j \text{ odd} \\ m(\lceil \frac{n}{2} \rceil + \frac{j}{2}) + (\frac{m^2-m}{2})n;\ j \text{ even.} \end{cases}$$

Since $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ for n odd, and $\lceil \frac{n}{2} \rceil = \frac{n}{2}$ for n even, it is easy to see that $\mathcal{P}^n_{m,d_1}(j) = \{\frac{mn}{2} + \frac{m^2 n}{2} - mn + m, \frac{mn}{2} + \frac{m^2 n}{2} - mn + 2m, \ldots, \frac{m^2 n}{2} + \frac{mn}{2}\}$ form an arithmetic sequence of difference $d_1 = m$. □

Lemma 4. *Let n, m be positive integers. For $1 \le j \le n$, the sum of*

$$\mathcal{P}^n_{m,d_2}(i,j) = \begin{cases} (\frac{j-1}{2})m + i;\ 1 \le i \le m;\ j \text{ odd} \\ m\lceil \frac{n}{2} \rceil + i + (\frac{j-2}{2})m;\ 1 \le i \le m;\ j \text{ even} \end{cases}$$

forms an arithmetic sequence of difference $d_2 = m^2$.

Proof. By simple calculation. It gives $\sum_{i=1}^{m} \mathcal{P}^n_{m,d_2}(i,j) = \mathcal{P}^n_{m,d_2}(j)$, where

$$\mathcal{P}^n_{m,d_2}(j) = \begin{cases} \frac{m}{2}(mj+1);\ j \text{ odd} \\ m^2\lceil \frac{n}{2} \rceil + \frac{m^2}{2}(j-1) + \frac{m}{2};\ j \text{ even.} \end{cases}$$

Similarly, since $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ for n odd, and $\lceil \frac{n}{2} \rceil = \frac{n}{2}$ for n even, it is easy to see that $\mathcal{P}^n_{m,d_2}(j) = \{\frac{m^2}{2} + \frac{m}{2}, \frac{3m^2}{2} + \frac{m}{2}, \ldots, m^2 n - \frac{m^2}{2} + \frac{m}{2}\}$ form an arithmetic sequence of difference $d_2 = m^2$. It concludes the proof. □

Now, we are ready to present our main theorem related to the existence of super (a,d)-$P_2 \rhd H$-antimagic total labeling of $G = L \rhd H$ when $L = C_n$.

Theorem 2. *Let $K = P_2 \rhd H$, and let $p_H = m_1 + m_2$ and $q_H = r_1 + r_2$ be the number of vertices and edges of graph H, respectively. For odd integer $n \ge 3$, if we assign the linear combination of $\mathcal{P}^n_{m,m}$ and $\mathcal{P}^n_{m,m^2}$ as a label of all elements in G, then $G = C_n \rhd H$ admits a super (a,d)-$P_2 \rhd H$ antimagic total labeling with $d = m_1 + m_2^2 + r_1 + r_2^2 + 1$.*

Proof. The graph $G = C_n \rhd H$ is a connected graph with vertex set and edge set of the graph $G = C_n \rhd H$ can be split in the following sets: $V(G) = \{x_j; 1 \le j \le n\} \cup \{x_{ij}; 1 \le i \le p_H - 1, 1 \le j \le n\}$ and $E(G) = \{x_j x_{j+1}, x_1 x_n; 1 \le j \le n-1\} \cup \{e_{lj}; 1 \le l \le q_H, 1 \le j \le n\}$. Thus $p_G = |V(G)| = np_H$ and $q_G = |E(G)| = nq_H + n$. Since the cover is $K = P_2 \rhd H$, and let $p_H = m_1 + m_2$ and $q_H = r_1 + r_2$, we can define the vertex labeling $f_1 : V(G) \cup E(G) \to \{1, 2, \ldots, p_G + q_G\}$ by using the linear combination of $\mathcal{P}^n_{m,m}$ and $\mathcal{P}^n_{m,m^2}$. By Lemmas 3 and 4, we use m_1 and r_1 for the partition $\mathcal{P}^n_{m,m}(i,j)$ and we use m_2 and r_2 for the partition $\mathcal{P}^n_{m,m^2}(i,j)$. For $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n$, the total labels can be expressed as follows

