Far East Journal of Mathematical Sciences (FJMS)
© 2017 Pushpa Publishing House, Allahabad, India
http://www.pphmj.com
http://dx.doi.org/10.17654/MS102050965
Volume 102, Number 5, 2017, Pages 965-978 **ISSN: 0972-0871**

ON METRIC DIMENSION OF EDGE-CORONA GRAPHS

Rinurwati[1,2], Herry Suprajitno[1] and Slamin[3]

[1]Mathematics Department
Universitas Airlangga
Indonesia
e-mail: rinur@matematika.its.ac.id
herrysuprajitno.math.unair@gmail.com

[2]Mathematics Department
Sepuluh Nopember Institute of Technology
Indonesia

[3]Information System Study Program
University of Jember
Indonesia
e-mail: slamin@unej.ac.id

Abstract

Given graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, an ordered set $U \subseteq V_G$ is called a resolving set of G if coordinate of distances of every vertex in G to vertices in U is different. Metric dimension of G is the minimal cardinality of a resolving set of G. An edge-corona graph $G \Diamond H$ is obtained by joining end vertices of $e_j \in E_g$, $j \in \{1, 2, ..., |E_G|\}$ with all vertices from jth-copy of H. This paper discusses some characterization and exact values for metric dimension of edge-corona from a connected graph not tree G with an arbitrary nontrivial graph H.

Received: April 3, 2017; Revised: June 7, 2017; Accepted: June 20, 2017
2010 Mathematics Subject Classification: 05C12.
Keywords and phrases: edge-corona, resolving set, metric dimension.

1. Introduction

Throughout we use simple graph and finite graph. Given a graph $G = (V_G, E_G)$, let V_G be a vertex set of G and let E_G be an edge set of G. For a further reference, we can see Chartrand et al. [2].

For an ordered set $U = \{u_1, u_2, ..., u_k\} \subseteq V_G$, a representation of $t \in V_G$ to $Ur(t|U)$ is defined to be $r(t|U) := (d(t, u_1), d(t, u_2), ..., d(t, u_k))$, where $d(t, u_i)$ is a distance from a vertex t to a vertex u_i. U is called a *resolving set* for G if for arbitrary two vertices $s, t \in V_G$, $r(s|U) \neq r(t|U)$. A resolving set for G with minimum cardinality is called *basis* for G. *Metric dimension* of G, denoted by $\dim(G)$, is the cardinality of a basis in G.

The results by Chartrand et al. [2] are used in this paper to mention the research on metric dimension of graphs obtained by operation of graphs. Previous researches on metric dimensions of corona graphs have been done, for example, by Iswadi et al. [5] and Yero et al. [7]. In [4], Hou and Shiu defined edge-corona operation of graphs and gave some results about spectrums of edge-corona graphs, but there is no result about metric dimension of edge-corona graphs yet. Recently, in [6], Rinurwati et al. studied about local metric dimension of edge-corona graphs.

Motivated by above results, we study further on edge-corona graphs. An *edge-corona graph* $G \Diamond H$ is obtained by joining end vertices of edge $e_j \in E_G$, $j \in \{1, 2, ..., |E(G)|\}$ with all vertices from jth-copy of H. The following figures are some examples of edge-corona graphs:

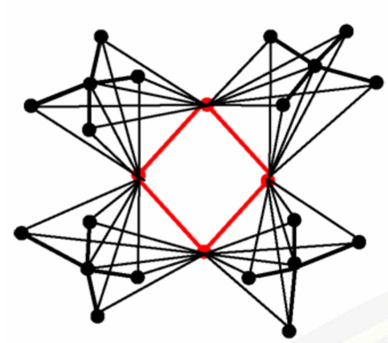

Figure 1. $C_4 \Diamond S_4$.

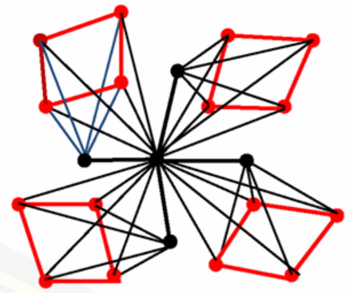

Figure 2. $S_4 \Diamond C_4$.

