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KOMUTATOR OPERATOR MOMENTUM SUDUT DALAM KOORDINAT BOLA DENGAN FUNGSI GELOMBANG ATOM HIDROGEN

SKRIPSI

Oleh
Abdul Rafie Nugraha
NIM 130210102054

PROGRAM STUDI PENDIDIKAN FISIKA
JURUSAN PENDIDIKAN MIPA
FAKULTAS KEGURUAN DAN ILMU PENDIDIKAN
UNIVERSITAS JEMBER
2017

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SKRIPSI

Diajukan guna melengkapi tugas akhir dan memenuhi salah satu syarat untuk
menyelesaikan Program Studi Pendidikan Fisika (S1)
dan mencapai gelar Sarjana Pendidikan

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2017

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PERSEMBAHAN

Skripsi ini saya persembahkan untuk:

1. Ibunda tercinta Panca Nurwasih dan Ayahanda (Alm) Abdul Kadir;
2. Guru-guruku sejak sekolah dasar sampai dengan perguruan tinggi;
3. Almamater Fakultas Keguruan dan Ilmu Pendidikan Universitas Jember;
4. Keluarga besar Kelas Unggulan Pendidikan Fisika 2013.

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MOTO

"Sang ilmuwan tidak mempelajari alam karena manfaatnya; ia mempelajarinya karena ia menyukainya, dan ia menyukainya karena keindahannya. Jika alam tidak indah, maka alam tidak patut untuk dipelajari, dan jika alam tidak patut untuk dipelajari, maka kehidupan menjadi tidak patut dijalani."*)

*) Serway, R.A. dan J.W. Jewett, Jr. 2010. *Fisika untuk Sains dan Teknik Buku 3 Edisi 6*. Jakarta: Salemba Teknika

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PERNYATAAN

Saya yang bertanda tangan di bawah ini:

nama : Abdul Rafie Nugraha

NIM : 130210102054

menyatakan dengan sesungguhnya bahwa karya ilmiah yang berjudul "Komutator Operator Momentum Sudut dalam Koordinat Bola dengan Fungsi Gelombang Atom Hidrogen" adalah benar-benar hasil karya sendiri, kecuali kutipan yang sudah saya sebutkan sumbernya, belum pernah diajukan pada institusi mana pun, dan bukan karya jiplakan. Saya bertanggung jawab atas keabsahan dan kebenaran isinya sesuai dengan sikap ilmiah yang harus dijunjung tinggi.

Demikian pernyataan ini saya buat dengan sebenarnya tanpa adanya tekanan dan paksaan dari pihak mana pun serta bersedia mendapat sanksi akademik jika ternyata di kemudian hari pernyataan ini tidak benar.

Jember, Maret 2017
Yang menyatakan,

Abdul Rafie Nugraha
NIM 130210102054

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SKRIPSI

KOMUTATOR OPERATOR MOMENTUM SUDUT DALAM KOORDINAT BOLA DENGAN FUNGSI GELOMBANG ATOM HIDROGEN

Oleh

Abdul Rafie Nugraha
NIM 130210102054

Pembimbing

Dosen Pembimbing Utama : Drs. Bambang Supriadi, M.Sc.

Dosen Pembimbing Anggota : Drs. Sri Handono Budi Prastowo, M.Si.

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PENGESAHAN

Skripsi berjudul "Komutator Operator Momentum Sudut dalam Koordinat Bola dengan Fungsi Gelombang Atom Hidrogen" telah diuji dan disahkan pada:

Hari, tanggal : Jumat, 31 Maret 2017

Tempat : Fakultas Keguruan dan Ilmu Pendidikan Universitas Jember

Tim Penguji:

Ketua,

Drs. Bambang Supriadi, M.Sc.
NIP 196807101993021001

Sekretaris,

Drs. Sri Handono Budi P., M.Si.
NIP 195803181985031004

Anggota I,

Dr. Yushardi, S.Si., M.Si.
NIP 196504201995121001

Anggota II,

Drs. Trapsilo Prihandono, M.Si.
NIP 196204011987021001

Mengesahkan
Dekan,

Prof. Drs. Dafik, M.Sc., Ph.D.
NIP. 196808021993031004

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RINGKASAN

Komutator Operator Momentum Sudut dalam Koordinat Bola dengan Fungsi Gelombang Atom Hidrogen; Abdul Rafie Nugraha; 130210102054; 2017: 51 Halaman; Program Studi Pendidikan Fisika, Jurusan Pendidikan Matematika dan Ilmu Pengetahuan Alam, Fakultas Keguruan dan Ilmu Pendidikan, Universitas Jember.

Mekanika kuantum merupakan bahasan fisika yang bersifat mikroskopik sehingga jika ingin mengukur suatu besaran, maka alat ukur yang digunakan terbatas dan hasil ukur yang didapat hanya probabilitas dari besaran tersebut. Walaupun, besaran fisika kuantum hanya memiliki alat ukur yang terbatas, namun besarannya dapat dihitung dengan seperangkat persamaan matematis yaitu dengan menggunakan persamaan schrödinger dan hubungan komutasi dari beberapa operator. Komutator bersifat komut jika beberapa operator dapat diukur secara serempak, namun pada bahasan mekanika kuantum umumnya observabel tidak dapat diukur secara serempak sehingga komutator bersifat tidak komut. Pembahasan mengenai momentum sudut sangat penting dalam mekanika kuantum, karena dari momentum sudut dapat diterangkan lebih mendalam sifat-sifat atom, molekul, kemagnetan dan lain sebagainya. Tujuan penelitian ini di antaranya (1) menentukan persamaan matematis dari komutator operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen; (2) menentukan hubungan komutasi operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen.

Jenis penelitian ini adalah *basic research* pada bidang fisika teori berupa pengembangan teori mekanika kuantum. Penelitian ini dilakukan di Laboratorium Fisika Dasar, Program Studi Pendidikan Fisika, Universitas Jember. Sumber data yang digunakan pada penelitian ini berasal dari buku, jurnal, dan internet tentang operator momentum sudut dalam koordinat bola, komutator, dan fungsi gelombang atom hidrogen. Metode pengambilan data penelitian ini yaitu menghitung komutator operator momentum sudut dalam koordinat jika

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dioperasikan dengan fungsi harmonik bola atom hidrogen. Setelah didapat hasil dari komutator tersebut, kemudian datanya dianalisis untuk mengetahui komutator tersebut komut atau tidak komut. Desain penelitian yang digunakan di antaranya tahap persiapan, tahap pengembangan teori, tahap hasil pengembangan teori, tahap validasi hasil pengembangan teori, tahap pengambilan data, tahap pembahasan, dan tahap kesimpulan.

Pada penelitian ini, momentum sudut pada atom hidrogen yang diukur hanya gerakan elektron mengelilingi inti dan tidak meninjau gerakan elektron berputar pada porosnya. Metode yang digunakan untuk dapat mengamati momentum sudut elektron adalah metode operator. Operator momentum sudut pada penelitian ini dioperasikan dengan fungsi gelombang atom hidrogen dalam koordinat bola sehingga fungsi gelombang bagian radial dari atom hidrogen dapat diabaikan karena tidak mempengaruhinya dan hanya menggunakan fungsi harmonik bola yang terdiri dari fungsi bagian polar dan fungsi bagian azimut.

Berdasarkan data hasil penelitian, dapat disimpulkan bahwa nilai eigen operator momentum sudut dalam koordinat bola dipengaruhi oleh fungsi harmonik bola atom hidrogen. Pada fungsi Y_{00}, seluruh hasil komutatornya komut sedangkan pada fungsi Y_{10}, $Y_{1\pm1}$, Y_{20}, $Y_{2\pm1}$, dan $Y_{2\pm2}$ terdapat beberapa operator yang tidak komut sehingga taat asas pada prinsip ketidakpastian Heisenberg. Pada fungsi Y_{10}, $Y_{1\pm1}$, Y_{20}, $Y_{2\pm1}$, dan $Y_{2\pm2}$ terdapat beberapa komutator yang merupakan bukan persoalan eigen sehingga pada fungsi tersebut, operasi operatornya tidak *compatible* yang menyebabkan momentum sudut elektron tidak dapat diamati.

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PRAKATA

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Penyusunan skripsi ini tidak lepas dari bantuan berbagai pihak. Oleh karena itu, penulis menyampaikan terima kasih kepada:

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Jember, Maret 2017

Penulis

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BAB 1. PENDAHULUAN

1.1 Latar Belakang

Pengukuran besaran dalam fisika klasik dapat dilakukan dengan cara dan hasil yang pasti serta dapat dilakukan secara serempak. Namun dalam mekanika kuantum, untuk beberapa besaran tertentu misalnya posisi dan momentum, akurasi pengukuran dua besaran tersebut tidak dapat ditingkatkan secara serempak. Jika kita meningkatkan akurasi salah satu besaran, maka akurasi besaran yang lain akan berkurang. Hal ini sesuai dengan prinsip ketidakpastian Heisenberg, bahwa tidak ada satu pun besaran fisika yang dapat diukur secara serempak dengan hasil yang pasti untuk partikel yang sangat kecil semacam elektron.

Pada saat mengukur observabel fisika klasik, gangguan pada sistem tersebut terjadi namun tidak begitu berarti sehingga dapat diabaikan. Dalam sistem atom dan subatom, proses pengukuran yang dilakukan mengganggu sistem secara signifikan sehingga tidak dapat diabaikan. Sebagai ilustrasi, ketika mengukur posisi elektron hidrogenik maka elektron harus dibombardir dengan radiasi elektromagnetik (foton). Jika posisi ditentukan secara akurat, maka panjang gelombang radiasi harus cukup pendek yaitu kurang dari 10^{-10} m sehingga energi dari bombardir elektron dengan foton lebih besar dari 10^4 eV. Energi ionisasi atom hidrogen sekitar 13,5 eV sehingga berakibat posisi elektron yang diukur akan terganggu cukup besar (Zettili, 2001:172).

Mekanika kuantum merupakan bahasan fisika yang bersifat mikroskopik sehingga jika ingin mengukur suatu besaran, maka alat ukur yang digunakan terbatas dan hasil ukur yang didapat hanya probabilitas dari besaran tersebut. Berbeda hal dengan fisika klasik yang terdapat banyak alat ukur karena besaran yang diukur bersifat makroskopik dan hasil yang didapatkan adalah pasti. Walaupun, besaran fisika kuantum hanya memiliki alat ukur yang terbatas, namun besarannya dapat dihitung dengan seperangkat persamaan matematis yaitu dengan menggunakan persamaan schrödinger dan hubungan komutasi dari beberapa operator. Akibat dari objek yang diukur bersifat mikroskopik, sehingga hasil yang diperoleh melalui persamaan matematis tersebut tidak pasti dan hanya dapat

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mengetahui probabilitas dari partikel tersebut. Ketidakpastian tersebut bukan hanya pengaruh dari lingkungannya, tetapi juga dari sistem itu sendiri. Ketidakpastian Heisenberg mengemukakan bahwa alam menetapkan suatu batas ketelitian yang dapat digunakan untuk melakukan sejumlah percobaan, tidak perduli sebaik apa pun peralatan ukur yang dirancang (Krane, 1992:145).

Mekanika kuantum didasari atas beberapa postulat, yaitu representasi keadaan, representasi variabel dinamis, nilai rata-rata variabel dinamis, dan tetapan gerak. Postulat yang mendasari dari operator adalah representasi variabel dinamis yang menyatakan bahwa setiap variabel dinamis (observabel) direpresentasikan oleh operator linier. Operator tersebut bekerja pada fungsi-fungsi dari sistem, dan mengubahnya menjadi fungsi gelombang yang lain (Purwanto, 2006:110). Variabel dinamis dalam fisika klasik sering ditemui pada posisi, momentum linier, momentum sudut, dan energi dapat langsung diukur atau dihitung tanpa harus mengenai suatu fungsi seperti pada mekanika kuantum. Operator sangat penting dalam bahasan mekanika kuantum karena memuat suatu kesatuan perangkat matematis untuk menghitung observabel yang akan diteliti dan menjadi satu-satunya pilihan untuk menyajikan besaran fisika.

Operasi dari beberapa operator berkaitan dengan hubungan komutasi (komutator). Komutator bersifat komut jika beberapa operator dapat diukur secara serempak, namun pada bahasan mekanika kuantum umumnya observabel tidak dapat diukur secara serempak sehingga komutator bersifat tidak komut. Observabel yang tidak dapat diukur secara serempak berkaitan dengan prinsip ketidakpastian Heisenberg. Komutator dapat bekerja jika operator tersebut dikenai fungsi eigen yang dalam penelitian ini digunakan fungsi gelombang atom hidrogen ternomalisasi. Fungsi gelombang tersebut didapat dari persamaan schrödinger tidak bergantung waktu (keadaan tunak). Berdasarkan hal tersebut, penelitian ini menggunakan atom hidrogen karena susunan atomnya yang sederhana yaitu hanya mengandung satu proton dan satu elektron. Pauling (1935: 112) menyatakan bahwa studi struktur atom hidrogen adalah langkah yang penting untuk mempelajari lebih lanjut struktur atom kompleks dan molekul, bukan hanya karena atom hidrogen merupakan struktur atom yang paling sederhana melainkan juga

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sebagai basis dalam perlakuan terhadap struktur atom berelektron banyak maupun molekul kompleks. Pada penelitian ini atom hidrogen yang digunakan hanya sampai dengan n = 3, karena jika n semakin besar maka fungsi gelombangnya akan semakin kompleks sehingga dilakukan pembatasan dalam perhitungan. Di samping itu, probabilitas menemukan elektron akan semakin menurun jika jarak elektron dengan inti semakin besar. Hermanto (2016) berpendapat bahwa semakin besar bilangan kuantum mengakibatkan semakin kecil nilai probabilitasnya yang artinya pada bilangan kuantum utama semakin besar, elektron semakin tidak ditemukan.

Momentum sudut memainkan peran penting dalam mekanika klasik. Studi tentang dinamika sistemnya memiliki simetri tertentu, seperti rotasi invarian dalam ruang dibuat sederhana dengan menggunakan konsep momentum sudut misalnya momentum sudut dari sistem terisolasi adalah kekal (Zettili, 2001:269). Momentum sudut dalam mekanika kuantum konsepnya lebih kompleks daripada mekanika klasik. Di dalam mekanika kuantum terdapat momentum sudut orbital dan momentum sudut spin. Momentum sudut spin merupakan besaran intrinsik dari partikel elementer seperti elektron dan foton, dan tidak akan dijumpai pada bahasan mekanika klasik (Liboff, 1980:309). Pada fisika klasik, besarnya momentum sudut yaitu jumlah keadaan yang didapat tak terbatas dengan mengubah vektor momentum sudut. Tetapi pada mekanika kuantum, hanya ada jumlah keadaan yang terbatas, yaitu bilangan kuantisasi. Selain itu pada mekanika kuantum, tidak dapat menggambarkan keadaan dengan menetapkan arah dari vektor momentum sudut, melainkan dengan memberikan komponen dari momentum sudut bersama beberapa arah (Goswami, 1997:229).

Perkembangan ilmu fisika, khususnya dalam mekanika kuantum pembahasan mengenai momentum sudut sangat penting, karena dari momentum sudut dapat diterangkan lebih mendalam sifat-sifat atom, molekul, kemagnetan dan lain sebagainya (Ivan, 1996). Dalam atom hidrogen, selain mengorbit inti, elektron juga membawa bentuk lain dari momentum sudut yang tidak berkaitan dengan ruang. Dianalogikan bahwa elektron sebagai bumi, selain mengorbit matahari (inti atom), bumi (elektron) juga mengalami rotasi di sekitar pusat massa

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yang disebut dengan spin (Griffiths, 1995:154). Dengan lahirnya konsep dualisme gelombang-partikel, artinya elektron tidak hanya bersifat sebagai partikel namun juga dapat bersifat sebagai gelombang. Perilaku elektron sebagai gelombang diselesaikan menggunakan fungsi matematika yang disebut orbital elektron. Setiap orbital atom memiliki satu set bilangan kuantum, yaitu energi, momentum sudut, dan proyeksi momentum sudut. Penelitian kali ini hanya fokus pada momentum sudut orbital dan mengabaikan efek spin elektron karena spin akan tetap bernilai konstan jika tidak dikenai dengan medan magnet eksternal (Efek Zeeman).

Pada teori mekanika kuantum, setiap besaran fisis teramati direpresentasikan oleh suatu operator mekanika kuantum. Untuk suatu besaran fisika teramati, dalam fisika klasik direpresentasikan oleh $Q\ (x,\ p)$, operator mekanika kuantumnya $\hat{Q}\ (\hat{x}, \hat{p})$ (Sunarmi, 2009). Operator momentum sudut $(\hat{L})$ merupakan vektor sehingga memiliki nilai dan arah, dimana arah momentum sudut dalam koordinat kartesian terdiri atas $\hat{L}_x$, $\hat{L}_y$, dan $\hat{L}_z$. Momentum sudut dapat bergerak ke segala dimensi ruang, sehingga akan lebih tepat jika menggunakan koordinat bola dengan mentransformasikan dari koordinat kartesian. Metode yang dapat digunakan untuk transformasi koordinat pada penelitian kali ini adalah dengan melakukan diferensial total. Alasan peneliti menggunakan koordinat bola yaitu variabel antar komponennya bebas (tidak bergantung dengan yang lain). Sebagai ilustrasi, jika sebuah bola dikaitkan dengan tali yang pendek maka akan menghasilkan putaran yang cepat. Namun jika bola dikaitkan dengan tali yang panjang, maka putarannya akan semakin pelan. Penelitian kali ini tidak meninjau jarak sehingga koordinat bola dapat mengatasi hal tersebut karena komponen dari r, θ, dan φ tidak saling bergantung satu sama lain sehingga berapapun jarak (r) yang digunakan tidak mempengaruhi putaran dari bola tersebut.

Beberapa penelitian sebelumnya mengenai komutator operator momentum sudut, antara lain: Sunarmi (2009) dalam penelitiannya tentang komutator operator momentum sudut dalam koordinat bola menyimpulkan bahwa komponen operator momentum sudut berkomutasi dengan operator yang sama dan juga dengan kuadrat operator momentum angular total; Enk dan Nienhuis (1994) tentang *Commutation Rules and Eigenvalues of Spin and Orbital Angular*

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Momentum of Radiation Fields menyimpulkan bahwa arti fisis operator *L* dan *S* lebih lanjut mengklarifikasi dengan menentukan aturan komutasinya. Kita melihat bahwa tiga komponen *S* komut, dan bahwa *S* memiliki komutator tak lenyap dengan komponen *L*. Oleh karena itu, karena operator ini tidak mentaati aturan komutasi untuk komponen momentum sudut, mereka tidak menghasilkan rotasi, dan mereka tidak mewakili momentum sudut yang sebenarnya. Di sisi lain, komponen momentum sudut total *J* mematuhi aturan komutasi yang sebenarnya; Penelitian Mei dan Yu (2012) tentang *The Definition of Universal Momentum Operator of Quantum Mechanics and the Essence of Micro-Particle's Spin* menyimpulkan bahwa ketika operator dikenai pada fungsi non-eigen, nilai non-eigen dan nilai rata-rata operator momentum ialah bilangan kompleks secara umum; Penelitian Pegg dkk. (2005) tentang *Minimum States Uncertainty States of Angular Momentum and Angular Position* menyimpulkan bahwa keadaan batasan minimum hasil ketidakpastian (Constrained Minimum Uncertainty Product/CMUP) menghasilkan hasil ketidakpastian lebih kecil daripada keadaan *intelligent* karena memiliki rapat probabilitas lebih besar di tepi jarak 2π daripada keadaan *intelligent*; Santhanam (1975) meneliti tentang *Quantum Mechanics in Discrete Space and Angular Momentum* menyimpulkan bahwa jumlah operator dan fase juga memenuhi QMDS (Quantum Mechanics in Discrete Space) ketika *n* berhingga dan menjadi konjugat kanonik dalam arti biasa sebagai *n* mendekati tak hingga

Berdasarkan uraian tersebut, besaran momentum sudut pada mekanika kuantum merupakan besaran mikroskopik, sehingga alat ukur atau metode hitung yang digunakan hanya seperangkat persamaan matematis yaitu dalam penelitian ini menggunakan metode operator. Oleh karena itu, perlu dilakukan penelitian dengan mengambil judul Komutator Operator Momentum Sudut Dalam Koordinat Bola Dengan Fungsi Gelombang Atom Hidrogen.

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1.2 Rumusan Masalah

Berdasarkan latar belakang tersebut maka dapat dirumuskan beberapa permasalahan, antara lain:

a. Bagaimana persamaan matematis dari komutator operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen?
b. Bagaimana hubungan komutasi operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen?

1.3 Batasan Masalah

Agar penelitian lebih terfokus dan dapat menjawab permasalahan yang ada, maka penulis membatasi masalah sebagai berikut:

a. Persamaan schrodinger yang digunakan ialah tidak bergantung waktu (keadaan tunak) dalam koordinat bola.
b. Fungsi gelombang atom hidrogen memenuhi syarat normalisasi.
c. Fungsi gelombang atom hidrogen yang digunakan dalam penelitian ini hanya fungsi harmonik bola.
d. Bilangan kuantum yang digunakan dalam penelitian ini yaitu untuk $n \leq 3$.
e. Efek dari spin elektron diabaikan.
f. Hubungan komutasi operator momentum sudut hanya menggunakan masing-masing minimal dua buah operator.

1.4 Tujuan Penelitian

Adapun tujuan dari penelitian ini adalah sebagai berikut:

a. Menentukan persamaan matematis dari komutator operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen.
b. Menentukan hubungan komutasi operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen.

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1.5 Manfaat Penelitian

Manfaat yang dapat diperoleh dari penelitian ini adalah sebagai berikut:

a. Bagi peneliti, dapat menambah wawasan, pengetahuan, dan pengalaman tentang momentum sudut dalam aspek kajian menurut mekanika kuantum.

b. Bagi pembaca, dapat dijadikan sebagai sebagai salah satu sumber referensi dalam mempelajari teori kuantum mengenai pokok bahasan komutator operator momentum sudut menggunakan fungsi gelombang atom hidrogen ternormalisasi dan dapat melakukan penelitian lebih lanjut dengan tema serupa.

c. Bagi lembaga, dapat memberikan sumbangan penelitian dan bahan referensi tambahan dalam pembelajaran fisika kuantum dengan pokok bahasan operator momentum sudut.

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BAB 2. TINJAUAN PUSTAKA

2.1 Persamaan Schrodinger

Pada kasus fisika kuantum takrelativistik, persamaan utama yang harus dipecahkan adalah suatu persamaan diferensial parsial orde kedua, yang dikenal sebagai persamaan Schrodinger. Berbeda dari hukum Newton, pemecahan persamaan Schrodinger yang disebut fungsi gelombang memberikan informasi tentang perilaku gelombang dari partikel (Krane, 1992:170). Pemecahan persamaan Schrodinger harus memenuhi tiga syarat sebagai berikut:

a. Tidak melanggar hukum kekekalan energi

Hukum kekekalan energi merupakan penjumlahan antara energi kinetik dan energi potensial dari suatu partikel, di mana jumlah total energinya selalu bersifat kekal. Persamaan hukum kekekalan energi dapat dirumuskan sebagai berikut.

$$K + V = E \tag{2.1}$$

dengan *K*, *V*, dan *E* berturut-turut adalah energi kinetik, energi potensial, dan jumlah energi kinetik dan potensial. Pada kasus takrelativistik, maka persamaan (2.1) menjadi

$$\frac{p^2}{2m} + V = E \tag{2.2}$$

b. Taat asas terhadap hipotesis de Broglie

Bentuk persamaan diferensial apa pun, haruslah taat asas terhadap hipotesis de Broglie. Untuk memecahkan persamaan matematik bagi sebuah partikel dengan momentum *p*, maka pemecahan yang didapati haruslah berbentuk sebuah fungsi gelombang λ yang sama dengan *h/p*. Dengan menggunakan $p = \hbar k$ dimana *k* adalah bilangan gelombang, maka energi kinetik dari partikel bebas menjadi

$$K = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{2.3}$$

c. Persamaan yang dihasilkan harus bernilai tunggal dan linier

Pemecahan persamaan Schrodinger harus memberi informasi tentang probabilitas menemukan partikelnya. Meskipun terkadang ditemukan probabilitas berubah secara tidak kontinu yang berarti partikelnya menghilang secara tiba-tiba dari satu titik dan muncul kembali pada titik lainnya. Walaupun ditemukan

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probabilitas yang tidak kontinu, namun fungsinya harus bernilai tunggal yaitu tidak boleh ada dua probabilitas untuk menemukan partikel di satu titik yang sama. Fungsinya harus pula linier, agar gelombangnya memiliki sifat superposisi yang diharapkan sebagai milik gelombang yang berperilaku baik (Krane, 1992:172).

Gelombang de Broglie partikel bebas $\psi(x, t)$ memiliki bentuk matematik yang serupa dengan A sin (kx-ωt), yaitu bentuk dasar sebuah gelombang dengan amplitudo A yang merambat dalam arah x positif. Untuk mempermudah, waktunya diabaikan (t = 0) sehingga

$$\psi(x,t) = A\sin(kx - \omega t)$$

$$\psi(x,0) = A\sin(kx - \omega \times 0)$$

dari persamaan di atas, dapat diperoleh hasil fungsi gelombang tak bergantung waktu

$$\psi(x) = A\, sin\, kx \qquad (2.4)$$

Dari persamaan (2.1) dan (2.3), satu-satunya cara untuk memperoleh suku yang mengandung k^2 adalah dengan mengambil turunan kedua dari persamaan (2.4)

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}K\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V)\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad (2.5)$$

Persamaan (2.5) merupakan persamaan Schrodinger bebas waktu satu dimensi

2.2 Persamaan Schrodinger untuk Atom Hidrogen

Sebuah atom hidrogen terdiri dari sebuah proton yaitu partikel yang bermuatan listrik +e dan sebuah elektron mengelilingi proton yaitu partikel yang bermuatan –e dengan perbandingan massa proton jauh lebih besar dari massa elektron yaitu m_p = 1836 m_e. Selain elektron berputar mengelilingi inti, ternyata elektron juga bergetar di titik setimbangnya sehingga elektron selain dipandang

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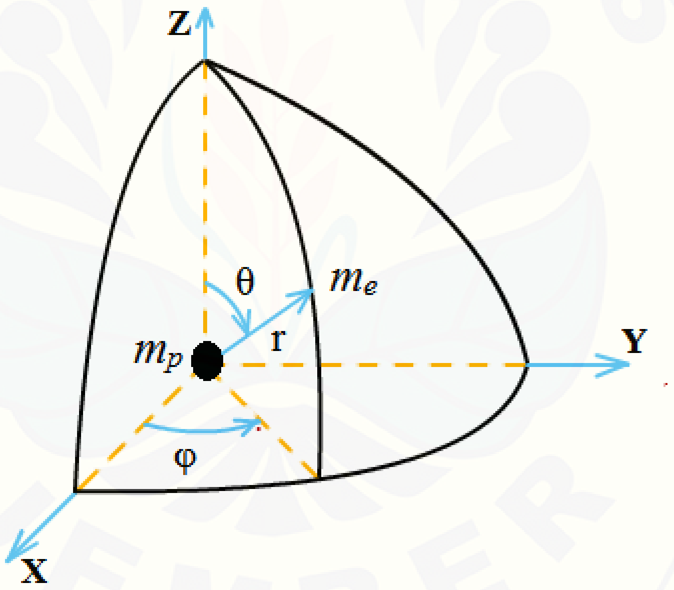
sebagai partikel, juga dipandang sebagai gelombang. Untuk kemudahan, dianggap protonnya diam dengan elektron bergerak disekelilingnya tetapi dicegah untuk melarikan diri oleh medan listrik proton. Massa tereduksi dari atom hidrogen yaitu

$$\mu = \frac{1}{\frac{1}{m_p} + \frac{1}{m_e}}$$

karena massa proton yang jauh lebih besar dari elektron, sehingga dilakukan pendekatan massa tereduksi atom hidrogen $\mu \approx m_e$. Persaman Schrodinger untuk elektron dalam tiga dimensi yang harus dipakai untuk persoalan atom hidrogen ialah

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m_e}{\hbar^2}(E - V)\psi = 0 \qquad (2.6)$$

(Beiser, 1983: 174)



Gambar 2.1. Posisi relatif antara proton dan elektron (Sumber: http://math.tutorcircle.com/analytical-geometry/polar-coordinates.html)

Karena proton dianggap diam, maka konstribusi energi sistem hanya diberikan oleh elektron yaitu energi kinetik

$$K = \frac{p^2}{2m_e} \qquad (2.7)$$

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dan energi potensial

$$V = -\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r} \tag{2.8}$$

yaitu

$$E \equiv H = K + V = \frac{p^2}{2m_e} - \frac{e^2}{4\pi\varepsilon_o}\frac{1}{r} \tag{2.9}$$

Dengan demikian, persamaan Schrodinger untuk atom hidrogen

$$-\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi - \left(\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi = E\psi$$

$$\left(-\frac{\hbar^2}{2m_e}\nabla^2 - \frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi\,(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \tag{2.10}$$

Pemecahan persamaan Schrodinger untuk atom hidrogen pada persamaan (2.10) bergantung pada besaran *r*. Dalam koordinat kartesian, *r* dinyatakan sebagai $\sqrt{x^2 + y^2 + z^2}$ yang sangat menyulitkan proses pemecahan persamaan. Oleh karena itu digunakan koordinat bola sebagai variabel *r, θ,* dan *φ* yang berarti tidak ada tanda akar kuadrat karena tidak perlu membagi *r* ke dalam bentuk *x*, *y*¸ *z* (Naresh, 2015). Di dalam koordinat bola (r, θ, φ), persamaan (2.10) menjadi

$$-\frac{\hbar^2}{2m_e}\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{\sin\theta}\frac{\partial\psi^2}{\partial\varphi^2}\right\} - \left(\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\psi\right) = E\psi \tag{2.11}$$

Selanjutnya, untuk mendapatkan solusi bagi persamaan (2.11) dilakukan pemisahan variabel $\psi(\vec{r}) = \psi\,(r, \theta, \varphi)$ sebagai berikut

$$\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi) \tag{2.12}$$

(Purwanto, 2006:154-155).

Alasan untuk menulis persamaan Schrodinger dalam koordinat bola untuk persoalan atom hidrogen ialah dalam bentuk ini persamaan dapat dipisahkan menjadi tiga persamaan yang bebas, masing-masing hanya mengandung satu koordinat saja. Fungsi *R* (*r*) memberikan bagaimana fungsi gelombang elektron *ψ* berubah sepanjang vektor jejari dari inti, dengan *θ* dan *φ* konstan. Fungsi *Θ* (*θ*) memberikan bagaimana fungsi gelombang elektron *ψ* berubah terhadap sudut zenit *θ* sepanjang meridian pada bola yang berpusat pada inti, dengan *r* dan *φ* konstan. Fungsi *Φ(φ)* memberikan bagaimana fungsi gelombang elektron *ψ* berubah terhadap sudut azimut *φ* sepanjang garis pada bola yang berpusat pada inti, dengan *r* dan *θ* konstan (Beiser, 1983: 176-177).

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Subtitusi ungkapan (2.12) ke dalam persamaan (2.11) kemudian dikalikan dengan $(2m_e\, r^2/\, \hbar^2)$, maka diperoleh

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\Theta\Phi+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)R\Phi+\frac{1}{sin^2\theta}\frac{d^2\Phi}{d\varphi^2}R\Theta$$
$$+\frac{2m_er^2}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)R\Theta\Phi=0$$

Persamaan diatas menjadi lebih sederhana jika tiap suku dibagi dengan $R\Theta\Phi$, sehingga didapat

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\phi sin^2\theta}\frac{d^2\Phi}{d\varphi^2}+\frac{2m_er^2}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)=0 \quad (2.13)$$

Dari persaman (2.13) ini tampak bahwa suku pertama dan keempat hanya bergantung jari - jari r, suku kedua dan ketiga hanya bergantung sudut θ dan φ. Penjumlahan suku-suku yang hanya bergantung pada jari-jari dan dua sudut ini akan selalu sama dengan nol untuk sembarang nilai r, θ, dan φ jika masing-masing suku sama dengan konstanta yang berharga $\pm\,\ell(\ell+1)$, maka suku yang hanya bergantung jari-jari akan menjadi

$$\frac{1}{R}\frac{d}{dr}\left(\mathrm{r}^2\frac{dR}{dr}\right)+\frac{2m_er^2}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)=\ell(\ell+1)$$

atau

$$\frac{d}{dr}\left(\mathrm{r}^2\frac{dR}{dr}\right)+\frac{2m_er^2}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)R=\ell(\ell+1)R \qquad (2.14)$$

sedangkan suku yang hanya mengandung sudut θ dan φ menjadi

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}=-\ell(\ell+1) \qquad (2.15)$$

Setelah dikalikan dengan $\sin^2\theta$**,** persamaan (2.15) menjadi

$$\frac{sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}+\ell(\ell+1)\sin^2\theta=0 \qquad (2.16)$$

Tampak bahwa persamaan (2.16) juga terpisah menjadi dua bagian yaitu bagian yang hanya bergantung pada sudut azimuth φ dan bagian yang bergantung pada θ. Selanjutnya tetapkan masing – masing bagian sama dengan konstanta $-m^2$ dan m^2**.** Persamaan (2.16) dapat juga dituliskan menjadi

$$\frac{sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}+\ell(\ell+1)\sin^2\theta=-m^2+m^2 \qquad (2.17)$$

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Melalui metode separasi variabel, persamaan (2.17) dapat dijabarkan sebagai berikut:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2$$

atau

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0 \quad (2.18)$$

Sehingga

$$\frac{sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta = m^2$$

atau, setelah dikalikan dengan $\Theta/\sin^2\theta$ diperoleh

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left\{\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0 \quad (2.19)$$

(Purwanto, 2006:155-157).

Dengan demikian, persamaan (2.11) dapat dipisah menjadi tiga persamaan deferensial biasa. Selanjutnya, persamaan (2.14), (2.18), dan (2.19) akan dijabarkan untuk memperoleh solusi gelombang radial, solusi gelombang polar, dan solusi gelombang azimuth.