$$\begin{aligned} f_1(x_j \cup x_{i,j}) &= \{\mathcal{P}^n_{m_1,m_1}\} \cup \{\mathcal{P}^n_{m_2,m_2^2} \oplus nm_1\} \\ f_1(x_1 x_n) &= \{mn+1\} \\ f_1(x_j x_{j+1}) &= \{mn+1+j;\ 1 \le j \le n-1\} \\ f_1(e_{l,j}) &= \{\mathcal{P}^n_{r_1,r_1} \oplus [mn+n]\} \cup \{\mathcal{P}^n_{r_2,r_2^2} \oplus [n(r_1)+mn+n]\}. \end{aligned}$$

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The vertex labeling f_1 is a bijective function $f_1 : V(G) \cup E(G) \to \{1, 2, \ldots, p_G + q_G\}$. The total edge-weights of $G = C_n \rhd H$ under the labeling f_1, for $1 \le j \le n-1$, constitute the following sets:

$$\begin{aligned}
w^1_{f_1} &= [\sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, j) + m_1 n + \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, j+1) + m_1 n] + [\sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i, j) \\
&\quad + m_2 n(m_1+1) + \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i, j+1) + m_2 n(m_1+1)] \\
&= [\mathcal{P}^n_{m_1,m_1}(j) + nm_1 + \mathcal{P}^n_{m_1,m_1}(j+1) + nm_1] + [\mathcal{P}^n_{m_2,m_2^2}(j) + nm_2(m_1+1) \\
&\quad + \mathcal{P}^n_{m_2,m_2^2}(j+1) + nm_2(m_1+1)] \\
&= \{[m_1(\frac{j+1}{2}) + (\frac{m_1+m_1{}^2}{2})n - m_1 n] + [m_1(\lceil\frac{n}{2}\rceil + \frac{j+1}{2}) + (\frac{m_1{}^2-m_1}{2})n]\} + \\
&\quad \{[\frac{m_2}{2}(m_2 j+1) + nm_1m_2] + [m_2{}^2\lceil\frac{n}{2}\rceil + \frac{m_2{}^2 j}{2} + \frac{m_2}{2} + nm_1m_2]\} \\
&= \{m_1\lceil\frac{n}{2}\rceil + m_1 j + m_1 + m_1{}^2 n - m_1 n\} + \{m_2{}^2\lceil\frac{n}{2}\rceil + m_2{}^2 j + m_2 + 2nm_2m_1\}
\end{aligned}$$

