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A survey of face-antimagic evaluations of graphs

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In memory of Mirka Miller

Abstract

The concept of face-antimagic labeling of plane graphs was introduced by Mirka Miller in 2003. This survey aims to give an overview of the recent results obtained in this topic.

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1 Introduction

Let $G = (V, E)$ be a finite connected graph without loops and multiple edges, where $V(G)$ and $E(G)$ are its vertex set and edge set, respectively. Let $|V(G)| = p$ and $|E(G)| = q$. A general reference for graph-theoretic notions is [44].

A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers or colors. Graph labelings provide valuable information that can be applied to different areas, see [30].

Sedláček [39] defined a graph to be *magic* if its edges can be labeled with a range of positive integers, such that all the vertex-weights are equal, where a vertex-weight of a vertex x is the sum of labels of all edges incident with x. A graph G is called *antimagic* if its edges can be labeled with the labels $1, 2, \dots, q$ in such a way that all vertex-weights are pairwise distinct. The concept of an antimagic graph was introduced by Hartsfield and Ringel [31]. They conjectured that every connected graph, except K_2, is antimagic. Alon et al. [5] used several probabilistic tools and some techniques from analytic number theory to show that this conjecture is true for all graphs having minimum degree $\Omega(\log |V(G)|)$. In [42] Wang proved that the Cartesian product of two or more cycles is antimagic. Cheng [28] proved that Cartesian products of two paths and of a cycle and a path are antimagic. In [29] Cheng generalized result in [42] and proved that all Cartesian products of two or more regular graphs are antimagic.

We investigate antimagic labelings of plane graphs with restrictions placed on the weights of faces. If we consider a *plane graph* (i.e., a planar graph drawn on the Euclidean plane without edge crossings), then in addition to vertices and edges we can also consider its faces, including the unique face of the infinite area. We denote a plane graph as $G = (V, E, F)$, where $F(G)$ is the set of all faces in G. If f denotes the number of internal faces then $|F(G)| = f + 1$.

A *labeling of type* $(1, 1, 1)$ assigns the labels from the set $\{1, 2, \dots, p + q + f + 1\}$ to the vertices, edges and faces of a plane graph G in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label.

A *labeling of type* $(1, 1, 0)$, is a bijection from the set $\{1, 2, \dots, p+q\}$ to the vertices and edges of plane graph G. This labeling is also called a *total labeling*. If we label only vertices (respectively edges) we call such a labeling a *vertex* (respectively *edge*) *labeling*; alternatively, this labeling is said to be labeling of type $(1, 0, 0)$ (respectively labeling of type $(0, 1, 0)$).

The *weight* of a face under a labeling is the sum of the labels (if present) carried by that face and the edges and vertices surrounding it.

A labeling of a plane graph is called *d-antimagic*, if for every number s, the set of s-sided face weights is $W_s = \{a_s, a_s + d, \dots, a_s + (f_s - 1)d\}$ for some integers a_s and d, $d \geq 0$, where f_s is the number of the s-sided faces. We allow different sets W_s for different s.

Somewhat related types of antimagic labelings were defined by Bodendiek and

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Walther in [27].

If $d = 0$ then Lih calls such labeling *magic* [35] and describes magic (0-antimagic) labelings of type $(1, 1, 0)$ for wheels, friendship graphs and prisms. The magic labelings of type $(1, 1, 1)$ for grid graphs and honeycomb are given in [8] and [9], respectively. Ali, Hussain, Ahmad, and Miller [4] studied magic labeling of type $(1, 1, 1)$ for wheels and subdivided wheels. Ahmad [1] proved that subdivided ladders admit magic labelings of type $(1, 1, 1)$ and admit consecutive magic labelings of type $(1, 1, 0)$. Kathiresan and Gokulakrishnan [34] provided magic labelings of type $(1, 1, 1)$ for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces, and one external infinite face.

The concept of d-antimagic labeling of plane graphs was defined in [22]. Lin et al. in [36] showed that prism D_n, $n \geq 3$, admits d-antimagic labelings of type $(1, 1, 1)$ for $d \in \{2, 4, 5, 6\}$. It is the case that d-antimagic labelings of type $(1, 1, 1)$ for D_n and for several $d \geq 7$ are described in [41], see also [14].