2.2.1 Solusi Persamaan Radial

Tampak pada persamaan (2.14) terdapat nilai atau energi eigen E. Untuk keadaan terikat yaitu keadaan dengan energi negatif $E = -|E|$, persamaan (2.14) diselesaikan dengan merubah variabel

$$\rho = \left(\frac{8m_e|E|}{\hbar^2}\right)^{1/2} r \quad (2.20)$$

Membuat persamaan (2.14) tereduksi menjadi

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{\ell(\ell+1)}{\rho^2}\mathrm{R} + \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)\mathrm{R} = 0 \quad (2.21)$$

dengan

$$\lambda = \frac{e^2}{4\pi\varepsilon_0\hbar^2}\left(\frac{m_e}{8|E|}\right)^{1/2} \quad (2.22)$$

Untuk menentukan solusi persamaan (2.21), diselidiki terlebih dahulu (perilaku) persamaan tersebut pada dua daerah ekstrim yaitu daerah jauh sekali dan daerah pusat koordinat. Untuk daerah jauh sekali dimana $\rho \to \infty$, persamaan (2.21) secara efektif menjadi

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$$\frac{d^2R}{d\rho^2} - \frac{1}{4}R = 0 \tag{2.23}$$

dengan solusi persamaan ini

$$R \approx e^{-\rho/2} \tag{2.24}$$

Sedangkan pada daerah titik asal, R ditulis sebagai

$$R(\rho) = \frac{U(\rho)}{\rho} \tag{2.25}$$

dan substitusikan ke dalam suku pertama persamaan (2.21) diperoleh

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left\{\rho^2\frac{d}{d\rho}\left(\frac{U}{\rho}\right)\right\} = \frac{d^2U}{\rho d\rho^2} \tag{2.26}$$

karena itu persamaan (2.21) tereduksi menjadi persamaan diferensial untuk U

$$\frac{d^2U}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}U + \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)U = 0 \tag{2.27}$$

Selanjutnya kalikan dengan ρ^2 dan ambil limit mendekati pusat koordinat

$$\lim_{\rho\to 0}\left\{\rho^2\frac{d^2U}{d\rho^2} - \ell(\ell+1)U + \lambda\rho U - \frac{1}{4}\rho^2 U\right\} = \left(\rho^2\frac{d^2U}{d\rho^2}\right) - \ell(\ell+1)U = 0 \tag{2.28}$$

Tampak bahwa suku dominannya adalah

$$\frac{d^2U}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}U = 0$$

Solusi yang memenuhi persarnaan suku dominan ini dan kondisi fisis keberhinggaan $\rho \to 0$ adalah

$$U \approx \rho^{\ell+1} \tag{2.29}$$

Karena itu solusi untuk daerah asal (koordinat), menggunakan hasil (2.29) dan hubungan (2.25) diberikan oleh

$$R \approx \rho^{\ell} \tag{2.30}$$

Pertimbangkan solusi-solusi untuk daerah ekstrim di depan, solusi umumnya diusulkan berbentuk perkalian antara solusi titik asal, posisi jauh sekali dan fungsi umum terhadap jarak

$$R(\rho) = \rho^{\ell}e^{-\rho/2}L(\rho) \tag{2.31}$$

Subtitusi ungkapan (2.31) ke dalam persamaan (2.21) didapatkan persamaan untuk L, yaitu

$$\rho\frac{d^2L}{d\rho^2} + \{2(\ell+1) - \rho\}\frac{dL}{d\rho} + \{\lambda - (\ell+1)\}L = 0 \tag{2.32}$$

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Solusi deret

$$L = \sum_{s=0}^{\infty} a_s \, \rho^s \tag{2.33}$$

akan memberi rumus rekursi

$$a_{s+1} = \frac{s + \ell + 1 - \lambda}{(s+1)(s+2\ell+2)} \, a_s \tag{2.34}$$

Tampak bahwa deret akan berhingga jika λ adalah bilangan bulat, misalkan $\lambda = n$, maka deret a_{s+1} dan seterusnya akan menjadi nol jika $s = n - \ell - 1$. Sehingga L(ρ) merupakan polinomial

$$L = \sum_{S=0}^{n-\ell-1} a_s \, \rho^s \tag{2.35}$$

Dengan menggunakan pemilihan $\lambda = n$, persamaan (2.32) menjadi

$$\rho \, \frac{d^2 L}{d\rho^2} + \{2\,(\ell + 1) - \rho\} \, \frac{dL}{d\rho} + \{n - (\ell + 1)\} \, L = 0 \tag{2.36}$$

Persamaan (2.36) ini tidak lain adalah persamaan diferensial Laguerre terasosiasi, yang mempunyai bentuk umum

$$\rho \, \frac{d^2 L_q^p}{d\rho^2} + \{p + 1 - \rho\} \, \frac{dL_q^p}{d\rho} + \{q - p\} \, L_q^p = 0 \tag{2.37}$$

Solusinya disebut polinom Laguerre terasosiasi L_q^p dapat diperoleh dari rumus Rodrigues

$$L_q^p(\rho) = \frac{q!}{(q-p)!} e^{\rho} \frac{d^q}{d\rho^q} \left(e^{-\rho} \rho^{q-p}\right)$$

Pada kasus ini, koefisien p dan q dihubungkan dengan bilangan kuantum orbital l dan bilangan bulat n yang disebut bilangan kuantum utama (Purwanto, 2006:160-163). Tinjau persamaan (2.36) dan (2.37), sehingga akan memberikan nilai $p = 2l + 1$ dan $q = n + l$ dan memperhatikan persamaan (2.24), (2.29), dan (2.31), Fungsi $U\,(\rho)$ diberikan oleh

$$U(\rho) \approx e^{-\rho/2} \rho^{\ell+1} L_{n+\ell}^{2\ell+1}(\rho)$$

dan dengan meninjau persamaan (2.25), maka persamaan di atas menjadi

$$R(\rho) \equiv R_{n\ell}(\rho) = N_{n\ell} e^{-\rho/2} \rho^{\ell} L_{n+\ell}^{2\ell+1}(\rho) \tag{2.38}$$

di mana N_{nl} merupakan konstanta normalisasi dan dengan melihat persamaan (2.20), nilai ρ dapat dimisalkan oleh $\frac{2r}{na_0}$ di mana a_0 merupakan radius Bohr. Untuk

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mencari nilai N_{nl}, digunakan syarat normalisasi dari persamaan radial yaitu $\int_0^\infty R_{n\ell}^2(r)r^2dr = 1$, sehingga akan menghasilkan

$$1 = \left(\frac{na_0}{2}\right)^3 N_{n\ell}^2 \int_0^\infty e^{-\rho}\rho^{2\ell+2}\left(L_{n+\ell}^{2\ell+1}\right)^2 d\rho \qquad (2.39)$$

dengan menggunakan tabel integral, nilai integral dari persamaan di atas diberikan oleh

$$\int_0^\infty e^{-\rho}\rho^{2\ell+2}\left(L_{n+\ell}^{2\ell+1}\right)^2 d\rho = \frac{2n[(n+1)!]^3}{(n-\ell-1)!}$$

sehingga nilai dari konstanta normalisasi N_{nl} adalah

$$N_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} \qquad (2.40)$$

dengan melihat persamaan (2.38) dan (2.38), didapat solusi umum persamaan radial ternormalisasi adalah

$$R_{n\ell}(\rho) = \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} e^{-\rho/2}\rho^\ell L_{n+\ell}^{2\ell+1}(\rho)$$

atau

$$R_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} e^{-\frac{r}{na_0}}\left(\frac{2r}{na_0}\right)^\ell L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right) \qquad (2.41)$$

(Rajasekar dan Velusamy, 2015:314-315).

2.2.2 Solusi Persamaan Polar

Persamaan diferensial (2.19) dengan konstanta $\ell(\ell+1)$ dan m^2 dikenal sebagai persamaan diferensial Legendre terasosiasi. Solusi dari persamaan ini dapat diperoleh menggunakan metode Frobenius dan diberikan oleh deret berhingga yang dikenal sebagai polinom Legendre terasosiasi. lnilah alasan pengambilan tetapan $\pm\,\ell(\ell+1)$ ketika menguraikan persamaan (2.13) menjadi persamaan (2.14) dan (2.10). Bila konstantanya bukan $\pm\,\ell(\ell+1)$ maka solusinya adalah deret takberhingga.

Solusi persamaan (2.19) diberikan oleh polinom Legendre $P_\ell^m(\cos\theta)$

$$\Theta(\theta) \equiv \Theta_{lm}(\theta) = N_{\ell m}P_\ell^m(\cos\theta) \qquad (2.42)$$

dengan N_{lm} merupakan konstanta normalisasi

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$$(\Theta_{\ell m}, \Theta_{\ell' m'}) = N_{\ell m}^{*} N_{\ell' m'} \int_0^{\pi} P_\ell^m(\cos\theta)\, P_{\ell'}^{m'}(\cos\theta) \sin\theta\, d\theta = \delta_{\ell\ell'}\delta_{mm'} \quad (2.43)$$

Mengingat sifat ortogonolitas $P_\ell^m(\cos\theta)$, sehingga

$$\int_0^{\pi} P_\ell^m(\cos\theta)\, P_{\ell}^{m'}(\cos\theta) \sin\theta\, d\theta = \frac{2}{2\ell+1}\frac{(\ell+|m|)!}{(\ell-|m|)!}\delta_{\ell\ell'}\delta_{mm'}$$

didapatkan nilai konstanta normalisasi

$$N_{\ell m} = \epsilon\sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}} \quad (2.44)$$

di mana $\epsilon = (-1)^m$ untuk $m > 0$ dan $\epsilon = 1$ untuk $m \leq 0$. Substitusi persamaan (2.44) ke dalam persamaan (2.42), sehingga diperoleh

$$\Theta_{\ell m}(\theta) = \sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^m(\cos\theta) \quad (2.45)$$

Bentuk eksplisit dari polinom $P_\ell^m(\cos\theta)$ dapat diperoleh melalui rumus Rodrigues

$$P_\ell^m(\cos\theta) = \frac{1}{2^\ell \ell!}(1-\cos^2\theta)^{m/2}\frac{d^{\ \ell+|m|}}{d\ cos^{\ \ell+|m|}\theta}(\cos^2\theta-1)^\ell \quad (2.46)$$

Dengan demikian persamaan (2.45) sebagai solusi umum persamaan polar dapat dituliskan menjadi

$$\Theta_{\ell m}(\theta) = \epsilon\sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\left[\frac{1}{2^\ell \ell!}(1-\cos^2\theta)^{|m|/2}\frac{d^{\ \ell+|m|}}{d\ cos^{\ \ell+|m|}\theta}(\cos^2\theta-1)^\ell\right] \quad (2.47)$$

(Purwanto, 2006:158-159).

2.2.3 Solusi Persamaan Azimuth

Persamaan (2.18) merupakan persamaan azimuth yang menggambarkan rotasi di sekitar sumbu *z*. Sudut rotasi di sekitar sumbu *z* ini adalah nol sampai 2π. Konstanta negatif $-m^2$ dipilih agar memberi solusi berupa fungsi sinusoidal yang periodik. Bila dipilih konstanta positif m^2 akan memberi solusi fungsi eksponensial (Purwanto, 2006:157-158).

Tinjau kembali persamaan (2.18) yang merupakan persamaan diferensial biasa yang pemecahannya dapat memisalkan $\frac{d}{d\varphi} = D$, sehingga akan menjadi

$$D^2\Phi + m^2\Phi = 0$$

$$(D^2 + m^2)\Phi = 0$$

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$$D = \pm im$$

kedua ruas dikalikan dengan Φ maka didapatkan

$$\frac{d\Phi}{\Phi} = \pm imd\varphi$$

dengan mengintegralkan kedua ruas, di mana ruas kiri dengan batas Φ_0 sampai Φ dan ruas kanan dengan batas 0 sampai φ, sehingga diperoleh hasil

$$\Phi = \Phi_0 e^{im\varphi} \tag{2.48}$$

Untuk menentukan besarnya nilai Φ_0 pada persamaan (2.48), maka fungsi Φ harus menggunakan syarat normalisasi sebagai berikut

$$\int_0^{2\pi} \Phi^* \, \Phi d\varphi = 1$$

sehingga dipenuhi oleh konstanta $\Phi_0 = 1/\sqrt{2\pi}$. Karena itu solusi persamaan azimuth adalah

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \tag{2.49}$$

dengan *m* merupakan bilangan bulat magnetik.

2.2.4 Solusi Gabungan

Solusi umum dari persamaan Schrodinger atom Hidrogen merupakan solusi gabungan dari solusi persamaan radial, polar, dan azimut. Solusi persamaan (2.12) dapat dinyatakan dalam bentuk lain dengan memasukkan solusi yang ada persamaan (2.41), (2.47), dan (2.49) sehingga akan diperoleh

$$\psi_{n\ell m}(r,\theta,\phi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \, \epsilon \sqrt{\frac{2\ell+1}{2} \frac{(\ell-m)!}{(\ell+m)!}} \left[\frac{1}{2^\ell \ell!} (1-\cos^2\theta)^{m/2} \frac{d^{\ell+m}}{d \cos^{\ell+m}\theta} (\cos^2\theta - 1)^\ell \right] \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3} \right]^{1/2} e^{-\frac{r}{na_0}} \left(\frac{2r}{na_0}\right)^\ell L_{n+\ell}^{2\ell+1} \left(\frac{2r}{na_0}\right) \tag{2.50}$$

dengan

n = 1, 2, 3, . . .

ℓ = 0, 1, 2, . . ., n-1

m = 0, ±1, ±2, . . ., ±ℓ

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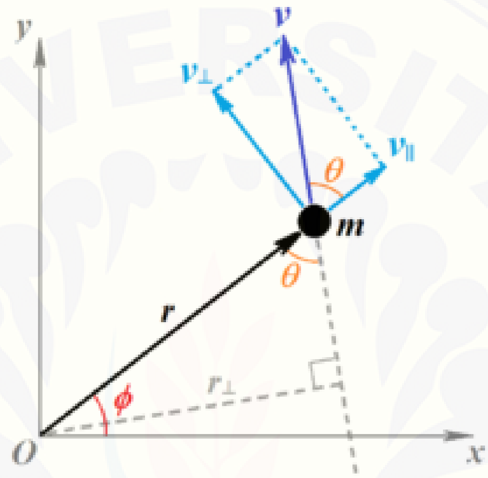
2.3 Momentum Sudut

Arti dari momentum sudut dalam fisika klasik (Liboff, 1980:310) adalah salah satu konstanta gerak fundamental (bersama dengan momentum linier dan energi) dari sistem terisolasi. Secara klasik, momentum sudut partikel adalah besaran yang bergantung pada momentum linier partikel **p** dan perpindahannya **r** dari titik asal. Hal ini diberikan dengan

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.51}$$



Gambar 2.2. Momentum Sudut *m* yaitu sebanding dengan komponen tegak lurus ($v_\perp$) dari kecepatan, atau ekuivalen terhadap tegak lurus perpindahan $r_\perp$ dari titik asal (sumber: https://en.wikipedia.org/wiki/Angular_momentum).

2.3.1 Komponen Koordinat Kartesian

Komponen kartesian klasik dari momentum sudut orbital **L** untuk partikel dengan momentum **p** = (p_x, p_y, p_z) di perpindahan **r** = (x, y, z) adalah

$$L_x = yp_z - zp_y \qquad L_y = zp_x - xp_z \qquad L_z = xp_y - yp_x \tag{2.52}$$

Operator mekanika kuantum $\hat{L}_x$, $\hat{L}_y$, dan $\hat{L}_z$ bersesuaian dengan observabel ini, yang diperoleh definisinya secara langsung dari persamaan (2.52), dengan **p** digantikan oleh operator gradien yang bersesuaian. Berikut ini adalah operator momentum sudut dalam mekanika kuantum

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \qquad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{2.53}$$

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Dalam hubungan 3 dimensi, vektor operator momentum linier dapat diuraikan menjadi

$$\hat{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) = -i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = -i\hbar\nabla \qquad (2.54)$$

Persamaan (2.53) dapat dituliskan sebagai persamaan single vector

$$\vec{L} = -i\hbar\vec{r} \times \vec{\nabla} \qquad (2.55)$$

(Liboff, 1980:310-311)

Kuadrat momentum sudut pada persamaan (2.55) adalah kuadrat dari masing-masing komponen operator momentum sudut, yang dapat ditulis sebagai berikut

$$L^2 = L_x^2 + L_y^2 + L_z^2 \qquad (2.56)$$

Selain ketiga operator momentum tersebut (L_x, L_y, dan L_z), ada juga operator momentum sudut lain yaitu pergeseran (*shift*) operator. Pergeseran operator akan terbukti sangat berguna untuk menentukan besaran momentum sudut dan untuk menilai elemen matriks operator momentum sudut. Satu operator, L_+ disebut operator naik dan yang lain, L_- disebut operator turun. Kedua operator didefinisikan sebagai berikut:

$$L_+ = L_x + iL_y \qquad\qquad L_- = L_x - iL_y \qquad (2.57)$$

Kebalikan hubungan dari persamaan (2.57) adalah

$$L_x = \frac{L_+ + L_-}{2} \qquad\qquad L_y = \frac{L_+ - L_-}{2i} \qquad (2.58)$$

(Atkins dan Friedman, 2005:101-102).

L_+ dan L_- bukanlah operator Hermitian, karenanya dapat dibuktikan bahwa

$$L_+ = L_-^\dagger \qquad (2.59)$$

selain itu,

$$L^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+) \qquad (2.60)$$

dan juga

$$L_+L_- = L^2 - L_z^2 + \hbar L_z \qquad (2.61)$$

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z \qquad (2.62)$$

Operator L_+ dan L_- memungkinkan untuk merepresentasikan semua fungsi eigen dari L^2 dan L_z menggunakan hanya satu fungsi eigen dan operator L_+ dan L_- (Peleg dkk, 1998:99).

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2.3.2 Komponen Koordinat Bola

Pada persamaan (2.55), operator momentum sudut berada pada koordinat kartesian karena operator laplace ∇ yang digunakan berada pada koordinat kartesian. Apabila operator momentum sudut akan diubah ke koordinat bola, maka operator ∇ harus diubah ke koordinat bola. Operator ∇ dalam koordinat bola dapat dituliskan sebagai

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \tag{2.63}$$

karena vektor $\vec{r} = r\hat{r}$, dan substitusikan operator ∇ dalam koordinat bola ke persamaan (2.55), sehingga diperoleh

$$\hat{L} = -i\hbar\left[r(\hat{r}\times\hat{r})\frac{\partial}{\partial r} + (\hat{r}\times\hat{\theta})\frac{\partial}{\partial\theta} + (\hat{r}\times\hat{\varphi})\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

selain itu, arah vektor satuan dari $(\hat{r}\times\hat{r}) = 0$, $(\hat{r}\times\hat{\theta}) = \hat{\varphi}$, dan $(\hat{r}\times\hat{\varphi}) = -\hat{\theta}$, maka diperoleh

$$\hat{L} = -i\hbar\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{2.64}$$

Vektor satuan $\hat{\theta}$ dan $\hat{\varphi}$ dapat diselesaikan ke dalam komponen kartesian

$$\hat{\theta} = (\cos\theta\cos\varphi)\hat{\imath} + (\cos\theta\sin\varphi)\hat{\jmath} - (\sin\theta)\hat{k}$$

$$\hat{\varphi} = -(\sin\varphi)\hat{\imath} + (\cos\varphi)\hat{\jmath}$$

Dengan memasukkan vektor satuan tersebut ke persamaan (2.64), sehingga diperoleh

$$\hat{L} = -i\hbar\left[(-\sin\varphi\hat{\imath} + \cos\varphi\hat{\jmath})\frac{\partial}{\partial\theta} - (\cos\theta\cos\varphi\hat{\imath} + \cos\theta\sin\varphi\,\hat{\jmath} - \sin\theta\hat{k})\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

Dengan mengambil vektor unit i, j, k secara berurut-urut untuk L_x, L_y, dan L_z sehingga dapat dituliskan menjadi

$$\hat{L}_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\varphi} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \tag{2.65}$$

$$\hat{L}_Y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\varphi} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \tag{2.66}$$

$$\hat{L}_Z = -i\hbar\frac{\partial}{\partial\varphi} \tag{2.67}$$

Operator naik dan operator turun dalam koordinat bola dapat dituliskan dengan memanfaatkan persamaan (2.57), sehingga didapat

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$$L_{\pm} = -i\hbar\left[(-\sin\varphi \pm i\cos\varphi)\frac{\partial}{\partial\theta} - (\cos\varphi \pm i\sin\varphi)\cot\theta\frac{\partial}{\partial\varphi}\right]$$

Karena $\cos\varphi \pm i\sin\varphi = e^{\pm i\varphi}$, sehingga dapat dituliskan menjadi

$$L_{\pm} = \pm\hbar e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right) \quad (2.68)$$

(Griffiths, 1995:150-151).

Operator yang banyak digunakan adalah kuadrat dari momentum sudut). Operator dari kuadrat momentum sudut dapat diselesaikan dengan perkalian titik (*dot product*), sehingga diperoleh

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \quad (2.69)$$

(Purwanto, 2006:175).

2.4 Hubungan Komutasi Operator Momentum Sudut

Di dalam mekanika kuantum (Purwanto, 2006:112), variabel-variabel dinamis pada umumnya tidak komut misalkan A dan B adalah dua variabel dinamis, umumnya berlaku:

$$AB \neq BA$$

atau

$$A_{op}B_{op}\psi \neq B_{op}A_{op}\psi \quad (2.70)$$

Selanjutnya didefinisikan hubungan komutasi atau komutator antara A dan B sebagai

$$AB - BA = [A, B] \quad (2.71)$$

Karena $\hat{x}$, $\hat{y}$, dan $\hat{z}$ komut secara bersamaan dan begitu juga $\hat{p}_x$, $\hat{p}_y$, dan $\hat{p}_z$ dan karena

$$[\hat{x}, \hat{p}_x] = i\hbar$$

$$[\hat{y}, \hat{p}_y] = i\hbar$$

$$[\hat{z}, \hat{p}_z] = i\hbar$$

sehingga komutator $\hat{L}_x$ dan $\hat{L}_y$ dapat diperoleh melalui

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \quad (2.72)$$

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Perhitungan yang sama menghasilkan dua komutasi yang lain; tapi hal itu jauh lebih sederhana untuk menyimpulkan komutasi dari (2.72) dengan permutasi siklis dari komponen *xyz*, $x \rightarrow y \rightarrow z \rightarrow x$:

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \quad (2.73)$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y \quad (2.74)$$

Karena $\hat{L}_x$, $\hat{L}_y$, dan $\hat{L}_z$ tidak komut, sehingga tidak dapat mengukurnya secara serentak untuk ketelitian yang berubah-ubah (Zettili, 2001:270).

2.5 Fungsi Eigen Momentum Sudut

Pada persamaan (2.67) dan (2.69), operator L_z dan L^2 hanya bergantung pada sudut θ dan φ, keadaan eigennya hanya bergantung pada θ dan φ. Penandaan keadaan eigennya bersama-sama dengan

$$\langle \theta\phi | \ell, m\rangle = Y_{\ell m}(\theta, \varphi) \quad (2.75)$$

di mana Y_{lm} ialah fungsi kontinyu dari θ dan φ (Zettili, 2001:302). Terlihat bahwa fungsi Y_{lm} separasi dari $\Theta_{lm}(\theta)$ dan $\Phi_m(\varphi)$. Dengan mensubstitusikan solusi dari persamaan (2.45) dan (2.49), sehingga

$$Y_{\ell m}(\theta, \varphi) = \epsilon \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^m(\cos\theta) e^{im\varphi} \quad (2.76)$$

Persamaan (2.75) disebut sebagai fungsi harmonik bola.

2.5.1 Fungsi Eigen Operator L_z

Operasikan persamaan (2.67) dengan menggunakan fungsi harmonik bola persamaan (2.75), sehingga dituliskan menjadi

$$L_z Y_{\ell m}(\theta, \varphi) = -i\hbar \frac{\partial}{\partial\varphi} Y_{\ell m}(\theta, \varphi)$$

Kemudian substitusikan persamaan (2.75), sehingga menjadi

$$L_z Y_{\ell m}(\theta, \varphi) = -i\hbar \frac{\partial}{\partial\varphi}\left(\epsilon \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^m(\cos\theta) e^{im\varphi}\right)$$

Turunan dari Y_{lm} hanya bergantung pada sudut φ, dengan mudah didapatkan

$$L_z Y_{\ell m}(\theta, \varphi) = -i\hbar(im)\epsilon \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^m(\cos\theta) e^{im\varphi}$$

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$$L_z Y_{\ell m}(\theta, \varphi) = m\hbar Y_{\ell m}(\theta, \varphi) \qquad (2.77)$$

Terlihat bahwa Y_{lm} (θ, ϕ) persamaan (2.76) merupakan fungsi eigen dari operator L_z dengan nilai eigen $(m\hbar)$.

2.5.2 Fungsi Eigen Operator L^2

Persamaan (2.69) dikalikan suatu fungsi eigen yang dalam hal ini fungsi harmonik bola agar diperoleh nilai eigen dari operator L^2

$$L^2 Y_{\ell m}(\theta, \varphi) = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_{\ell m}(\theta, \varphi) \qquad (2.78)$$

Nilai turunan kedua dari Y_{lm} adalah $-m^2 Y_{\ell m}$, dan memisahkan variabel Y_{lm} sehingga persamaan (2.77) menjadi

$$L^2 Y_{\ell m}(\theta, \varphi) = -\hbar^2 \left[\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right\}\Theta_{\ell m}\right]\Phi_m \qquad (2.79)$$

Selanjutnya gunakan persamaan (2.19) untuk Θ_{lm}, maka

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\ell m}}{d\theta}\right) + \left\{\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta_{\ell m} = 0$$

Maka

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\ell m}}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta_{\ell m} = -\ell(\ell+1)\Theta_{\ell m}$$

Dengan demikian diperoleh

$$L^2 Y_{\ell m}(\theta, \phi) = -\hbar^2[-\ell(\ell+1)\Theta_{\ell m}]\Phi_m$$

Karena Θ_{lm} dan Φ_m merupakan fungsi dari harmonik bola, sehingga

$$L^2 Y_{\ell m}(\theta, \phi) = \ell(\ell+1)\hbar^2 Y_{\ell m}(\theta, \phi) \qquad (2.80)$$

Artinya Y_{lm} (θ, ϕ) juga merupakan fungsi eigen dari L^2 dengan nilai eigen $\ell(\ell+1)\hbar^2$. Hal ini berarti Y_{lm} (θ, φ) merupakan fungsi eigen serempak dari L_z dan L^2, dan hasil ini memberikan konsekuensi lebih lanjut yaitu

$$[L^2, L_z] = 0 \qquad (2.81)$$

(Purwanto, 2006:175-176).

Operator L^2 juga komut dengan operator L_x dan L_y . Hal ini berarti kuadrat dari momentum sudut yaitu modulus, dapat memiliki nilai yang pasti pada saat yang sama sebagai salah satu komponennya (Landau dan Liftshitz, 1977:85). L^2 komut dengan operator L_x, L_y, dan L_z dilihat dari bentuk persamaan operator laplace

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dalam koordinat bola dan bentuk dari L_z pada persamaan (2.67), operasi $\frac{\partial}{\partial\varphi}$ tidak memiliki efek pada fungsi atau operator yang bergantung pada θ, tidak ada fungsi dari φ di operator L^2, dan $\frac{\partial}{\partial\varphi}$ komut dengan $\frac{\partial^2}{\partial\varphi^2}$ (Miller, 2008:251).

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BAB 3. METODE PENELITIAN

3.1 Jenis Penelitian

Jenis penelitian ini adalah *basic research*, dengan hasil pengembangan teori yang ada pada tinjauan pustaka. Penelitian ini dilakukan untuk mengetahui hubungan komutasi operator momentum sudut dalam sistem koordinat bola dapat bersifat komut atau tidak komut jika dikenai fungsi gelombang atom hidrogen.

3.2 Tempat dan Waktu Penelitian

Penelitian ini dilakukan di Laboratorium Fisika Dasar, Program Studi Pendidikan Fisika pada Semester 8 bulan Januari sampai bulan Februari 2017.

3.3 Objek Penelitian

Objek penelitian ini adalah pada materi fisika modern dan fisika kuantum yang berkaitan dengan persamaan schrodinger tiga dimensi tak bergantung waktu untuk atom hidrogen dan komutator operator momentum sudut dalam koordinat bola.

3.4 Definisi Operasional

Agar tidak terjadi kesalahan dalam mengartikan istilah-istilah dalam penelitian ini, maka diberikan definisi operasional mengenai variabel penelitian sebagai berikut:

a. Komutator Operator Momentum Sudut

Komutator operator momentum sudut adalah suatu metode yang digunakan untuk mengetahui hubungan komut atau tidak komut dari operator momentum sudut. Apabila operator momentum sudut yang dihitung bernilai nol (komut), berarti operator tersebut dapat diukur secara serempak dalam waktu yang bersamaan. Namun, jika operator tersebut tidak bernilai nol (tidak komut), berarti operator tersebut tidak dapat diukur secara serempak atau dengan kata lain taat asas ketidakpastian Heisenberg.

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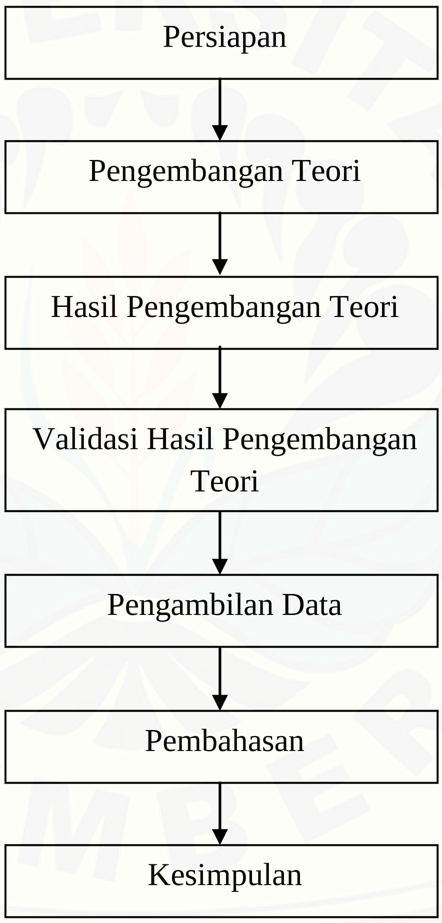
b. Fungsi Gelombang Atom Hidrogen

Fungsi gelombang atom hidrogen merupakan solusi dari persamaan schrodinger atom hidrogen yang berupa persamaan diferensial orde dua. Fungsi ini diperoleh dengan memasukkan potensial atom hidrogen ke dalam persamaan schrodinger, kemudian menerapkan metode separasi variabel dan syarat batas agar fungsinya menjadi ternormalisasi.

3.5 Langkah Penelitian



3.5.1 Persiapan

Tahap ini ialah tahap untuk mempersiapkan bahan-bahan yang akan dijadikan sumber referensi mengenai penelitian yang akan dikaji dengan cara mengumpulkan buku yang relevan, internet, dan jurnal berskala naasional ataupun internasional terkait dengan penelitian ini.

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3.5.2 Pengembangan Teori

Pada tahap ini, peneliti mengembangkan teori yang telah ada di berbagai sumber referensi mengenai komutator operator momentum sudut. Teori yang dikembangkan adalah komutator operator momentum sudut dalam sistem koordinat bola dengan menggunakan fungsi gelombang atom hidrogen. Langkah pertama yang dilakukan adalah menentukan fungsi gelombang atom hidrogen. Langkah selanjutnya yaitu mengubah bentuk operator momentum sudut dari koordinat kartesian ke koordinat bola dengan metode diferensial total. Langkah terakhir yaitu mencari hubungan komutasi dari beberapa operator tersebut yang telah ditransformasi ke koordinat bola dengan menggunakan fungsi gelombang atom hidrogen.

3.5.3 Hasil Pengembangan Teori

Dari pengembangan teori yang telah dilakukan, dapat diperoleh persamaan matematis komutator operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen. Hasil pengembangan teori dapat digunakan untuk menentukan hubungan komutasi dari operator momentum sudut jika dikenai fungsi gelombang atom hidrogen.

3.5.4 Validasi Hasil Pengembangan Teori

Pada tahap ini peneliti bertujuan untuk membandingkan persamaan matematis operator momentum sudut dalam koordinat bola dan persamaan matematis komutator operator momentum sudut dalam koordinat bola antara hasil pengembangan dengan hasil penelitian yang diperoleh dari buku, internet, atau jurnal. Adapun peneliti memvisualisasikan grafik fungsi radial dan fungsi harmonik bola menggunakan bola dengan menggunakan Matlab. Grafik tersebut kemudian dicocokkan dengan grafik fungsi radial dan fungsi harmonik bola di berbagai referensi.

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3.5.5 Pengambilan Data

Tahap ini adalah tahap perhitungan secara manual untuk menentukan komutator operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen.

3.5.6 Pembahasan

Hasil dari pengambilan data akan dibahas secara rinci dalam kaitannya mengenai komutator operator momentum sudut dalam koordinat bola dengan menggunakan fungsi gelombang atom hidrogen

3.5.7 Kesimpulan

Hasil dari pembahasan kemudian disimpulkan untuk menjawab rumusan permasalahan dalam penelitian.

3.6 Teknik Analisis Data

3.6.3 Analisis Data

a. Operator momentum sudut dalam koordinat bola

1) Operator $\hat{L}_x$

$$\hat{L}_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$$

2) Operator $\hat{L}_y$

$$\hat{L}_y = -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)$$

3) Operator $\hat{L}_z$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

4) Operator $\hat{L}_+$

$$\hat{L}_+ = \hbar e^{+i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

5) Operator $\hat{L}_-$

$$\hat{L}_- = -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

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6) Operator $\hat{L}^2$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

b. Fungsi harmonik bola dari atom hidrogen

$$Y_{\ell m}(\theta,\varphi) = \Theta_{\ell m}(\theta)\Phi_{\mathrm{m}}(\varphi)$$

c. Komutator operator dengan fungsi harmonik bola

$$[A,B]Y_{\ell m}(\theta,\varphi) = (AB - BA)Y_{\ell m}(\theta,\varphi)$$

3.6.4 Teknik Penyajian

Teknik penyajian menampilkan tabel data simulasi untuk menentukan komutator operator momentum sudut dengan fungsi harmonik bola atom hidrogen.