$$\begin{aligned}
w^2_{f_1} &= [\sum_{l=1}^{r} \mathcal{P}^n_{r_1,r_1}(l, j) + r_1(mn+2n) + \sum_{l=1}^{r} \mathcal{P}^n_{r_1,r_1}(l, j+1) + r_1(mn+2n)] \\
&\quad + [\sum_{l=1}^{r} \mathcal{P}^n_{r_2,r_2^2}(l, j) + r_2(nr_1+mn+2n) + \sum_{l=1}^{r} \mathcal{P}^n_{r_2,r_2^2}(l, j+1) + r_2(nr_1 \\
&\quad + mn + 2n)] \\
&= [\mathcal{P}^n_{r_1,r_1}(j) + r_1(mn+2n) + \mathcal{P}^n_{r_1,r_1}(j+1) + r_1(mn+2n)] + [\mathcal{P}^n_{r_2,r_2^2}(j) \\
&\quad + r_2(nr_1+mn+2n) + \mathcal{P}^n_{r_2,r_2^2}(j+1) + r_2(nr_1+mn++2n)] \\
&= \{[r_1(\frac{j+1}{2}) + (\frac{r_1+r_1{}^2}{2})n - r_1 n + r_1(mn+n)] + [r_1(\lceil\frac{n}{2}\rceil + \frac{j+1}{2}) + (\frac{r_1{}^2-r_1}{2})n + \\
&\quad r_1(mn+n)]\} + \{[\frac{r_2}{2}(r_2 j+1) + r_2(nr_1+mn+n)] + [r_2{}^2\lceil\frac{n}{2}\rceil + \frac{r_2{}^2 j}{2} + \frac{r_2}{2} + \\
&\quad r_2(nr_1+mn+n)]\} = \{r_1\lceil\frac{n}{2}\rceil + r_1 j + r_1 + r_1{}^2 n - r_1 n + 2r_1(mn+n)\} + \\
&\quad \{r_2{}^2\lceil\frac{n}{2}\rceil + r_2{}^2 j + r_2 + 2r_2(nr_1+mn+n)\} \\
W^1_{f_1} &= w^1_{f_1} + f_1(x_j x_{j+1}) + w^2_{f_1} = w^1_{f_1} + mn + j + 1 + w^2_{f_1} = \mathcal{C}_1 + j[m_1 + m_2{}^2 + r_1 + r_2{}^2 + 1]
\end{aligned}$$

where $\mathcal{C}_1 = \{m_1\lceil\frac{n}{2}\rceil + m_1 + m_1{}^2 n - m_1 n\} + \{m_2{}^2\lceil\frac{n}{2}\rceil + m_2 + 2nm_2m_1\} + mn + \{r_1\lceil\frac{n}{2}\rceil + r_1 + r_1{}^2 n - r_1 n + 2r_1(mn + n)\} + \{r_2{}^2\lceil\frac{n}{2}\rceil + r_2 + 2r_2(nr_1 + mn + n)\} + 1$. While the total K-weight for $j = 1, n$ is as follows:

$$\begin{aligned}
w^1_{f_1} &= [\sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, 1) + m_1 n + \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, n) + m_1 n] + [\sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i, 1) \\
&\quad + m_2 n(m_1+1) + \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i, n) + m_2 n(m_1+1)] \\
&= [\mathcal{P}^n_{m_1,m_1}(1) + nm_1 + \mathcal{P}^n_{m_1,m_1}(n) + nm_1] + [\mathcal{P}^n_{m_2,m_2^2}(1) + nm_2(m_1+1) \\
&\quad + \mathcal{P}^n_{m_2,m_2^2}(n) + nm_2(m_1+1)] \\
&= \{[m_1(\frac{j+1}{2}) + (\frac{m_1+m_1{}^2}{2})n - m_1 n] + [m_1(\frac{n+1}{2}) + (\frac{m_1+m_1{}^2}{2})n - m_1 n]\} + \\
&\quad \{[\frac{m_2}{2}(m_2+1) + nm_1m_2] + [\frac{m_2}{2}(m_2 n+1) + nm_1m_2]\}
\end{aligned}$$

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$$= \{(m_1^2+m_1)n - \frac{3}{2}m_1 n + \frac{3}{2}m_1\} + \{\frac{m_2}{2}(m_2 n + m_2 + 2) + 2nm_1m_2\}$$