This paper discusses some characterization and exact values for metric dimension of edge-corona from a connected graph not tree G with an arbitrary nontrivial graph H.

2. Results

We begin with the following:

Lemma 1. *Let the order and size of a connected graph G and a graph H be, respectively, $p_1 \geq 3$, $q_1 \geq 2$ and $p_2 \geq 2$, $q_2 \geq 0$. If jth-copy of H, $H_j = (V_{H_j}, E_{H_j})$; $j \in \{1, 2, ..., q_1\}$, is a subgraph of $G \Diamond H$, then the following hold*:

(a) *If* $s, t \in V_{H_j}$, *then* $d_{G \Diamond H}(s, u) = d_{G \Diamond H}(t, u)$ *for every* $u \in V_{G \Diamond H} - V_{H_j}$.

(b) *If* $V_{H_j} \cap U = \varnothing$ *for some j, then U is not a resolving set for* $G \Diamond H$.

(c) *If U is a basis for* $G \Diamond H$, *then* $V_G \cap U$ *is empty.*

(d) *For every connected graph H and resolving set U of* $G \Diamond H$, $U_j = U \cap V_{H_j}$ *is a resolving set for* H_j.

Proof. (a) We know that $s, t \in V_{H_j}$. Let $z = s_j \in V_G$, and take any $u \in V_{G \lozenge H}$ and $u \notin V_{H_j}$. The result can be followed directly from the fact that

$$d_{G \lozenge H}(s, u) = d_{G \lozenge H}(s, z) + d_{G \lozenge H}(z, u) = d_{G \lozenge H}(t, z) + d_{G \lozenge H}(z, u)$$

$$= d_{G \lozenge H}(t, u).$$

(b) We suppose that $V_{H_j} \cap U = \varnothing$ for some $j \in \{1, ..., q_1\}$. Let $s, t \in V_{H_j}$. By (a), we have $d_{G \lozenge H}(s, y) = d_{G \lozenge H}(t, y)$ for every vertex $y \in U$, which is a contradiction.

(c) We suppose that $V_G \cap U \neq \varnothing$. We will present that $U_1 = U - V_G$ is a resolving set for $G \lozenge H$. Let $s, t \in V_{G \lozenge H}$, with $s \neq t$. There are four cases to be considered:

Case 1. $s, t \in V_{H_j}$.

Using (a), we conclude that there exists a vertex $x \in V_{H_j} \cap U_1$ such that $d_{G \lozenge H}(s, x)$ is not equal to $d_{G \lozenge H}(t, x)$.

Case 2. $s \in V_{H_j}$, $t \in V_{H_h}$ and $j \neq h$.

Let $u \in V_{H_j} \cap U_1$. Then we have $d_{G \lozenge H}(s, u) \leq 2 \leq d_{G \lozenge H}(t, u)$.

Case 3. $s, t \in V_G$.

Let $s = s_{h_j}$ be a vertex of $e_j = s_{i_j} s_{h_j}$ of G, where $i \neq h \in \{1, 2, ..., p_1\}$ for some $j \in \{1, 2, ..., q_1\}$, and let $t \neq s_{i_j}$. Let $z \in V_{H_j} \cap U'$. Thus, we have

$$d_{G \lozenge H}(s, z) = 1 < 1 + d_{G \lozenge H}(t, s) = d_{G \lozenge H}(t, z).$$

Case 4. $s \in V_{H_j}$, $t \in V_G$. There are two subcases:

(1) If s is adjacent to t, then $t = s_{i_j}$ or $t = s_{h_j}$ of $e_j = s_{i_j} s_{h_j}$ of G, for some $j \in \{1, 2, ..., q_1\}$. Let $e_k = s_{i_k} s_{h_k} \in E_G$; $j \neq k \in \{1, 2, ..., q_1\}$ and $i \neq h \in \{1, 2, ..., p_1\}$. Let $a \in V_{H_k} \cap U_1$. Let $t = s_{h_j} = s_{i_k}$. Then we have

$$d_{G \Diamond H}(s, a) = d_{G \Diamond H}(s, t) + d_{G \Diamond H}(t, a) = 1 + d_{G \Diamond H}(t, a) > d_{G \Diamond H}(t, a).$$

(2) If s is not adjacent to t, then $t = s_{i_k} \neq s_{h_j}$ or $t = s_{h_k}$. We have

$$d_{G \Diamond H}(s, a) = d_{G \Diamond H}(s, t) + d_{G \Diamond H}(t, a) > d_{G \Diamond H}(t, a).$$

So, U_1 is a resolving set for $G \Diamond H$.