Tabel 3.1 Contoh data simulasi untuk menentukan komutator operator momentum sudut dengan fungsi harmonik bola hidrogen

No	Komutator Operator Momentum Sudut	Fungsi Harmonik Bola								
		Y_{00}	Y_{10}	Y_{1-1}	Y_{11}	Y_{20}	Y_{2-1}	Y_{21}	Y_{2-2}	Y_{22}
1	$[\hat{L}_x,\hat{L}_y]$									
2	$[\hat{L}_y,\hat{L}_z]$									
3	$[\hat{L}_z,\hat{L}_x]$									
4	$[\hat{L}_z,\hat{L}_+]$									
5	$[\hat{L}_z,\hat{L}_-]$									
6	$[\hat{L}_x,\hat{L}^2]$									
7	$[\hat{L}_y,\hat{L}^2]$									
8	$[\hat{L}_z,\hat{L}^2]$									

3.7 Validasi Hasil Pengembangan Teori

Validasi dalam penelitian ini menggunakan data dari buku atau jurnal yang relevan mengenai operator momentum sudut dalam koordinat bola, hasil beberapa komutator operator momentum sudut dalam koordinat bola, dan fungsi gelombang atom hidrogen untuk $n \leq 3$ (Pembuktian terlampir). Adapun fungsi gelombang atom hidrogen kemudian disimulasikan dengan menggunakan *software* Matlab2014a.

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3.7.1 Validasi Operator Momentum Sudut dalam Koordinat bola

Berikut ini merupakan tabel perbandingan operator momentum sudut dalam koordinat bola dari buku teks (Tabel 3.2) dan perhitungan matematis manual (Tabel 3.3).

Tabel 3.2 Operator momentum sudut dalam koordinat bola

No	Operator Momentum Sudut	Operator Momentum Sudut dalam Koordinat Bola
1	L_x	$-i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$
2	L_y	$-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$
3	L_z	$-i\hbar\frac{\partial}{\partial\varphi}$
4	L_+	$\hbar e^{+i\varphi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\varphi}\right)$
5	L_-	$-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right)$
6	L^2	$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$

(Rajasekar dan Velusamy, 2015:267).

Tabel 3.3 Validasi operator momentum sudut dalam koordinat bola

No	Operator Momentum Sudut	Operator Momentum Sudut dalam Koordinat Bola
1	L_x	$-i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$
2	L_y	$-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$
3	L_z	$-i\hbar\frac{\partial}{\partial\varphi}$
4	L_+	$\hbar e^{+i\varphi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\varphi}\right)$
5	L_-	$-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right)$
6	L^2	$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$

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3.7.2 Validasi Hasil Beberapa Komutator Operator Momentum Sudut dalam Koordinat Bola

Berikut ini merupakan tabel perbandingan hasil beberapa komutator operator momentum sudut dalam koordinat bola dari penelitian terdahulu (Tabel 3.4) dan perhitungan matematis manual (Tabel 3.5).

Tabel 3.4 Hasil beberapa komutator operator momentum sudut dalam koordinat bola

No	Komutator Operator Momentum Sudut	Komutator Operator Momentum Sudut dalam Koordinat Bola
1	$[\hat{L}_x, \hat{L}_x]$	0
2	$[\hat{L}_y, \hat{L}_y]$	0
3	$[\hat{L}_z, \hat{L}_z]$	0
4	$[\hat{L}_x, \hat{L}_y]$	$\hbar^2 \frac{\partial}{\partial\varphi}$
5	$[\hat{L}_y, \hat{L}_z]$	$-\hbar^2 \left(sin\,\varphi \frac{\partial}{\partial\theta} + cos\,\varphi\, cot\,\theta \frac{\partial}{\partial\varphi}\right)$
6	$[\hat{L}_z, \hat{L}_x]$	$\hbar^2 \left(cos\,\varphi \frac{\partial}{\partial\theta} - sin\,\varphi\, cot\,\theta \frac{\partial}{\partial\varphi}\right)$
7	$[\hat{L}_x, \hat{L}_+]$	$i\hbar^2 \frac{\partial}{\partial\varphi}$
8	$[\hat{L}_y, \hat{L}_+]$	$-\hbar^2 \frac{\partial}{\partial\varphi}$
9	$[\hat{L}_z, \hat{L}_+]$	$\hbar^2 e^{i\varphi} \left(\frac{\partial}{\partial\theta} + icot\,\theta \frac{\partial}{\partial\varphi}\right)$
10	$[\hat{L}_x, \hat{L}_-]$	$i\hbar^2 \frac{\partial}{\partial\varphi}$
11	$[\hat{L}_y, \hat{L}_-]$	$-\hbar^2 \frac{\partial}{\partial\varphi}$
12	$[\hat{L}_z, \hat{L}_-]$	$-\hbar^2 e^{-i\varphi} \left(-\frac{\partial}{\partial\theta} + icot\,\theta \frac{\partial}{\partial\varphi}\right)$
13	$[\hat{L}_x, \hat{L}^2]$	0
14	$[\hat{L}_y, \hat{L}^2]$	0
15	$[\hat{L}_z, \hat{L}^2]$	0
16	$[\hat{L}_+, \hat{L}^2]$	0
17	$[\hat{L}_-, \hat{L}^2]$	0
18	$[\hat{L}_+, \hat{L}_-]$	$-2i\hbar^2 \frac{\partial}{\partial\varphi}$
19	$[\hat{L}_-, \hat{L}_+]$	$2i\hbar^2 \frac{\partial}{\partial\varphi}$

(Sunarmi, 2009)

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Tabel 3.5 Validasi hasil beberapa komutator operator momentum sudut dalam koordinat bola

No	Komutator Operator Momentum Sudut	Komutator Operator Momentum Sudut dalam Koordinat Bola
1	$[\hat{L}_x, \hat{L}_x]$	0
2	$[\hat{L}_y, \hat{L}_y]$	0
3	$[\hat{L}_z, \hat{L}_z]$	0
4	$[\hat{L}_x, \hat{L}_y]$	$\hbar^2 \frac{\partial}{\partial \varphi}$
5	$[\hat{L}_y, \hat{L}_z]$	$-\hbar^2 \left(sin\, \varphi \frac{\partial}{\partial \theta} + cos\, \varphi\, cot\, \theta \frac{\partial}{\partial \varphi} \right)$
6	$[\hat{L}_z, \hat{L}_x]$	$\hbar^2 \left(cos\, \varphi \frac{\partial}{\partial \theta} - sin\, \varphi\, cot\, \theta \frac{\partial}{\partial \varphi} \right)$
7	$[\hat{L}_x, \hat{L}_+]$	$i\hbar^2 \frac{\partial}{\partial \varphi}$
8	$[\hat{L}_y, \hat{L}_+]$	$-\hbar^2 \frac{\partial}{\partial \varphi}$
9	$[\hat{L}_z, \hat{L}_+]$	$\hbar^2 e^{i\varphi} \left(\frac{\partial}{\partial \theta} + icot\, \theta \frac{\partial}{\partial \varphi} \right)$
10	$[\hat{L}_x, \hat{L}_-]$	$i\hbar^2 \frac{\partial}{\partial \varphi}$
11	$[\hat{L}_y, \hat{L}_-]$	$-\hbar^2 \frac{\partial}{\partial \varphi}$
12	$[\hat{L}_z, \hat{L}_-]$	$\hbar^2 e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - icot\, \theta \frac{\partial}{\partial \varphi} \right)$
13	$[\hat{L}_x, \hat{L}^2]$	0
14	$[\hat{L}_y, \hat{L}^2]$	0
15	$[\hat{L}_z, \hat{L}^2]$	0
16	$[\hat{L}_+, \hat{L}^2]$	0
17	$[\hat{L}_-, \hat{L}^2]$	0
18	$[\hat{L}_+, \hat{L}_-]$	$-2i\hbar^2 \frac{\partial}{\partial \varphi}$
19	$[\hat{L}_-, \hat{L}_+]$	$2i\hbar^2 \frac{\partial}{\partial \varphi}$

Berdasarkan Tabel 3.4 dan 3.5, dapat diperoleh makna bahwa komutator operator momentum sudut dalam koordinat bola akan komut (bernilai nol) apabila antar komponen operator momentum sudutnya sama dan kuadrat operator momentum sudut beroperasi dengan semua komponen operator momentum sudut. Operator penaik $(\hat{L}_+)$ dan operator penurun $(\hat{L}_-)$ tidak merubah nilai dari komponen operator momentum sudut $\hat{L}_x$ dan $\hat{L}_y$, tetapi merubah nilai dan $\hat{L}_z$.

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3.7.3 Validasi Fungsi Gelombang Atom Hidrogen

Berikut ini merupakan tabel perbandingan fungsi gelombang atom hidrogen $n \leq 3$ dari buku teks (Tabel 3.6) dan perhitungan matematis manual (Tabel 3.7).

Tabel 3.6 Fungsi gelombang atom hidrogen

n	*l*	*m*	R_{nl} (r)	Θ_{lm} (θ)	Φ_m (φ)
1	0	0	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
2	0	0	$\frac{1}{(2a_0)^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
	1	0	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
		±1	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\mp\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\varphi}$
3	0	0	$\frac{2}{(3a_0)^{3/2}}\left(1-\frac{2r}{3a_0}+\frac{2r^2}{27a_0^2}\right)e^{-r/3a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
	1	0	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0}-\frac{r^2}{6a_0^2}\right)e^{-r/3a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
		±1	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0}-\frac{r^2}{6a_0^2}\right)e^{-r/3a_0}$	$\mp\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\varphi}$
	2	0	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\sqrt{\frac{5}{8}}(3cos^2\theta-1)$	$\frac{1}{\sqrt{2\pi}}$
		±1	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\mp\sqrt{\frac{15}{4}}\sin\theta\cos\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\varphi}$
		±2	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\frac{\sqrt{15}}{4}sin^2\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm 2i\varphi}$

(Krane, 2012:205).

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Tabel 3.7 Validasi fungsi gelombang atom hidrogen

n	*l*	*m*	R_{nl} (r)	Θ_{lm} (θ)	Φ_m (φ)
1	0	0	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
2	0	0	$\frac{1}{(2a_0)^{3/2}}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
	1	0	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
		±1	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\mp\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}} e^{\pm i\varphi}$
3	0	0	$\frac{2}{(3a_0)^{3/2}}\left(1-\frac{2r}{3a_0}+\frac{2r^2}{27a_0^2}\right) e^{-r/3a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
	1	0	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0}-\frac{r^2}{6a_0^2}\right) e^{-r/3a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
		±1	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0}-\frac{r^2}{6a_0^2}\right) e^{-r/3a_0}$	$\pm\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}} e^{\pm i\varphi}$
	2	0	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\sqrt{\frac{5}{8}}(3cos^2\theta-1)$	$\frac{1}{\sqrt{2\pi}}$
		±1	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\mp\sqrt{\frac{15}{4}}\sin\theta\cos\theta$	$\frac{1}{\sqrt{2\pi}} e^{\pm i\varphi}$
		±2	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{\sqrt{15}}{4} sin^2\theta$	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\varphi}$

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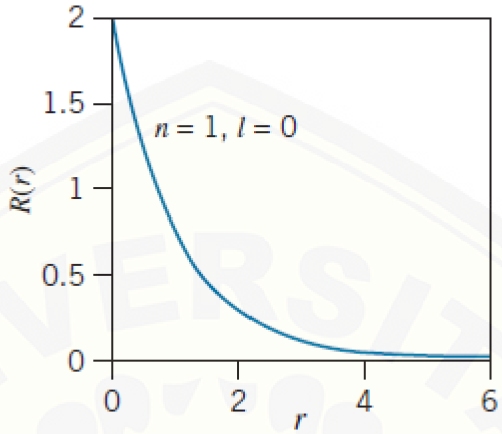
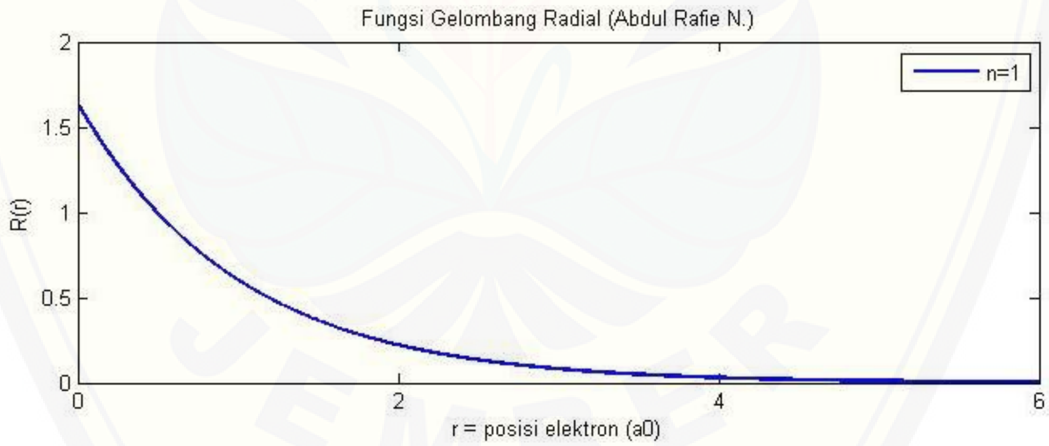
a. Fungsi gelombang radial atom hidrogen

Berikut ini adalah perbandingan grafik fungsi radial untuk atom hidrogen dari hasil Matlab2014 terhadap buku teks (Krane, 2012: 206).





Gambar 3.1 Grafik fungsi gelombang radial atom hidrogen n=1 dari buku teks (gambar atas) dan hasil simulasi Matlab2014a (gambar bawah).

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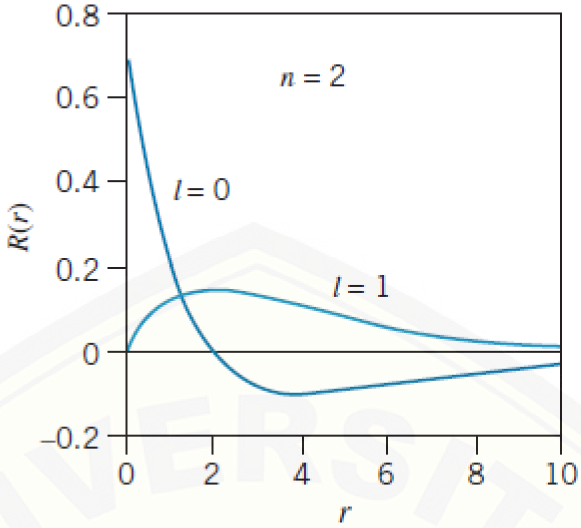

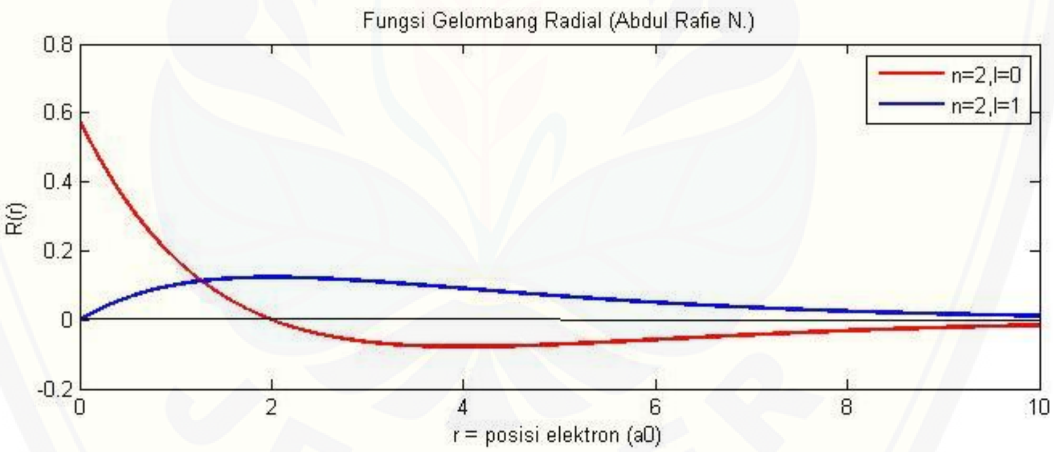

Gambar 3.2 Grafik fungsi gelombang radial atom hidrogen n=2 dari buku teks (gambar atas) dan hasil simulasi Matlab2014a (gambar bawah).

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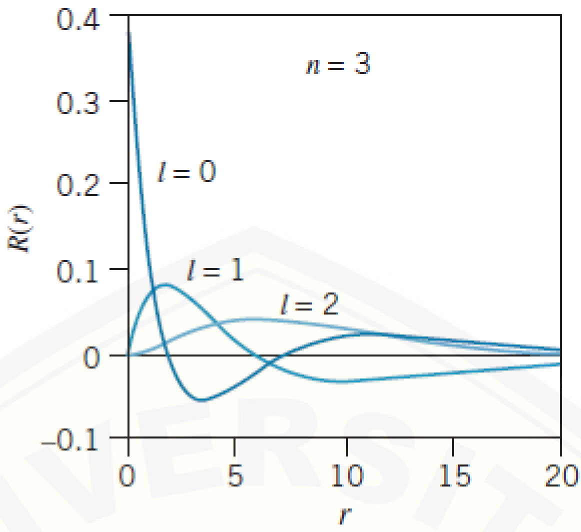

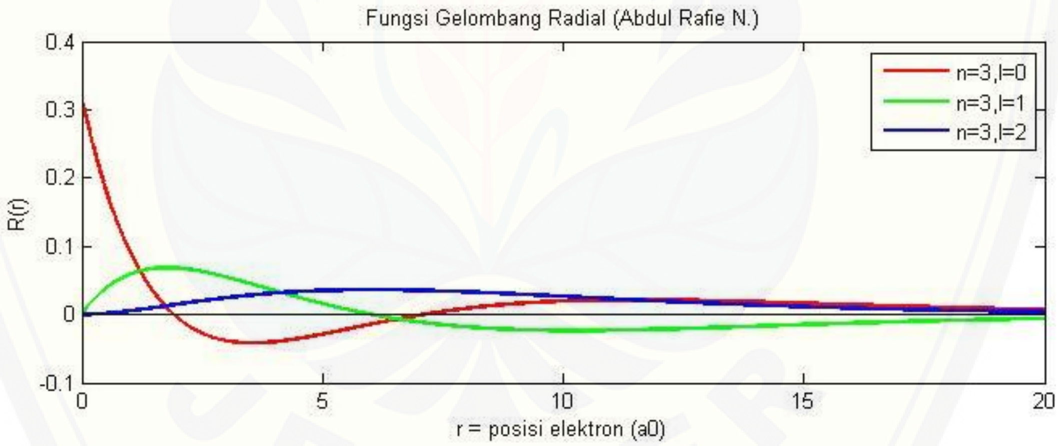

Gambar 3.3 Grafik fungsi gelombang radial atom hidrogen n=3 dari buku teks (gambar atas) dan hasil simulasi Matlab2014a (gambar bawah).

b. Fungsi harmonik bola atom hidrogen

Berikut ini adalah perbandingan grafik fungsi harmonik bola untuk atom hidrogen dari hasil Matlab2014a terhadap buku teks (Levi, 2006: 498).

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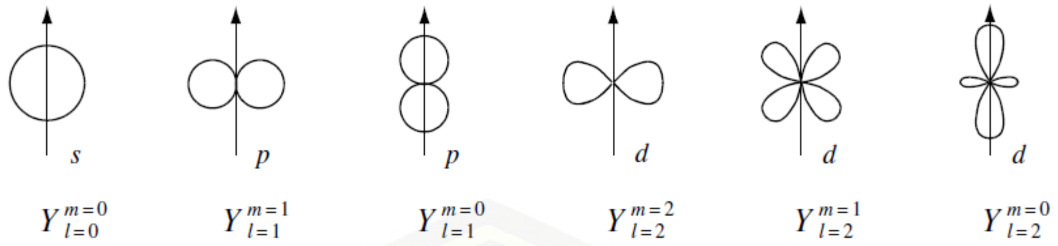
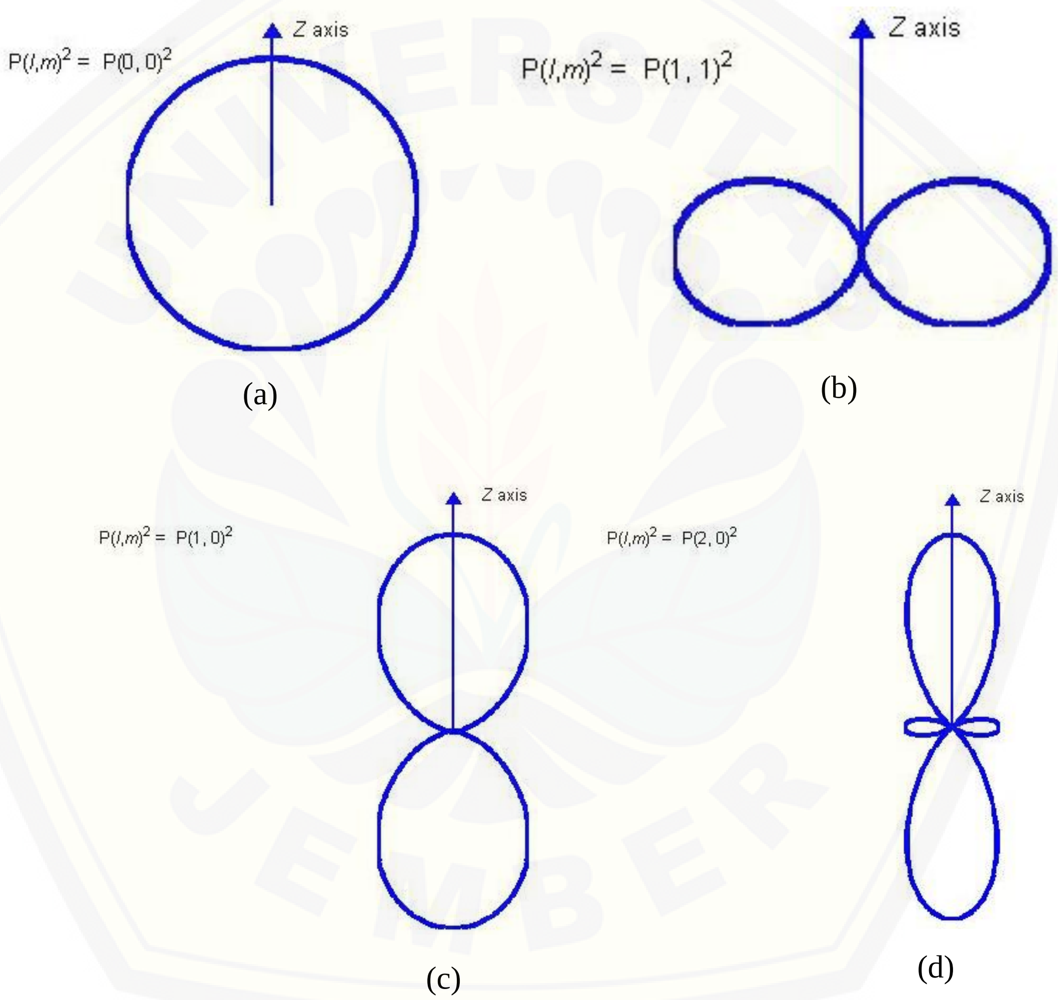


Gambar 3.4 Grafik fungsi harmonik bola atom hidrogen dari buku teks



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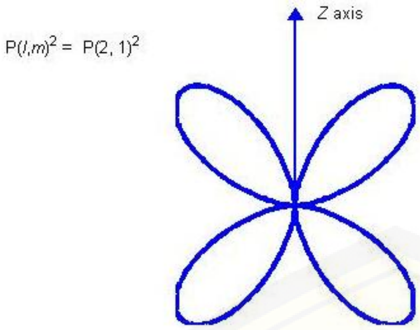
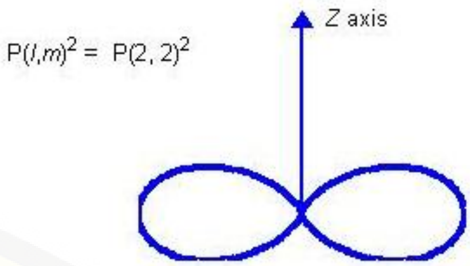


(e)



(f)

Gambar 3.5 Grafik fungsi harmonik bola atom hidrogen dari simulasi Matlab2014a di mana (a) Y_{00}^{2}, (b) Y_{11}^{2}, (c) Y_{10}^{2}, (d) Y_{20}^{2}, (e) Y_{21}^{2}, dan (f) Y_{22}^{2}

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BAB 5. PENUTUP

5.1 Kesimpulan

Pada penelitian ini telah dikaji komutator operator momentum sudut dalam koordinat bola terhadap fungsi gelombang atom hidrogen. Berdasarkan data hasil penelitian dapat disimpulkan:

a. Pada fungsi Y_{00} diketahui bahwa fungsinya sesuai terhadap semua komutator operator momentum sudut sehingga termasuk ke dalam persoalan eigen. Pada fungsi Y_{10}, $Y_{1\pm1}$, Y_{20}, $Y_{2\pm1}$, dan $Y_{2\pm2}$ terdapat beberapa komutator operator momentum sudut yang tidak ada nilai eigen karena operatornya mengubah fungsi tersebut menjadi fungsi lain sehingga tidak termasuk ke dalam persoalan eigen. Komutator $[\hat{L}_x, \hat{L}_y]$ dan komutator kuadrat momentum sudut dengan ketiga komponen momentum sudut merupakan persoalan eigen karena menghasilkan nilai eigen sehingga seluruh fungsi harmonik bola atom hidrogen merupakan fungsi eigen untuk komutator tersebut. Komutator $[\hat{L}_z, \hat{L}_-]$ termasuk persoalan eigen untuk beberapa fungsi tertentu seperti Y_{00}, Y_{1-1}, dan Y_{2-2}. Komutator $[\hat{L}_z, \hat{L}_+]$ juga termasuk persoalan eigen untuk beberapa fungsi seperti Y_{00}, Y_{11}, dan Y_{22}. Pada prinsipnya, operator penurun bekerja pada bilangan kuantum magnetik minimum (negatif) dan operator penaik bekerja pada bilangan kuantum magnetik maksimum (positif).

b. Komutator $[\hat{L}_x, \hat{L}_y]$ komut di fungsi Y_{00}, Y_{10}, dan Y_{20} dan tidak komut di fungsi $Y_{1\pm1}$, $Y_{2\pm1}$, dan $Y_{2\pm2}$. Pada prinsipnya, bilangan kuantum magnetik memainkan peran penting pada komutator tersebut karena jika m = 0 menghasilkan nilai nol, m = ±1 menghasilkan nilai $\pm i\hbar$, dan m = ±2 menghasilkan nilai $\pm 2i\hbar$. Komutator $[\hat{L}_y, \hat{L}_z]$ dan $[\hat{L}_z, \hat{L}_x]$ hanya dapat ditentukan pada keadaan dengan fungsi gelombang yang simetri bola yaitu bernilai nol sehingga saat menentukan komutator tersebut pada keadaan fungsi gelombang yang bergantung pada sudut, komutator tersebut tidak dapat ditentukan sebab elektron berputar sangat cepat. Komutator $[\hat{L}_z, \hat{L}_+]$ komut di fungsi Y_{00}, Y_{11}, dan Y_{22} sehingga selain fungsi tersebut komutatornya tidak dapat ditentukan

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karena bukan persoalan eigen. Komutator $[\hat{L}_z, \hat{L}_-]$ komut di fungsi Y_{00}, Y_{1-1}, dan Y_{2-2} sehingga selain fungsi tersebut komutatornya tidak dapat ditentukan karena bukan persoalan eigen. $[\hat{L}_x, \hat{L}^2]$, $[\hat{L}_y, \hat{L}^2]$, dan $[\hat{L}_z, \hat{L}^2]$ komut di semua fungsi harmonik bola atom hidrogen karena L^2 merupakan *magnitude* (besar) momentum sudut dari ketiga komponen momentum sudut L_x, L_y, dan L_z.

5.2 Saran

Berdasarkan kesimpulan tersebut, penulis memberikan saran agar diadakan penelitian lebih lanjut seperti mengubah koordinat atom hidrogen menjadi koordinat yang lain, menyelesaikan dengan numerik dan memvisualisasikan hasil komutator operator momentum sudut, serta dapat juga menggunakan fungsi gelombang atom yang lain.

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LAMPIRAN A. MATRIKS PENELITIAN

Judul	Rumusan Masalah	Variabel	Indikator	Sumber Data	Metode Penelitian
KOMUTATOR OPERATOR MOMENTUM SUDUT DALAM KOORDINAT BOLA DENGAN FUNGSI GELOMBANG ATOM HIDROGEN	1. Bagaimana persamaan matematis dari komutator operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen? 2. Bagaimana hubungan antar komponen operator momentum sudut dalam koordinat bola dengan fungsi gelombang atom hidrogen?	1. Variabel bebas: Fungsi harmoik bola atom hidrogen 2. Variabel terikat: Komutator operator momentum sudut dengan fungsi gelombang atom hidrogen	1 Komutator operator momentum sudut 2 Koordinat bola 3 Fungsi gelombang atom hidrogen	Literatur yang sesuai: a Buku b Jurnal c Internet	1. Jenis Penelitian: *basic research* 2. Analisis: Persamaan teori yang dilakukan dalam penelitian ini adalah: a Persamaan Schrodinger pada atom hidrogen tak bergantung waktu b Operator momentum sudut c Transformasi koordinat d Sifat komutator

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LAMPIRAN B. PERSAMAAN SCHRODINGER UNTUK ATOM HIDROGEN

Persamaan Schrdinger tiga dimensi dalam koordinat kartesian adalah sebagai berikut:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{2m}{\hbar^2}(E-V)\psi=0$$

dengan

$$V=-\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}$$

sehingga

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{2m_e}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi=0$$

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi-\frac{2m_e}{\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi=\frac{2m_e}{\hbar^2}E\psi$$

$$-\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi-\left(\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi=E\psi$$

$$\left(-\frac{\hbar^2}{2m_e}\nabla^2-\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi(\boldsymbol{r})=E\psi(\boldsymbol{r}) \qquad \text{(B.1)}$$

Jika persamaan (B.1) diubah ke dalam koordinat bola, maka operator ∇^2 diubah ke koordinat bola menjadi

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right)$$

sehingga persamaan (B.1) atau (2.11) menjadi

$$-\frac{\hbar^2}{2m_e}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right)\right)\psi-\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\psi=E\psi$$

$$-\frac{\hbar^2}{2m_e}\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right\}-\left(\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\psi\right)=E\psi \qquad \text{(B.2)}$$

Selanjutnya, untuk mendapatkan solusi bagi persamaan (2.11) dilakukan pemisahan variabel $\psi(\vec{r})=\psi(r,\theta,\varphi)$ sebagai berikut

$$\psi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi) \qquad \text{(B.3)}$$

dengan memanfaatkan turunan parsial, maka persamaan (B.3) jika diturunkan terhadap r, θ, dan φ akan menjadi

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$$\frac{\partial\psi}{\partial r}=\frac{\partial}{\partial r}R\Theta\Phi=\Theta\Phi\frac{dR}{dr} \qquad \text{(B.4a)}$$

$$\frac{\partial\psi}{\partial\theta}=\frac{\partial}{\partial\theta}R\Theta\Phi=R\Phi\frac{d\Theta}{d\theta} \qquad \text{(B.4b)}$$

$$\frac{\partial\psi}{\partial\varphi}=\frac{\partial}{\partial\varphi}R\Theta\Phi=R\Theta\frac{d\Phi}{d\varphi}$$

$$\frac{\partial^2\psi}{\partial\varphi^2}=\frac{\partial}{\partial\varphi}R\Theta\frac{d\Phi}{d\varphi}=\ R\Theta\frac{d^2\Phi}{d\varphi^2} \qquad \text{(B.4c)}$$

$$-\frac{\hbar^2}{2m_e}\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right\}-\left(E+\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi=0$$

$$\frac{\hbar^2}{2m_e}\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right\}+\left(E+\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)\psi=0 \qquad \text{(B.5)}$$

dengan memasukkan turunan fungsi ψ yang telah didapat dari persamaan (B.4a), (B.4b), (B.4c) ke dalam persamaan (B.5), maka akan diperoleh

$$\frac{\hbar^2}{2m_e}\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin\theta}\frac{d^2\Phi}{d\varphi^2}\right\}+\left(E+\frac{e^2}{4\pi\varepsilon_o}\frac{1}{r}\right)R\Theta\Phi=0 \qquad \text{(B.6)}$$

persamaan (B.6) dapat diselesaikan dengan mengalikan $(2m_e\, r^2/\,\hbar^2)$ dan membagi $R\Theta\Phi$ ke semua ruas, sehingga akan didapatkan hasil

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}+\frac{2m_e r^2}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)=0 \qquad \text{(B.7)}$$

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LAMPIRAN C. SOLUSI RADIAL ATOM HIDROGEN

$$\frac{d}{dr}\left(\mathrm{r}^2\frac{dR}{dr}\right)+\frac{2m_e r^2}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)R=\ell(\ell+1)R$$

dengan menggunakan pemisalan

$$\rho=\left(\frac{8m_e|E|}{\hbar^2}\right)^{1/2} r$$

$$d\rho=\left(\frac{8m_e|E|}{\hbar^2}\right)^{1/2} dr$$

$$\frac{d}{dr}\left(\mathrm{r}^2\frac{dR}{dr}\right)+\frac{2m_e r^2}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)R-\ell(\ell+1)R=0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\mathrm{r}^2\frac{dR}{dr}\right)+\frac{2m_e}{\hbar^2}\left[E+\frac{e^2}{4\pi\varepsilon_o r}-\frac{\ell(\ell+1)\hbar^2}{2m_e r^2}\right]R=0$$

$$\frac{d}{\left(\frac{8m_e|E|}{\hbar^2}\right)^{-1}\rho^2 d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{2m_e}{\hbar^2}R\left[E+\frac{e^2}{4\pi\varepsilon_o\left(\frac{8m_e|E|}{\hbar^2}\right)^{-1/2}\rho}-\frac{\ell(\ell+1)\hbar^2}{2m_e\left(\frac{8m_e|E|}{\hbar^2}\right)^{-1}\rho^2}\right]=0 \quad \text{(C.1)}$$