$$w_{f_1}^2 = [\sum_{l=1}^{r}\mathcal{P}_{r_1,r_1}^n(l,1) + r_1(mn+2n) + \sum_{l=1}^{r}\mathcal{P}_{r_1,r_1}^n(l,n) + r_1(mn+2n)]$$
$$+[\sum_{l=1}^{r}\mathcal{P}_{r_2,r_2^2}^n(l,1) + r_2(nr_1+mn+2n) + \sum_{l=1}^{r}\mathcal{P}_{r_2,r_2^2}^n(l,n) + r_2(nr_1 + mn + 2n)]$$
$$= [\mathcal{P}_{r_1,r_1}^n(1) + r_1(mn+2n) + \mathcal{P}_{r_1,r_1}^n(n) + r_1(mn+2n)] + [\mathcal{P}_{r_2,r_2^2}^n(1) + r_2(nr_1+mn+2n) + \mathcal{P}_{r_2,r_2^2}^n(n) + r_2(nr_1+mn+2n)]$$
$$= \{[r_1(\frac{j+1}{2}) + (\frac{r_1+{r_1}^2}{2})n - r_1 n + r_1(mn+n)] + [r_1(\frac{n+1}{2}) + (\frac{r_1+{r_1}^2}{2})n - r_1 n + r_1(mn+n)]\} + \{[\frac{r_2}{2}(r_2+1) + r_2(nr_1+mn+n)] + [\frac{m_2}{2}(m_2 n+1) + r_2(nr_1+mn+n)]\}$$
$$= \{(r_1^2+r_1)n - \frac{3}{2}r_1 n + \frac{3}{2}r_1 + 2r_1(mn+n)\} + \{\frac{r_2}{2}(r_2 n + r_2 + 2) + 2r_2(nr_1+mn+n)\}$$
$$W_{f_1}^2 = w_{f_1}^1 + f_1(x_1x_n) + w_{f_1}^2 = w_{f_1}^1 + mn + 1 + w_{f_1}^2 = \mathcal{C}_1, \text{ is the smallest value.}$$

From the two K-weights, we have the following

$$\bigcup_{t=1}^{2} W_{f_1}^t = \{\mathcal{C}_1, \mathcal{C}_1 + [m_1 + {m_2}^2 + r_1 + {r_2}^2 + 1], \mathcal{C}_1 + 2[m_1 + {m_2}^2 + r_1 + {r_2}^2 + 1], \ldots, \mathcal{C}_1 + (n-1)[m_1 + {m_2}^2 + r_1 + {r_2}^2 + 1]\}.$$

It is easy to see that all total K-weight elements form an arithmetic sequence with the smallest value $\mathcal{C}_1$ and the difference $d = m_1 + m_2^2 + r_1 + r_2^2 + 1$. Since the biggest d will be achieved when $d = m^2 + r^2$, for $m = \frac{p_K}{2}$ and $r = \frac{q_K}{2}$, it gives $d \le ({p_K}^2 + {q_K}^2) - (\frac{n}{n-1})(\frac{1}{2}{p_K}^2 + \frac{1}{2}{q_K}^2 - q_K)$. It concludes the proof. □

To emphasis our general theorem, we will take $H = W_s$. Thus, we will have the following corollary.

Corollary 2. *Let $K = P_2 \rhd W_s$, and let $p_{W_s} = m_1 + m_2$ and $q_{W_s} = r_1 + r_2$ be the number of vertices and edges of graph W_s, respectively. For odd integer $n \ge 3$, if we assign the linear combination of $\mathcal{P}_{m,m}^n$ and $\mathcal{P}_{m,m^2}^n$ as a label of all elements in G, then $G = C_n \rhd W_s$ admits a super (a,d)-$P_2 \rhd W_s$ antimagic total labeling with $d = m_1 + m_2^2 + r_1 + r_2^2 + 1$.*