A d-antimagic labeling is called *super* if the smallest possible labels appear on the vertices. The super d-antimagic labelings of type $(1, 1, 1)$ for antiprisms and for $d \in \{0, 1, 2, 3, 4, 5, 6\}$ are described in [11], and the existence of such labelings for Jahangir graphs for certain different values of d is shown in [40]. The existence of super d-antimagic labelings of type $(1, 1, 1)$ for toroidal fullerenes is examined in [26], for generalized prism is investigated in [24] and for uniform subdivisions of wheels is studied in [32].

In the next section let us focus on the recent results obtained in this topic.

2 Recent results on d-antimagic labelings

Kathiresan and Ganesan [33] define a class of plane graphs denoted by P_a^b, $a \geq 3$, $b \geq 2$, as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1, 2, \ldots, a-1$ joining v_i and v_{i+1} with b internally disjoint paths of length $i + 1$. They prove that P_a^b has d-antimagic labelings of type $(1, 1, 1)$ for $d \in \{0, 1, 2, 3, 4, 6\}$. Lin and Sugeng [37] prove that P_a^b has a d-antimagic labeling of type $(1, 1, 1)$ for $d \in \{5, a-7, a-4, a-3, a-1, a+1, a+2, a+3, a+5, 2a-3, 2a-1, 2a+1, 2a+3, 3a-3, 3a-1, 3a+1, 4a-3, 4a-1, 5a-5, 5a-3, 6a-7, 6a-5, 7a-7, 7a-2\}$.

Similarly, Bača, Baskoro and Cholily define a class of plane graphs denoted by C_a^b as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1, 2, \ldots, a$ joining v_i and v_{i+1} with b internally disjoint paths of length $i + 1$, where indices are taken modulo a. In [12] and [13] they prove that for $a \geq 3$ and $b \geq 2$, the graph C_a^b admits a d-antimagic labeling of type $(1, 1, 1)$ for $d \in \{0, 1, 2, 3, a-2, a-1, a+1, a+2\}$.

The *generalized Petersen graph* $P(n, m)$, $n \geq 3$ and $1 \leq m \leq \lfloor (n-1)/2 \rfloor$, consists of an outer n-cycle $y_0, y_1, \ldots, y_{n-1}$, a set of n spokes $y_i x_i$, $0 \leq i \leq n-1$, and n edges $x_i x_{i+m}$, $0 \leq i \leq n-1$, with indices taken modulo n. The standard Petersen graph is the instance $P(5, 2)$. By definition, $P(n, m)$ is a 3-regular graph which has $2n$

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vertices and $3n$ edges. Generalized Petersen graphs were first defined by Watkins [43].

If $m = 1$ and $n \geq 3$ or $m = 2$ and n is even, $n \geq 6$, then the generalized Petersen graph $P(n,m)$ is plane. Note that $P(n,1)$ is the prism D_n. The next two theorems give necessary conditions for $P(n,2)$ to possess a d-antimagic labeling of type $(1,1,1)$.

Theorem 2.1. [20] *For every generalized Petersen graph $P(n,2)$, $n \geq 6$, there is no d-antimagic vertex labeling with $d \geq 10$ and no d-antimagic edge labeling with $d \geq 15$.*

In light of Theorem 2.1 and the fact that under a d-antimagic face labeling $F(P(n,2)) \to \{1,2,\dots,n\}$ the parameter d is no more than 1, we get the following upper bound for the difference of antimagic labeling of type $(1,1,1)$.

Theorem 2.2. [20] *Let $P(n,2)$, $n \geq 6$, be a generalized Petersen graph which admits d_1-antimagic vertex labeling φ_1, d_2-antimagic edge labeling φ_2 and 1-antimagic face labeling φ_3, $d_1 \geq 0$, $d_2 \geq 0$. If the labelings φ_1, $p+\varphi_2$ and $p+q+\varphi_3$ combine to a d-antimagic labeling of type $(1,1,1)$ then the parameter $d \leq 24$.*

In [20] it is proved that $P(n,2)$ has a 1-antimagic labeling of type $(1,1,1)$ for n even, $n \geq 6$, and a d-antimagic labeling of type $(1,1,1)$ for $n \equiv 2 \pmod 4$, $n \geq 6$, $n \neq 10$ and $d \in \{0,2,3\}$. Moreover, there is a description of a 2-antimagic labeling of type $(1,1,1)$ for the dodecahedron $P(10,2)$. For $n \equiv 0 \pmod 4$ the following results have been obtained.