(d) Let $U_j = U \cap V_{H_j}$. For $s \in U_j$ or $t \in U_j$, it is obvious that $r(s|U_j) \neq r(t|U_j)$. We suppose that $s, t \in (V_{H_j} - U_j)$. It is known that U is a resolving set for $G \Diamond H$. We obtain that $r(s|U)$ is not the same as $r(t|U)$. Using (a), we obtain $d_{G \Diamond H}(s, a) = d_{G \Diamond H}(t, a)$ for every $a \in (V_{G \Diamond H} - V_{H_j})$. So, there is a vertex $a \in U_j$ with $d_{G \Diamond H}(s, a) \neq d_{G \Diamond H}(t, a)$. Therefore, either a is adjacent to s and a is not adjacent to t or a is not adjacent to s and a is adjacent to t. In Case 1, we have $d_{G \Diamond H}(s, a) = d_{H_j}(s, a) = 1$ and $d_{G \Diamond H}(t, a) = 2 \leq d_{H_j}(t, a)$. It is analogous for cases when a is not adjacent to s and a is adjacent to t. Thus, U_j is a resolving set for H_j. □

Theorem 2. *Let the order and size of a connected graph* (*not a tree*) G *and a graph* H *be, respectively,* $p_1 \geq 3$, $q_1 \geq 3$ *and* $p_2 \geq 2$, $q_2 \geq 1$. *Then*

$$\dim(G \Diamond H) \geq q_1 \dim(H).$$

Proof. We consider that W is a minimum resolving set for $G \Diamond H$. From Lemma 1(c), we obtain $W \cap V_G = \emptyset$. Then, by Lemma 1(b), we have for every $j \in \{1, 2, ..., q_1\}$, there is a set $W_j \neq \varnothing \subseteq W$ such that $W = \bigcup_{j=1}^{q_1} W_j$. Moreover, by Lemma 1(d), we obtain that W_j is a resolving set for H_j. Therefore,

$$\dim(G \lozenge H) = |W| = \sum_{j=1}^{q_1} |W_j| = \sum_{j=1}^{q_1} \dim(H) \geq q_1 \dim(H).$$

Then we obtain the lower bound on $\dim(G \lozenge H)$. □

Theorem 3. *Let the order and size of a connected graph (not a tree) G and a graph H be, respectively,* $p_1 \geq 3$, $q_1 \geq 3$ *and* $p_2 \geq 2$, $q_2 \geq 1$. *If diameter of* $HD(H)$ *is smaller than or equal to two, then*

$$\dim(G \lozenge H) = q_1 \dim(H).$$

Proof. Let $W_j \subset V_{H_j}$ be a resolving set for H_j and let $W = \bigcup_{j=1}^{q_1} W_j$. We will prove that W is a resolving set for $G \lozenge H$. There are four cases to be considered:

Case 1. $s, t \in V_{H_j}$. Because $D(H) \leq 2$, we obtain $r(s|W_j) \neq r(t|W_j)$ for every $j \in \{1, 2, ..., q_1\}$, so $r(s|W) \neq r(t|W)$.

Case 2. $s \in V_{H_j}$, $t \in V_{H_h}$, and $j \neq h$. Let $a \in V_{H_j}$. Thus, we obtain $d(s, a) \leq 2 \leq d(t, a)$.

Case 3. $s, t \in V_G$. For every $a \in V_{H_j}$, we find $d(s, a) = 1 \leq d(t, s) + d(s, a) = d(t, a)$.