Persamaan (C.1) dikalikan dengan $\left(\frac{8m_e|E|}{\hbar^2}\right)^{-1}$ sehingga akan diperoleh

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\left[\frac{E}{4|E|}+\frac{e^2}{4\pi\varepsilon_o}\frac{1}{\rho}\frac{2m_e}{\hbar^2}\left(\frac{8m_e|E|}{\hbar^2}\right)^{-1/2}-\frac{\ell(\ell+1)}{\rho^2}\right]R=0$$

karena $E=-|E|$, sehingga persamaan di atas dapat dituliskan menjadi

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\left[-\frac{1}{4}+\frac{e^2}{2\pi\varepsilon_o}\frac{1}{\rho}\frac{m_e}{\hbar}(8m_e|E|)^{-1/2}-\frac{\ell(\ell+1)}{\rho^2}\right]R=0$$

persamaan diatas dapat disederhanakan dengan pemisalan $\lambda=\frac{e^2}{4\pi\varepsilon_0\hbar^2}\left(\frac{m_e}{8|E|}\right)^{1/2}$, sehingga persamaannya akan menjadi

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\left[-\frac{1}{4}+\frac{\lambda}{\rho}-\frac{\ell(\ell+1)}{\rho^2}\right]R=0$$

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\frac{\ell(\ell+1)}{\rho^2}\mathrm{R}+\left(\frac{\lambda}{\rho}-\frac{1}{4}\right)\mathrm{R}=0 \quad \text{(C.2)}$$

Untuk daerah tak hingga (jauh sekali)

$$\frac{d^2R}{d\rho^2}-\frac{1}{4}\mathrm{R}=0$$

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dengan solusi

$R \approx e^{-\rho/2}$

Untuk daerah asal (pusat koordinat)

$$R\ (\rho) = \frac{U\ (\rho)}{\rho}$$

$$\frac{dR}{d\rho} = \frac{d}{d\rho}\left(\frac{U}{\rho}\right) = \frac{1}{\rho}\frac{dU}{d\rho} - \frac{1}{\rho^2}U$$

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) = \frac{d}{d\rho}\left[\rho^2\left(\frac{1}{\rho}\frac{dU}{d\rho} - \frac{1}{\rho^2}U\right)\right] = \rho\,\frac{d^2U}{d\rho^2}$$

Persamaan (C.2) akan menjadi

$$\frac{d^2U}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}\,\mathrm{U} + \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)U = 0 \qquad \text{(C.3)}$$

kalikan persamaan di atas dengan ρ^2 dan ambil limit mendekati nol (pusat koordinat)

$$\lim_{\rho\to 0}\left\{\rho^2\frac{d^2U}{d\rho^2} - \ell(\ell+1)U + \lambda\rho U - \frac{1}{4}\rho^2 U\right\} = \left(\rho^2\frac{d^2U}{d\rho^2}\right) - \ell(\ell+1)U = 0$$

$$\frac{d^2U}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}\,\mathrm{U}$$

solusi persamaan di atas adalah

$\mathrm{U} \approx \rho^{\ell+1}$

$$R\ (\rho) = \frac{U\ (\rho)}{\rho} = \rho^{\ell}$$

dengan menggabungkan solusi R pada daerah tak hingga dan daerah pusat koordinat, serta fungsi umum terhadap jarak, maka diperoleh

$$\mathrm{R}\ (\rho) = \rho^{\ell} e^{-\rho/2}\ \mathrm{L}(\rho) \qquad \text{(C.4)}$$

atau $\mathrm{U}\ (\rho) = \rho^{\ell+1} e^{-\rho/2}\ \mathrm{L}(\rho)$

di mana $L\ (\rho)$ merupakan sebuah polinomial yang bentuknya

$$\mathrm{L}(\rho) = a_0 + a_1\rho + a_2\rho^2 + \cdots + a_s\rho^s \text{ atau } L = \sum_{s=0}^{\infty} a_s\,\rho^s \qquad \text{(C.5)}$$

dengan $a_0 \neq 0$

Untuk suku pertama persamaan (C.3), penyelesaiannya

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$$\frac{d^2U}{d\rho^2} = \mathrm{L}(\rho)\left[\ell(\ell+1)\rho^{\ell-1}e^{-\frac{\rho}{2}} - (\ell+1)\rho^{\ell}e^{-\frac{\rho}{2}} + \frac{1}{4}\rho^{\ell+1}e^{-\frac{\rho}{2}}\right]$$

$$+\frac{d^2\mathrm{L}(\rho)}{d\rho^2}\left[2(\ell+1)\rho^{\ell}e^{-\rho/2} - \rho^{\ell+1}e^{-\rho/2}\right] + \frac{d^2\mathrm{L}(\rho)}{d\rho^2}\left[\rho^{\ell+1}e^{-\rho/2}\right]$$

Persamaan (C.3) menjadi

$$\mathrm{L}(\rho)\left[\ell(\ell+1)\rho^{\ell-1}e^{-\frac{\rho}{2}} - (\ell+1)\rho^{\ell}e^{-\frac{\rho}{2}} + \tfrac{1}{4}\rho^{\ell+1}e^{-\frac{\rho}{2}}\right] + \tfrac{d^2\mathrm{L}(\rho)}{d\rho^2}\left[2(\ell+1)\rho^{\ell}e^{-\rho/2} - \rho^{\ell+1}e^{-\rho/2}\right] + \tfrac{d^2\mathrm{L}(\rho)}{d\rho^2}\left[\rho^{\ell+1}e^{-\rho/2}\right] - \tfrac{\ell(\ell+1)}{\rho^2}\,\mathrm{U} + \left(\tfrac{\lambda}{\rho} - \tfrac{1}{4}\right)U = 0 \quad \text{(C.6)}$$

kedua ruas terakhir pada persamaan (C.6) tampak konstan, sehingga dapat diselesaikan dengan

$$\frac{\ell(\ell+1)}{\rho^2}\,\mathrm{U} = \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)U = konstan$$

$$\frac{\ell(\ell+1)}{\rho^2} + \frac{1}{4} = \frac{\lambda}{\rho}$$

$$\frac{\ell(\ell+1)}{\rho} + \frac{\rho}{4} = \lambda$$

Persamaan (C.6) dapat disederhanakan dengan mengalikan semua ruas dengan $\rho^{-\ell}e^{-\frac{\rho}{2}}$, sehingga didapat

$$\rho\frac{d^2L}{d\rho^2} + \{2(\ell+1) - \rho\}\frac{dL}{d\rho} + \{\lambda - \ell - 1\}L = 0 \quad \text{(C.7)}$$

Substitusi persamaan (C.5) ke dalam (C.7) dan koefisien ρ^s sama dengan nol

$$(s+1)sa_{s+1} + 2(\ell+1)(s+1)a_{s+1} - 2sa_s + [\lambda - 2(\ell+1)]a_s = 0$$

$$a_{s+1}[s^2 + s + 2\ell s + 2\ell + 2s + 2] - a_s[2s - \lambda + 2\ell + 2] = 0$$

$$a_{s+1}[s^2 + s + 2\ell s + 2\ell + 2s + 2] = a_s[2s - \lambda + 2\ell + 2]$$

$$\frac{a_{s+1}}{a_s} = \frac{[2s - \lambda + 2\ell + 2]}{[s^2 + s + 2\ell s + 2\ell + 2s + 2]}$$

$$\frac{a_{s+1}}{a_s} = \frac{2(s + \ell + 1) - \lambda}{(s+1)(s+2\ell+2)}$$

Jadi secara umum rumus rekursi didapatkan

$$a_{s+1} = \frac{s + \ell + 1 - \lambda}{(s+1)(s+2\ell+2)}a_s$$

Deret (C.5) harus bernilai terbatas dari *s*. Dengan kata lain, pada rumus rekursi ketika $s \to \infty$ dan $a_{s+1} \to a_s/s$, maka $\rho \to \infty$ dan $L(\rho) \to e^{\rho}$ akan divergen. Untuk

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membatasi ekspansi deret (C.5) setelah suku $s + 1$, maka yang memenuhi deret batasnya adalah $\lambda = s + l + 1 = n$, dengan n adalah bilangan bulat positif. Nilai s dapat diasumsikan nol, sehingga nilai $n \geq l + 1$. Dengan substitusi nilai $\lambda = n$, maka persamaan (C.7) menjadi

$$\frac{d^2L}{d\rho^2} + \{2\,(\ell+1) - \rho\}\ \frac{dL}{d\rho} + \{n - \ (\ell+1)\}\,L \ = \ 0 \qquad \text{(C.8)}$$

Persamaan (C.8) merupakan bentuk dari polinom *Lagurre Terasosiasi*.

$$\rho\ \frac{d^2L_q^p}{d\rho^2} + \{p + 1 - \rho\}\ \frac{dL_q^p}{d\rho} + \{q - \ p\}\,L_q^p \ = \ 0$$

Polinom Laguerre Terasosiasi dapat diselesaikan dengan rumus Rodrigues

$$L_q^p(\rho) = \frac{q!}{(q-p)!}e^{\rho}\,\frac{d^q}{d\rho^q}\left(e^{-\rho}\rho^{q-p}\right)$$

Fungsi $U\,(\rho)$ diberikan oleh $U(\rho) \approx e^{-\rho/2}\rho^{\ell+1}L_{n+\ell}^{2\ell+1}(\rho)$ dan karena $R\,(\rho) = \frac{U\,(\rho)}{\rho}$, sehingga

$$R(\rho) \equiv R_{n\ell}(\rho) = N_{n\ell}e^{-\rho/2}\rho^{\ell}L_{n+\ell}^{2\ell+1}(\rho)$$

N_{nl} merupakan konstanta yang dicari melalui syarat normalisasi gelombang

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} R_{n\ell}^2(r)\Theta_{\ell m}^2(\theta)\Phi_m^2(\varphi)r^2\sin\theta\, d\varphi\, d\theta\, dr = 1$$

karena fungsi $R\,(\rho)$ hanya bergantung pada ρ yang didalamnya mengandung variabel r, sehingga normalisasi yang digunakan hanya pada integral bagian radial

$$\int_0^{\infty} R_{n\ell}^2(r)r^2dr = 1$$

$$\int_0^{\infty} R_{n\ell}^2(\rho)\left(\rho\frac{na_0}{2}\right)^2\frac{na_0}{2}d\rho = 1$$

$$\left(\frac{na_0}{2}\right)^3 N_{n\ell}^2\int\limits_0^{\infty} e^{-\rho}\rho^{2\ell+2}\left(L_{n+\ell}^{2\ell+1}\right)^2 d\rho = 1$$

$$\left(\frac{na_0}{2}\right)^3 N_{n\ell}^2\left[\frac{2n[(n+1)!]^3}{(n-\ell-1)!}\right] = 1$$

$$N_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2}$$

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$$R_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} e^{-\rho/2}\rho^{\ell} L_{n+\ell}^{2\ell+1}(\rho)$$

$$R_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} e^{-\frac{r}{na_0}} \left(\frac{2r}{na_0}\right)^{\ell} L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right) \quad \text{(C.8)}$$

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LAMPIRAN D. SOLUSI AZIMUT ATOM HIDROGEN

Alasan penggunaan konstanta $-m^2$ karena pada konstanta m^2 didapatkan bahwa untuk posisi yang sama, nilainya akan berbeda.

Konstanta m^2

$$\frac{d^2\Phi}{d\varphi^2} = m^2\Phi$$

$$D^2\Phi = m^2\Phi$$

$$D = m$$

$$\frac{d}{d\varphi} = m$$

$$\frac{d\Phi}{d\varphi} = m\Phi$$

$$\frac{d\Phi}{\Phi} = m d\varphi$$

$$\int_{\Phi_0}^{\Phi} \frac{d\Phi}{\Phi} = m \int_0^{\varphi} d\varphi$$

$$\ln\frac{\Phi}{\Phi_0} = m\varphi$$

$$\Phi = \Phi_0 e^{\pm m\varphi} \qquad \text{(D.1)}$$

Dengan memasukkan sudut $\varphi = \frac{\pi}{6}$ ke solusi (D.1), sehingga

$$\Phi(\varphi) = \Phi_0 e^{\pm m\left(\frac{\pi}{6}\right)}$$

kemudian ketika sudut berputar sejauh 2π, maka

$$\Phi(\varphi + 2\pi) = \Phi_0 e^{\pm m\left(\frac{\pi}{6}+2\pi\right)}$$

Terlihat bahwa nilai dari $\Phi(\varphi) \neq \Phi(\varphi + 2\pi)$, sehingga konstanta m^2 tidak cocok sebagai solusi persamaan azimuth.

Konstanta $-m^2$

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi$$

$$D^2\Phi = -m^2\Phi$$

$$D = \pm im$$

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$$\frac{d\Phi}{\mathrm{d}\varphi} = \pm im\Phi$$

$$\frac{d\Phi}{\Phi} = \pm imd\varphi$$

$$\int_{\Phi_0}^{\Phi} \frac{d\Phi}{\Phi} = \pm im \int_0^{\varphi} d\varphi$$

$$ln\frac{\Phi}{\Phi_0} = \pm im\varphi$$

$$\Phi = \Phi_0 \mathrm{e}^{\pm \mathrm{im}\,\varphi}$$

$$\Phi = \Phi_0 \mathrm{e}^{\mathrm{im}\,\varphi} \pm \Phi_0 \mathrm{e}^{-\mathrm{im}\,\varphi}$$

Ambil operasi positif, sehingga

$$\Phi = \left(\Phi_0 \mathrm{e}^{\mathrm{im}\,\varphi} + \Phi_0 \mathrm{e}^{-\mathrm{im}\,\varphi}\right) \times \frac{2}{2}$$

$$\Phi = 2\Phi_0 \times \left(\frac{\mathrm{e}^{\mathrm{im}\,\varphi} + \mathrm{e}^{-\mathrm{im}\,\varphi}}{2}\right)$$

$$\Phi = 2\Phi_0 \cos(\mathrm{m}\varphi)$$

$$\Phi = A\cos(\mathrm{m}\varphi) \qquad \text{(D.2)}$$

Dengan memasukkan sudut $\varphi = \frac{\pi}{6}$ ke solusi (D.2), sehingga

$$\Phi(\varphi) = A\cos\mathrm{m}\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$$

kemudian ketika sudut berputar sejauh 2π, maka

$$\Phi(\varphi + 2\pi) = A\cos\mathrm{m}\left(\frac{\pi}{6} + 2\pi\right) = \frac{1}{2}\sqrt{3}$$

Tampak bahwa nilai yang didapat sama antara $\Phi(\varphi)$ dan $\Phi(\varphi + 2\pi)$, walaupun sudut telah berputar sejauh 2π sehingga konstanta $-m^2$ cocok untuk solusi azimut. Untuk menentukan besarnya Φ_0, maka solusinya harus dinormalisasi

$$\int_0^{2\pi} \Phi^* \,\Phi \mathrm{d}\varphi = 1$$

$$\int_0^{2\pi} \Phi_0 \mathrm{e}^{-\mathrm{im}\,\varphi}\, \Phi_0 \mathrm{e}^{\mathrm{im}\,\varphi} \mathrm{d}\varphi = 1$$

$$\int_0^{2\pi} \Phi_0^2 \,\mathrm{d}\varphi = 1$$

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$$\Phi_0^2[\varphi]_0^{2\pi} = 1$$

$$\Phi_0^2[2\pi] = 1$$

$$\Phi_0 = \frac{1}{\sqrt{2\pi}}$$

sehingga solusi azimut ternormalisasi adalah

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \qquad \text{(D.3)}$$

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LAMPIRAN E. PEMBUKTIAN FUNGSI GELOMBANG ATOM HIDROGEN

Berikut ini akan dijabarkan mengenai pembuktian fungsi gelombang atom hidrogen yang ada pada Tabel 3.2

1. Fungsi gelombang untuk n = 1

Fungsi Azimut

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

m = 0

$$\Phi_0(\varphi) = \frac{1}{\sqrt{2\pi}} e^{i.0.\varphi} = \frac{1}{\sqrt{2\pi}}$$

Fungsi Polar

$$\Theta_{\ell m}(\theta) = \epsilon\sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\left[\frac{1}{2^\ell \ell!}(1-\cos^2\theta)^{|m|/2}\frac{d^{\ \ell+|m|}}{d\cos^{\ \ell+|m|}\theta}(\cos^2\theta-1)^\ell\right]$$

l = 0

$$\Theta_{00}(\theta) = \sqrt{\frac{2.0+1}{2}\frac{(0-0)!}{(0+0)!}}\left[\frac{1}{2^0 0!}(1-\cos^2\theta)^{0/2}\frac{d^{\ 0+0}}{d\cos^{\ 0+0}\theta}(\cos^2\theta-1)^0\right]$$

$$= \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

Fungsi Radial

$$R_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} e^{-\frac{r}{na_0}}\left(\frac{2r}{na_0}\right)^\ell L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right)$$

$$L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right) = \frac{(-1)^{2\ell+1}(n+\ell)!}{(n-\ell-1)!} e^{\frac{2r}{na_0}} \frac{d^{n+\ell}}{d\left(\frac{2r}{na_0}\right)^{n+\ell}}\left[e^{-\frac{2r}{na_0}}\left(\frac{2r}{na_0}\right)^{n-\ell-1}\right]$$

n = 1

$$L_1^1\left(\frac{2r}{a_0}\right) = \frac{(-1)^{2.0+1}(1+0)!}{(1-0-1)!} e^{\frac{2r}{a_0}} \frac{d^{1+0}}{d\left(\frac{2r}{a_0}\right)^{1+0}}\left[e^{-\frac{2r}{a_0}}\left(\frac{2r}{a_0}\right)^{1-0-1}\right]$$

$$= (-1)e^{\frac{2r}{a_0}} \frac{d}{d\left(\frac{2r}{a_0}\right)} e^{-\frac{2r}{a_0}}$$

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$$L_1^1\left(\frac{2r}{a_0}\right) = -e^{\frac{2r}{a_0}}\left(e^{-\frac{2r}{a_0}}\right) = 1$$

$$R_{10} = \left[\left(\frac{2}{a_0}\right)^3 \frac{(1-0-1)!}{2[(1+1)!]^3}\right]^{1/2} e^{-\frac{r}{a_0}}\left(\frac{2r}{a_0}\right)^0 L_1^1\left(\frac{2r}{a_0}\right)$$

$$= \left[\left(\frac{2}{a_0}\right)^3 \frac{1}{2}\right]^{1/2} e^{-\frac{r}{a_0}}(1)$$

$$= \left[\frac{8}{a_0^3}\frac{1}{2}\right]^{1/2} e^{-\frac{r}{a_0}}$$

$$= \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

2. Fungsi gelombang untuk $n = 2$

Fungsi Azimut

$$\Phi_\mathrm{m}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

m = 0

$$\Phi_0(\varphi) = \frac{1}{\sqrt{2\pi}} e^{i.0.\varphi} = \frac{1}{\sqrt{2\pi}}$$

m = ±1

$$\Phi_{\pm 1}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{\pm i\varphi}$$

Fungsi Polar

$$\Theta_{\ell m}(\theta) = \epsilon\sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\left[\frac{1}{2^\ell \ell!}(1-\cos^2\theta)^{|m|/2}\frac{d^{\ \ell+|m|}}{d\cos^{\ \ell+|m|}\theta}(\cos^2\theta-1)^\ell\right]$$

$l = 0$ dan $m = 0$

$$\Theta_{00}(\theta) = \sqrt{\frac{2.0+1}{2}\frac{(0-0)!}{(0+0)!}}\left[\frac{1}{2^0 0!}(1-\cos^2\theta)^{0/2}\frac{d^{\ 0+0}}{d\cos^{\ 0+0}\theta}(\cos^2\theta-1)^0\right]$$

$$= \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

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l = 1 dan *m* = 0

$$\Theta_{10}(\theta) = \sqrt{\frac{2.1+1}{2}\frac{(1-0)!}{(1+0)!}}\left[\frac{1}{2^1 1!}(1-\cos^2\theta)^{0/2}\frac{d^{\,1+0}}{d\,cos^{\,1+0}\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{2.1+1}{2}\frac{(1-0)!}{(1+0)!}}\left[\frac{1}{2^1 1!}(1-\cos^2\theta)^{0/2}\frac{d^{\,1+0}}{d\,cos^{\,1+0}\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{3}{2}}\left[\frac{1}{2}\times 1\times\frac{d}{d\cos\theta}(\cos^2\theta-1)\right]$$

$$= \sqrt{\frac{3}{2}}\left[\frac{1}{2}\times 1\times 2\cos\theta\right]$$

$$= \sqrt{\frac{3}{2}}\cos\theta$$

l = 1 dan *m* = 1

$$\Theta_{11}(\theta) = (-1)^1\sqrt{\frac{2.1+1}{2}\frac{(1-1)!}{(1+1)!}}\left[\frac{1}{2^1 1!}(1-\cos^2\theta)^{1/2}\frac{d^{\,1+1}}{d\,cos^{\,1+1}\theta}(\cos^2\theta-1)^1\right]$$

$$= -\sqrt{\frac{3}{2}\frac{1}{2}}\left[\frac{1}{2^1 1!}(1-\cos^2\theta)^{1/2}\frac{d^{\,2}}{d\,cos^{\,2}\theta}(\cos^2\theta-1)^1\right]$$

$$= -\sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta-1)^1\right]$$

$$= -\sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}2\cos\theta\right]$$

$$= -\sqrt{\frac{3}{4}}\left[(1-\cos^2\theta)^{1/2}\right]$$

$$= -\sqrt{\frac{3}{4}}\left[(\sin^2\theta)^{1/2}\right]$$

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$$\Theta_{11}(\theta) = -\frac{\sqrt{3}}{2}\sin\theta$$

l = 1 dan *m* = -1

$$\Theta_{1-1}(\theta) = \epsilon\sqrt{\frac{2.1+1}{2}\frac{(1-|-1|)!}{(1+|-1|)!}}\left[\frac{1}{2^1 1!}(1 - \cos^2\theta)^{|-1|/2}\frac{d^{\,1+|-1|}}{d\,cos^{\,1+|-1|}\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{3}{2}\frac{1}{2}}\left[\frac{1}{2^1 1!}(1-\cos^2\theta)^{1/2}\frac{d^{\,2}}{d\,cos^{\,2}\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}2\cos\theta\right]$$

$$= \sqrt{\frac{3}{4}}\left[(1-\cos^2\theta)^{1/2}\right]$$

$$= \sqrt{\frac{3}{4}}\left[(\sin^2\theta)^{1/2}\right]$$

$$\Theta_{1-1}(\theta) = \frac{\sqrt{3}}{2}\sin\theta$$

Fungsi Radial

$$R_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} e^{-\frac{r}{na_0}}\left(\frac{2r}{na_0}\right)^{\ell} L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right)$$

$$L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right) = \frac{(-1)^{2\ell+1}(n+\ell)!}{(n-\ell-1)!}e^{\frac{2r}{na_0}}\frac{d^{n+\ell}}{d\left(\frac{2r}{na_0}\right)^{n+\ell}}\left[e^{-\frac{2r}{na_0}}\left(\frac{2r}{na_0}\right)^{n-\ell-1}\right]$$

n = 2 dan *l* = 0

$$L_2^1\left(\frac{r}{a_0}\right) = \frac{(-1)^1(2+0)!}{(2-0-1)!}e^{\frac{r}{a_0}}\frac{d^{2+0}}{d\left(\frac{r}{a_0}\right)^{2+0}}\left[e^{-\frac{r}{a_0}}\left(\frac{r}{a_0}\right)^{2-0-1}\right]$$

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$$L_2^1\left(\frac{r}{a_0}\right) = -2e^{\frac{r}{a_0}}\frac{d}{d\left(\frac{r}{a_0}\right)}\frac{d}{d\left(\frac{r}{a_0}\right)}\left[e^{-\frac{r}{a_0}}\left(\frac{r}{a_0}\right)^1\right]$$

$$= -2e^{\frac{r}{a_0}}\frac{d}{d\left(\frac{r}{a_0}\right)}\left[e^{-\frac{r}{a_0}} - e^{-\frac{r}{a_0}}\left(\frac{r}{a_0}\right)\right]$$

$$= -2e^{\frac{r}{a_0}}\left[-e^{-\frac{r}{a_0}} - e^{-\frac{r}{a_0}} - e^{-\frac{r}{a_0}}\left(\frac{r}{a_0}\right)\right]$$

$$= -2e^{\frac{r}{a_0}}\left[-2e^{-\frac{r}{a_0}} - e^{-\frac{r}{a_0}}\left(\frac{r}{a_0}\right)\right]$$

$$= -2e^{\frac{r}{a_0}}\left[-2 + \left(\frac{r}{a_0}\right)\right]e^{-\frac{r}{a_0}}$$

$$= 2\left[2 - \left(\frac{r}{a_0}\right)\right]$$

$$R_{20} = \left[\left(\frac{1}{a_0}\right)^3 \frac{(2-0-1)!}{2.2[(2+0)!]^3}\right]^{1/2} e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)^0 L_2^1\left(\frac{r}{a_0}\right)$$

$$= \left[\left(\frac{1}{a_0}\right)^3 \frac{1}{4\times 2^3}\right]^{1/2} e^{-\frac{r}{2a_0}} \times 2\left[2 - \left(\frac{r}{a_0}\right)\right]$$

$$= \frac{1}{2}2\left[\left(\frac{1}{a_0}\right)^3 \frac{1}{2^3}\right]^{1/2} e^{-\frac{r}{2a_0}} \times 2\left[2 - \left(\frac{r}{a_0}\right)\right]$$

$$= \frac{1}{(2a_0)^{3/2}}\left[2 - \left(\frac{r}{a_0}\right)\right]e^{-\frac{r}{2a_0}}$$

n = 2 dan l = 1

$$L_3^3\left(\frac{r}{a_0}\right) = \frac{(-1)^{2.1+1}(2+1)!}{(2-1-1)!}e^{\frac{r}{a_0}}\frac{d^{2+1}}{d\left(\frac{r}{a_0}\right)^{2+1}}\left[e^{-\frac{r}{a_0}}\left(\frac{r}{a_0}\right)^{2-1-1}\right]$$

$$= \frac{(-1)^3(3)!}{(0)!}e^{\frac{r}{a_0}}\frac{d^3}{d\left(\frac{r}{a_0}\right)^3}\left[e^{-\frac{r}{a_0}}\left(\frac{r}{a_0}\right)^0\right]$$

$$= -6e^{\frac{r}{a_0}}\frac{d}{d\left(\frac{r}{a_0}\right)}\frac{d}{d\left(\frac{r}{a_0}\right)}\frac{d}{d\left(\frac{r}{a_0}\right)}e^{-\frac{r}{a_0}}$$

$$L_3^3\left(\frac{r}{a_0}\right) = 6e^{\frac{r}{a_0}}e^{-\frac{r}{a_0}} = 6$$

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$$R_{21}=\left[\left(\frac{1}{a_0}\right)^3\frac{(2-1-1)!}{2.2[(2+1)!]^3}\right]^{1/2}e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)^1L_3^3\left(\frac{r}{a_0}\right)$$

$$=\left[\left(\frac{1}{a_0}\right)^3\frac{(0)!}{2.2[(3)!]^3}\right]^{1/2}e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)6$$

$$=\left[\left(\frac{1}{a_0}\right)^3\frac{1}{4[6]^3}6^2\right]^{1/2}e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)$$

$$=\left[\left(\frac{1}{a_0}\right)^3\frac{1}{4.6}\right]^{1/2}e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)$$

$$=\left[\left(\frac{1}{a_0}\right)^3\frac{1}{2.2.2.3}\right]^{1/2}e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)$$

$$=\left[\left(\frac{1}{a_0}\right)^3\frac{1}{2^3 3}\right]^{1/2}e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)$$

$$=\left[\left(\frac{1}{2a_0}\right)^3\frac{1}{3}\right]^{1/2}e^{-\frac{r}{2a_0}}\left(\frac{r}{a_0}\right)$$

$$=\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$$

3. Fungsi gelombang untuk n = 3

Fungsi Azimut

$$\Phi_m(\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}$$

m = 0

$$\Phi_0(\varphi)=\frac{1}{\sqrt{2\pi}}e^{i.0.\varphi}=\frac{1}{\sqrt{2\pi}}$$

m = ±1

$$\Phi_{\pm1}(\varphi)=\frac{1}{\sqrt{2\pi}}e^{\pm i\varphi}$$

m = ±2

$$\Phi_{\pm2}(\varphi)=\frac{1}{\sqrt{2\pi}}e^{\pm i2\varphi}$$

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Fungsi Polar

$$\Theta_{\ell m}(\theta) = \epsilon\sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\left[\frac{1}{2^{\ell}\ell!}(1-\cos^2\theta)^{|m|/2}\frac{d^{\ \ell+|m|}}{d\ cos^{\ \ell+|m|}\theta}(\cos^2\theta-1)^{\ell}\right]$$

l = 0 dan *m* = 0

$$\Theta_{00}(\theta) = \epsilon\sqrt{\frac{2.0+1}{2}\frac{(0-0)!}{(0+0)!}}\left[\frac{1}{2^{0}0!}(1-\cos^2\theta)^{0/2}\frac{d^{\ 0+0}}{d\ cos^{\ 0+0}\theta}(\cos^2\theta-1)^{0}\right]$$

$$\Theta_{00}(\theta) = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

l = 1 dan *m* = 0

$$\Theta_{10}(\theta) = \sqrt{\frac{2.1+1}{2}\frac{(1-0)!}{(1+0)!}}\left[\frac{1}{2^{1}1!}(1-\cos^2\theta)^{0/2}\frac{d^{\ 1+0}}{d\ cos^{\ 1+0}\theta}(\cos^2\theta-1)^{1}\right]$$

$$= \sqrt{\frac{2.1+1}{2}\frac{(1-0)!}{(1+0)!}}\left[\frac{1}{2^{1}1!}(1-\cos^2\theta)^{0/2}\frac{d^{\ 1+0}}{d\ cos^{\ 1+0}\theta}(\cos^2\theta-1)^{1}\right]$$

$$= \sqrt{\frac{3}{2}}\left[\frac{1}{2}\times 1\times\frac{d}{d\cos\theta}(\cos^2\theta-1)\right]$$

$$= \sqrt{\frac{3}{2}}\left[\frac{1}{2}\times 1\times 2\cos\theta\right]$$

$$= \sqrt{\frac{3}{2}}\cos\theta$$

l = 1 dan *m* = 1

$$\Theta_{11}(\theta) = (-1)^1\sqrt{\frac{2.1+1}{2}\frac{(1-1)!}{(1+1)!}}\left[\frac{1}{2^{1}1!}(1-\cos^2\theta)^{1/2}\frac{d^{\ 1+1}}{d\ cos^{\ 1+1}\theta}(\cos^2\theta - 1)^{1}\right]$$

$$= -\sqrt{\frac{3}{2}\frac{1}{2}}\left[\frac{1}{2^{1}1!}(1-\cos^2\theta)^{1/2}\frac{d^{\ 2}}{d\ cos^{\ 2}\theta}(\cos^2\theta-1)^{1}\right]$$

$$= -\sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta-1)^{1}\right]$$

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$$= -\sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}2\cos\theta\right]$$

$$= -\sqrt{\frac{3}{4}}\left[(1-\cos^2\theta)^{1/2}\right]$$

$$= -\sqrt{\frac{3}{4}}\left[(\sin^2\theta)^{1/2}\right]$$

$$\Theta_{11}(\theta) = -\frac{\sqrt{3}}{2}\sin\theta$$

l = 1 dan *m* = -1

$$\Theta_{1-1}(\theta) = \epsilon\sqrt{\frac{2.1+1}{2}\frac{(1-|-1|)!}{(1+|-1|)!}}\left[\frac{1}{2^1 1!}(1\right.$$

$$\left. -\cos^2\theta)^{|-1|/2}\frac{d^{\,1+|-1|}}{d\cos^{\,1+|-1|}\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{3}{2}\frac{1}{2}}\left[\frac{1}{2^1 1!}(1-\cos^2\theta)^{1/2}\frac{d^{\,2}}{d\cos^{\,2}\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta-1)^1\right]$$

$$= \sqrt{\frac{3}{4}}\left[\frac{1}{2}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}2\cos\theta\right]$$

$$= \sqrt{\frac{3}{4}}\left[(1-\cos^2\theta)^{1/2}\right]$$

$$= \sqrt{\frac{3}{4}}\left[(\sin^2\theta)^{1/2}\right]$$

$$\Theta_{1-1}(\theta) = \frac{\sqrt{3}}{2}\sin\theta$$

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l = 2 dan m = 0

$$\Theta_{20}(\theta) = \sqrt{\frac{2.2+1}{2}\frac{(2-0)!}{(2+0)!}}\left[\frac{1}{2^2 2!}(1-\cos^2\theta)^{0/2}\frac{d^{\,2+0}}{d\,cos^{\;2+0}\theta}(\cos^2\theta-1)^2\right]$$

$$= \sqrt{\frac{5}{2}}\left[\frac{1}{8}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}\;(\cos^2\theta-1)^2\right]$$

$$= \sqrt{\frac{5}{2}}\left[\frac{1}{8}\frac{d}{d\cos\theta}4\cos\theta\,(\cos^2\theta-1)\right]$$

$$= \sqrt{\frac{5}{2}}\left[\frac{1}{2}(3\cos^2\theta-1)\right]$$

$$\Theta_{20}(\theta) = \sqrt{\frac{5}{8}}[(3\cos^2\theta-1)\,]$$

l = 2 dan m = 1

$$\Theta_{21}(\theta) = (-1)^1\sqrt{\frac{2.2+1}{2}\frac{(2-1)!}{(2+1)!}}\left[\frac{1}{2^2 2!}(1-\cos^2\theta)^{1/2}\frac{d^{\,2+1}}{d\,cos^{\;2+1}\theta}(\cos^2\theta\right.$$
$$\left.-1)^2\right]$$

$$= -\sqrt{\frac{5}{2}\frac{1}{6}}\left[\frac{1}{8}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta-1)^2\right]$$

$$= -\sqrt{\frac{5}{2}\frac{1}{2.3}}\left[\frac{1}{8}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\left(\frac{d^{\,2}}{dcos^{\;2}\theta}(\cos^2\theta-1)^2\right)\right]$$

$$= -\frac{1}{2}\sqrt{\frac{5}{3}}\left[\frac{1}{8}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}(12\cos^2\theta-1)\right]$$

$$= -\frac{1}{2}\sqrt{\frac{5}{3}}\,12\frac{1}{8}\left[(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}(\cos^2\theta-1)\right]$$

$$= -\frac{6}{8}\sqrt{\frac{5}{3}}\left[(1-\cos^2\theta)^{1/2}(2\cos\theta)\right]$$