Theorem 2.3. [20] *If $n \equiv 0 \pmod 4$, $n \geq 8$ and $d \in \{2,3,6,9\}$, then the generalized Petersen graph $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$.*

It has been conjectured in [20] that

Conjecture 1. [20] *There is a d-antimagic labeling of type $(1,1,1)$ for the generalized Petersen graph $P(n,2)$ for $n \equiv 2 \pmod 4$, $n \geq 6$ and $d \in \{6,9\}$.*

The upper bound for difference d in Theorem 2.2 is too large, therefore the authors in [20] proposed the following open problem.

Open Problem 1. [20] *Find other possible values of the parameter d and the corresponding d-antimagic labeling of type $(1,1,1)$ for the generalized Petersen graph $P(n,2)$.*

For $n \geq 1$, $m \geq 1$, we denote by H_n^m (*honeycomb*) the hexagonal plane map with m rows and n columns of hexagons. The face set $F(H_n^m)$ contains mn 6-sided faces and one external infinite face. It was proved in [15] that if n is even, $n \geq 2$ and $m \geq 1$, then the plane map H_n^m supports 2-antimagic and 4-antimagic labelings of type $(1,1,1)$. For n odd, it was obtained the following result.

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Theorem 2.4. [16] *If n is odd, $n \geq 1$, $m \geq 1$, $mn > 1$ and $d \in \{1, 2, 3, 4\}$, then the hexagonal plane map H_n^m admits a d-antimagic labeling of type $(1, 1, 1)$.*

In [16], it is proposed the following open problem.

Open Problem 2. [16] *Find other possible values of the parameter d and the corresponding d-antimagic labeling of type $(1, 1, 1)$ for the hexagonal plane map H_n^m.*

For $n \geq 1$ and $m \geq 1$, let G_n^m be the *grid graph* which can be defined as the Cartesian product $P_{m+1} \times P_{n+1}$ of a path on $(m+1)$ vertices with a path on $(n+1)$ vertices embedded in the plane.

Necessary conditions for grids to bear d-antimagic labelings of types $(1, 0, 0)$ and $(0, 1, 0)$ as listed in [21] are given in the following theorem.

Theorem 2.5. [21] *For every grid graph G_n^m, $m, n > 7$, there is no d-antimagic vertex labeling with $d \geq 5$ and no d-antimagic edge labeling with $d \geq 9$.*

Applying the previous theorem, and the fact that under a d-antimagic face labeling $F(G_n^m) \to \{1, 2, \dots, f\}$ the parameter d is no more than 1, we get the following upper bound for the difference of antimagic labeling of type $(1, 1, 1)$.

Theorem 2.6. [21] *Let G_n^m, $m, n > 7$, be a graph which admits a d_1-antimagic vertex labeling σ_1, a d_2-antimagic edge labeling σ_2 and a 1-antimagic face labeling σ_3, $d_1 \geq 0$, $d_2 \geq 0$. If the labelings σ_1, $p + \sigma_2$ and $p + q + \sigma_3$ combine to a d-antimagic labeling of type $(1, 1, 1)$, then the parameter $d \leq 13$.*

Bača, Lin and Miller proved the following theorem:

Theorem 2.7. [21] *For $m \geq 1$, $n \geq 1$, $n + m \neq 2$ and $d \in \{1, 2, 3, 4, 6\}$, the grid graph G_n^m has a d-antimagic labeling of type $(1, 1, 1)$.*

This theorem led the authors to propose

Conjecture 2. [21] *There is a 5-antimagic labeling of type $(1, 1, 1)$ for the plane graph G_n^m and for all $m \geq 1$, $n \geq 1$, $m + n \neq 2$.*

From the necessary conditions it follows that $d \leq 13$. Therefore, it is a natural step to formulate the following open problem.