Proof. The graph $G = C_n \rhd W_s$ is a connected graph with vertex set and edge set of the graph $G = C_n \rhd W_s$ can be split in the following sets: $V(G) = \{x_j; 1 \le j \le n\} \cup \{x_{ij}; 1 \le i \le (s+1)-1, 1 \le j \le n\}$ and $E(G) = \{x_j x_{j+1}, x_1 x_n; 1 \le j \le n-1\} \cup \{x_i^j x_{i+1}^j, x_j x_{s-1}, x_j x_1^j, x_j x_s; 1 \le j \le n; 1 \le i \le s-2\} \cup \{x_i^j x_s; 1 \le j \le n; 1 \le i \le s-1\}$. Thus $p_G = |V(G)| = n(s+1)$ and $q_G = |E(G)| = 2ns$. Since the cover is $K = P_2 \rhd W_s$, and let $p_{W_s} = m_1 + m_2$ and $q_{W_s} = r_1 + r_2$, we can define the vertex labeling $f_1 : V(G) \cup E(G) \to \{1, 2, \ldots, p_G + q_G\}$ by using the linear combination of $\mathcal{P}_{m,m}^n$ and $\mathcal{P}_{m,m^2}^n$. By Lemmas 3 and 4, we use m_1 and r_1 for the partition $\mathcal{P}_{m,m}^n(i,j)$ and we use m_2 and r_2 for the partition $\mathcal{P}_{m,m^2}^n(i,j)$. For $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n$, the total labels can be expressed as follows

$$f_2(x_j \cup x_{i,j}) = \{\mathcal{P}_{m_1,m_1}^n\} \cup \{\mathcal{P}_{m_2,m_2^2}^n \oplus nm_1\}$$
$$f_2(x_1x_n) = \{mn+1\}$$
$$f_2(x_jx_{j+1}) = \{mn+1+j;\ 1 \le j \le n-1\}$$
$$f_2(x_i^j x_{i+1}^j \cup x_j x_{s-1} \cup x_j x_1^j \cup x_j x_s \cup x_i^j x_s) = \{\mathcal{P}_{r_1,r_1}^n \oplus [mn+n]\} \cup \{\mathcal{P}_{r_2,r_2^2}^n \oplus [n(r_1)+mn+n]\}.$$

The vertex labeling f_2 is a bijective function $f_2 : V(G) \cup E(G) \to \{1, 2, \ldots, p_G + q_G\}$. The total edge-weights of $G = C_n \rhd W_s$ under the labeling f_2, for $1 \le j \le n-1$, constitute the following sets:

$$\begin{aligned}
w^1_{f_2} &= [\sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i,j) + m_1 n + \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i,j+1) + m_1 n] + [\sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i,j) \\
&\quad + m_2 n(m_1+1) + \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i,j+1) + m_2 n(m_1+1)] \\
&= [\mathcal{P}^n_{m_1,m_1}(j) + nm_1 + \mathcal{P}^n_{m_1,m_1}(j+1) + nm_1] + [\mathcal{P}^n_{m_2,m_2^2}(j) + nm_2(m_1+1) \\
&\quad + \mathcal{P}^n_{m_2,m_2^2}(j+1) + nm_2(m_1+1)] \\
&= \{[m_1(\frac{j+1}{2}) + (\frac{m_1+{m_1}^2}{2})n - m_1 n] + [m_1(\lceil\frac{n}{2}\rceil + \frac{j+1}{2}) + (\frac{{m_1}^2-m_1}{2})n]\} + \\
&\quad \{[\frac{m_2}{2}(m_2 j+1) + nm_1m_2] + [{m_2}^2\lceil\frac{n}{2}\rceil + \frac{{m_2}^2 j}{2} + \frac{m_2}{2} + nm_1m_2]\} \\
&= \{m_1\lceil\frac{n}{2}\rceil + m_1 j + m_1 + {m_1}^2 n - m_1 n\} + \{{m_2}^2\lceil\frac{n}{2}\rceil + {m_2}^2 j + m_2 + 2nm_2m_1\} \\
w^2_{f_2} &= [\sum_{l=1}^{r} \mathcal{P}^n_{r_1,r_1}(l,j) + r_1(mn+2n) + \sum_{l=1}^{r} \mathcal{P}^n_{r_1,r_1}(l,j+1) + r_1(mn+2n)] \\
&\quad + [\sum_{l=1}^{r} \mathcal{P}^n_{r_2,r_2^2}(l,j) + r_2(nr_1+mn+2n) + \sum_{l=1}^{r} \mathcal{P}^n_{r_2,r_2^2}(l,j+1) + r_2(nr_1 \\
&\quad + mn + 2n)] \\
&= [\mathcal{P}^n_{r_1,r_1}(j) + r_1(mn+2n) + \mathcal{P}^n_{r_1,r_1}(j+1) + r_1(mn+2n)] + [\mathcal{P}^n_{r_2,r_2^2}(j) \\
&\quad + r_2(nr_1+mn+2n) + \mathcal{P}^n_{r_2,r_2^2}(j+1) + r_2(nr_1+mn++2n)] \\
&= \{[r_1(\frac{j+1}{2}) + (\frac{r_1+{r_1}^2}{2})n - r_1 n + r_1(mn+n)] + [r_1(\lceil\frac{n}{2}\rceil + \frac{j+1}{2}) + (\frac{{r_1}^2-r_1}{2})n + \\
&\quad r_1(mn+n)]\} + \{[\frac{r_2}{2}(r_2 j+1) + r_2(nr_1+mn+n)] + [{r_2}^2\lceil\frac{n}{2}\rceil + \frac{{r_2}^2 j}{2} + \frac{r_2}{2} + \\
&\quad r_2(nr_1+mn+n)]\} = \{r_1\lceil\frac{n}{2}\rceil + r_1 j + r_1 + {r_1}^2 n - r_1 n + 2r_1(mn+n)\} + \\
&\quad \{{r_2}^2\lceil\frac{n}{2}\rceil + {r_2}^2 j + r_2 + 2r_2(nr_1+mn+n)\} \\
W^1_{f_1} &= w^1_{f_1} + f_1(x_j x_{j+1}) + w^2_{f_1} = w^1_{f_1} + mn + j + 1 + w^2_{f_1} = \mathcal{C}_1 + j[m_1 + {m_2}^2 + r_1 + {r_2}^2 + 1]
\end{aligned}$$