Case 4. $s \in V_{H_j}$ and $t \in V_G$. If s is adjacent to t, then $t = t_j = s_{i_j}$ or $t = t_h = s_{h_j}$ of $e_j = s_{i_j} s_{h_j}$ of G. Let $a \in W_h$, for some $j \neq h$. Hence, we obtain $d(s, a) = 1 + d(t, a) > d(t, a)$. Furthermore, if s is not adjacent to t, for $a \in W_h$, then we obtain $d(s, a) > d(t, s) + d(s, a) = d(t, a)$. Therefore, for every $s \neq t \in V_{G \lozenge H}$, we have $r(s|W) \neq r(t|W)$, so $\dim(G \lozenge H) \leq q_1 \dim(H)$. Combining with Theorem 2, we conclude that $\dim(G \lozenge H) = q_1 \dim(H)$. □

We recall a well known lemma below to present a consequence of Theorem 3. We note that $K_{m,n}$ is a complete bipartite graph of order $m + n$, K_m is a complete graph of order m, and $\overline{K}_n$ is an empty-graph of order n.

Lemma 4 [2]. *Given a connected graph G, order of G is* $p \geq 4$. $\dim(G) = (p - 2)$ *if and only if* $G = K_{a,b}$ $(a, b \geq 1)$; $G = (K_a + \overline{K}_b)$, $(a \geq 1, b \geq 2)$, *or* $G = (K_a + (K_1 \cup K_b))$, $(a, b \geq 1)$.

Corollary 5. *Let the order and size of a connected graph (not a tree) G and a connected graph H be, respectively,* $p_1 \geq 3$, $q_1 \geq 3$ *and* $p_2 \geq 4$, $q_2 \geq 4$. *Diameter of H is smaller than or equal to two.*

$\dim(G \lozenge H) = q_1(p_2 - 2)$ *if and only if*

$$H = K_{a,b} \ (a, b \geq 1); \quad H = (K_a + \overline{K}_b), (a \geq 1, b \geq 2)$$

$$\text{or } H = (K_a + (K_1 \cup K_b)), (a, b \geq 1).$$

A special condition of Theorem 3 is given in the following results.

Corollary 6. *Let the order and size of a connected graph (not a tree) be, respectively,* $p_1 \geq 3$ *and* $q_2 \geq 3$. *If* $H \cong F_{1,p_2}$ *or* $H \cong Q_{1,p_2}$ *with* $p_2 \geq 7$, *then*

$$\dim(G \lozenge H) = q_1 \left\lfloor \frac{2p_2 + 2}{5} \right\rfloor.$$

A fan graph, denoted by F_{1,p_2}, is a joint graph $K_1 + P_{p_2}$, where K_1 is a trivial graph with one vertex and P_{p_2} is a path graph with p_2 vertices [4]. A wheel graph, denoted by W_{1,p_2}, is a joint graph $K_1 + C_{p_2}$, where K_1 is a trivial graph with one vertex and C_{p_2} is a cycle graph with p_2 vertices [6].

Theorem 7. *Let the order and size of a connected graph (not a tree) G and a graph H be, respectively,* $p_1 \geq 3,\ q_1 \geq 3$ *and* $p_2 \geq 2,\ q_2 \geq 1$. *Let* ω *be the cardinality of isolated vertices in H. Let* λ *be the cardinality of connected components in H with order greater than or equal to two*:

$$\dim(G \lozenge H) \leq \begin{cases} q_1(p_2 - \lambda - 1) & \text{for } \lambda \geq 1 \text{ and } \omega \geq 1, \\ q_1(p_2 - \lambda) & \text{for } \lambda \geq 1 \text{ and } \omega = 0, \\ q_1(p_2 - 1) & \text{for } \lambda = 0. \end{cases}$$