Open Problem 3. [21] *Find other possible values of the parameter d and corresponding d-antimagic labelings of type $(1, 1, 1)$ for G_n^m.*

The *friendship graph* F_n is a set of n triangles having a common center vertex. So, the face set of F_n contains n 3-sided faces and one external infinite face. Bača, Brankovic and Semaničová–Feňovčíková [19] proved that

Theorem 2.8. [19] *The friendship graph F_n, $n \geq 2$, has a super d-antimagic labeling of type $(1, 1, 1)$ for $d \in \{0, 2, 4, \dots, 20\}$.*
Moreover, if $n \equiv 1 \pmod 2$ then the graph F_n also admits a super d-antimagic labeling of type $(1, 1, 1)$ for $d \in \{1, 3, 5, 7, 9, 11, 15, 17\}$.

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A *triangular snake* is a connected graph whose blocks are cycles C_3 and its block-cutpoint graph is a path, see [38]. By E_n we denote the triangular snake embedded in the plane with the vertex set $V(E_n) = \{v_1, v_2, \dots, v_{n+1}, u_1, u_2, \dots, u_n\}$ and the edge set $E(E_n) = \{v_iv_{i+1}, v_iu_i, u_iv_{i+1} : i = 1, 2, \dots, n\}$. The face set of E_n contains n 3-sided faces and one external infinite face. For the triangular snake E_n it was shown that

Theorem 2.9. [19] *The graph E_n, $n \geq 2$, has a super d-antimagic labeling of type $(1,1,1)$ for $d \in \{0, 1, 2, \dots, 19\}$.*

Let B_n be a graph consisting of a set of n cycles C_4 having a common center vertex. Consider the graph B_n embedded in the plane with the vertex set $V(B_n) = \{v_i, u_i, w_i, v : i = 1, 2, \dots, n\}$ and the edge set $E(B_n) = \{v_iv, w_iv, v_iu_i, u_iw_i : i = 1, 2, \dots, n\}$. The face set of B_n contains n 4-sided faces and one external infinite face. It is known that

Theorem 2.10. [19] *The graph B_n, $n \geq 2$, has a super d-antimagic labeling of type $(1,1,1)$ for $d \in \{0, 2, 4, 6, \dots, 28, 30, 34\}$.*
Moreover, if $n \equiv 1 \pmod 2$ then the graph B_n also admits a super d-antimagic labeling of type $(1,1,1)$ for $d \in \{1, 3, 5, \dots, 27\}$.

A quadrilateral snake is a connected graph whose blocks are the cycles C_4. By G_n we denote the quadrilateral snake embedded in the plane with vertex set $V(G_n) = \{v_1, v_2, \dots, v_{n+1}, u_1, u_2, \dots, u_n, w_1, w_2, \dots w_n\}$ and the edge set $E(G_n) = \{v_iu_i, u_iv_{i+1}, v_iw_i, w_iv_{i+1} : i = 1, 2, \dots, n\}$. The face set of G_n contains n 4-sided faces and one external infinite face. It was proved that

Theorem 2.11. [19] *The graph G_n, $n \geq 2$, has a super d-antimagic labeling of type $(1,1,1)$ for $d \in \{0, 1, 2, \dots, 31\}$.*

In [18], it is examined the existence of super d-antimagic labelings of type $(1,1,1)$ for the plane graphs containing a special Hamilton path.

Theorem 2.12. [18] *Let G be a plane graph. If there exists in G a Hamilton path such that for every face except the external face, the Hamilton path contains all but one edges surrounding that face, then G is super d-antimagic of type $(1,1,1)$ for $d = 0, 1, 2, 3, 5$.*

Moreover, if $2(|F(G)| - 1) \leq |V(G)|$, then it is also possible to find a super d-antimagic labeling of type $(1,1,1)$ for $d = 4$ and $d = 6$.

Theorem 2.13. [18] *Let G be a plane graph. If there exists in G a Hamilton path such that for every face except the external face, the Hamilton path contains all but one edge surrounding that face and if $2(|F(G)| - 1) \leq |V(G)|$, then G is super d-antimagic of type $(1,1,1)$ for $d = 0, 1, 2, 3, 4, 5, 6$.*

The results from Theorems 2.12 and 2.13 are generalized as follows:

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Theorem 2.14. [18] *Let G be a plane graph and let $M = \lfloor |V(G)|/(|F(G)|-1) \rfloor$. Suppose that there exists in G a Hamilton path such that for every face, except the external face, the Hamilton path contains all but one edge surrounding that face.*

i) If $M = 1$, then G admits a super d-antimagic labeling of type $(1,1,1)$ for $d = 0,1,2,3,5$.

ii) If $M \geq 2$, then G admits a super d-antimagic labeling of type $(1,1,1)$ for $d = 0,1,2,3,\ldots, M+4$.