where $\mathcal{C}_1 = \{m_1\lceil\frac{n}{2}\rceil + m_1 + {m_1}^2 n - m_1 n\} + \{{m_2}^2\lceil\frac{n}{2}\rceil + m_2 + 2nm_2m_1\} + mn + \{r_1\lceil\frac{n}{2}\rceil + r_1 + {r_1}^2 n - r_1 n + 2r_1(mn + n)\} + \{{r_2}^2\lceil\frac{n}{2}\rceil + r_2 + 2r_2(nr_1 + mn + n)\} + 1$. While the total K-weight for $j = 1, n$ is as follows:

$$\begin{aligned}
w^1_{f_2} &= [\sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i,1) + m_1 n + \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i,n) + m_1 n] + [\sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i,1) \\
&\quad + m_2 n(m_1+1) + \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2^2}(i,n) + m_2 n(m_1+1)] \\
&= [\mathcal{P}^n_{m_1,m_1}(1) + nm_1 + \mathcal{P}^n_{m_1,m_1}(n) + nm_1] + [\mathcal{P}^n_{m_2,m_2^2}(1) + nm_2(m_1+1) \\
&\quad + \mathcal{P}^n_{m_2,m_2^2}(n) + nm_2(m_1+1)] \\
&= \{[m_1(\frac{j+1}{2}) + (\frac{m_1+{m_1}^2}{2})n - m_1 n] + [m_1(\frac{n+1}{2}) + (\frac{m_1+{m_1}^2}{2})n - m_1 n]\} + \\
&\quad \{[\frac{m_2}{2}(m_2+1) + nm_1m_2] + [\frac{m_2}{2}(m_2 n+1) + nm_1m_2]\}
\end{aligned}$$

$$= \{(m_1^2 + m_1)n - \frac{3}{2}m_1 n + \frac{3}{2}m_1\} + \{\frac{m_2}{2}(m_2 n + m_2 + 2) + 2nm_1 m_2\}$$