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$$= -\frac{6}{4}\sqrt{\frac{5}{3}}(\sin^2\theta)^{1/2}\cos\theta$$

$$= -\frac{3}{2}\sqrt{\frac{5}{3}}\sin\theta\cos\theta$$

$$= -\sqrt{\frac{5}{3}\frac{9}{4}}\sin\theta\cos\theta$$

$$= -\sqrt{5\frac{3}{4}}\sin\theta\cos\theta$$

$$\Theta_{21}(\theta) = -\sqrt{\frac{15}{4}}\sin\theta\cos\theta$$

l = 2 dan m = -1

$$\Theta_{2-1}(\theta) = \sqrt{\frac{2.2+1}{2}\frac{(2-|-1|)!}{(2+|-1|)!}}\left[\frac{1}{2^2 2!}(1-\cos^2\theta)^{|-1|/2}\frac{d^{\,2+|-1|}}{d\cos^{\,2+|-1|}\theta}(\cos^2\theta - 1)^2\right]$$

$$= \sqrt{\frac{5}{2}\frac{1}{6}}\left[\frac{1}{8}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta - 1)^2\right]$$

$$= \sqrt{\frac{5}{2}\frac{1}{2.3}}\left[\frac{1}{8}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}\left(\frac{d^{\,2}}{dcos^{\,2}\theta}(\cos^2\theta - 1)^2\right)\right]$$

$$= \frac{1}{2}\sqrt{\frac{5}{3}}\left[\frac{1}{8}(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}(12\cos^2\theta - 1)\right]$$

$$= \frac{1}{2}\sqrt{\frac{5}{3}}\,12\frac{1}{8}\left[(1-\cos^2\theta)^{1/2}\frac{d}{d\cos\theta}(\cos^2\theta - 1)\right]$$

$$= \frac{6}{8}\sqrt{\frac{5}{3}}\left[(1-\cos^2\theta)^{1/2}(2\cos\theta)\right]$$

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$$= \frac{6}{4}\sqrt{\frac{5}{3}}(\sin^2\theta)^{1/2}\cos\theta$$

$$= \frac{3}{2}\sqrt{\frac{5}{3}}\sin\theta\cos\theta$$

$$= \sqrt{\frac{5}{3}\frac{9}{4}}\sin\theta\cos\theta$$

$$= \sqrt{5\frac{3}{4}}\sin\theta\cos\theta$$

$$\Theta_{2-1}(\theta) = \sqrt{\frac{15}{4}}\sin\theta\cos\theta$$

l = 2 dan m = 2

$$\Theta_{22}(\theta) = (-1)^2\sqrt{\frac{2.2+1}{2}\frac{(2-2)!}{(2+2)!}}\left[\frac{1}{2^2 2!}(1-\cos^2\theta)^{2/2}\frac{d^{\,2+2}}{dcos^{2+2}\theta}(\cos^2\theta - 1)^2\right]$$

$$= \sqrt{\frac{5}{2}\frac{1}{(4)!}}\left[\frac{1}{8}(1-\cos^2\theta)\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta-1)^2\right]$$

$$= \frac{1}{8}\sqrt{\frac{5}{2}\frac{1}{4.3.2}}\left[(1-\cos^2\theta)\frac{d}{d\cos\theta}\left(\frac{d^3}{dcos^3\theta}(\cos^2\theta-1)^2\right)\right]$$

$$= \frac{1}{8}\frac{1}{4}\sqrt{\frac{5}{3}}\left[(1-\cos^2\theta)\frac{d}{d\cos\theta}(24\cos\theta)\right]$$

$$= 24\frac{1}{8}\frac{1}{4}\sqrt{\frac{5}{3}}(\sin^2\theta)$$

$$= \frac{3}{4}\sqrt{\frac{5}{3}}(\sin^2\theta)$$

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$$= \frac{1}{4}\sqrt{\frac{5}{3}9}(\sin^2\theta)$$

$$= \frac{1}{4}\sqrt{15}(\sin^2\theta)$$

$$\Theta_{22}(\theta) = \frac{\sqrt{15}}{4}(\sin^2\theta)$$

l = 2 dan m = -2

$$\Theta_{2-2}(\theta) = \sqrt{\frac{2.2+1}{2}\frac{(2-|-2|)!}{(2+|-2|)!}}\left[\frac{1}{2^2 2!}(1-\cos^2\theta)^{|-2|/2}\frac{d^{\,2+|-2|}}{d\,cos^{\,2+|-2|}\theta}\,(\cos^2\theta - 1)^2\right]$$

$$= \sqrt{\frac{5}{2}\frac{1}{(4)!}}\left[\frac{1}{8}(1-\cos^2\theta)\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}\frac{d}{d\cos\theta}(\cos^2\theta - 1)^2\right]$$

$$= \frac{1}{8}\sqrt{\frac{5}{2}\frac{1}{4.3.2}}\left[(1-\cos^2\theta)\frac{d}{d\cos\theta}\left(\frac{d^3}{dcos^3\theta}(\cos^2\theta - 1)^2\right)\right]$$

$$= \frac{1}{8}\frac{1}{4}\sqrt{\frac{5}{3}}\left[(1-\cos^2\theta)\frac{d}{d\cos\theta}(24\cos\theta)\right]$$

$$= 24\frac{1}{8}\frac{1}{4}\sqrt{\frac{5}{3}}(\sin^2\theta)$$

$$= \frac{3}{4}\sqrt{\frac{5}{3}}(\sin^2\theta)$$

$$= \frac{1}{4}\sqrt{\frac{5}{3}9}(\sin^2\theta)$$

$$= \frac{1}{4}\sqrt{15}(\sin^2\theta)$$

$$\Theta_{2-2}(\theta) = \frac{\sqrt{15}}{4}(\sin^2\theta)$$

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Fungsi Radial

$$R_{n\ell} = \left[\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}\right]^{1/2} e^{-\frac{r}{na_0}} \left(\frac{2r}{na_0}\right)^{\ell} L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right)$$

$$L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right) = \frac{(-1)^{2\ell+1}(n+\ell)!}{(n-\ell-1)!} e^{\frac{2r}{na_0}} \frac{d^{n+\ell}}{d\left(\frac{2r}{na_0}\right)^{n+\ell}}\left[e^{-\frac{2r}{na_0}}\left(\frac{2r}{na_0}\right)^{n-\ell-1}\right]$$

n = 3 dan l = 0

$$L_3^1\left(\frac{2r}{3a_0}\right) = \frac{(-1)^{2.0+1}(3+0)!}{(3-0-1)!} e^{\frac{2r}{3a_0}} \frac{d^{3+0}}{d\left(\frac{2r}{3a_0}\right)^{3+0}}\left[e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^{3-0-1}\right]$$

$$= \frac{(-1)^1 3\times 2!}{2!} e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)} \frac{d}{d\left(\frac{2r}{3a_0}\right)} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^2\right]$$

$$L_3^1\left(\frac{2r}{3a_0}\right) = -3e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[2\left(\frac{2r}{3a_0}\right)e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^2\right]$$

$$= -3e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[2e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}}2\left(\frac{2r}{3a_0}\right) + e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^2 - 2\left(\frac{2r}{3a_0}\right)e^{-\frac{2r}{3a_0}}\right]$$

$$= -3e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[2e^{-\frac{2r}{3a_0}} - 4\left(\frac{2r}{3a_0}\right)e^{-\frac{2r}{3a_0}} + \left(\frac{2r}{3a_0}\right)^2 e^{-\frac{2r}{3a_0}}\right]$$

$$= -3e^{\frac{2r}{3a_0}}\left[-2e^{-\frac{2r}{3a_0}} - 4e^{-\frac{2r}{3a_0}} + 4\left(\frac{2r}{3a_0}\right)e^{-\frac{2r}{3a_0}} + 2\left(\frac{2r}{3a_0}\right)e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^2\right]$$

$$= -3e^{\frac{2r}{3a_0}}\left[-2-4+4\left(\frac{2r}{3a_0}\right) + 2\left(\frac{2r}{3a_0}\right) - \left(\frac{2r}{3a_0}\right)^2\right]e^{-\frac{2r}{3a_0}}$$

$$= -3\left[-6+6\left(\frac{2r}{3a_0}\right) - \left(\frac{2r}{3a_0}\right)^2\right]$$

$$R_{30} = \left[\left(\frac{2}{3a_0}\right)^3 \frac{(3-0-1)!}{2.3[(3+0)!]^3}\right]^{1/2} e^{-\frac{r}{3a_0}} \left(\frac{2r}{3a_0}\right)^0 L_3^1\left(\frac{2r}{3a_0}\right)$$

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$$= \left[\left(\frac{2}{3a_0}\right)^3 \frac{2}{2.3.3^3.2^3}\right]^{1/2} e^{-\frac{r}{3a_0}} \times -3\left[-6+6\left(\frac{2r}{3a_0}\right)-\left(\frac{2r}{3a_0}\right)^2\right]$$

$$= \frac{1}{9}\left[\left(\frac{1}{3a_0}\right)^3\right]^{1/2} e^{-\frac{r}{3a_0}} \times -3\left[-6+\left(\frac{4r}{a_0}\right)-\left(\frac{4r^2}{9a_0^2}\right)\right]$$

$$= \frac{1}{3}\left[\left(\frac{1}{3a_0}\right)^3\right]^{1/2} e^{-\frac{r}{3a_0}} \times -6\left[-1+\left(\frac{4r}{6a_0}\right)-\left(\frac{4r^2}{54a_0^2}\right)\right]$$

$$= 2\left[\left(\frac{1}{3a_0}\right)^3\right]^{1/2} e^{-\frac{r}{3a_0}}\left[1-\left(\frac{4r}{6a_0}\right)+\left(\frac{4r^2}{54a_0^2}\right)\right]$$

$$R_{30} = \frac{2}{(3a_0)^{3/2}}\left(1-\frac{2r}{3a_0}+\frac{2r^2}{27a_0^2}\right)e^{-r/3a_0}$$

n = 3 dan l = 1

$$L_4^3\left(\frac{2r}{3a_0}\right) = \frac{(-1)^{2.1+1}(3+1)!}{(3-1-1)!} e^{\frac{2r}{3a_0}} \frac{d^{3+1}}{d\left(\frac{2r}{3a_0}\right)^{3+1}}\left[e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^{3-1-1}\right]$$

$$= -24e^{\frac{2r}{3a_0}} \frac{d^4}{d\left(\frac{2r}{3a_0}\right)^4}\left[e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^1\right]$$

$$= -24e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\frac{d}{d\left(\frac{2r}{3a_0}\right)}\frac{d}{d\left(\frac{2r}{3a_0}\right)}\frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)\right]$$

$$= -24e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\frac{d}{d\left(\frac{2r}{3a_0}\right)}\frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)\right]$$

$$= -24e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[-e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}} + e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)\right]$$

$$= -24e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[-2e^{-\frac{2r}{3a_0}} + e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)\right]$$

$$= -24e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[2e^{-\frac{2r}{3a_0}} + e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)\right]$$

$$= -24e^{\frac{2r}{3a_0}} \frac{d}{d\left(\frac{2r}{3a_0}\right)}\left[3e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)\right]$$

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$$= -24e^{\frac{2r}{3a_0}}\left[-3e^{-\frac{2r}{3a_0}} - e^{-\frac{2r}{3a_0}} + e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)\right]$$

$$= -24e^{\frac{2r}{3a_0}}\left[-4 + \left(\frac{2r}{3a_0}\right)\right]e^{-\frac{2r}{3a_0}}$$

$$L_4^3\left(\frac{2r}{3a_0}\right) = 24\left[4 - \left(\frac{2r}{3a_0}\right)\right]$$

$$R_{31} = \left[\left(\frac{2}{3a_0}\right)^3 \frac{(3-1-1)!}{2.3[(3+1)!]^3}\right]^{1/2} e^{-\frac{r}{3a_0}}\left(\frac{2r}{3a_0}\right)^1 L_4^3\left(\frac{2r}{3a_0}\right)$$

$$= \left[\left(\frac{2}{3a_0}\right)^3 \frac{1}{2.3[(4)!]^3}\right]^{1/2} e^{-\frac{r}{3a_0}}\left(\frac{2r}{3a_0}\right) 24\left[4 - \left(\frac{2r}{3a_0}\right)\right]$$

$$= \left[\left(\frac{2}{3a_0}\right)^3 \frac{1}{2\times 3\times 4^3\times 3^3\times 2^3\times 1^3}\right]^{1/2} e^{-\frac{r}{3a_0}}\left(\frac{2r}{3a_0}\right) 24\left[4 - \left(\frac{2r}{3a_0}\right)\right]$$

$$= \frac{24}{3^2}\left[\left(\frac{1}{3a_0}\right)^3 \frac{1}{2\times 4^2\times 4}\right]^{1/2}\left(\frac{2r}{3a_0}\right)\left[4 - \left(\frac{2r}{3a_0}\right)\right]e^{-\frac{r}{3a_0}}$$

$$= \frac{24}{3^2}\frac{1}{4}\frac{1}{2}\left[\left(\frac{1}{3a_0}\right)^3 \frac{1}{2}\right]^{1/2}\left(\frac{8r}{3a_0} - \frac{4r^2}{9a_0^2}\right)e^{-\frac{r}{3a_0}}$$

$$= \frac{3}{9}\left[\left(\frac{1}{3a_0}\right)^3 \frac{1}{2}\right]^{1/2}\frac{8}{3}\left(\frac{r}{a_0} - \frac{\frac{1}{2}r^2}{3a_0^2}\right)e^{-\frac{r}{3a_0}}$$

$$= \frac{8}{9}\left[\left(\frac{1}{3a_0}\right)^3 \frac{1}{2}\right]^{1/2}\left(\frac{r}{a_0} - \frac{r^2}{6a_0^2}\right)e^{-\frac{r}{3a_0}}$$

$$R_{31} = \frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0} - \frac{r^2}{6a_0^2}\right)e^{-r/3a_0}$$

n = 3 dan l = 2

$$L_5^5\left(\frac{2r}{3a_0}\right) = \frac{(-1)^{2.2+1}(3+2)!}{(3-2-1)!}e^{\frac{2r}{3a_0}}\frac{d^{3+2}}{d\left(\frac{2r}{3a_0}\right)^{3+2}}\left[e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^{3-2-1}\right]$$

$$= \frac{(-1)^5(5)!}{(0)!}e^{\frac{2r}{3a_0}}\frac{d^5}{d\left(\frac{2r}{3a_0}\right)^5}\left[e^{-\frac{2r}{3a_0}}\left(\frac{2r}{3a_0}\right)^0\right]$$

$$= -\frac{5.4.3.2.1}{1}e^{\frac{2r}{3a_0}}\frac{d^5}{d\left(\frac{2r}{3a_0}\right)^5}\left[e^{-\frac{2r}{3a_0}}\right]$$

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$$= -120e^{\frac{2r}{3a_0}} \times -e^{-\frac{2r}{3a_0}}$$

$$L_5^5\left(\frac{2r}{3a_0}\right) = 120$$

$$R_{32} = \left[\left(\frac{2}{3a_0}\right)^3 \frac{(3-2-1)!}{2.3[(3+2)!]^3}\right]^{1/2} e^{-\frac{r}{3A_0}} \left(\frac{2r}{3a_0}\right)^{\ell} L_5^5\left(\frac{2r}{3a_0}\right)$$

$$= \left[\left(\frac{2}{3a_0}\right)^3 \frac{(0)!}{2.3[(5)!]^3}\right]^{1/2} e^{-\frac{r}{3A_0}} \left(\frac{2r}{3a_0}\right)^2 \times 120$$

$$= 120\left[\left(\frac{2}{3a_0}\right)^3 \frac{1}{2 \times 3 \times 5^3 \times 4^3 \times 3^3 \times 2^3 \times 1^3}\right]^{1/2} e^{-\frac{r}{3A_0}} \left(\frac{2r}{3a_0}\right)^2$$

$$= \frac{120}{5 \times 3^2 \times 4 \times 2}\left[\left(\frac{1}{3a_0}\right)^3 \frac{1}{2 \times 5}\right]^{1/2} \frac{4r^2}{9a_0^2} e^{-\frac{r}{3A_0}}$$

$$= \frac{120}{360}\frac{4}{9}\left[\left(\frac{1}{3a_0}\right)^3 \frac{1}{10}\right]^{1/2} \frac{r^2}{a_0^2} e^{-\frac{r}{3A_0}}$$

$$= \frac{1}{3}\frac{4}{9}\left[\left(\frac{1}{3a_0}\right)^3 \frac{1}{10}\right]^{1/2} \frac{r^2}{a_0^2} e^{-\frac{r}{3A_0}}$$

$$R_{32} = \frac{4}{27\sqrt{10}(3a_0)^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$$

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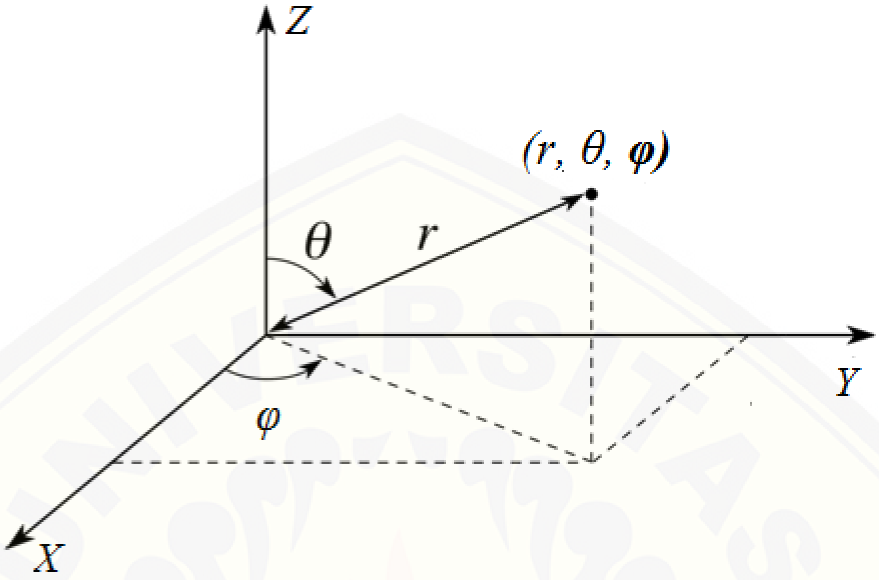
LAMPIRAN F. OPERATOR MOMENTUM SUDUT DALAM KOORDINAT BOLA



Gambar 1. Koordinat Bola (Sumber: http://www.met.reading.ac.uk/pplato2/h-flap/phys11_3.html).

Transformasi koordinat kartesian ke bola adalah

$$x = rsin\theta cos\varphi \quad \text{(G.1)}$$

$$y = rsin\theta sin\varphi \quad \text{(G.2)}$$

$$z = rcos\theta \quad \text{(G.3)}$$

Jika dikembalikan persamaan (G.1), (G.2), dan (G.3) ke koordinat kartesian, maka akan menjadi

$$r = (x^2 + y^2 + z^2)^{1/2} \quad \text{(G.4)}$$

$$\cos\theta = \left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right) \quad \text{(G.5)}$$

$$\tan\varphi = \left(\frac{y}{x}\right) \quad \text{(G.6)}$$

Transformasi koordinat dari koordinat kartesian ke koordinat bola dapat dilakukan dengan metode diferensial total. Jika terdapat fungsi ψ yang bergantung pada r, $cos\ \theta$, $tan\ \varphi$, maka bentuk diferensial totalnya menjadi

$$d\psi = \frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial\cos\theta}dcos\theta + \frac{\partial\psi}{\partial tan\varphi}dtan\varphi \quad \text{(G.7)}$$

Persamaan (G.4) dapat dituliskan sebagai berikut jika masing-masing diturunkan terhadap x, y, z

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$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial\psi}{\partial\cos\theta}\frac{\partial cos\theta}{\partial x}+\frac{\partial\psi}{\partial tan\varphi}\frac{\partial tan\varphi}{\partial x} \quad (G.8)$$

$$\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial\psi}{\partial\cos\theta}\frac{\partial\cos\theta}{\partial y}+\frac{\partial\psi}{\partial tan\varphi}\frac{\partial tan\varphi}{\partial y} \quad (G.9)$$

$$\frac{\partial\psi}{\partial z}=\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial\psi}{\partial\cos\theta}\frac{\partial\cos\theta}{\partial z}+\frac{\partial\psi}{\partial tan\varphi}\frac{\partial tan\varphi}{\partial z} \quad (G.10)$$

Persamaan (G.8), (G.9), dan (G.10) merupakan dasar untuk merubah operator momentum sudut pada koordinat kartesian ke koordinat bola

1. Transformasi $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, dan $\frac{\partial}{\partial z}$ ke koordinat bola

1.1 Transformasi $\frac{\partial}{\partial x}$

Dengan menggunakan persamaan (G.8) kemudian fungsi ψ berada diluar ruas perkalian sehingga

$$\frac{\partial}{\partial x}\psi=\left[\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial cos\theta}{\partial x}\frac{\partial}{\partial\cos\theta}+\frac{\partial tan\varphi}{\partial x}\frac{\partial}{\partial tan\varphi}\right]\psi \quad (G.11)$$

Untuk menghitung $\frac{\partial}{\partial\cos\theta}$ didapat dengan menurunkannya terhadap $\partial\theta$

$$\frac{\partial\cos\theta}{\partial\theta}=-\sin\theta$$

$$\partial\cos\theta=-\sin\theta\,\partial\theta$$

$$\frac{\partial}{\partial\cos\theta}=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \quad (G.12)$$

Hal serupa dapat digunakan untuk mencari $\frac{\partial}{\partial tan\varphi}$

$$\frac{\partial tan\varphi}{\partial\varphi}=\frac{1}{cos^2\varphi}$$

$$\partial tan\varphi=\frac{1}{cos^2\varphi}\partial\varphi$$

$$\frac{\partial}{\partial tan\varphi}=cos^2\varphi\frac{\partial}{\partial\varphi} \quad (G.13)$$

dengan memasukkan persamaan (G.12) dan (G.13) ke dalam persamaan (G.11), sehingga menjadi

$$\frac{\partial}{\partial x}\psi=\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}\frac{\partial}{\partial r}+\left(\frac{\partial}{\partial x}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)\times-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial x}\left(\frac{y}{x}\right)cos^2\varphi\frac{\partial}{\partial\varphi}\right]\psi$$

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$$= \left[\frac{x}{r}\frac{\partial}{\partial r} + \left(-\frac{xz}{r^3} \times -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) - \frac{y}{x^2} cos^2\varphi \frac{\partial}{\partial\varphi}\right]\psi$$

dengan memasukkan nilai *x*, *y*, dan *z* ke persamaan (G.1), (G.2), dan (G.3) didapatkan hasil sebagai

$$\frac{\partial}{\partial x}\psi = \left[\frac{rsin\theta cos\varphi}{r}\frac{\partial}{\partial r} + \frac{rsin\theta cos\varphi \times rcos\theta}{r^3}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{rsin\theta sin\varphi}{r^2 sin^2\theta cos^2\varphi} cos^2\varphi \frac{\partial}{\partial\varphi}\right]\psi$$

$$= \left[sin\theta cos\varphi \frac{\partial}{\partial r} + \frac{cos\varphi cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]\psi$$

$$\frac{\partial}{\partial x} = sin\theta cos\varphi \frac{\partial}{\partial r} + \frac{cos\varphi cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi} \qquad \text{(G.14)}$$

1.2 Transformasi $\frac{\partial}{\partial y}$ ke koordinat bola

Dengan menggunakan persamaan (G.9) kemudian fungsi ψ berada diluar ruas perkalian sehingga

$$\frac{\partial}{\partial y}\psi = \left[\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \cos\theta}{\partial y}\frac{\partial}{\partial\cos\theta} + \frac{\partial tan\varphi}{\partial y}\frac{\partial}{\partial tan\varphi}\right]\psi \qquad \text{(G.15)}$$

Substitusikan persamaan (G.12) dan (G.13) ke dalam persamaan (G.15), sehingga menjadi

$$\frac{\partial}{\partial y}\psi = \left[\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}\frac{\partial}{\partial r} + \left(\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right) \times -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x}\right) cos^2\varphi \frac{\partial}{\partial\varphi}\right]\psi$$

$$= \left[\frac{y}{r}\frac{\partial}{\partial r} + \left(-\frac{yz}{r^3} \times -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{1}{x} cos^2\varphi \frac{\partial}{\partial\varphi}\right]\psi$$

dengan memasukkan nilai *x*, *y*, dan *z* ke persamaan (G.1), (G.2), dan (G.3) didapatkan hasil sebagai

$$\frac{\partial}{\partial y}\psi = \left[\frac{rsin\theta sin\varphi}{r}\frac{\partial}{\partial r} + \frac{rsin\theta sin\varphi \times rcos\theta}{r^3}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{cos^2\varphi}{rsin\theta cos\varphi}\frac{\partial}{\partial\varphi}\right]\psi$$

$$= \left[sin\theta sin\varphi \frac{\partial}{\partial r} + \frac{sin\varphi cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]\psi$$

$$\frac{\partial}{\partial y} = sin\theta sin\varphi \frac{\partial}{\partial r} + \frac{sin\varphi cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi} \qquad \text{(G.16)}$$

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1.3 Transformasi $\frac{\partial}{\partial z}$ ke koordinat bola

Dengan menggunakan persamaan (G.10) kemudian fungsi ψ berada diluar ruas perkalian sehingga

$$\frac{\partial}{\partial Z}\psi = \left[\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \cos\theta}{\partial z}\frac{\partial}{\partial \cos\theta} + \frac{\partial tan\varphi}{\partial z}\frac{\partial}{\partial tan\varphi}\right]\psi \qquad \text{(G.17)}$$

Substitusikan persamaan (G.12) dan (G.13) ke dalam persamaan (G.17), sehingga menjadi

$$\frac{\partial}{\partial z}\psi = \left[\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}\frac{\partial}{\partial r} + \left(\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)\times -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right.$$

$$\left. + \frac{\partial}{\partial z}\left(\frac{y}{x}\right)cos^2\varphi\frac{\partial}{\partial\varphi}\right]\psi$$

$$= \left[\frac{z}{r}\frac{\partial}{\partial r} + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta} + 0\right]\psi$$

$$= \left[\frac{rcos\theta}{r}\frac{\partial}{\partial r} + \frac{1}{r}\left(1 - \left(\frac{z}{r}\right)^2\right)\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\right]\psi$$

Jika melihat pada persamaan (G.5), maka nilai $\left(\frac{z}{r}\right)^2 = cos^2\theta$ sehingga didapatkan hasil

$$\frac{\partial}{\partial z}\psi = \left[cos\theta\frac{\partial}{\partial r} + \frac{1}{r}(1 - cos^2\theta)\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\right]\psi$$

$$= \left[cos\theta\frac{\partial}{\partial r} + \frac{1}{r}(sin^2\theta)\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\right]\psi$$

$$= \left[cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right]\psi$$

$$\frac{\partial}{\partial z} = cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} \qquad \text{(G.18)}$$

2. Transformasi operator momentum sudut ke koordinat bola

Setelah mendapat transformasi $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, dan $\frac{\partial}{\partial z}$ pada persamaan (G.14), (G.16), dan (G.18) serta memasukkan nilai x, y, dan z pada persamaan (G.1), (G.2), dan (G.3). Langkah selanjutnya yaitu menentukan operator momentum sudut dalam koordinat bola dengan memanfaatkan ketiga persamaan tersebut.

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2.1 Transformasi $\hat{L}_x$

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$= -i\hbar\left[rsin\theta sin\varphi\left(cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\right.$$

$$\left. - rcos\theta\left(sin\theta sin\varphi\frac{\partial}{\partial r} + \frac{sin\varphi cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right)\right]$$

$$= -i\hbar\left[rsin\theta sin\varphi cos\theta\frac{\partial}{\partial r} - \frac{1}{r}rsin\theta sin\varphi\sin\theta\frac{\partial}{\partial\theta} - rcos\theta sin\theta sin\varphi\frac{\partial}{\partial r}\right.$$

$$\left. - rcos\theta\frac{sin\varphi cos\theta}{r}\frac{\partial}{\partial\theta} - rcos\theta\frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= -i\hbar\left[(sin\theta sin\varphi cos\theta - rcos\theta sin\theta sin\varphi)\frac{\partial}{\partial r}\right.$$

$$\left. - (sin^2\theta sin\varphi + cos^2\theta sin\varphi)\frac{\partial}{\partial\theta} - \frac{cos\theta cos\varphi}{sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= -i\hbar\left[-sin\varphi(sin^2\theta + cos^2\theta)\frac{\partial}{\partial\theta} - \frac{cos\theta cos\varphi}{sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= -i\hbar\left[-sin\varphi\frac{\partial}{\partial\theta} - \frac{cos\theta cos\varphi}{sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\hat{L}_x = i\hbar\left[sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right] \quad \text{(G.19)}$$

2.2 Transformasi $\hat{L}_y$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= -i\hbar\left[rcos\theta\left(sin\theta cos\varphi\frac{\partial}{\partial r} + \frac{cos\varphi cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left. - rsin\theta cos\varphi\left(cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\right]$$

$$= -i\hbar\left[rcos\theta sin\theta cos\varphi\frac{\partial}{\partial r} + rcos\theta\frac{cos\varphi cos\theta}{r}\frac{\partial}{\partial\theta} - rcos\theta\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right.$$

$$\left. - rsin\theta cos\varphi cos\theta\frac{\partial}{\partial r} + rsin\theta cos\varphi\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right]$$

$$= -i\hbar\left[(rcos\theta sin\theta cos\varphi - rsin\theta cos\varphi cos\theta)\frac{\partial}{\partial r}\right.$$

$$\left. + (cos^2\theta cos\varphi + sin^2\theta cos\varphi)\frac{\partial}{\partial\theta} - \frac{cos\theta sin\varphi}{sin\theta}\frac{\partial}{\partial\varphi}\right]$$

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$$= -i\hbar\left[\cos\varphi(\sin^2\theta+\cos^2\theta)\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\hat{L}_y = -i\hbar\left[\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right] \quad \text{(G.20)}$$

2.3 Transformasi $\hat{L}_z$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -i\hbar\left[r\sin\theta\cos\varphi\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\sin\varphi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right.$$
$$\left. - r\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]$$

$$= -i\hbar\left[r\sin\theta\cos\varphi\sin\theta\sin\varphi\frac{\partial}{\partial r} + r\sin\theta\cos\varphi\frac{\sin\varphi\cos\theta}{r}\frac{\partial}{\partial\theta}\right.$$
$$+ r\sin\theta\cos\varphi\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} - r\sin\theta\sin\varphi\sin\theta\cos\varphi\frac{\partial}{\partial r}$$
$$\left. - r\sin\theta\sin\varphi\frac{\cos\varphi\cos\theta}{r}\frac{\partial}{\partial\theta} + r\sin\theta\sin\varphi\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= -i\hbar\left[(r\sin^2\theta\cos\varphi\sin\varphi - r\sin^2\theta\sin\varphi\cos\varphi)\frac{\partial}{\partial r}\right.$$
$$+ \left(\sin\theta\cos\varphi\sin\varphi\cos\theta - \sin\theta\sin\varphi\cos\varphi\cos\theta\frac{\partial}{\partial\theta}\right.$$
$$\left. + (\cos^2\varphi + \sin^2\varphi)\frac{\partial}{\partial\varphi}\right)\Bigg]$$

$$= -i\hbar\left[0 + 0 + (\cos^2\varphi + \sin^2\varphi)\frac{\partial}{\partial\varphi}\right]$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi} \quad \text{(G.21)}$$

2.4 Transformasi $\hat{L}_+$

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$

dengan memasukkan $\hat{L}_x$ dan $\hat{L}_y$ yang didapat dari persamaan (G.19) dan (G.20), sehingga akan menjadi

$$\hat{L}_+ = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) + i \times -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$= i\hbar\left[\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) + i\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\right]$$

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$$= i\hbar\left[(sin\varphi - icos\varphi)\frac{\partial}{\partial\theta} + cot\theta(cos\varphi + isin\varphi)\frac{\partial}{\partial\varphi}\right]$$

karena nilai $cos\varphi + isin\varphi = e^{i\varphi}$ sehingga

$$\hat{L}_+ = i\hbar\left[(sin\varphi - icos\varphi)\frac{\partial}{\partial\theta} + cot\theta e^{i\varphi}\frac{\partial}{\partial\varphi}\right]$$

$$= \hbar\left[(cos\varphi + isin\varphi)\frac{\partial}{\partial\theta} + icot\theta e^{i\varphi}\frac{\partial}{\partial\varphi}\right]$$

$$= \hbar\left[e^{i\varphi}\frac{\partial}{\partial\theta} + icot\theta e^{i\varphi}\frac{\partial}{\partial\varphi}\right]$$

$$\hat{L}_+ = \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + icot\theta\frac{\partial}{\partial\varphi}\right] \quad (G.22)$$

2.5 Transformasi $\hat{L}_-$

$\hat{L}_- = \hat{L}_x - i\hat{L}_y$

dengan memasukkan $\hat{L}_x$ dan $\hat{L}_y$ yang didapat dari persamaan (G.19) dan (G.20), sehingga akan menjadi

$$\hat{L}_- = i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) - i \times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$= i\hbar\left[\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) - i\left(-cos\varphi\frac{\partial}{\partial\theta} + cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right]$$

$$= i\hbar\left[(sin\varphi + icos\varphi)\frac{\partial}{\partial\theta} + cot\theta(cos\varphi - isin\varphi)\frac{\partial}{\partial\varphi}\right]$$

karena nilai $cos\varphi - isin\varphi = e^{-i\varphi}$ sehingga

$$\hat{L}_- = i\hbar\left[(sin\varphi + icos\varphi)\frac{\partial}{\partial\theta} + cot\theta e^{-i\varphi}\frac{\partial}{\partial\varphi}\right]$$

$$= \hbar\left[(-cos\varphi + isin\varphi)\frac{\partial}{\partial\theta} + icot\theta e^{-i\varphi}\frac{\partial}{\partial\varphi}\right]$$

$$= -\hbar\left[(cos\varphi - isin\varphi)\frac{\partial}{\partial\theta} - icot\theta e^{-i\varphi}\frac{\partial}{\partial\varphi}\right]$$

$$= -\hbar\left[e^{-i\varphi}\frac{\partial}{\partial\theta} - icot\theta e^{-i\varphi}\frac{\partial}{\partial\varphi}\right]$$

$$\hat{L}_- = -\hbar e^{-i\varphi}\left[\frac{\partial}{\partial\theta} - icot\theta\frac{\partial}{\partial\varphi}\right] \quad (G.23)$$