In [18] it is noted that if the plane graph contains s_1-sided, s_2-sided, ..., s_t-sided faces and their number is "almost the same" then it is possible to label the graph with the super d-antimagic labeling of type $(1,1,1)$ for differences up to $t+4$, where t is the number of different sided faces.

Immediately from the Theorem 2.12 it follows that the grid graph $P_n \times P_2$, $n \geq 3$ also admits a super 5-antimagic labeling of type $(1,1,1)$. This result supports Conjecture 2. Moreover, it is shown how to find other feasible values of the parameter d for a super d-antimagic labeling of type $(1,1,1)$ of $P_n \times P_2$.

Theorem 2.15. [18] *The Cartesian product $P_n \times P_2$, $n \geq 3$, admits a super d-antimagic labeling of type $(1,1,1)$ for $d \in \{0,1,2,\ldots,15\}$.*

In addition to the lattices having 4-sided faces, the lattices having 3-sided faces have also been studied. The L_n^m can be obtained from the grid graph $P_n \times P_m$ by adding a new edge in every 4-sided face such that the added edges are "parallel". In [17] it is proved that the graph L_n^m, $n \geq 2$, $2 \leq m \leq 5$, admits a super d-antimagic labeling of type $(1,1,1)$ for $d = 0,2,4$ (see also [6, 7]). From Theorem 2.12 it immediately follows that the graph L_n^2, $n \geq 2$, also admits a super d-antimagic labeling of type $(1,1,1)$ for $d = 0,1,2,3,5$. In [18], other feasible values of parameter d are found for a super d-antimagic labeling of type $(1,1,1)$ of L_n^2.

Theorem 2.16. [18] *The graph L_n^2, $n \geq 2$, admits a super d-antimagic labeling of type $(1,1,1)$ for $d \in \{0,1,2,\ldots,9\}$.*

Several authors investigated the existence of super d-antimagic labelings for disconnected plane graphs. There was studied in the following problem: If a graph G admits a (super) d-antimagic labeling, does the disjoint union of m copies of the graph G, denoted by mG, admit a (super) d-antimagic labeling as well?

Ahmad et al. in [2] investigated super d-antimagicness of type $(1,1,0)$. They proved that if there exists a super 0-antimagic labeling of type $(1,1,0)$ of a plane graph G then, for every positive integer m, the graph mG also admits a super 0-antimagic labeling of type $(1,1,0)$. Moreover, if a plane graph G with 3-sided inner faces admits a super d-antimagic labeling of type $(1,1,0)$ for $d = 0,6$ then, for every positive integer m, the graph mG also admits a super d-antimagic labeling of type $(1,1,0)$. They also proved that if a plane tripartite graph G with 3-sided inner faces admits a super d-antimagic labeling of type $(1,1,0)$ for $d = 2,4$ then for

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every positive integer m, the graph mG admits a super d-antimagic labeling of type $(1, 1, 0)$. In [2] it is shown that if a plane graph G with 4-sided inner faces admits a super d-antimagic labeling of type $(1, 1, 0)$ for $d = 0, 4, 8$ then the disjoint union of an arbitrary number of copies of G also admits a super d-antimagic labeling of type $(1, 1, 0)$.

Super d-antimagic labelings of type $(1, 1, 1)$ for disjoint union of prisms and for d belonging to $\{0, 1, 2, 3, 4, 5\}$ are given in [3] and for d belonging to $\{6, 7\}$ are given in [10]. The existence of the super d-antimagic labeling of type $(1, 1, 1)$ for the disjoint union of m copies of antiprism and for $d \in \{1, 2, 3, 5, 6\}$ is proved in [25].

The next theorem gives a result concerning (super) 1-antimagic labeling for an arbitrary plane graph. Note that the symbol z_{ext} is used to denote the unique external face in the plane graph.