Proof. We consider that $\omega \geq 1$ and $\lambda \geq 1$. Let $(A_\ell)_j$, $\ell \in \{1, 2, ..., \lambda\}$ be ℓth-connected component of H_j, $(p_\ell)_j$ is one of vertices of $(A_\ell)_j$ and $P_j \subseteq V_{G \lozenge H}$ with $P_j = \{(p_\ell)_j; \ell \in \{1, 2, ..., \lambda\}\}$, $j \in \{1, 2, ..., q_1\}$. If $\omega \geq 2$, let b_j be one of isolated vertices of H_j, and $Q_j = \{b_j\}$. If $\omega = 1$, then we consider $Q_j = \varnothing$. Now, we will show that $W = \bigcup_{h=1}^{q_1}(P_h \cup Q_h)$ resolves a graph $G \lozenge H$. Let $s, t \in V_{G \lozenge H}$, $s \neq t$. We consider $s, t \notin W$. So, there are four cases to be considered:

Case 1. $s = s_j \in V_G$ with $s = s_{i_j}$ or $s = s_{h_j}$ of edge $e_i = s_{i_j}s_{h_j}$; $s_{i_j}, s_{h_j} \in V_G$, $i, h \in \{1, 2, ..., p_1\}$; $j \in \{1, 2, ..., q_1\}$; and $t \in V_{H_j}$. Then for every vertex $a \in V_{H_h} \cap W$ with $h \neq j$, we obtain $d(t, a) = d(t, s) + d(s, a) > d(s, a)$.

Case 2. $s = s_j \in V_G$ with $s = s_{i_j}$ or $s = s_{h_j}$ of edge $e_j = s_{i_j}s_{h_j}$; $s_{i_j}, s_{h_j} \in V_G$, $i, h \in \{1, 2, ..., p_1\}$; $j \in \{1, 2, ..., q_1\}$; and $t \notin V_{H_j}$. For every vertex $a \in (W \cap V_{H_j})$, we have $d(s, a) = 1 < d(t, a)$.

Case 3. $s \in V_{H_j}$ and $t \in V_{H_h}$, $h \neq j$. For every vertex $a \in W \cap V_{H_j}$, we obtain $d(s, a) \leq 2 \leq (t, a)$.

Case 4. $s, t \in V_{H_j}$. Let s be not an isolated vertex in V_{H_j}. There exists a vertex $a \in W \cap V_{H_j}$ such that a is adjacent to s, thus $d(s, a) = 1 < 2 = d(t, a)$. So, for any two vertices $t, s \in V_{G \lozenge H}$ with $t \neq s$, we get $r(s|W) \neq r(t|W)$, and consequently, $\dim(G \lozenge H) \leq q_1(p_2 - \lambda - 1)$. If $\omega = 0$, then we take $W = \bigcup_{j=1}^{q_1} P_j$ and we get $\dim(G \lozenge H) \leq q_1(p_2 - \lambda)$. If $\lambda = 0$, then we take $W = \bigcup_{j=1}^{q_1} Q_j$ so that we get $\dim(G \lozenge H) \leq q_1(p_2 - 1)$. So, this completes the proof. □

Corollary 8. *Let the order and size of a connected graph (not a tree) G and a connected graph H be, respectively,* $p_1 \geq 3$, $q_1 \geq 3$ *and* $p_2 \geq 2$, $q_2 \geq 0$. *Then*

$$\dim(G \lozenge H) = q_1(p_2 - 1) \text{ if and only if } H \cong \overline{K_{p_2}}.$$

Theorem 9. *Let the order and size of a connected graph (not a tree) G and a connected graph H be, respectively,* $p_1 \geq 3$, $q_1 \geq 3$ *and* $p_2 \geq 2$, $q_2 \geq 1$. *Then*

$$\dim(G \lozenge H) = q_1(p_2 - 1) \text{ if and only if } H \text{ is isomorphic with } K_{p_2}.$$

Furthermore, if H is not isomorphic to K_{p_2}, *then* $\dim(G \lozenge H) \leq q_1(p_2 - 2)$.