2.6 Transformasi $\hat{L}^2$

$\hat{L}^2 = \hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar\hat{L}_z$

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$$= \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta}+icot\theta\frac{\partial}{\partial\varphi}\right]\times -\hbar e^{-i\varphi}\left[\frac{\partial}{\partial\theta}-icot\theta\frac{\partial}{\partial\varphi}\right]+\left(-i\hbar\frac{\partial}{\partial\varphi}\right)^2 - \hbar\left(-i\hbar\frac{\partial}{\partial\varphi}\right)$$

$$= -\hbar^2\left\{e^{i\varphi}e^{-i\varphi}\frac{\partial^2}{\partial\theta^2}-ie^{i\varphi}e^{-i\varphi}\left(-csc^2\theta\frac{\partial}{\partial\varphi}+cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}\right) + e^{i\varphi}i\cot\theta\left(-ie^{-i\varphi}\frac{\partial}{\partial\theta}+e^{-i\varphi}\frac{\partial^2}{\partial\varphi\partial\theta}\right) + e^{i\varphi}cot^2\theta\left(-ie^{-i\varphi}\frac{\partial}{\partial\varphi}+e^{-i\varphi}\frac{\partial^2}{\partial\varphi^2}\right)\right\}-\hbar^2\frac{\partial^2}{\partial\varphi^2}+i\hbar^2\frac{\partial}{\partial\varphi}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}+i\,csc^2\theta\frac{\partial}{\partial\varphi}+\ cot\,\theta\frac{\partial}{\partial\theta}-i\,cot^2\theta\frac{\partial}{\partial\varphi}+cot^2\theta\frac{\partial^2}{\partial\varphi^2}\right\}-\hbar^2\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}+i\,(csc^2\theta-cot^2\theta)\frac{\partial}{\partial\varphi}-i\ cot\,\theta\frac{\partial}{\partial\theta}+cot^2\theta\frac{\partial^2}{\partial\varphi^2}\right\}-\hbar^2\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}+i\,\frac{\partial}{\partial\varphi}+cot\,\theta\frac{\partial}{\partial\theta}+cot^2\theta\frac{\partial^2}{\partial\varphi^2}\right\}-\hbar^2\frac{\partial^2}{\partial\varphi^2}+i\hbar^2\frac{\partial}{\partial\varphi}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}+i\,\frac{\partial}{\partial\varphi}+cot\,\theta\frac{\partial}{\partial\theta}+cot^2\theta\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial\varphi^2}-i\frac{\partial}{\partial\varphi}\right\}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}+cot\,\theta\frac{\partial}{\partial\theta}+cot^2\theta\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial\varphi^2}\right\}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}+cot\,\theta\frac{\partial}{\partial\theta}+(cot^2\theta+1)\frac{\partial^2}{\partial\varphi^2}\right\}$$

$$= -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}+cot\,\theta\frac{\partial}{\partial\theta}+\ csc^2\theta\frac{\partial^2}{\partial\varphi^2}\right\}$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad (G.24)$$

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LAMPIRAN G. HUBUNGAN KOMUTASI OPERATOR MOMENTUM SUDUT DALAM KOORDINAT KARTESIAN

1. Hubungan Komutasi Operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$$

$$[\hat{L}_x, \hat{L}_y] = \left[-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]$$

$$[\hat{L}_x, \hat{L}_y]\psi = \left[\left(-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \times -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right) - \left(-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \times -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right)\right]\psi$$

$$= (-i\hbar)^2\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \times \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\psi - \left[\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \times \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]\psi$$

$$= (-i\hbar)^2\left[\left(y\frac{\partial}{\partial z}z\frac{\partial\psi}{\partial x} - y\frac{\partial}{\partial z}x\frac{\partial\psi}{\partial z} - z\frac{\partial}{\partial y}z\frac{\partial\psi}{\partial x} + z\frac{\partial}{\partial y}x\frac{\partial\psi}{\partial z}\right) - \left(z\frac{\partial}{\partial x}y\frac{\partial\psi}{\partial z} - z\frac{\partial}{\partial x}z\frac{\partial\psi}{\partial y} - x\frac{\partial}{\partial z}y\frac{\partial\psi}{\partial z} + x\frac{\partial}{\partial z}z\frac{\partial\psi}{\partial y}\right)\right]$$

$$= (-i\hbar)^2\left[\left(y\frac{\partial\psi}{\partial x} + yz\frac{\partial^2\psi}{\partial z\partial x} - yx\frac{\partial^2\psi}{\partial z^2} - z^2\frac{\partial^2\psi}{\partial y\partial x} + zx\frac{\partial^2\psi}{\partial y\partial z}\right) - \left(zy\frac{\partial^2\psi}{\partial x\partial z} - z^2\frac{\partial^2\psi}{\partial x\partial y} - xy\frac{\partial^2\psi}{\partial z^2} + x\frac{\partial\psi}{\partial y} + xz\frac{\partial^2\psi}{\partial z\partial y}\right)\right]$$

$$= (-i\hbar)^2\left[y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right]$$

$$= (-i\hbar)^2\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]\psi$$

$$= (-i\hbar)(-i\hbar)\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]\psi$$

$$= (i\hbar)(-i\hbar)\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]\psi$$

$$= (i\hbar)\hat{L}_z\psi$$

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$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

2. Hubungan Komutasi Operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z] = [\hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,]$$

$$[\hat{L}_y, \hat{L}_z] = \left[-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]$$

$$[\hat{L}_y, \hat{L}_z]\psi = \left[\left(-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right.\right.$$

$$\left.\times -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right) - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)$$

$$\left.\times -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\psi$$

$$= (-i\hbar)^2\left[\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \times \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi$$

$$- \left[\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \times \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\psi$$

$$= (-i\hbar)^2\left[\left(z\frac{\partial}{\partial x}x\frac{\partial\psi}{\partial y} - z\frac{\partial}{\partial x}y\frac{\partial\psi}{\partial x} - x\frac{\partial}{\partial z}x\frac{\partial\psi}{\partial y} + x\frac{\partial}{\partial z}y\frac{\partial\psi}{\partial x}\right)\right.$$

$$\left.- \left(x\frac{\partial}{\partial y}z\frac{\partial\psi}{\partial x} - x\frac{\partial}{\partial y}x\frac{\partial\psi}{\partial z} - y\frac{\partial}{\partial x}z\frac{\partial\psi}{\partial x} + y\frac{\partial}{\partial x}x\frac{\partial\psi}{\partial z}\right)\right]$$

$$= (-i\hbar)^2\left[z\frac{\partial\psi}{\partial y} + zx\frac{\partial^2\psi}{\partial x\partial y} - zy\frac{\partial^2\psi}{\partial x^2} - x^2\frac{\partial^2\psi}{\partial z\partial y} + xy\frac{\partial^2\psi}{\partial z\partial x}\right]$$

$$- \left[xz\frac{\partial^2\psi}{\partial y\partial x} + x^2\frac{\partial^2\psi}{\partial y\partial z} - yz\frac{\partial^2\psi}{\partial x^2} + y\frac{\partial\psi}{\partial z} + yx\frac{\partial^2\psi}{\partial x\partial z}\right]$$

$$= (-i\hbar)^2\left[z\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial z}\right]$$

$$= (i\hbar)(-i\hbar)\left[y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y}\right]$$

$$= (i\hbar)(-i\hbar)\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]\psi$$

$$= i\hbar\hat{L}_x\,\psi$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$$

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3. Hubungan Komutasi Operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y]$$

$$[\hat{L}_z, \hat{L}_x] = \left[-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

$$[\hat{L}_z, \hat{L}_x]\psi = \left[\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \times -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right)\right.$$
$$\left. - \left(-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \times -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\right]\psi$$

$$= (-i\hbar)^2\left[\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \times \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]\psi$$
$$- \left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \times \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi$$

$$= (-i\hbar)^2\left[\left(x\frac{\partial}{\partial y}y\frac{\partial\psi}{\partial z} - x\frac{\partial}{\partial y}z\frac{\partial\psi}{\partial y} - y\frac{\partial}{\partial x}y\frac{\partial\psi}{\partial z} + y\frac{\partial}{\partial x}z\frac{\partial\psi}{\partial y}\right)\right.$$
$$\left. - \left(y\frac{\partial}{\partial z}x\frac{\partial\psi}{\partial y} - y\frac{\partial}{\partial z}y\frac{\partial\psi}{\partial x} - z\frac{\partial}{\partial y}x\frac{\partial\psi}{\partial y} + z\frac{\partial}{\partial y}y\frac{\partial\psi}{\partial x}\right)\right]$$

$$= (-i\hbar)^2\left[\left(x\frac{\partial\psi}{\partial z} + xy\frac{\partial^2\psi}{\partial y\partial z} - xz\frac{\partial^2\psi}{\partial y^2} - y^2\frac{\partial^2\psi}{\partial x\partial z} + yz\frac{\partial^2\psi}{\partial x\partial y}\right)\right.$$
$$\left. - \left(yx\frac{\partial^2\psi}{\partial z\partial y} - y^2\frac{\partial^2\psi}{\partial z\partial x} - zx\frac{\partial^2\psi}{\partial y^2} + z\frac{\partial\psi}{\partial x} + zy\frac{\partial^2\psi}{\partial y\partial x}\right)\right]$$

$$= (-i\hbar)^2\left[x\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial x}\right]$$

$$= (-i\hbar)(-i\hbar)\left[x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right]\psi$$

$$= (i\hbar)(-i\hbar)\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]\psi$$

$$= i\hbar\hat{L}_y\,\psi$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

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LAMPIRAN H. PEMBUKTIAN KOMUTATOR OPERATOR MOMENTUM SUDUT DALAM KOORDINAT BOLA

1. Komutator operator $\hat{L}_x$ dan $\hat{L}_x$

$$[\hat{L}_x, \hat{L}_x]\psi = (\hat{L}_x\hat{L}_x - \hat{L}_x\hat{L}_x)\psi$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$- \left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= 0$$

2. Komutator operator $\hat{L}_y$ dan $\hat{L}_y$

$$[\hat{L}_y, \hat{L}_y]\psi = (\hat{L}_y\hat{L}_y - \hat{L}_y\hat{L}_y)\psi$$

$$= \left[\left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}$$

$$- \left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= 0$$

3. Komutator operator $\hat{L}_z$ dan $\hat{L}_z$

$$[\hat{L}_z, \hat{L}_z]\psi = (\hat{L}_z\hat{L}_z - \hat{L}_z\hat{L}_z)\psi$$

$$= \left[\left(-i\hbar\frac{\partial}{\partial\varphi} \times -i\hbar\frac{\partial}{\partial\varphi}\right) - \left(-i\hbar\frac{\partial}{\partial\varphi} \times -i\hbar\frac{\partial}{\partial\varphi}\right)\right]\psi$$

$$= 0$$

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4. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y]\psi = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)\psi$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$-\left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right.$$

$$\left.+\, cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right\}\psi$$

$$-\, \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right.$$

$$\left.-\, cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= \hbar^2\left\{sin\varphi cos\varphi\frac{\partial^2\psi}{\partial\theta^2} - sin^2\varphi\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$+\, cot\theta cos\varphi\left(-sin\varphi\frac{\partial\psi}{\partial\theta} + cos\varphi\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)$$

$$\left.-\, cot^2\theta cos\varphi\left(cos\varphi\frac{\partial\psi}{\partial\varphi} + sin\varphi\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$-\, \hbar^2\left\{cos\varphi sin\varphi\frac{\partial^2\psi}{\partial\theta^2} + cos^2\varphi\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi}cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$-\, cot\theta sin\varphi\left(cos\varphi\frac{\partial\psi}{\partial\theta} + sin\varphi\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left.-\, cot^2\theta sin\varphi\left(-sin\varphi\frac{\partial\psi}{\partial\varphi} + cos\varphi\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$= \hbar^2\left\{\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)(-sin^2\varphi - cos^2\varphi)\right.$$

$$\left.+\, cot\theta(cos^2\varphi + sin^2\varphi)\frac{\partial^2\psi}{\partial\theta\partial\varphi} - cot^2\theta(cos^2\varphi + sin^2\varphi)\frac{\partial\psi}{\partial\varphi}\right\}$$

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$$= \hbar^2 \left\{csc^2\theta \frac{\partial\psi}{\partial\varphi} - cot^2\theta \frac{\partial\psi}{\partial\varphi}\right\}$$

$$= \hbar^2\{csc^2\theta - cot^2\theta\}\frac{\partial\psi}{\partial\varphi}$$

$$= \hbar^2 \frac{\partial\psi}{\partial\varphi}$$

$$[\hat{L}_x, \hat{L}_y] = \hbar^2 \frac{\partial}{\partial\varphi}$$

5. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z]\psi = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)\psi$$

$$= \left[\left\{i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right.$$

$$\left. - \left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$- \hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial^2\psi}{\partial\theta\partial\varphi} + \sin\varphi\cot\theta\frac{\partial^2\psi}{\partial\varphi^2}\right\}$$

$$- \hbar^2\left\{sin\varphi\frac{\partial\psi}{\partial\theta} - cos\varphi\frac{\partial^2\psi}{\partial\theta\partial\varphi} + \cot\theta\left(cos\varphi\frac{\partial\psi}{\partial\varphi} + sin\varphi\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$= \hbar^2\left(-sin\varphi\frac{\partial\psi}{\partial\theta} - \cot\theta cos\varphi\frac{\partial\psi}{\partial\varphi}\right)$$

$$[\hat{L}_y, \hat{L}_z] = \hbar^2\left(-sin\varphi\frac{\partial}{\partial\theta} - \cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)$$

6. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x]\psi = (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)\psi$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]\psi$$

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$$= \hbar^2 \left\{\frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right\} \psi$$

$$- \hbar^2 \left\{sin\varphi \frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\} \psi$$

$$= \hbar^2 \left\{cos\varphi \frac{\partial\psi}{\partial\theta} + sin\varphi \frac{\partial^2\psi}{\partial\varphi\partial\theta} + cot\theta \left(-sin\varphi \frac{\partial\psi}{\partial\varphi} + cos\varphi \frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$- \hbar^2 \left\{sin\varphi \frac{\partial^2\psi}{\partial\theta\partial\varphi} + cot\theta cos\varphi \frac{\partial^2\psi}{\partial\varphi^2}\right\}$$

$$= \hbar^2 \left(cos\varphi \frac{\partial\psi}{\partial\theta} - sin\varphi cot\theta \frac{\partial\psi}{\partial\varphi}\right)$$

$$[\hat{L}_z, \hat{L}_x] = \hbar^2 \left(cos\varphi \frac{\partial}{\partial\theta} - sin\varphi cot\theta \frac{\partial}{\partial\varphi}\right)$$

7. Komutator operator $\hat{L}_x$ dan $\hat{L}_+$

$$[\hat{L}_x, \hat{L}_+]\psi = (\hat{L}_x\hat{L}_+ - \hat{L}_+\hat{L}_x)\psi$$

$$= \left[\left\{i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right) \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{\hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi}\right) \times i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right)\right\}\right] \psi$$

$$= i\hbar^2 \left\{sin\varphi \frac{\partial}{\partial\theta} e^{i\varphi} \frac{\partial}{\partial\theta} + sin\varphi \frac{\partial}{\partial\theta} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial}{\partial\theta}\right.$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi}\right\} \psi$$

$$- i\hbar^2 \left\{e^{i\varphi} \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} + e^{i\varphi} \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right.$$

$$\left. + e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} + e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right\} \psi$$

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$$= i\hbar^2 \left\{ sin\varphi e^{i\varphi} \frac{\partial^2\psi}{\partial\theta^2} + isin\varphi e^{i\varphi} \left(-csc^2\theta \frac{\partial\psi}{\partial\varphi} + cot\,\theta \frac{\partial^2\psi}{\partial\theta\partial\varphi} \right) \right.$$

$$+ cot\theta cos\varphi \left(ie^{i\varphi} \frac{\partial\psi}{\partial\theta} + e^{i\varphi} \frac{\partial^2\psi}{\partial\varphi\partial\theta} \right)$$

$$\left. + icot^2\theta\, cos\varphi \left(ie^{i\varphi} \frac{\partial\psi}{\partial\varphi} + e^{i\varphi} \frac{\partial^2\psi}{\partial\varphi^2} \right) \right\}$$

$$- i\hbar^2 \left\{ e^{i\varphi} sin\varphi \frac{\partial^2\psi}{\partial\theta^2} + e^{i\varphi} cos\varphi \left(-csc^2\theta \frac{\partial\psi}{\partial\varphi} + cot\theta \frac{\partial^2\psi}{\partial\theta\partial\varphi} \right) \right.$$

$$+ ie^{i\varphi}\, cot\,\theta \left(cos\varphi \frac{\partial\psi}{\partial\varphi} + sin\varphi \frac{\partial^2\psi}{\partial\varphi\partial\theta} \right)$$

$$\left. + ie^{i\varphi} cot^2\theta \left(-sin\varphi \frac{\partial\psi}{\partial\varphi} + cos\varphi \frac{\partial^2\psi}{\partial\varphi^2} \right) \right\}$$

$$= i\hbar^2 \left\{ -isin\varphi e^{i\varphi} csc^2\theta \frac{\partial\psi}{\partial\varphi} - cot^2\theta\, cos\varphi e^{i\varphi} \frac{\partial\psi}{\partial\varphi} + e^{i\varphi} cos\varphi csc^2\theta \frac{\partial\psi}{\partial\varphi} \right.$$

$$\left. + ie^{i\varphi} cot^2\theta sin\varphi \frac{\partial\psi}{\partial\varphi} \right\}$$

$$= i\hbar^2 \left\{ isin\varphi e^{i\varphi} (cot^2\theta - csc^2\theta) \frac{\partial\psi}{\partial\varphi} + cos\varphi e^{i\varphi} (csc^2\theta - cot^2\theta) \frac{\partial\psi}{\partial\varphi} \right\}$$

$$= i\hbar^2 e^{i\varphi} (cos\varphi - isin\varphi) \frac{\partial\psi}{\partial\varphi}$$

$$= i\hbar^2 e^{i\varphi} e^{-i\varphi} \frac{\partial\psi}{\partial\varphi}$$

$$= i\hbar^2 \frac{\partial\psi}{\partial\varphi}$$

$$[\hat{L}_x, \hat{L}_+] = i\hbar^2 \frac{\partial}{\partial\varphi}$$

8. Komutator operator $\hat{L}_y$ dan $\hat{L}_+$

$$[\hat{L}_y, \hat{L}_+]\psi = (\hat{L}_y \hat{L}_+ - \hat{L}_+ \hat{L}_y)\psi$$

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$$= \left[\left\{i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \times \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\,cot\,\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$-\left\{\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\,cot\,\theta\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= i\hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}e^{i\varphi}\frac{\partial}{\partial\theta} - \cos\varphi\frac{\partial}{\partial\theta}e^{i\varphi}\,i\,cot\,\theta\frac{\partial}{\partial\varphi}\right.$$

$$\left.+\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}e^{i\varphi}\,i\,cot\,\theta\frac{\partial}{\partial\varphi}\right\}\psi$$

$$-\,i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta} + e^{i\varphi}\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$\left.-\,e^{i\varphi}\,i\,cot\,\theta\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\theta} + e^{i\varphi}\,i\,cot\,\theta\frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= i\hbar^2\left\{-\cos\varphi e^{i\varphi}\frac{\partial^2\psi}{\partial\theta^2} - i\cos\varphi\,e^{i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$+\sin\varphi\cot\theta\left(ie^{i\varphi}\frac{\partial\psi}{\partial\theta} + e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left.+\,isin\varphi cot^2\theta\left(ie^{i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$-\,i\hbar^2\left\{-e^{i\varphi}\cos\varphi\frac{\partial^2\psi}{\partial\theta^2} + e^{i\varphi}\sin\varphi\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$-\,ie^{i\varphi}\,cot\,\theta\left(-sin\varphi\frac{\partial\psi}{\partial\theta} + \cos\varphi\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left.+\,icot^2\theta e^{i\varphi}\left(cos\varphi\frac{\partial\psi}{\partial\varphi} + sin\varphi\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$= i\hbar^2\left\{i\cos\varphi\,e^{i\varphi}\,csc^2\theta\frac{\partial\psi}{\partial\varphi}\right.$$

$$\left.-\sin\varphi\,cot^2\theta\,e^{i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{i\varphi}\sin\varphi\,csc^2\theta\frac{\partial\psi}{\partial\varphi} - icot^2\theta e^{i\varphi}cos\varphi\frac{\partial\psi}{\partial\varphi}\right\}$$

$$= i\hbar^2\left\{i\cos\varphi\,e^{i\varphi}\left(csc^2\theta - cot^2\theta\right)\frac{\partial\psi}{\partial\varphi}\right.$$

$$\left.-\,e^{i\varphi}\sin\varphi\left(cot^2\theta - csc^2\theta\right)\frac{\partial\psi}{\partial\varphi}\right\}$$

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$$= i\hbar^2 e^{i\varphi}(i\cos\varphi + \sin\varphi)\frac{\partial\psi}{\partial\varphi}$$

$$= \hbar^2 e^{i\varphi}(-\cos\varphi + isin\varphi)\frac{\partial\psi}{\partial\varphi}$$

$$= -\hbar^2 e^{i\varphi}(\cos\varphi - isin\varphi)\frac{\partial\psi}{\partial\varphi}$$

$$= -\hbar^2 e^{i\varphi} e^{-i\varphi}\frac{\partial\psi}{\partial\varphi}$$

$$= -\hbar^2\frac{\partial\psi}{\partial\varphi}$$

$$[\hat{L}_y, \hat{L}_+] = -\hbar^2\frac{\partial}{\partial\varphi}$$

9. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+]\psi = (\hat{L}_z\hat{L}_+ - \hat{L}_+\hat{L}_z)\psi$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left.-\left\{\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]\psi$$

$$= i\hbar^2\left\{-\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right\}\psi$$

$$- i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= i\hbar^2\left\{-ie^{i\varphi}\frac{\partial\psi}{\partial\theta} - e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta} - i\cot\theta\left(ie^{i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$- i\hbar^2\left\{-e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta} - e^{i\varphi}i\cot\theta\frac{\partial^2\psi}{\partial\varphi^2}\right\}$$

$$= i\hbar^2\left(-ie^{i\varphi}\frac{\partial\psi}{\partial\theta} + \cot\theta e^{i\varphi}\frac{\partial\psi}{\partial\varphi}\right)$$

$$= i\hbar^2 e^{i\varphi}\left(-i\frac{\partial\psi}{\partial\theta} + \cot\theta\frac{\partial\psi}{\partial\varphi}\right)$$

$$= \hbar^2 e^{i\varphi}\left(\frac{\partial\psi}{\partial\theta} + icot\theta\frac{\partial\psi}{\partial\varphi}\right)$$

$$[\hat{L}_z, \hat{L}_+] = \hbar^2 e^{i\varphi}\left(\frac{\partial}{\partial\theta} + icot\theta\frac{\partial}{\partial\varphi}\right)$$

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10. Komutator operator $\hat{L}_x$ dan $\hat{L}_-$

$$[\hat{L}_x, \hat{L}_-]\psi = (\hat{L}_x\hat{L}_- - \hat{L}_-\hat{L}_x)\psi$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= i\hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}e^{-i\varphi}\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$\left. - cot\theta cos\varphi\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right\}\psi$$

$$- i\hbar^2\left\{-e^{-i\varphi}\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} - e^{-i\varphi}\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right.$$

$$\left. + e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial}{\partial\theta} + e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= i\hbar^2\left\{e^{-i\varphi}sin\varphi\frac{\partial^2\psi}{\partial\theta^2} + isin\varphi e^{-i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)\right.$$

$$- cot\theta cos\varphi\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\theta} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left. + icot^2\theta cos\varphi\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$- i\hbar^2\left\{-e^{-i\varphi}sin\varphi\frac{\partial^2\psi}{\partial\theta^2} - e^{-i\varphi}cos\varphi\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)\right.$$

$$+ i\cot\theta e^{-i\varphi}\left(cos\varphi e^{-i\varphi} + sin\varphi\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left. + icot^2\theta\, e^{-i\varphi}\left(-sin\varphi\frac{\partial\psi}{\partial\varphi} + cos\varphi\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$= i\hbar^2\left\{-isin\varphi e^{-i\varphi}csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot^2\theta cos\varphi e^{-i\varphi}\frac{\partial\psi}{\partial\varphi}\right.$$

$$\left. - e^{-i\varphi}cos\varphi csc^2\theta\frac{\partial\psi}{\partial\varphi} + icot^2\theta e^{-i\varphi}sin\varphi\frac{\partial\psi}{\partial\varphi}\right\}$$

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$$= i\hbar^2 \left\{ isin\varphi e^{-i\varphi} (-csc^2\theta + cot^2\theta) \frac{\partial\psi}{\partial\varphi} + cos\varphi e^{-i\varphi} (cot^2\theta - csc^2\theta) \frac{\partial\psi}{\partial\varphi} \right\}$$

$$= i\hbar^2 e^{-i\varphi} (-cos\varphi - isin\varphi) \frac{\partial\psi}{\partial\varphi}$$

$$= -i\hbar^2 e^{-i\varphi} (cos\varphi + isin\varphi) \frac{\partial\psi}{\partial\varphi}$$

$$= -i\hbar^2 e^{-i\varphi} e^{i\varphi} \frac{\partial\psi}{\partial\varphi}$$

$$= -i\hbar^2 \frac{\partial\psi}{\partial\varphi}$$

$$[\hat{L}_x, \hat{L}_-] = -i\hbar^2 \frac{\partial}{\partial\varphi}$$

11. Komutator operator $\hat{L}_y$ dan $\hat{L}_-$

$$[\hat{L}_y, \hat{L}_-]\psi = (\hat{L}_y \hat{L}_- - \hat{L}_- \hat{L}_y)\psi$$

$$= \left[\left\{-i\hbar \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi}\right) \times -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i\,cot\,\theta \frac{\partial}{\partial\varphi}\right)\right\} - \left\{-\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i\,cot\,\theta \frac{\partial}{\partial\varphi}\right) \times -i\hbar \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi}\right)\right\}\right] \psi$$

$$= i\hbar^2 \left\{\cos\varphi \frac{\partial}{\partial\theta} e^{-i\varphi} \frac{\partial}{\partial\theta} - \cos\varphi \frac{\partial}{\partial\theta} e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi}\right\} \psi$$

$$- i\hbar^2 \left\{e^{-i\varphi} \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} - e^{-i\varphi} \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} - e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi} \cos\varphi \frac{\partial}{\partial\theta} + e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi}\right\} \psi$$

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$$= i\hbar^2 \left\{ \cos\varphi e^{-i\varphi} \frac{\partial^2\psi}{\partial\theta^2} - i\cos\varphi\, e^{-i\varphi} \left(-csc^2\theta \frac{\partial\psi}{\partial\varphi} + cot\theta \frac{\partial^2\psi}{\partial\theta\partial\varphi} \right) \right.$$

$$- \sin\varphi \cot\theta \left(-ie^{-i\varphi} \frac{\partial\psi}{\partial\theta} + e^{-i\varphi} \frac{\partial^2\psi}{\partial\varphi\partial\theta} \right)$$

$$\left. + isin\varphi cot^2\theta \left(-ie^{-i\varphi} \frac{\partial\psi}{\partial\varphi} + e^{-i\varphi} \frac{\partial^2\psi}{\partial\varphi^2} \right) \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \cos\varphi \frac{\partial^2\psi}{\partial\theta^2} - e^{-i\varphi} \sin\varphi \left(-csc^2\theta \frac{\partial\psi}{\partial\varphi} + cot\theta \frac{\partial^2\psi}{\partial\theta\partial\varphi} \right) \right.$$

$$- ie^{-i\varphi} \, cot\,\theta \left(-sin\varphi \frac{\partial\psi}{\partial\theta} + \cos\varphi \frac{\partial^2\psi}{\partial\varphi\partial\theta} \right)$$

$$\left. + icot^2\theta e^{-i\varphi} \left(cos\varphi \frac{\partial\psi}{\partial\varphi} + sin\varphi \frac{\partial^2\psi}{\partial\varphi^2} \right) \right\}$$

$$= i\hbar^2 \left\{ i\cos\varphi\, e^{-i\varphi}\, csc^2\theta \frac{\partial\psi}{\partial\varphi} \right.$$

$$+ \sin\varphi\, cot^2\theta\, e^{-i\varphi} \frac{\partial\psi}{\partial\varphi}$$

$$\left. - e^{-i\varphi} \sin\varphi\, csc^2\theta \frac{\partial\psi}{\partial\varphi} - icot^2\theta e^{-i\varphi} cos\varphi \frac{\partial\psi}{\partial\varphi} \right\}$$

$$= i\hbar^2 \left\{ i\cos\varphi\, e^{-i\varphi} (csc^2\theta - cot^2\theta) \frac{\partial\psi}{\partial\varphi} \right.$$

$$\left. + e^{-i\varphi} \sin\varphi\, (cot^2\theta - csc^2\theta) \frac{\partial\psi}{\partial\varphi} \right\}$$

$$= i\hbar^2 e^{-i\varphi} (i\cos\varphi - \sin\varphi) \frac{\partial\psi}{\partial\varphi}$$

$$= \hbar^2 e^{-i\varphi} (-\cos\varphi - isin\varphi) \frac{\partial\psi}{\partial\varphi}$$

$$= -\hbar^2 e^{-i\varphi} (\cos\varphi + isin\varphi) \frac{\partial\psi}{\partial\varphi}$$

$$= -\hbar^2 e^{-i\varphi} e^{i\varphi} \frac{\partial\psi}{\partial\varphi}$$

$$= -\hbar^2 \frac{\partial\psi}{\partial\varphi}$$

$$[\hat{L}_y, \hat{L}_-] = -\hbar^2 \frac{\partial}{\partial\varphi}$$

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12. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-]\psi = (\hat{L}_z\hat{L}_- - \hat{L}_-\hat{L}_z)\psi$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]\psi$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right\}\psi$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= i\hbar^2\left\{-ie^{-i\varphi}\frac{\partial\psi}{\partial\theta} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta} - i\cot\theta\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta} - e^{-i\varphi}i\cot\theta\frac{\partial^2\psi}{\partial\varphi^2}\right\}$$

$$= i\hbar^2\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\theta} - \cot\theta e^{-i\varphi}\frac{\partial\psi}{\partial\varphi}\right)$$

$$= i\hbar^2 e^{-i\varphi}\left(-i\frac{\partial\psi}{\partial\theta} - \cot\theta\frac{\partial\psi}{\partial\varphi}\right)$$

$$= \hbar^2 e^{-i\varphi}\left(\frac{\partial\psi}{\partial\theta} - i\cot\theta\frac{\partial\psi}{\partial\varphi}\right)$$

$$[\hat{L}_z, \hat{L}_-] = \hbar^2 e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

13. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2]\psi = (\hat{L}_x\hat{L}^2 - \hat{L}^2\hat{L}_x)\psi$$

$$= \left[\left\{i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$- \left\{-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

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$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times\, i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right] \psi$$

$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} + sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} + sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right.$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} \psi$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right\} \psi$$

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$$= -i\hbar^3\left\{sin\varphi\left(-csc^2\theta\frac{\partial\psi}{\partial\theta}+cot\theta\frac{\partial^2\psi}{\partial\theta^2}\right)+sin\varphi\frac{\partial^3\psi}{\partial\theta^3}\right.$$

$$+\,sin\varphi\left(-\frac{2\cot\theta}{sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{1}{sin^2\theta}\frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2}\right)+cos\varphi\,cot^2\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}$$

$$\left.+\,cos\varphi\,cot\theta\frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2}+cos\varphi\frac{\cot\theta}{sin^2\theta}\frac{\partial^3\psi}{\partial\varphi^3}\right\}$$

$$+\,i\hbar^3\left\{cot\theta sin\varphi\frac{\partial^2\psi}{\partial\theta^2}+cos\varphi\,cot\theta\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi}+cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$+\,sin\varphi\frac{\partial^3\psi}{\partial\theta^3}+cos\varphi\left(\frac{2\cot\theta}{sin^2\theta}\frac{\partial\psi}{\partial\varphi}-2csc^2\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}+cot\theta\frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2}\right)$$

$$+\,\frac{1}{sin^2\theta}\left(-sin\varphi\frac{\partial\psi}{\partial\theta}+2cos\varphi\frac{\partial^2\psi}{\partial\theta\partial\varphi}+sin\varphi\frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2}\right)$$

$$\left.+\,\frac{\cot\theta}{sin^2\theta}\left(-cos\varphi\frac{\partial\psi}{\partial\varphi}-2sin\varphi\frac{\partial^2\psi}{\partial\varphi^2}+cos\varphi\frac{\partial^3\psi}{\partial\varphi^3}\right)\right\}$$

$$[\hat{L}_x,\hat{L}^2]=0$$

14. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$$[\hat{L}_y,\hat{L}^2]\psi=(\hat{L}_y\hat{L}^2-\hat{L}^2\hat{L}_y)\psi$$

$$=\left[\left\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

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$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta}+\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2}+\cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}$$

$$\left.-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}\psi$$

$$-i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}+\frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial}{\partial\theta}\right.$$

$$-\frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$\left.-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= i\hbar^3\left\{\cos\varphi\left(-csc^2\theta\frac{\partial\psi}{\partial\theta}+cot\theta\frac{\partial^2\psi}{\partial\theta^2}\right)+\cos\varphi\frac{\partial^3\psi}{\partial\theta^3}\right.$$

$$+\cos\varphi\left(-\frac{2\cot\theta}{sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{1}{sin^2\theta}\frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2}\right)-sin\varphi\,cot^2\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}$$

$$\left.-sin\varphi\,cot\theta\frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2}-sin\varphi\frac{\cot\theta}{sin^2\theta}\frac{\partial^3\psi}{\partial\varphi^3}\right\}$$

$$-i\hbar^3\left\{cot\theta cos\varphi\frac{\partial^2\psi}{\partial\theta^2}-sin\varphi\,cot\theta\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi}+cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$+\cos\varphi\frac{\partial^3\psi}{\partial\theta^3}-sin\varphi\left(\frac{2\cot\theta}{sin^2\theta}\frac{\partial\psi}{\partial\varphi}-2csc^2\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}+cot\theta\frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2}\right)$$

$$+\frac{1}{sin^2\theta}\left(-cos\varphi\frac{\partial\psi}{\partial\theta}-2sin\varphi\frac{\partial^2\psi}{\partial\theta\partial\varphi}+cos\varphi\frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2}\right)$$

$$\left.-\frac{\cot\theta}{sin^2\theta}\left(-sin\varphi\frac{\partial\psi}{\partial\varphi}+2cos\varphi\frac{\partial^2\psi}{\partial\varphi^2}+sin\varphi\frac{\partial^3\psi}{\partial\varphi^3}\right)\right\}$$