$$w_{f_2}^2 = [\sum_{l=1}^{r} \mathcal{P}_{r_1,r_1}^n(l,1) + r_1(mn+2n) + \sum_{l=1}^{r} \mathcal{P}_{r_1,r_1}^n(l,n) + r_1(mn+2n)]$$
$$+ [\sum_{l=1}^{r} \mathcal{P}_{r_2,r_2^2}^n(l,1) + r_2(nr_1+mn+2n) + \sum_{l=1}^{r} \mathcal{P}_{r_2,r_2^2}^n(l,n) + r_2(nr_1 + mn + 2n)]$$
$$= [\mathcal{P}_{r_1,r_1}^n(1) + r_1(mn+2n) + \mathcal{P}_{r_1,r_1}^n(n) + r_1(mn+2n)] + [\mathcal{P}_{r_2,r_2^2}^n(1) + r_2(nr_1+mn+2n) + \mathcal{P}_{r_2,r_2^2}^n(n) + r_2(nr_1+mn+2n)]$$
$$= \{[r_1(\frac{j+1}{2}) + (\frac{r_1+{r_1}^2}{2})n - r_1 n + r_1(mn+n)] + [r_1(\frac{n+1}{2}) + (\frac{r_1+{r_1}^2}{2})n - r_1 n + r_1(mn+n)]\} + \{[\frac{r_2}{2}(r_2+1) + r_2(nr_1+mn+n)] + [\frac{m_2}{2}(m_2 n+1) + r_2(nr_1+mn+n)]\}$$
$$= \{(r_1^2 + r_1)n - \frac{3}{2}r_1 n + \frac{3}{2}r_1 + 2r_1(mn+n)\} + \{\frac{r_2}{2}(r_2 n + r_2 + 2) + 2r_2(nr_1+mn+n)\}$$
$$W_{f_2}^2 = w_{f_2}^1 + f_2(x_1x_n) + w_{f_2}^2 = w_{f_2}^1 + mn + 1 + w_{f_2}^2 = \mathcal{C}_2, \text{ is the smallest value.}$$

From the two K-weights, we have the following

$$\bigcup_{t=1}^{2} W_{f_2}^t = \{\mathcal{C}_2, \mathcal{C}_2 + [m_1 + {m_2}^2 + r_1 + {r_2}^2 + 1], \mathcal{C}_2 + 2[m_1 + {m_2}^2 + r_1 + {r_2}^2 + 1], \ldots, \mathcal{C}_2 + (n-1)[m_1 + {m_2}^2 + r_1 + {r_2}^2 + 1]\}.$$

It is easy to see that all total K-weight elements form an arithmetic sequence with the smallest value $\mathcal{C}_2$ and the difference $d = m_1 + m_2^2 + r_1 + r_2^2 + 1$. It concludes the proof. □

Fig. 1 shows an example of super (a,d)-*antimagic total covering* of graph $G = C_5 \rhd W_5$ using a linear combination of $\mathcal{P}_{m,m}^n(i,j)$ and $\mathcal{P}_{m,m^2}^n(i,j)$. We use linear combination $\mathcal{P}_{4,4}^5(i,j)$ and $\mathcal{P}_{2,2^2}^5(i,j)$ for vertex labeling and linear combination $\mathcal{P}_{5,5}^5(i,j)$ and $\mathcal{P}_{5,5^2}^5(i,j)$ for edge labeling. Thus the value of $d = 4 + 2^2 + 5 + 5^2 + 1 = 39$ and the smallest value is $a = 1351$.

We have shown the theorem above, the question now, how many feasible values of $d = m_1 + m_2^2 + r_1 + r_2^2$ can we have? The following theorem will describe its number of possibility feasible d.