Proof. Since $\dim(K_{p_2}) = p_2 - 1$, by Theorem 3, we obtain $\dim(G \lozenge H) = q_1(p_2 - 1)$. On the other hand, we consider $H \ncong K_{p_2}$. Let $A \subseteq V_H$, $x \in V_H$, and $N_A(x) := \{y \in A | y \text{ is adjacent to } x\}$. Let $b, c \in V_H$ and $A_{b,c} = V_H \setminus \{b, c\}$. Because the graph H is connected and $H \ncong K_{p_2}$, there exist at least two vertices b, c of V_H such that $N_{A_{b,c}}(b) \neq N_{A_{b,c}}(c)$. Let $b_j, c_j \in V_{H_j}$ be jth-copy of $b, c \in V_H$, respectively. Let $W =$

$\bigcup_{j=1}^{p_2}(V_{H_j} - \{b_j, c_j\})$. Now, we show that W resolves a graph $G \lozenge H$. Let $f \neq g \in V_{G \lozenge H}$ but $f, g \notin W$. We have three cases as follows:

Case 1. $f = b_j$ and $g = c_j$. Because $N_{A_{b,c}}(b) \neq N_{A_{b,c}}(c)$, we obtain $r(f|W) \neq r(g|W)$.

Case 2. $f = s_j \in V_G$ with $f = s_{i_j}$ or $f = s_{h_j}$ of edge $e_j = s_{i_j}s_{h_j}$; $s_{i_j}, s_{h_j} \in V_G$, $i, h \in \{1, 2, ..., p_1\}$, $j \in \{1, 2, ..., q_1\}$, and $g \in V_{H_j}$. We have $d(g, s) = (d(g, f) + d(f, s)) = (1 + d(f, s)) > d(f, s)$. If $f \in V_{H_j}$ and $g \in V_{H_h}$ with $h \neq j$, then for every vertex $s \in (V_{H_j} - \{b_j, c_j\})$, we have $d(f, s) \leq 2 \leq d(g, s)$.

Case 3. $f, g \in V_G$ with $f = s_j = s_{i_j}$ or $f = s_j = s_{h_j}$ of edge $e_j = s_{i_j}s_{h_j}$; $s_{i_j}, s_{h_j} \in V_G$; $i, h \in \{1, 2, ..., p_1\}$; $j \in \{1, 2, ..., q_1\}$; and $f \neq g$. Let $f = s_j$. Then, for every vertex $s \in (V_{H_j} - \{b_j, c_j\})$, we have $d(f, s) = 1 < d(g, s)$. Therefore, for any two different vertices $f, g \in V_{G \lozenge H}$, we get $r(f|W) \neq r(g|W)$. So, $\dim(G \lozenge H) \leq q_1(p_2 - 2)$. □

The above bound is tight as we have presented in Corollary 5.

Theorem 10. *Let the order and size of a connected graph* (*not a tree*) *G and a connected graph H be, respectively,* $p_1 \geq 3$, $q_1 \geq 3$ *and* $p_2 \geq 2$, $q_2 \geq 1$. *Then*

$$\dim(G \lozenge H) \leq q_1(\dim(K_2 + H) - 1).$$

Proof. Let $K_2 + H_j$ be a subgraph of $G \lozenge H$ graph which is formed by connecting every end vertices s_{i_j} and s_{h_j} of edge $e_j = S_{i_j}S_{h_j} \in E_G$ with all vertices in H_j. For every edge $e_j \in E_G$, let U_j be a basis of $K_2 + H_j$ and $U = \bigcup_{j=1}^{q_1} U_j$. By Lemma 1(c), we obtain that s_{i_j} and s_{h_i} do not belong to any basis for $K_2 + H_j$. Hence, there is no vertex from G that contained in U. In other words, $U \cap V_G \neq \emptyset$. Now, we will prove that U is a resolving set for $G \lozenge H$. Given two vertices $s, t \in V_{G \lozenge H}$, there are four cases to be considered:

Case 1. $s, t \in V_{H_j}$. There exists $x \in U_j$ such that $d_{K_2+H_j}(s, x) \neq d_{K_2+H_j}(t, x)$. This leads to $d_{G \lozenge H}(s, x) \neq d_{G \lozenge H}(t, x)$.

Case 2. $s \in V_{H_j}$ and $t \in V_{H_h}$, $j \neq h$. Let $y \in U_j$. We obtain $d_{G \lozenge H}(s, y) \leq 2 < 3 \leq d_{G \lozenge H}(t, y)$.