Theorem 2.17. [23] *Let $G(V, E, F)$ be a plane graph. If there exists a (super) 1-antimagic labeling h of type $(1, 1, 1)$ of G such that $h(z_{ext}) = p + q + f + 1$ then, for every positive integer m, the graph mG also admits a (super) 1-antimagic labeling of type $(1, 1, 1)$.*

The next theorem presents results on (super) d-antimagic labelings of type $(1, 1, 1)$ for plane graphs containing 3-sided inner faces.

Theorem 2.18. [23] *Let $G(V, E, F)$ be a plane graph with 3-sided inner faces. Let h be a (super) d-antimagic labeling of type $(1, 1, 1)$ of G such that $h(z_{ext}) = p+q+f+1$.*

i) If $d = 1, 5, 7$, then for every positive integer m, the graph mG also admits a (super) d-antimagic labeling of type $(1, 1, 1)$.

ii) If G is a tripartite graph and $d = 3$, then for every positive integer m, the graph mG also admits a (super) d-antimagic labeling of type $(1, 1, 1)$.

iii) If G is a tripartite graph and $d = 0, 2, 4, 6$, then for every odd positive integer m, the graph mG also admits a (super) d-antimagic labeling of type $(1, 1, 1)$.

According to Theorems 2.8 and 2.9, from Theorem 2.18, we get

Corollary 2.1. *Let m, n, d be nonnegative integers, $n \geq 2$, $m \geq 1$. Then for m odd, the graphs mF_n and mE_n admit a super d-antimagic labeling of type $(1, 1, 1)$, for $d = 0, 2, 4, 6$. Moreover, if $n \equiv 1 \pmod 2$ then mF_n and mE_n admit a super d-antimagic labeling of type $(1, 1, 1)$, for $d = 1, 3, 5, 7$.*

For the plane graphs with all inner 4-sided faces, the following theorem was proved.

Theorem 2.19. [23] *Let $G(V, E, F)$ be a plane graph with 4-sided inner faces. Let h be a (super) d-antimagic labeling of type $(1, 1, 1)$ of G such that $h(z_{ext}) = p+q+f+1$. If $d = 1, 3, 5, 7, 9$, then for every positive integer m, the graph mG also admits a (super) d-antimagic labeling of type $(1, 1, 1)$.*

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Combining Theorems 2.10 and 2.11 with Theorem 2.19 we obtain the next two corollaries.

Corollary 2.2. [23] *Let* m, n, d *be nonnegative integers,* $3 \leq n \equiv 1 \pmod 2$*,* $m \geq 1$*. Then the graph* mB_n *admits a super* d*-antimagic labeling of type* $(1, 1, 1)$*, for* $d = 1, 3, 5, 7, 9$.

Corollary 2.3. [23] *Let* m, n, d *be nonnegative integers,* $n \geq 2$*,* $m \geq 1$*. Then the graph* mG_n *admits super* d*-antimagic labeling of type* $(1, 1, 1)$*, for* $d = 1, 3, 5, 7, 9$.

It is possible to generalize the results in Theorems 2.18 and 2.19 for plane graphs containing only k-sided faces except the external face, where k is a positive integer, $k \geq 3$.

Theorem 2.20. [23] *Let* k, d *be positive integers,* $k \geq 3$*. Let* $G(V, E, F)$ *be a plane graph with* k*-sided inner faces. Let* h *be a (super)* d*-antimagic labeling of type* $(1, 1, 1)$ *of* G *such that* $h(z_{ext}) = p + q + f + 1$.

i) If $d = 1, 2k \pm 1$*, then for every positive integer* m*, the graph* mG *also admits a (super)* d*-antimagic labeling of type* $(1, 1, 1)$.

ii) Moreover, if k *is even and* $d = k \pm 1$*, then for every positive integer* m*, the graph* mG *also admits a (super)* d*-antimagic labeling of type* $(1, 1, 1)$.

We can also formulate similar results for plane graphs with two kinds of inner faces.

Theorem 2.21. [23] *Let* k *be a positive integer,* $k \geq 3$*. Let* $G(V, E, F)$ *be a plane graph with* k*-sided and* $(k+1)$*-sided inner faces. Let* h *be a (super)* $(2k+1)$*-antimagic labeling of type* $(1, 1, 1)$ *of* G *such that* $h(z_{ext}) = p+q+f+1$*. Then, for every positive integer* m*, the graph* mG *also admits a (super)* $(2k + 1)$*-antimagic labeling of type* $(1, 1, 1)$.

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