Theorem 3. *Let m and r be positive integer of $m = m_1 + m_2$ and $r = r_1 + r_2$. If $d = m_1 + m_2^2 + r_1 + r_2^2$ then the number of possible different d is at least m for $m > r$, at least r for $r > m$, and at most mr.*

Proof. Let $d_1 = m_1 + m_2^2$ and $d_2 = r_1 + r_2^2$. Based on Theorem 1, the equation $m_1 + m_2 = m$ has $\binom{m+2-1}{m}$ number of solutions. When we substitute all the possible solutions it will possibly gives the same d_1. Take $m_2 = 1, m_1 = m - 1$ and $m_1 = m, m_2 = 0$, and substitute into d_1 yields the following:

$$d_1 = m_1 + m_2^2 = m - 1 + (1)^2 = m, \text{ or}$$
$$d_1 = m_1 + m_2^2 = m + (0)^2 = m.$$

Thus, the number of possible solution is less than one. It implies that the number of possible solution $m_1 + m_2 = m$ satisfying for different $d_1 = m_1 + m_2^2$ is the following

$$\binom{m+2-1}{m} - 1 = \binom{m+1}{m} - 1 = \frac{(m+1)!}{m!1!} - 1 = \frac{(m+1)(m!)}{m!1} - 1 = m.$$

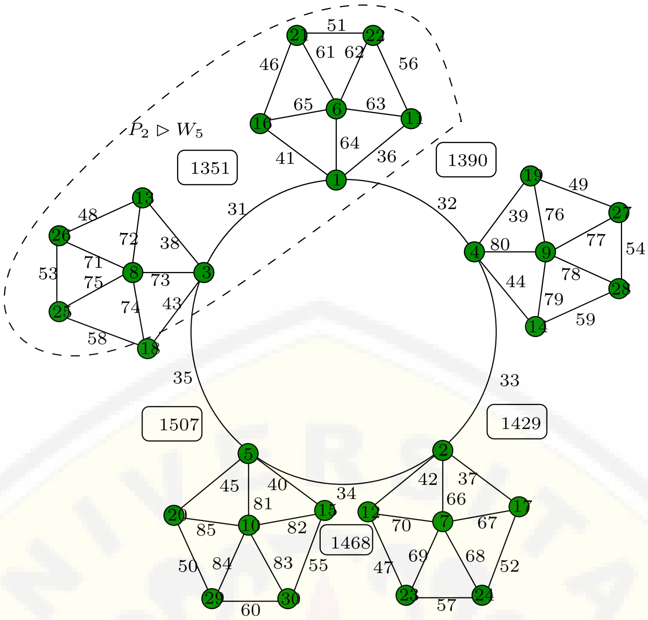

Fig. 1. Super $(1351, 39)$-$P_2 \triangleright W_5$-*antimagic* total *covering* of graph $G = C_5 \triangleright W_5$.

By the same manner, we will get the number of solution such that the feasible d_2 has different r. Since $d = d_1 + d_2$ and we consider an optimal parameter d_1 or d_2, with number of possible d_1 and d_2 are respectively m and r, the number of different solution of d, for $m > r$ and for $r > m$ are m and r respectively. Furthermore, since d_1 and d_2 has respectively at most m and r solutions, $d = d_1 + d_2$ has at most mr solutions. □

3. Concluding remarks

We have shown the existence of super antimagicness of comb product of any graphs $G = L \triangleright H$ when $L = C_n$ and $K = P_2 \triangleright H$. By using a partition technique we can prove that, for odd $n \geq 3$, $G = C_n \triangleright H$ admits a super(a, d)-$P_2 \triangleright H$-antimagic total labeling with difference $d = m_1^2 + m_2 + r_1^2 + r_2 + 1$. For more illustration of our general theorem, we have taken a special $H = W_s$. For odd $n \geq 3$, $G = C_n \triangleright W_s$ admits a super(a, d)-$P_2 \triangleright W_s$-antimagic total labeling. However, for n is even we have not found any result yet. Thus, we propose the following open problem.

Open Problem 1. For even $n \geq 3$, do the graphs $G = C_n \triangleright H$ admit a super (a, d)-$P_2 \triangleright H$-antimagic total labeling with all feasible d?

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