Case 3. Let $e_j = s_{i_j} s_{h_j}$, $e_k = s_{i_h} s_{h_k} \in E_G$ with $j \neq k$. Let $s_{i_j} = s$, $s_{h_j} = t$, $s_{i_k} = u$ and $s_{h_k} = v$. Suppose s and t are adjacent to the vertices of V_{H_j}. There are two subcases:

Subcase 1. $t = u$ and $s \neq v$. Thus, for every vertex $x \in U_j$, we obtain

$$d_{G \lozenge H}(s, x) = 1 < d_{G \lozenge H}(s, v) + 1 = d_{G \lozenge H}(v, x)$$

and $d_{G \lozenge H}(s, x) = d_{G \lozenge H}(t, x)$.

Subcase 2. $t \neq u$ and $s \neq v$. Hence, for every vertex $x \in U_j$, we obtain

$$d_{G \lozenge H}(s, x) = 1 < d_{G \lozenge H}(t, u) + 1 \leq d_{G \lozenge H}(s, u) + 1$$

$$= d_{G \lozenge H}(x, u) < d_{G \lozenge H}(x, v)$$

and $d_{G \lozenge H}(s, x) = d_{G \lozenge H}(t, x)$.

Case 4. $s \in V_{H_j}$ and $t \in V_G$. There are two subcases:

Subcase 1. s is adjacent to t. Then $t = s_{i_j}$ or $t = s_{h_j}$. For every vertex $x \in U_k$, $k \neq j$, $k \in \{1, 2, ..., q_1\}$, we obtain

$$d_{G \lozenge H}(s, x) = d_{G \lozenge H}(t, x) + 1 > d_{G \lozenge H}(t, x).$$

Subcase 2. s is not adjacent to t. Then $t = s_{i_k}$ or $t = s_{h_k}$ with $j \neq k$ for some $k \in \{1, 2, ..., q_1\}$. Hence, there exists $x \in U_k$ adjacent to t. Thus, we have

$$d_{G \lozenge H}(s, x) = d_{G \lozenge H}(s, t) + 1 > d_{G \lozenge H}(t, x).$$

Therefore, for any two vertices $s, t \in V_{G \lozenge H}$ with $s \neq t$, we obtain $r(s|U) \neq r(t|U)$, consequently, $\dim(G \lozenge H) \leq q_1 \dim(K_2 + H)$. □

Theorem 11. *Let the order and size of a connected graph (not a tree) G and a connected graph H be, respectively,* $p_1 \geq 3$, $q_1 \geq 3$ *and* $p_2 \geq 7$, $q_2 \geq 6$. *Diameter of H is greater than or equal to six or* $H \cong C_{p_2}$:

$$\dim(G \lozenge H) = q_1([\dim(K_2 + H)] - 1).$$

Proof. We consider W is a basis for $G \lozenge H$. By Lemma 1(c), we obtain $W \cap V_G = \varnothing$, so $W = \bigcup_{j=1}^{q_1} W_j$, with $W_j \subset V_{H_j}$. By using Lemma 1(b), we have $W_j \neq \varnothing$ for every $j \in \{1, 2, ..., q_1\}$. Furthermore, we will show that for every $u \in V_{H_j} - W_j$ holds $r(u|W_j) \neq (1, 1, ..., 1)$. There are two cases:

Case 1. H is a cycle graph of order $p_2 \geq 7$. If $r(x|W_j) = (1, 1)$ for some $x \in V_{H_j} - W_j$, then as $p_2 \geq 7$, there are two vertices $s, t \in V_{H_j} - W_j$ such that $d_{H_j}(s, u) > 1$ and $d_{H_j}(t, u) > 1$, for every $u \in W_j$. So, $d_{G \lozenge H}(t, u) = d_{G \lozenge H}(s, u) = 2$ for every $u \in W_j$. This is a contradiction, since by using Lemma 1(a), we have $d_{G \lozenge H}(s, u) = d_{G \lozenge H}(t, u)$ for every $x \in W - W_j$.