$[\hat{L}_y,\hat{L}^2]=0$

15. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$[\hat{L}_z,\hat{L}^2]\psi=(\hat{L}_z\hat{L}^2-\hat{L}^2\hat{L}_z)\psi$

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$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}\right.$$

$$\left. - \left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]\psi$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}\psi$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= i\hbar^3\left\{\cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi} + \frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^3\psi}{\partial\varphi^3}\right\}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi} + \frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^3\psi}{\partial\varphi^3}\right\}$$

$$[\hat{L}_z, \hat{L}^2] = 0$$

16. Komutator operator $\hat{L}_+$ dan $\hat{L}^2$

$$[\hat{L}_+, \hat{L}^2]\psi = (\hat{L}_+\hat{L}^2 - \hat{L}^2\hat{L}_+)\psi$$

$$= \left[\left\{\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}\right.$$

$$- \left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left. \times\, \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

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$$= -\hbar^3 \left\{ e^{i\varphi} \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} + e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} + e^{i\varphi} \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right.$$

$$+ i \cot\theta\, e^{i\varphi} \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} + i \cot\theta\, e^{i\varphi} \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2}$$

$$\left. + i \cot\theta\, e^{i\varphi} \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} \psi$$

$$+ \hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} e^{i\varphi} \frac{\partial}{\partial\theta} + \cot\theta \frac{\partial}{\partial\theta} i \cot\theta\, e^{i\varphi} \frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} e^{i\varphi} \frac{\partial}{\partial\theta} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} i \cot\theta\, e^{i\varphi} \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} e^{i\varphi} \frac{\partial}{\partial\varphi}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} i \cot\theta\, e^{i\varphi} \frac{\partial}{\partial\varphi} \right\} \psi$$

$$= -\hbar^3 \left\{ e^{i\varphi} \left(-csc^2\theta \frac{\partial\psi}{\partial\theta} + cot\theta \frac{\partial^2\psi}{\partial\theta^2} \right) + e^{i\varphi} \frac{\partial^3\psi}{\partial\theta^3} \right.$$

$$+ e^{i\varphi} \left(-\frac{2\cot\theta}{sin^2\theta} \frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{sin^2\theta} \frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2} \right) + i\, cot^2\theta e^{i\varphi} \frac{\partial^2\psi}{\partial\theta\partial\varphi}$$

$$\left. + i \cot\theta\, e^{i\varphi} \frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2} + i \frac{\cot\theta}{sin^2\theta} e^{i\varphi} \frac{\partial^3\psi}{\partial\varphi^3} \right\}$$

$$+ \hbar^3 \left\{ \cot\theta e^{i\varphi} \frac{\partial^2\psi}{\partial\theta^2} + i \cot\theta\, e^{i\varphi} \left(-csc^2\theta \frac{\partial\psi}{\partial\varphi} + cot\theta \frac{\partial^2\psi}{\partial\theta\partial\varphi} \right) \right.$$

$$+ e^{i\varphi} \frac{\partial^3\psi}{\partial\theta^3} + ie^{i\varphi} \left(\frac{2\cot\theta}{sin^2\theta} \frac{\partial\psi}{\partial\varphi} - 2csc^2\theta \frac{\partial^2\psi}{\partial\theta\partial\varphi} + cot\theta \frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2} \right)$$

$$+ \frac{1}{\sin^2\theta} \left(-e^{i\varphi} \frac{\partial\psi}{\partial\theta} + 2ie^{i\varphi} \frac{\partial^2\psi}{\partial\theta\partial\varphi} + e^{i\varphi} \frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2} \right)$$

$$\left. + \frac{i\cot\theta}{sin^2\theta} \left(-e^{i\varphi} \frac{\partial\psi}{\partial\varphi} + 2ie^{i\varphi} \frac{\partial^2\psi}{\partial\varphi^2} + e^{i\varphi} \frac{\partial^3\psi}{\partial\varphi^3} \right) \right\}$$

$[\hat{L}_+, \hat{L}^2] = 0$

17. Komutator operator $\hat{L}_-$ dan $\hat{L}^2$

$[\hat{L}_-, \hat{L}^2]\psi = (\hat{L}_-\hat{L}^2 - \hat{L}^2\hat{L}_-)\psi$

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$$= \left[\left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}\right.$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= \hbar^3\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$- i\cot\theta\, e^{-i\varphi}\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} - i\cot\theta\, e^{-i\varphi}\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left.- i\cot\theta\, e^{-i\varphi}\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}\psi$$

$$- \hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}e^{-i\varphi}\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\theta}i\cot\theta\, e^{-i\varphi}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}e^{-i\varphi}\frac{\partial}{\partial\theta}\right.$$

$$- \frac{\partial^2}{\partial\theta^2}i\cot\theta\, e^{-i\varphi}\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}e^{-i\varphi}\frac{\partial}{\partial\varphi}$$

$$\left.- \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}i\cot\theta\, e^{-i\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= \hbar^3\left\{e^{-i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\theta} + cot\theta\frac{\partial^2\psi}{\partial\theta^2}\right) + e^{-i\varphi}\frac{\partial^3\psi}{\partial\theta^3}\right.$$

$$+ e^{-i\varphi}\left(-\frac{2\cot\theta}{sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{sin^2\theta}\frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2}\right) - i\,cot^2\theta e^{-i\varphi}\frac{\partial^2\psi}{\partial\theta\partial\varphi}$$

$$\left.- i\cot\theta\, e^{-i\varphi}\frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2} - i\frac{\cot\theta}{sin^2\theta}e^{-i\varphi}\frac{\partial^3\psi}{\partial\varphi^3}\right\}$$

$$- \hbar^3\left\{\cot\theta e^{-i\varphi}\frac{\partial^2\psi}{\partial\theta^2} - i\cot\theta\, e^{-i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$+ e^{-i\varphi}\frac{\partial^3\psi}{\partial\theta^3} - ie^{-i\varphi}\left(\frac{2\cot\theta}{sin^2\theta}\frac{\partial\psi}{\partial\varphi} - 2csc^2\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi} + cot\theta\frac{\partial^3\psi}{\partial\varphi\,\partial\theta^2}\right)$$

$$+ \frac{1}{\sin^2\theta}\left(e^{-i\varphi}\frac{\partial\psi}{\partial\theta} - 2ie^{-i\varphi}\frac{\partial^2\psi}{\partial\theta\partial\varphi} + e^{-i\varphi}\frac{\partial^3\psi}{\partial\theta\,\partial\varphi^2}\right)$$

$$\left.- \frac{i\cot\theta}{sin^2\theta}\left(e^{-i\varphi}\frac{\partial\psi}{\partial\varphi} - 2ie^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi^2} + e^{-i\varphi}\frac{\partial^3\psi}{\partial\varphi^3}\right)\right\}$$

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$[\hat{L}_-, \hat{L}^2] = 0$

18. Komutator operator $\hat{L}_+$ dan $\hat{L}_-$

$$[\hat{L}_+, \hat{L}_-]\psi = (\hat{L}_+\hat{L}_- - \hat{L}_-\hat{L}_+)\psi$$

$$= \left[\left\{\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= -\hbar^2\left\{e^{i\varphi}\frac{\partial}{\partial\theta}e^{-i\varphi}\frac{\partial}{\partial\theta} - e^{i\varphi}\frac{\partial}{\partial\theta}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$\left. + e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta} + e^{i\varphi}\cot^2\theta\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$+ \hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}e^{i\varphi}\frac{\partial}{\partial\theta} + e^{-i\varphi}\frac{\partial}{\partial\theta}e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$\left. - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta} + e^{-i\varphi}\cot^2\theta\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= -\hbar^2\left\{e^{i\varphi}e^{-i\varphi}\frac{\partial^2\psi}{\partial\theta^2} - ie^{i\varphi}e^{-i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$+ e^{i\varphi}i\cot\theta\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\theta} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left. + e^{i\varphi}\cot^2\theta\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$+ \hbar^2\left\{e^{-i\varphi}e^{i\varphi}\frac{\partial^2\psi}{\partial\theta^2} + ie^{-i\varphi}e^{i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$- e^{-i\varphi}i\cot\theta\left(ie^{i\varphi}\frac{\partial\psi}{\partial\theta} + e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left. + e^{-i\varphi}\cot^2\theta\left(ie^{i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$= \hbar^2\left\{-2i\,csc^2\theta\frac{\partial\psi}{\partial\varphi} + 2i\cot^2\theta\frac{\partial\psi}{\partial\varphi}\right\}$$

$$= 2i\hbar^2\{\cot^2\theta - csc^2\theta\}\frac{\partial\psi}{\partial\varphi}$$

$$= -2i\hbar^2\frac{\partial\psi}{\partial\varphi}$$

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$$[\hat{L}_+, \hat{L}_-] = -2i\hbar^2 \frac{\partial}{\partial\varphi}$$

19. Komutator operator $\hat{L}_-$ dan $\hat{L}_+$

$$[\hat{L}_-, \hat{L}_+]\psi = (\hat{L}_-\hat{L}_+ - \hat{L}_+\hat{L}_-)\psi$$

$$= \left[\left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]\psi$$

$$= -\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}e^{i\varphi}\frac{\partial}{\partial\theta} + e^{-i\varphi}\frac{\partial}{\partial\theta}e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$\left. - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta} + e^{-i\varphi}\cot^2\theta\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$+ \hbar^2\left\{e^{i\varphi}\frac{\partial}{\partial\theta}e^{-i\varphi}\frac{\partial}{\partial\theta} - e^{i\varphi}\frac{\partial}{\partial\theta}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$\left. + e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta} + e^{i\varphi}\cot^2\theta\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\varphi}\right\}\psi$$

$$= -\hbar^2\left\{e^{-i\varphi}e^{i\varphi}\frac{\partial^2\psi}{\partial\theta^2} + ie^{-i\varphi}e^{i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$- e^{-i\varphi}i\cot\theta\left(ie^{i\varphi}\frac{\partial\psi}{\partial\theta} + e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left. + e^{-i\varphi}\cot^2\theta\left(ie^{i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$+ \hbar^2\left\{e^{i\varphi}e^{-i\varphi}\frac{\partial^2\psi}{\partial\theta^2} - ie^{i\varphi}e^{-i\varphi}\left(-csc^2\theta\frac{\partial\psi}{\partial\varphi} + cot\theta\frac{\partial^2\psi}{\partial\theta\partial\varphi}\right)\right.$$

$$+ e^{i\varphi}i\cot\theta\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\theta} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi\partial\theta}\right)$$

$$\left. + e^{i\varphi}\cot^2\theta\left(-ie^{-i\varphi}\frac{\partial\psi}{\partial\varphi} + e^{-i\varphi}\frac{\partial^2\psi}{\partial\varphi^2}\right)\right\}$$

$$= \hbar^2\left\{2i\,csc^2\theta\frac{\partial\psi}{\partial\varphi} - 2i\cot^2\theta\frac{\partial\psi}{\partial\varphi}\right\}$$

$$= 2i\hbar^2\{csc^2\theta - cot^2\theta\}\frac{\partial\psi}{\partial\varphi}$$

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$$= 2i\hbar^2 \frac{\partial \psi}{\partial \varphi}$$

$$[\hat{L}_-, \hat{L}_+] = 2i\hbar^2 \frac{\partial}{\partial \varphi}$$

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LAMPIRAN I. SIMULASI FUNGSI GELOMBANG ATOM HIDROGEN DENGAN PROGRAM MATLAB VERSI R2014a

I.1 Fungsi Radial

a. Bilangan $n = 1$

```
a0=5.3;
n=0:0.0001:6;
r=n.*a0;
pr=r./a0;
R10=2.*((1./a0)^1.5).*exp(-r./a0).*10;
plot(pr,R10,'linewidth',2)
xlabel('r = posisi elektron (a0)')
ylabel('R(r)')
title('Fungsi Gelombang Radial (Abdul Rafie N.)')
legend('n=1')
axis ([0 6 0 2])
```

b. Bilangan $n = 2$

```
a0=5.3;
n=0:0.0001:10;
r=n.*a0;
pr=r./a0;
R20=(1./(2*(sqrt(2)))).*((1./a0).^1.5).*(2-(r/a0)).*exp(-
r/(2*a0)).*10;
R21=(1./(2*(sqrt(6)))).*((1./a0).^1.5).*(r./a0).*exp(-
r/(2.*a0)).*10;
plot(pr,R20,'r-',pr,R21,'linewidth',2)
xlabel('r = posisi elektron (a0)')
ylabel('R(r)')
title('Fungsi Gelombang Radial (Abdul Rafie N.)')
legend('n=2,l=0','n=2,l=1')
axis ([0 10 -0.2 0.8])
```

c. Bilangan $n = 3$

```
a0=5.3;
n=0:0.001:20;
r=n.*a0;
pr=r./a0;
R30=(2./(81.*(sqrt(3)))).*((1./a0).^1.5).*(27-
(18.*(r./a0))+(2.*(r.^2)./(a0.^2))).*exp(-r./(3.*a0)).*10;
R31=(4./(81.*(sqrt(6)))).*((1./a0).^1.5).*(r./a0).*(6-
(r./a0)).*exp(-r./(3.*a0)).*10;
R32=(4./(81.*(sqrt(30)))).*((1./a0).^1.5).*((r.^2)/(a0.^2)).*exp(-
r/(3.*a0)).*10;
plot(pr,R30,'r-',pr,R31,'g-',pr,R32,'linewidth',2)
xlabel('r = posisi elektron (a0)')
ylabel('R(r)')
title('Fungsi Gelombang Radial (Abdul Rafie N.)')
legend('n=3,l=0','n=3,l=1','n=3,l=2')
```

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```
axis ([0 20 -0.1 0.4])
```

I.2 Fungsi Harmonik

```
L = input('Enter the orbital L (L = 0, 1, 2, 3, ...)  ')
m_L = input('Enter the magnetic quantum number mL (mL = 0, 1, 2,
3, ... L)  ')

tmin = -pi;
tmax = pi;
numt = 511;
dt = (tmax-tmin)/(numt-1);
t = tmin : dt : tmax;

%define cos(t)
z = cos(t);

%definite associated legendre polynomials

Plm = legendre(L,z);

A = Plm(m_L+1,:)/max(abs(Plm(m_L+1,:)));
B = -A;

fs = 10;

xA = A.^2 .* sin(t);
yA = A.^2 .* cos(t);
xB = B.^2 .* sin(t);
yB = B.^2 .* cos(t);

tm1 = 'P({\itl},{\itm})^2 =  P(';
tm2 = num2str(L);
tm3 = ', ';
tm4 = num2str(m_L);
tm5 = ')^2';
tm = [tm1 tm2 tm3 tm4 tm5];

%set(gcf,'Units','centimeters','position',[16,6,12,11]);
subplot(1,2,2);
plot(xA,yA,'b', 'linewidth',3)
hold on
plot(xB,yB,'b', 'linewidth',3)
plot([0 0],[0,1.2],'b','linewidth',2);
h_plot = plot([0 0],[1.18,1.18],'^');
set(h_plot,'MarkerSize',8,'MarkerFaceColor','b',
'MarkerEdgeColor','b');
text(0.15,1.2,'{\itZ} axis','Fontsize',fs);
text(-1.8,1,tm,'Fontsize',fs);