Case 2. Diameter of H is greater than or equal to six. Let s and t be two vertices in $V_{H_j} - W_j$. Because W is a resolving set for $G \lozenge H$, we obtain that $r(s|W)$ is not equal to $r(t|W)$. By Lemma 1(a), it is mentioned that

$d_{G \Diamond H}(s, u) = d_{G \Diamond H}(t, u)$ for every vertex $u \in V_{G \Diamond H} - V_{H_j}$. So, there is a vertex $x \in W_j$ such that $d_{G \Diamond H}(s, x)$ is not equal to $d_{G \Diamond H}(t, x)$, consequently, either x is adjacent to t and x is not adjacent to s or x is not adjacent to t and x is adjacent to s. Moreover, we consider that there is a vertex $y \in V_{H_j} - W_j$ such that $r(y|W)$ is equal to $(1, 1, ..., 1)$. If there exists vertex $u \in W_j$ such that $d_{H_j}(z, u) > 1$, then for every $c \in V_{H_j} - (W_j \cup \{y, z\})$, there is a vertex $x \in W_j$ such that z is adjacent to x. Therefore, the diameter of H_j is lower than or equal to five. Furthermore, if for every $z \in V_{H_j} - W_j$, there is a vertex $x_z \in W_j$ such that x_z is adjacent to z, then the diameter of H is lower than or equal to four. So, if diameter of H is greater than or equal to six, then for every vertex $y \in V_{H_j} - W_j$ holds $r(y|W_j) \neq (1, 1, ..., 1)$.

We use $K_2 + H_j$ as the subgraph of $G \Diamond H$ which is formed by connecting every end vertices s_i, s_h of edge $e_j = s_i s_h$ of G, to every vertex in H_j. From Cases 1 and 2 above, we obtain that for every vertex $y \in V_{H_j} - W_j$ holds $r(s_j|W_j) = r(y|W_j) \neq (1, 1, ..., 1)$. Therefore, W_j is a resolving set for $K_2 + H_j$. So, for every $j \in \{1, 2, ..., q_1\}$ holds $\dim(K_2 + H_j) - 1 \leq |W_j|$. Therefore, $\dim(G \Diamond H) \leq q_1(\dim(K_2 + H_j) - 1)$. The proof is complete. □

The following is a consequence of Theorem 11.

Corollary 12. *Let the order and size of a connected graph* (*not a tree*) G *be, respectively,* $p_1 \geq 3$ *and* $q_2 \geq 3$. *The following hold*:

(i) *If* $p_2 \geq 7$, *then* $\dim(G \Diamond P_{p_2}) = q_1\left(\left\lfloor \frac{2p_2 + 2}{5} \right\rfloor - 1\right)$.

(ii) *If* $p_2 \geq 7$, *then* $\dim(G \Diamond C_{p_2}) = q_1\left(\left\lfloor \frac{2p_2 + 2}{5} \right\rfloor - 1\right)$.

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions and constructive criticism which improved the presentation of the paper.

References

[1] P. S. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, On *k*-dimensional graphs and their bases, Period. Math. Hungar. 46(1) (2003), 9-15.

[2] G. Chartrand L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000), 99-113.

[3] J. Caceres, C. Hernando, M. Mora, M. L. Puertas, I. M. Pelayo and C. Seara, On the metric dimension of some families of graphs, Electron. Notes Discrete Math. 22 (2005), 129-133.

[4] Y. Hou and W. C. Shiu, The spectrum of the edge corona of two graphs, Electron. J. Linear Algebra 20 (2010), 586-594.

[5] H. Iswadi, E. T. Baskoro and R. Simanjuntak, On the metric dimension of corona product of graphs, Far East J. Math. Sci. (FJMS) 52(2) (2011), 155-170.

[6] Rinurwati, Slamin and H. Suprajitno, General results of local metric dimensions of edge-corona of graphs, Int. Math. Forum 11(16) (2016), 793-799.

[7] I. G. Yero, D. Kuziak and J. A. Rodriguez-Velazquez, On the metric dimension of corona product graphs, Comput. Math. Appl. 61(9) (2011), 2793-2798.