axis equal tight
axis off
```

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LAMPIRAN J. KOMUTATOR OPERATOR MOMENTUM SUDUT DENGAN FUNGSI HARMONIK BOLA ATOM HIDROGEN

I.1 Harmonik Bola Y_{00}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y]Y_{00} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)Y_{00}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$- \left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{00}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial Y_{00}}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial Y_{00}}{\partial\varphi}\right.$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial Y_{00}}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial Y_{00}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{00}}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{00}}{\partial\varphi}\right.$$

$$\left. - cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{00}}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{00}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sqrt{4\pi}} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}\right.$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sqrt{4\pi}} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}\right\}$$

$$- \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sqrt{4\pi}} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}\right.$$

$$\left. - cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sqrt{4\pi}} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}\right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi(0) - sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi(0) + cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi(0) - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi(0) \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi(0) + cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi(0) - cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi(0) - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi(0) \right\}$$

$[\hat{L}_x, \hat{L}_y]Y_{00} = 0$

b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z]Y_{00} = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)Y_{00}$$

$$= \left[\left\{ i\hbar \left(-\cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} - \left\{ -i\hbar \frac{\partial}{\partial\varphi} \times i\hbar \left(-\cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{00}$$

$$= \hbar^2 \left\{ -\cos\varphi \frac{\partial}{\partial\theta} \frac{\partial Y_{00}}{\partial\varphi} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{00}}{\partial\varphi} \right\} - \hbar^2 \left\{ -\frac{\partial}{\partial\varphi} \cos\varphi \frac{\partial Y_{00}}{\partial\theta} + \frac{\partial}{\partial\varphi} \sin\varphi \cot\theta \frac{\partial Y_{00}}{\partial\varphi} \right\}$$

$$= \hbar^2 \left\{ -\cos\varphi \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}} \right\} - \hbar^2 \left\{ -\frac{\partial}{\partial\varphi} \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} + \frac{\partial}{\partial\varphi} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}} \right\}$$

$$= \hbar^2 \{ -\cos\varphi\,(0) + \sin\varphi \cot\theta\,(0) \} - \hbar^2 \left\{ -\frac{\partial}{\partial\varphi} \cos\varphi\,(0) + \frac{\partial}{\partial\varphi} \sin\varphi \cot\theta\,(0) \right\}$$

$[\hat{L}_y, \hat{L}_z]Y_{00} = 0$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x]Y_{00} = (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)Y_{00}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} - \left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{00}$$

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$$= \hbar^2 \left\{ \frac{\partial}{\partial \varphi} sin\varphi \frac{\partial Y_{00}}{\partial \theta} + \frac{\partial}{\partial \varphi} cot\theta cos\varphi \frac{\partial Y_{00}}{\partial \varphi} \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial \theta} \frac{\partial Y_{00}}{\partial \varphi} + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \frac{\partial Y_{00}}{\partial \varphi} \right\}$$

$$= \hbar^2 \left\{ \frac{\partial}{\partial \varphi} sin\varphi \frac{\partial}{\partial \theta} \frac{1}{\sqrt{4\pi}} + \frac{\partial}{\partial \varphi} cot\theta cos\varphi \frac{\partial}{\partial \varphi} \frac{1}{\sqrt{4\pi}} \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \frac{1}{\sqrt{4\pi}} + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \frac{1}{\sqrt{4\pi}} \right\}$$

$$= \hbar^2 \left\{ \frac{\partial}{\partial \varphi} sin\varphi (0) + \frac{\partial}{\partial \varphi} cot\theta cos\varphi (0) \right\}$$

$$- \hbar^2 \{ sin\varphi (0) + cot\theta cos\varphi (0) \}$$

$$[\hat{L}_z, \hat{L}_x] Y_{00} = 0$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+] Y_{00} = (\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z) Y_{00}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial \varphi} \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi} \right) \right\} \right.$$

$$\left. - \left\{ \hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi} \right) \times -i\hbar \frac{\partial}{\partial \varphi} \right\} \right] Y_{00}$$

$$= i\hbar^2 \left\{ - \frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \right\} Y_{00}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right\} Y_{00}$$

$$= i\hbar^2 \left\{ - \frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial Y_{00}}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \frac{\partial Y_{00}}{\partial \varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial Y_{00}}{\partial \varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial Y_{00}}{\partial \varphi} \right\}$$

$$= i\hbar^2 \left\{ - \frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} \frac{1}{\sqrt{4\pi}} - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{1}{\sqrt{4\pi}} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \frac{1}{\sqrt{4\pi}} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \frac{1}{\sqrt{4\pi}} \right\}$$

$$= i\hbar^2 \left\{ - \frac{\partial}{\partial \varphi} e^{i\varphi} (0) - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \, (0) \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} (0) - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} (0) \right\}$$

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$[\hat{L}_z, \hat{L}_+]Y_{00} = 0$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-]Y_{00} = (\hat{L}_z\hat{L}_- - \hat{L}_-\hat{L}_z)Y_{00}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right] Y_{00}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right\}Y_{00}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}Y_{00}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial Y_{00}}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial Y_{00}}{\partial\varphi}\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial Y_{00}}{\partial\varphi} - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{00}}{\partial\varphi}\right\}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{1}{\sqrt{4\pi}} - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}} - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}\right\}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}e^{i\varphi}(0) - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\, e^{i\varphi}(0)\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}e^{i\varphi}(0) - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}e^{i\varphi}(0)\right\}$$

$[\hat{L}_z, \hat{L}_-]Y_{00} = 0$

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f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2]Y_{00} = (\hat{L}_x\hat{L}^2 - \hat{L}^2\hat{L}_x)Y_{00}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{00}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{00}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{00}$$

$$+ i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left.+ \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\}Y_{00}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial Y_{00}}{\partial\theta} + sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{00}}{\partial\theta^2} + sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{00}}{\partial\varphi^2} \right.$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{00}}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{00}}{\partial\theta^2}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{00}}{\partial\varphi^2} \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial Y_{00}}{\partial\theta} + \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial Y_{00}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial Y_{00}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial Y_{00}}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial Y_{00}}{\partial\theta}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial Y_{00}}{\partial\varphi} \right\}$$

$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} + sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \frac{1}{\sqrt{4\pi}} \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{1}{\sqrt{4\pi}} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \frac{1}{\sqrt{4\pi}} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{1}{\sqrt{4\pi}} \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} + \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} + \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}} \right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta\,(0) + sin\varphi \frac{\partial}{\partial\theta}(0) + sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta}(0) \right.$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta\,(0) + cot\theta cos\varphi \frac{\partial}{\partial\varphi}(0)$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta}(0) \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi(0) + \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi(0) + \frac{\partial^2}{\partial\theta^2} sin\varphi(0) \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi(0) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi(0)$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi(0) \right\}$$

$$[\hat{L}_x, \hat{L}^2]Y_{00} = 0$$

g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$$[\hat{L}_y, \hat{L}^2]Y_{00} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{00}$$

$$= \left[\left\{ -i\hbar \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times -i\hbar \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{00}$$

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$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta}+\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2}+\cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}$$

$$\left.-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{00}$$

$$-i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}+\frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial}{\partial\theta}\right.$$

$$-\frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$\left.-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\}Y_{00}$$

$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial Y_{00}}{\partial\theta}+\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{00}}{\partial\theta^2}+\cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{00}}{\partial\varphi^2}\right.$$

$$-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial Y_{00}}{\partial\theta}$$

$$\left.-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{00}}{\partial\theta^2}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{00}}{\partial\varphi^2}\right\}$$

$$-i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial Y_{00}}{\partial\theta}-\cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial Y_{00}}{\partial\varphi}\right.$$

$$+\frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial Y_{00}}{\partial\theta}-\frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial Y_{00}}{\partial\varphi}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial Y_{00}}{\partial\theta}$$

$$\left.-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial Y_{00}}{\partial\varphi}\right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \frac{1}{\sqrt{4\pi}} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{1}{\sqrt{4\pi}} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \frac{1}{\sqrt{4\pi}} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{1}{\sqrt{4\pi}} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sqrt{4\pi}} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sqrt{4\pi}} \right\}$$

$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta\,(0) + \cos\varphi \frac{\partial}{\partial\theta}(0) + \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta}(0) \right.$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta\,(0)$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi}(0) - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta}(0) \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi\,(0) - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta\,(0) \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi\,(0) - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta\,(0) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi\,(0)$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta\,(0) \right\}$$

$$[\hat{L}_y, \hat{L}^2] Y_{00} = 0$$

h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z, \hat{L}^2] Y_{00} = (\hat{L}_z \hat{L}^2 - \hat{L}^2 \hat{L}_z) Y_{00}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{00}$$

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$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{00}$$

$$-i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}+\frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\right\}Y_{00}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial Y_{00}}{\partial\theta}+\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{00}}{\partial\theta^2}+\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{00}}{\partial\varphi^2}\right\}$$

$$-i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial Y_{00}}{\partial\varphi}+\frac{\partial^2}{\partial\theta^2}\frac{\partial Y_{00}}{\partial\varphi}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial Y_{00}}{\partial\varphi}\right\}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}\frac{1}{\sqrt{4\pi}}+\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}\frac{1}{\sqrt{4\pi}}+\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{1}{\sqrt{4\pi}}\right\}$$

$$-i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}+\frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\frac{1}{\sqrt{4\pi}}\right\}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}\cot\theta\,(0)+\frac{\partial}{\partial\varphi}(0)+\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}(0)\right\}$$

$$-i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}(0)+\frac{\partial^2}{\partial\theta^2}(0)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}(0)\right\}$$

$$[\hat{L}_z,\hat{L}^2]Y_{00} = 0$$

I.2 Harmonik Bola Y_{10}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x,\hat{L}_y]Y_{10} = (\hat{L}_x\hat{L}_y-\hat{L}_y\hat{L}_x)Y_{10}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta}+cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta}-cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$-\left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta}-cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta}+cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{10}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial Y_{10}}{\partial\theta} - sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial Y_{10}}{\partial\varphi} \right.$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial Y_{10}}{\partial\theta} - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial Y_{10}}{\partial\varphi} \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial Y_{10}}{\partial\theta} + cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial Y_{10}}{\partial\varphi} \right.$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial Y_{10}}{\partial\theta} - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial Y_{10}}{\partial\varphi} \right\}$$

$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} \cos\theta \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} \cos\theta \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} \cos\theta \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} \cos\theta \right\}$$

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$$= \hbar^2 \left\{ -\sqrt{\frac{3}{4\pi}} sin\varphi cos\varphi \cos\theta - sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi (0) \right.$$

$$+ \sqrt{\frac{3}{4\pi}} cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi (-\sin\theta)$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi (0) \right\}$$

$$- \hbar^2 \left\{ -\sqrt{\frac{3}{4\pi}} cos\varphi sin\varphi \cos\theta + cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi (0) \right.$$

$$- \sqrt{\frac{3}{4\pi}} cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi (-\sin\theta)$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi (0) \right\}$$

$$= \hbar^2 \left\{ -\sqrt{\frac{3}{4\pi}} sin\varphi cos\varphi \cos\theta - 0 - \sqrt{\frac{3}{4\pi}} cot\theta cos\varphi \sin\theta \, (-sin\varphi) \right.$$

$$\left. - 0 \right\}$$

$$- \hbar^2 \left\{ -\sqrt{\frac{3}{4\pi}} cos\varphi sin\varphi \cos\theta + 0 + \sqrt{\frac{3}{4\pi}} cot\theta sin\varphi \sin\theta \, (cos\varphi) \right.$$

$$\left. - 0 \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{3}{4\pi}} cos\theta cos\varphi sin\varphi - \sqrt{\frac{3}{4\pi}} cos\theta sin\varphi cos\varphi \right\}$$

$$= \hbar^2 \{ cos\varphi sin\varphi - sin\varphi cos\varphi \} \sqrt{\frac{3}{4\pi}} cos\theta$$

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$[\hat{L}_x, \hat{L}_y]Y_{10} = (0)Y_{10}$

$[\hat{L}_x, \hat{L}_y] = 0$

b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z]Y_{10} = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)Y_{10}$$

$$= \left[\left\{i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right.$$

$$\left. - \left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{10}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{10}}{\partial\varphi} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{10}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial Y_{10}}{\partial\theta} + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial Y_{10}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{4\pi}}cos\theta + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{4\pi}}cos\theta\right\}$$

$$- \hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\theta}\sqrt{\frac{3}{4\pi}}cos\theta + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{4\pi}}cos\theta\right\}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}(0) + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}(0)\right\}$$

$$- \hbar^2\left\{-\sqrt{\frac{3}{4\pi}}\frac{\partial}{\partial\varphi}\cos\varphi\,(-sin\theta) + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\,(0)\right\}$$

$$[\hat{L}_y, \hat{L}_z]Y_{10} = \hbar^2\sqrt{\frac{3}{4\pi}}sin\theta(\sin\varphi)$$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x]Y_{10} = (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)Y_{10}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right] Y_{10}$$

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$$= \hbar^2 \left\{ \frac{\partial}{\partial \varphi} sin\varphi \frac{\partial Y_{10}}{\partial \theta} + \frac{\partial}{\partial \varphi} cot\theta cos\varphi \frac{\partial Y_{10}}{\partial \varphi} \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial \theta} \frac{\partial Y_{10}}{\partial \varphi} + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \frac{\partial Y_{10}}{\partial \varphi} \right\}$$

$$= \hbar^2 \left\{ \frac{\partial}{\partial \varphi} sin\varphi \frac{\partial}{\partial \theta} \sqrt{\frac{3}{4\pi}} cos\theta + \frac{\partial}{\partial \varphi} cot\theta cos\varphi \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{4\pi}} cos\theta + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{3}{4\pi}} \frac{\partial}{\partial \varphi} sin\varphi(-sin\theta) + \frac{\partial}{\partial \varphi} cot\theta cos\varphi(0) \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial \theta}(0) + cot\theta cos\varphi \frac{\partial}{\partial \varphi}(0) \right\}$$

$$[\hat{L}_z, \hat{L}_x] Y_{10} = -\hbar^2 \sqrt{\frac{3}{4\pi}} sin\theta cos\varphi$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+] Y_{10} = (\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z) Y_{10}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial \varphi} \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi} \right) \right\} \right.$$

$$\left. - \left\{ \hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi} \right) \times -i\hbar \frac{\partial}{\partial \varphi} \right\} \right] Y_{10}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \right\} Y_{10}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right\} Y_{10}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial Y_{10}}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \frac{\partial Y_{10}}{\partial \varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial Y_{10}}{\partial \varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial Y_{10}}{\partial \varphi} \right\}$$

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$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} \sqrt{\frac{3}{4\pi}} cos\theta - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{4\pi}} cos\theta - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$= i\hbar^2 \left\{ -\sqrt{\frac{3}{4\pi}} \frac{\partial}{\partial \varphi} e^{i\varphi} (-sin\theta) - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \, (0) \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} (0) - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} (0) \right\}$$

$$= i\hbar^2 \sqrt{\frac{3}{4\pi}} i e^{i\varphi} sin\theta$$

$$[\hat{L}_z, \hat{L}_+] Y_{10} = -\hbar^2 \sqrt{\frac{3}{4\pi}} e^{i\varphi} sin\theta$$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-] Y_{10} = (\hat{L}_z \hat{L}_- - \hat{L}_- \hat{L}_z) Y_{10}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial \varphi} \times -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \varphi} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \varphi} \right) \times -i\hbar \frac{\partial}{\partial \varphi} \right\} \right] Y_{10}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial \varphi} e^{-i\varphi} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \right\} Y_{10}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right\} Y_{10}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial \varphi} e^{-i\varphi} \frac{\partial Y_{10}}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{-i\varphi} i \cot\theta \frac{\partial Y_{10}}{\partial \varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial \theta} \frac{\partial Y_{10}}{\partial \varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial Y_{10}}{\partial \varphi} \right\}$$

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$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta}\sqrt{\frac{3}{4\pi}}cos\theta - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{4\pi}}cos\theta\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{4\pi}}cos\theta - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{4\pi}}cos\theta\right\}$$

$$= i\hbar^2\left\{\sqrt{\frac{3}{4\pi}}\frac{\partial}{\partial\varphi}e^{-i\varphi}(-sin\theta) - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\,(0)\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}(0) - e^{-i\varphi}i\cot\theta\,(0)\right\}$$

$$= -i\hbar^2\sqrt{\frac{3}{4\pi}} \times -ie^{-i\varphi}sin\theta$$

$$[\hat{L}_z, \hat{L}_-]Y_{10} = -\hbar^2\sqrt{\frac{3}{4\pi}}e^{-i\varphi}sin\theta$$

f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2]Y_{10} = (\hat{L}_x\hat{L}^2 - \hat{L}^2\hat{L}_x)Y_{10}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{10}$$

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$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{10}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} Y_{10}$$

$$+ i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left. + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\} Y_{10}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial Y_{10}}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{10}}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{10}}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial Y_{10}}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{10}}{\partial\theta^2}$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{10}}{\partial\varphi^2}\right\}$$

$$+ i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{10}}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{10}}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial Y_{10}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial Y_{10}}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial Y_{10}}{\partial\theta}$$

$$\left. + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial Y_{10}}{\partial\varphi}\right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta + sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{4\pi}} cos\theta \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta \right.$$

$$+ \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta + \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

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$$= -i\hbar^3 \left\{ \sqrt{\frac{3}{4\pi}} sin\varphi \frac{\partial}{\partial\theta} \cot\theta\,(-sin\theta) + \sqrt{\frac{3}{4\pi}} sin\varphi(sin\theta) \right.$$

$$+ \sqrt{\frac{3}{4\pi}} sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta}(0) + \sqrt{\frac{3}{4\pi}} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta\,(-sin\theta)$$

$$\left. + \sqrt{\frac{3}{4\pi}} cot\theta cos\varphi(0) + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta}(0) \right\}$$

$$+ i\hbar^3 \left\{ \sqrt{\frac{3}{4\pi}} \cot\theta \frac{\partial}{\partial\theta} sin\varphi(-sin\theta) + \sqrt{\frac{3}{4\pi}} \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi(0) \right.$$

$$+ \sqrt{\frac{3}{4\pi}} \frac{\partial^2}{\partial\theta^2} sin\varphi(-sin\theta) + \sqrt{\frac{3}{4\pi}} \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi(0)$$

$$\left. + \sqrt{\frac{3}{4\pi}} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi(-sin\theta) + \sqrt{\frac{3}{4\pi}} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi(0) \right\}$$

$$= -i\hbar^3 \left\{ \sqrt{\frac{3}{4\pi}} sin\varphi \frac{\partial}{\partial\theta}(-\cos\theta) + \sqrt{\frac{3}{4\pi}} sin\varphi(sin\theta) + 0 \right.$$

$$\left. + \sqrt{\frac{3}{4\pi}} cot\theta cos\varphi \frac{\partial}{\partial\varphi}(-\cos\theta) + 0 + 0 \right\}$$

$$+ i\hbar^3 \left\{ \sqrt{\frac{3}{4\pi}} sin\varphi \cot\theta\,(-cos\theta) + 0 + \sqrt{\frac{3}{4\pi}} sin\varphi(sin\theta) + 0 \right.$$

$$\left. + \sqrt{\frac{3}{4\pi}} \frac{1}{\sin^2\theta}(-sin\theta)(-sin\varphi) + 0 \right\}$$

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$$= -i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}sin\varphi(\sin\theta) + \sqrt{\frac{3}{4\pi}}sin\varphi(sin\theta) + \sqrt{\frac{3}{4\pi}}cot\theta cos\varphi(0)\right\}$$

$$+ i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}sin\varphi\cot\theta\,(-cos\theta) + \sqrt{\frac{3}{4\pi}}sin\varphi(sin\theta)\right.$$

$$\left.+ \sqrt{\frac{3}{4\pi}}\frac{1}{\sin^2\theta}(-sin\theta)(-sin\varphi)\right\}$$

$$= -i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}sin\varphi(\sin\theta)\right\} + i\hbar^3\left\{-\cos^2\theta + 1\left(\sqrt{\frac{3}{4\pi}}sin\varphi\frac{1}{sin\theta}\right)\right\}$$

$$= -i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}sin\varphi(\sin\theta)\right\} + i\hbar^3\left\{\sin^2\theta\left(\sqrt{\frac{3}{4\pi}}sin\varphi\frac{1}{sin\theta}\right)\right\}$$

$$= -i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}sin\varphi(\sin\theta)\right\} + i\hbar^3\left\{\sin\theta\left(\sqrt{\frac{3}{4\pi}}sin\varphi\right)\right\} \times \left(\frac{cos\theta}{cos\theta}\right)$$

$$= -i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}sin\varphi(\sin\theta)\frac{cos\theta}{cos\theta}\right\} + i\hbar^3\left\{\sin\theta\left(\sqrt{\frac{3}{4\pi}}sin\varphi\frac{cos\theta}{cos\theta}\right)\right\}$$

$$= -i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}cos\theta sin\varphi(\sin\theta)tan\theta\right\}$$

$$+ i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}cos\theta sin\varphi(\sin\theta)tan\theta\right\}$$

$$= i\hbar^3\{-sin\varphi(\sin\theta)tan\theta + sin\varphi(\sin\theta)tan\theta\}\sqrt{\frac{3}{4\pi}}cos\theta$$

$$[\hat{L}_x, \hat{L}^2]Y_{10} = 0Y_{10}$$

$$[\hat{L}_x, \hat{L}^2] = 0$$

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g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$$[\hat{L}_y,\hat{L}^2]Y_{10} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{10}$$

$$= \Bigg[\Bigg\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Bigg\}$$

$$-\Bigg\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$\times -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\Bigg\}\Bigg]Y_{10}$$

$$= i\hbar^3\Bigg\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + \cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + \cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}$$

$$-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Bigg\}Y_{10}$$

$$-i\hbar^3\Bigg\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$-\frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\Bigg\}Y_{10}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial Y_{10}}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{10}}{\partial\theta^2} + \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{10}}{\partial\varphi^2} \right.$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{10}}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{10}}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{10}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial Y_{10}}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial Y_{10}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial Y_{10}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial Y_{10}}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial Y_{10}}{\partial\theta}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial Y_{10}}{\partial\varphi} \right\}$$

$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{4\pi}} cos\theta \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{4\pi}} cos\theta - \sin\varphi\cot\theta \frac{\partial}{\partial\varphi} cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta + \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$- \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

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$$= i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\,(-sin\theta) + \sqrt{\frac{3}{4\pi}}\cos\varphi\,(sin\theta)\right.$$

$$+\sqrt{\frac{3}{4\pi}}\cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}(0) - \sqrt{\frac{3}{4\pi}}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\,(-sin\theta)$$

$$\left.-\sqrt{\frac{3}{4\pi}}\sin\varphi\cot\theta\,(0) - \sqrt{\frac{3}{4\pi}}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}(0)\right\}$$

$$- i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\,(-sin\theta) - \sqrt{\frac{3}{4\pi}}\cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\,(0)\right.$$

$$+\sqrt{\frac{3}{4\pi}}\frac{\partial^2}{\partial\theta^2}\cos\varphi\,(-sin\theta) - \sqrt{\frac{3}{4\pi}}\frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\,(0)$$

$$+\sqrt{\frac{3}{4\pi}}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\,(-sin\theta)$$

$$\left.-\sqrt{\frac{3}{4\pi}}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\,(0)\right\}$$

$$= i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}\cos\varphi\,(sin\theta) + \sqrt{\frac{3}{4\pi}}\cos\varphi\,(sin\theta)\right\}$$

$$- i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}\cot\theta\cos\varphi\,(-cos\theta) + \sqrt{\frac{3}{4\pi}}\cos\varphi\,(sin\theta)\right.$$

$$\left.+\sqrt{\frac{3}{4\pi}}\frac{1}{\sin^2\theta}\cos\varphi\,(sin\theta)\right\}$$

$$= i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}\cos\varphi\,(sin\theta)\right\}$$

$$- i\hbar^3\left\{-\sqrt{\frac{3}{4\pi}}\cos^2\theta\cos\varphi\frac{1}{sin\theta} + \sqrt{\frac{3}{4\pi}}\frac{1}{sin\theta}\cos\varphi\right\}$$

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$$= i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}\cos\varphi\,(sin\theta)\right\} - i\hbar^3\left\{-\cos^2\theta + 1\left(\sqrt{\frac{3}{4\pi}}\frac{1}{sin\theta}\cos\varphi\right)\right\}$$

$$= i\hbar^3\left\{\sqrt{\frac{3}{4\pi}}\cos\varphi\,(sin\theta)\right\} - i\hbar^3\left\{\sin^2\theta\left(\sqrt{\frac{3}{4\pi}}\frac{1}{sin\theta}\cos\varphi\right)\right\}$$

$$= i\hbar^3\{\cos\varphi\,(sin\theta) - sin\theta\cos\varphi\}\sqrt{\frac{3}{4\pi}}\times\frac{cos\theta}{cos\theta}$$

$$= \frac{i\hbar^3}{cos\theta}\{\cos\varphi\,(sin\theta) - sin\theta\cos\varphi\}\sqrt{\frac{3}{4\pi}}cos\theta$$

$$[\hat{L}_y, \hat{L}^2]Y_{10} = 0\,Y_{10}$$

$$[\hat{L}_y, \hat{L}^2] = 0$$

h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z, \hat{L}^2]Y_{10} = (\hat{L}_z\hat{L}^2 - \hat{L}^2\hat{L}_z)Y_{10}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi}\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}\right.$$

$$\left. - \left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]Y_{10}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{10}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\right\}Y_{10}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial Y_{10}}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{10}}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{10}}{\partial\varphi^2}\right\}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial Y_{10}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial Y_{10}}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial Y_{10}}{\partial\varphi}\right\}$$

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$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{4\pi}} cos\theta + \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{4\pi}} cos\theta \right.$$

$$\left. + \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta + \frac{\partial^2}{\partial\theta^2} \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right.$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{4\pi}} cos\theta \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{3}{4\pi}} \frac{\partial}{\partial\varphi} \cot\theta\,(-sin\theta) + \sqrt{\frac{3}{4\pi}}(0) + \sqrt{\frac{3}{4\pi}} \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta}(0) \right\}$$

$$- i\hbar^3 \left\{ \sqrt{\frac{3}{4\pi}} \cot\theta \frac{\partial}{\partial\theta}(0) + \sqrt{\frac{3}{4\pi}} \frac{\partial^2}{\partial\theta^2}(0) + \sqrt{\frac{3}{4\pi}} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2}(0) \right\}$$

$$= -\, i\hbar^3 \sqrt{\frac{3}{4\pi}} \frac{\partial}{\partial\varphi} \cot\theta\,(sin\theta) \times \frac{cos\theta}{cos\theta}$$

$$= \frac{-\, i\hbar^3}{cos\theta} \frac{\partial}{\partial\varphi} \cot\theta\,(sin\theta) \sqrt{\frac{3}{4\pi}} cos\theta$$

$$= \frac{-\, i\hbar^3}{cos\theta}(0) \sqrt{\frac{3}{4\pi}} cos\theta$$

$$[\hat{L}_z, \hat{L}^2] Y_{10} = 0\; Y_{10}$$

$$[\hat{L}_z, \hat{L}^2] = 0$$

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I.3 Harmonik Bola Y_{1-1}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y]Y_{1-1} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)Y_{1-1}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}$$

$$-\left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{1-1}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial Y_{1-1}}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial Y_{1-1}}{\partial\varphi}\right.$$

$$\left.+\, cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial Y_{1-1}}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

$$-\,\hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{1-1}}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{1-1}}{\partial\varphi}\right.$$

$$\left.-\, cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{1-1}}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi} \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi} \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi} \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi} \right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} e^{-i\varphi} sin\varphi cos\varphi(-\sin\theta) \right.$$

$$- \sqrt{\frac{3}{8\pi}} sin\varphi sin\varphi(-sin\theta)\left(-ie^{-i\varphi}\right)$$

$$+ \sqrt{\frac{3}{8\pi}} cot\theta cos\varphi\left(-sin\varphi e^{-i\varphi} - icos\varphi e^{-i\varphi}\right)(\cos\theta)$$

$$\left. - \sqrt{\frac{3}{8\pi}} cot\theta cos\varphi cos\theta\left(-icos\varphi e^{-i\varphi} - sin\varphi e^{-i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} cos\varphi sin\varphi e^{-i\varphi} (-\sin\theta) \right.$$

$$+ \sqrt{\frac{3}{8\pi}} cos\varphi cos\varphi(-\sin\theta)\left(-ie^{-i\varphi}\right)$$

$$- \sqrt{\frac{3}{8\pi}} cot\theta sin\varphi cos\theta\left(cos\varphi e^{-i\varphi} - isin\varphi e^{-i\varphi}\right)$$

$$\left. - \sqrt{\frac{3}{8\pi}} cot\theta sin\varphi cos\theta\left(isin\varphi e^{-i\varphi} - cos\varphi e^{-i\varphi}\right) \right\}$$

$$= \hbar^2 \left\{ - \sqrt{\frac{3}{8\pi}} sin^2\varphi(-sin\theta)\left(-ie^{-i\varphi}\right) \right.$$

$$\left. - \sqrt{\frac{3}{8\pi}} cos^2\varphi(-sin\theta)\left(-ie^{-i\varphi}\right) \right\}$$

$$= -\hbar^2 ie^{-i\varphi} sin\theta \sqrt{\frac{3}{8\pi}} \{sin^2\varphi + cos^2\varphi\}$$

$$= -i\hbar^2 \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$\left[\hat{L}_x, \hat{L}_y\right] Y_{1-1} = -i\hbar^2 Y_{1-1}$$

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$[\hat{L}_x, \hat{L}_y] = -i\hbar^2$

b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z]Y_{1-1} = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)Y_{1-1}$$

$$= \left[\left\{i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right.$$

$$\left.-\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{1-1}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{1-1}}{\partial\varphi} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

$$-\hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial Y_{1-1}}{\partial\theta} + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}\right.$$

$$\left.+\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}\right\}$$

$$-\hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\theta}\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}\right.$$

$$\left.+\frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}\right\}$$

$$= \hbar^2\left\{-\sqrt{\frac{3}{8\pi}}\cos\varphi\cos\theta\,\left(-ie^{-i\varphi}\right) + \sqrt{\frac{3}{8\pi}}\sin\varphi\cos\theta\left(-e^{-i\varphi}\right)\right\}$$

$$-\hbar^2\left\{-\sqrt{\frac{3}{8\pi}}\cos\theta\,\left(-\sin\varphi e^{-i\varphi} - i\cos\varphi e^{-i\varphi}\right)\right.$$

$$\left.+\sqrt{\frac{3}{8\pi}}\cos\theta\left(-icos\varphi e^{-i\varphi} - \sin\varphi\, e^{-i\varphi}\right)\right\}$$

$$= \hbar^2\left\{\sqrt{\frac{3}{8\pi}}e^{-i\varphi}\,cos\,\theta\,\,(icos\varphi) - \sqrt{\frac{3}{8\pi}}e^{-i\varphi}\,cos\,\theta\,(sin\varphi)\right\}$$

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$$= \hbar^2 \cos\theta \{icos\varphi - sin\varphi\} \frac{i}{i} \sqrt{\frac{3}{8\pi}} e^{-i\varphi}$$

$$= -\frac{\hbar^2}{i} \cos\theta \{cos\varphi + isin\varphi\} \sqrt{\frac{3}{8\pi}} e^{-i\varphi}$$

$$[\hat{L}_y, \hat{L}_z] Y_{1-1} = -\frac{\hbar^2}{i} \cos\theta \{e^{i\varphi}\} \sqrt{\frac{3}{8\pi}} e^{-i\varphi}$$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x] Y_{1-1} = (\hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z) Y_{1-1}$$

$$= \left[\left\{-i\hbar \frac{\partial}{\partial\varphi} \times i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right) \times -i\hbar \frac{\partial}{\partial\varphi}\right\}\right] Y_{1-1}$$

$$= \hbar^2 \left\{\frac{\partial}{\partial\varphi} sin\varphi \frac{\partial Y_{1-1}}{\partial\theta} + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

$$- \hbar^2 \left\{sin\varphi \frac{\partial}{\partial\theta} \frac{\partial Y_{1-1}}{\partial\varphi} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

$$= \hbar^2 \left\{\frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi}\right.$$

$$\left. + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi}\right\}$$

$$- \hbar^2 \left\{sin\varphi \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi}\right.$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi}\right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} \cos\theta \, \left(cos\varphi e^{-i\varphi} - isin\varphi e^{-i\varphi}\right) \right.$$

$$\left. + \sqrt{\frac{3}{8\pi}} cos\theta\left(isin\varphi e^{-i\varphi} - cos\varphi e^{-i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} sin\varphi \cos\theta \, \left(-ie^{-i\varphi}\right) + \sqrt{\frac{3}{8\pi}} cos\theta cos\varphi\left(-e^{-i\varphi}\right) \right\}$$

$$= -\hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} \cos\theta \; e^{-i\varphi}(-isin\varphi - cos\varphi) \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} \cos\theta \; e^{-i\varphi}(isin\varphi + cos\varphi) \right\}$$

$$[\hat{L}_z, \hat{L}_x]Y_{1-1} = \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} \cos\theta \; e^{-i\varphi}\left(e^{i\varphi}\right) \right\}$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+]Y_{1-1} = (\hat{L}_z\hat{L}_+ - \hat{L}_+\hat{L}_z)Y_{1-1}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\,cot\,\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\,cot\,\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right] Y_{1-1}$$

$$= i\hbar^2\left\{-\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{i\varphi}\,i\,cot\,\theta\frac{\partial}{\partial\varphi}\right\}Y_{1-1}$$

$$- i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - e^{i\varphi}\,i\,cot\,\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}Y_{1-1}$$

$$= i\hbar^2\left\{-\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial Y_{1-1}}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{i\varphi}\,i\,cot\,\theta\frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

$$- i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial Y_{1-1}}{\partial\varphi} - e^{i\varphi}\,i\,cot\,\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

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$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$\left. - \frac{\partial}{\partial \varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$\left. - e^{i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$= i\hbar^2 \left\{ -\sqrt{\frac{3}{8\pi}} cos\theta(0) - \sqrt{\frac{3}{8\pi}} \cos\theta\,(0) \right\}$$

$$- i\hbar^2 \left\{ -\sqrt{\frac{3}{8\pi}} e^{i\varphi} cos\theta\left(-ie^{-i\varphi}\right) - \sqrt{\frac{3}{8\pi}} e^{i\varphi} i \cos\theta \left(-e^{-i\varphi}\right) \right\}$$

$$= -i\hbar^2 \left\{ 2\sqrt{\frac{3}{8\pi}} e^{i\varphi} cos\theta\left(ie^{-i\varphi}\right) \right\}$$

$$[\hat{L}_z, \hat{L}_+] Y_{1-1} = 2\hbar^2 \{e^{i\varphi} cos\theta\} \sqrt{\frac{3}{8\pi}} e^{-i\varphi}$$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-] Y_{1-1} = (\hat{L}_z \hat{L}_- - \hat{L}_- \hat{L}_z) Y_{1-1}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial \varphi} \times -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \varphi} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \varphi} \right) \times -i\hbar \frac{\partial}{\partial \varphi} \right\} \right] Y_{1-1}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial \varphi} e^{-i\varphi} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \right\} Y_{1-1}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right\} Y_{1-1}$$

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$$= i\hbar^2 \left\{ \frac{\partial}{\partial \varphi} e^{-i\varphi} \frac{\partial Y_{1-1}}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{-i\varphi} i \cot\theta \frac{\partial Y_{1-1}}{\partial \varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial \theta} \frac{\partial Y_{1-1}}{\partial \varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial Y_{1-1}}{\partial \varphi} \right\}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial \varphi} e^{-i\varphi} \frac{\partial}{\partial \theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$\left. - \frac{\partial}{\partial \varphi} e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$\left. - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$= i\hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} cos\theta \left(e^{-2i\varphi}\right) - \sqrt{\frac{3}{8\pi}} \cos\theta \left(e^{-2i\varphi}\right) \right\}$$

$$- i\hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} cos\theta \left(-ie^{-2i\varphi}\right) + \sqrt{\frac{3}{8\pi}} i \cos\theta \left(e^{-2i\varphi}\right) \right\}$$

$$\left[\hat{L}_z, \hat{L}_-\right] Y_{1-1} = 0$$

f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$\left[\hat{L}_x, \hat{L}^2\right] Y_{1-1} = \left(\hat{L}_x \hat{L}^2 - \hat{L}^2 \hat{L}_x\right) Y_{1-1}$$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial \theta} + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right) \right.$$

$$\left. \left. \times i\hbar \left(sin\varphi \frac{\partial}{\partial \theta} + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \right) \right\} \right] Y_{1-1}$$

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$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{1-1}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right\} Y_{1-1}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left.+ \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\} Y_{1-1}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial Y_{1-1}}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{1-1}}{\partial\theta^2}\right.$$

$$+ sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{1-1}}{\partial\varphi^2} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial Y_{1-1}}{\partial\theta}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{1-1}}{\partial\theta^2} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{1-1}}{\partial\varphi^2}\right\}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{1-1}}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{1-1}}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial Y_{1-1}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial Y_{1-1}}{\partial\varphi}$$

$$\left.+ \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial Y_{1-1}}{\partial\theta} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial Y_{1-1}}{\partial\varphi}\right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$+ \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} + \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ \sqrt{\frac{3}{8\pi}} e^{-i\varphi} sin\varphi \left(-\frac{1}{sin^2\theta} cos\,\theta - cos\,\theta \right) \right.$$

$$+ \sqrt{\frac{3}{8\pi}} e^{-i\varphi} sin\varphi(- cos\,\theta) + \sqrt{\frac{3}{8\pi}} sin\varphi \frac{1}{sin^2\theta} cos\theta e^{-i\varphi}$$

$$- \sqrt{\frac{3}{8\pi}} ie^{-i\varphi} cot\theta cos\varphi\, cot\,\theta\, cos\theta + \sqrt{\frac{3}{8\pi}} cos\theta cos\varphi ie^{-i\varphi}$$

$$\left. + \sqrt{\frac{3}{8\pi}} cos\theta cos\varphi \frac{1}{sin^2\theta} ie^{-i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ -\sqrt{\frac{3}{8\pi}} cos\,\theta\, sin\varphi e^{-i\varphi} \right.$$

$$- \sqrt{\frac{3}{8\pi}} cot\,\theta cos\varphi ie^{-i\varphi} \left(-\frac{1}{sin\theta} + cot\,\theta\, cos\theta \right)$$

$$- \sqrt{\frac{3}{8\pi}} sin\varphi cos\theta e^{-i\varphi} + \sqrt{\frac{3}{8\pi}} cos\theta cos\varphi ie^{-i\varphi}$$

$$+ \sqrt{\frac{3}{8\pi}} cos\theta \frac{1}{sin^2\theta} \left(-2sin\varphi e^{-i\varphi} - 2icos\varphi e^{-i\varphi} \right)$$

$$\left. + \sqrt{\frac{3}{8\pi}} cos\theta \frac{1}{sin^2\theta} \left(2icos\varphi e^{-i\varphi} + 2sin\varphi e^{-i\varphi} \right) \right\}$$

$$[\hat{L}_x, \hat{L}^2] Y_{1-1} = 0$$

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g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$$[\hat{L}_y,\hat{L}^2]Y_{1-1} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{1-1}$$

$$= \left[\left\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{1-1}$$

$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + \cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + \cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}$$

$$\left.- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{1-1}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$\left.- \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\}Y_{1-1}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial Y_{1-1}}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{1-1}}{\partial\theta^2} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{1-1}}{\partial\varphi^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{1-1}}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{1-1}}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{1-1}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial Y_{1-1}}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial Y_{1-1}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial Y_{1-1}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial Y_{1-1}}{\partial\varphi}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial Y_{1-1}}{\partial\theta} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial Y_{1-1}}{\partial\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$\left. - sin\varphi cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$- \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{3}{8\pi}} e^{-i\varphi} \cos\varphi \left(-\frac{1}{\sin^2\theta} \cos\theta - \cos\theta \right) \right.$$

$$- \sqrt{\frac{3}{8\pi}} e^{-i\varphi} \cos\varphi \cos\theta + \sqrt{\frac{3}{8\pi}} \frac{1}{\sin^2\theta} \cos\varphi \cos\theta e^{-i\varphi}$$

$$+ i \sqrt{\frac{3}{8\pi}} \sin\varphi \cot\theta \cot\theta \cos\theta e^{-i\varphi}$$

$$\left. - i \sqrt{\frac{3}{8\pi}} \sin\varphi \cos\theta \, e^{-i\varphi} - \sqrt{\frac{3}{8\pi}} \sin\varphi \cos\theta \frac{1}{\sin^2\theta} \left(i e^{-i\varphi} \right) \right\}$$

$$- i\hbar^3 \left\{ - \sqrt{\frac{3}{8\pi}} \cos\theta \cos\varphi \, e^{-i\varphi} \right.$$

$$- \sqrt{\frac{3}{8\pi}} \cot\theta \, i e^{-i\varphi} \sin\varphi \left(-\frac{1}{\sin\theta} + \cot\theta \cos\theta \right)$$

$$- \sqrt{\frac{3}{8\pi}} e^{-i\varphi} \cos\varphi \cos\theta - \sqrt{\frac{3}{8\pi}} \sin\varphi \cos\theta \left(i e^{-i\varphi} \right)$$

$$+ \sqrt{\frac{3}{8\pi}} \frac{1}{\sin^2\theta} \cos\theta \left(-2\cos\varphi e^{-i\varphi} + 2i\sin\varphi \, e^{-i\varphi} \right)$$

$$\left. - \sqrt{\frac{3}{8\pi}} \frac{1}{\sin^2\theta} \cos\theta \left(-2\cos\varphi \, e^{-i\varphi} + 2\, i\sin\varphi \, e^{-i\varphi} \right) \right\}$$

$$[\hat{L}_y, \hat{L}^2] Y_{1-1} = 0$$

h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z, \hat{L}^2] Y_{1-1} = (\hat{L}_z \hat{L}^2 - \hat{L}^2 \hat{L}_z) Y_{1-1}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{1-1}$$

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$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right\} Y_{1-1}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi} \right\} Y_{1-1}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{1-1}}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{1-1}}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{1-1}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta}\frac{\partial Y_{1-1}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial Y_{1-1}}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial Y_{1-1}}{\partial\varphi} \right\}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta}\sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}\sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$\left. + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right.$$

$$\left. + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\sqrt{\frac{3}{8\pi}} sin\theta e^{-i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{3}{8\pi}} \cot\theta\, cos\theta\left(-ie^{-i\varphi}\right) - \sqrt{\frac{3}{8\pi}} sin\theta\left(-ie^{-i\varphi}\right) \right.$$

$$\left. + \sqrt{\frac{3}{8\pi}}\frac{1}{\sin^2\theta} sin\theta\left(ie^{-i\varphi}\right) \right\}$$

$$- i\hbar^3 \left\{ \sqrt{\frac{3}{8\pi}} cos\theta\cot\theta\left(-ie^{-i\varphi}\right) - \sqrt{\frac{3}{8\pi}} sin\theta\left(-ie^{-i\varphi}\right) \right.$$

$$\left. + \sqrt{\frac{3}{8\pi}}\frac{1}{\sin^2\theta} sin\theta\left(ie^{-i\varphi}\right) \right\}$$

$$[\hat{L}_z, \hat{L}^2] Y_{1-1} = 0$$

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I.4 Harmonik Bola Y_{11}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y]Y_{11} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)Y_{11}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$- \left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{11}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial Y_{11}}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial Y_{11}}{\partial\varphi}\right.$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial Y_{11}}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial Y_{11}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{11}}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{11}}{\partial\varphi}\right.$$

$$\left.- cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{11}}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{11}}{\partial\varphi}\right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right) \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right)$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right) \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right)$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right)$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\,\theta\; e^{i\varphi} \right) \right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} sin\varphi cos\varphi \left(sin\,\theta\; e^{i\varphi} \right) + \sqrt{\frac{3}{8\pi}} sin\varphi sin\varphi \left(-sin\,\theta\; ie^{i\varphi} \right) \right.$$

$$- \sqrt{\frac{3}{8\pi}} cot\theta\, cos\,\theta\;\; cos\varphi \left(-sin\varphi\; e^{i\varphi} + icos\varphi e^{i\varphi} \right)$$

$$\left. + \sqrt{\frac{3}{8\pi}} cot\theta cos\theta cos\varphi \left(icos\varphi\; e^{i\varphi} - sin\varphi e^{i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} cos\varphi sin\varphi e^{i\varphi} \left(sin\,\theta\; \right) + \sqrt{\frac{3}{8\pi}} cos\varphi cos\varphi \left(sin\,\theta\; ie^{i\varphi} \right) \right.$$

$$+ \sqrt{\frac{3}{8\pi}} cot\theta sin\varphi\; cos\,\theta\;\; \left(cos\varphi\; e^{i\varphi} + isin\varphi e^{i\varphi} \right)$$

$$\left. + \sqrt{\frac{3}{8\pi}} cot\theta sin\varphi cos\theta \left(-isin\varphi\; e^{i\varphi} - cos\varphi ie^{i\varphi} \right) \right\}$$

$$= \hbar^2 \left\{ - \sqrt{\frac{3}{8\pi}} sin\varphi sin\varphi \left(sin\,\theta\; ie^{i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} cos\varphi cos\varphi \left(sin\,\theta\; ie^{i\varphi} \right) \right\}$$

$$= \hbar^2 \left\{ - \sqrt{\frac{3}{8\pi}} sin^2\varphi (sin\theta) \left(ie^{i\varphi} \right) - \sqrt{\frac{3}{8\pi}} cos^2\varphi (sin\theta) \left(ie^{i\varphi} \right) \right\}$$

$$= -\hbar^2 ie^{i\varphi} sin\theta \sqrt{\frac{3}{8\pi}} \{ sin^2\varphi + cos^2\varphi \}$$

$$= -i\hbar^2 \sqrt{\frac{3}{8\pi}} e^{i\varphi} sin\theta$$

$$[\hat{L}_x, \hat{L}_y] Y_{11} = i\hbar^2 Y_{1-1}$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar^2$$

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b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z]Y_{11} = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)Y_{11}$$

$$= \left[\left\{i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right.$$

$$\left. - \left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{11}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{11}}{\partial\varphi} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{11}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial Y_{11}}{\partial\theta} + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial Y_{11}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}\right)\right.$$

$$\left. + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}\right)\right\}$$

$$- \hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\theta}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}\right)\right.$$

$$\left. + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}\right)\right\}$$

$$= \hbar^2\left\{\sqrt{\frac{3}{8\pi}}\cos\varphi\left(\cos\theta\, ie^{i\varphi}\right) - \sqrt{\frac{3}{8\pi}}\sin\varphi\cos\theta\left(-e^{i\varphi}\right)\right\}$$

$$- \hbar^2\left\{\sqrt{\frac{3}{8\pi}}\cos\theta\ \left(-\sin\varphi\, e^{i\varphi} + i\cos\varphi\, e^{i\varphi}\right)\right.$$

$$\left. - \sqrt{\frac{3}{8\pi}}\cos\theta\frac{\partial}{\partial\varphi}\left(i\cos\varphi\, e^{i\varphi} - \sin\varphi\, e^{i\varphi}\right)\right\}$$

$$= \hbar^2\left\{\sqrt{\frac{3}{8\pi}}\cos\varphi\left(\cos\theta\, ie^{i\varphi}\right) - \sqrt{\frac{3}{8\pi}}\sin\varphi\cos\theta\left(-e^{i\varphi}\right)\right\}$$

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$$= \hbar^2 e^{i\varphi} \cos\theta \sqrt{\frac{3}{8\pi}} \{i\cos\varphi + \sin\varphi\} \frac{i}{i}$$

$$= -\frac{\hbar^2}{i} e^{i\varphi} \cos\theta \sqrt{\frac{3}{8\pi}} \{\cos\varphi - i\sin\varphi\}$$

$$[\hat{L}_y, \hat{L}_z] Y_{11} = -\frac{\hbar^2}{i} \cos\theta \{e^{-i\varphi}\} \sqrt{\frac{3}{8\pi}} e^{i\varphi}$$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x] Y_{11} = (\hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z) Y_{11}$$

$$= \left[\left\{-i\hbar \frac{\partial}{\partial\varphi} \times i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\right) \times -i\hbar \frac{\partial}{\partial\varphi}\right\}\right] Y_{11}$$

$$= \hbar^2 \left\{\frac{\partial}{\partial\varphi} sin\varphi \frac{\partial Y_{11}}{\partial\theta} + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial Y_{11}}{\partial\varphi}\right\}$$

$$- \hbar^2 \left\{sin\varphi \frac{\partial}{\partial\theta} \frac{\partial Y_{11}}{\partial\varphi} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial Y_{11}}{\partial\varphi}\right\}$$

$$= \hbar^2 \left\{\frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi}\right)\right.$$

$$\left. + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi}\right)\right\}$$

$$- \hbar^2 \left\{sin\varphi \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi}\right)\right.$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi}\right)\right\}$$

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$$= \hbar^2 \left\{ -\sqrt{\frac{3}{8\pi}} \cos\theta \ \left(cos\varphi\, e^{i\varphi} + isin\varphi e^{i\varphi} \right) \right.$$

$$\left. - \sqrt{\frac{3}{8\pi}} cos\theta \left(-sin\varphi\, ie^{i\varphi} - cos\varphi e^{i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ -\sqrt{\frac{3}{8\pi}} sin\varphi \cos\theta \ ie^{i\varphi} + \sqrt{\frac{3}{8\pi}} cos\theta cos\varphi e^{i\varphi} \right\}$$

$$= -\hbar^2 \left\{ -\sqrt{\frac{3}{8\pi}} sin\varphi \cos\theta \ ie^{i\varphi} + \sqrt{\frac{3}{8\pi}} cos\theta cos\varphi e^{i\varphi} \right\}$$

$$= -\hbar^2 \sqrt{\frac{3}{8\pi}} e^{i\varphi} \cos\theta \ \{ -isin\varphi + cos\varphi \}$$

$$[\hat{L}_z, \hat{L}_x] Y_{11} = -\hbar^2 \sqrt{\frac{3}{8\pi}} e^{i\varphi} \cos\theta \ \{ e^{-i\varphi} \}$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+] Y_{11} = (\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z) Y_{11}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{11}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{11}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \right\} Y_{11}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial Y_{11}}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial Y_{11}}{\partial\varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial Y_{11}}{\partial\varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{11}}{\partial\varphi} \right\}$$

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$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right.$$

$$\left. - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right.$$

$$\left. - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right\}$$

$$= i\hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} cos\theta 2ie^{2i\varphi} - \sqrt{\frac{3}{8\pi}} \cos\theta\, 2ie^{2i\varphi} \right\}$$

$$- i\hbar^2 \left\{ \sqrt{\frac{3}{8\pi}} e^{i\varphi} cos\theta \left(ie^{i\varphi} \right) - \sqrt{\frac{3}{8\pi}} e^{i\varphi} i \cos\theta \left(e^{i\varphi} \right) \right\}$$

$[\hat{L}_z, \hat{L}_+] Y_{11} = 0$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-] Y_{11} = (\hat{L}_z \hat{L}_- - \hat{L}_- \hat{L}_z) Y_{11}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{11}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{11}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \right\} Y_{11}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial Y_{11}}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i \cot\theta \frac{\partial Y_{11}}{\partial\varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial Y_{11}}{\partial\varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{11}}{\partial\varphi} \right\}$$

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$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right.$$

$$\left. - \frac{\partial}{\partial\varphi} e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right.$$

$$\left. - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right\}$$

$$= i\hbar^2 \left\{ -\sqrt{\frac{3}{8\pi}} cos\theta(0) - \sqrt{\frac{3}{8\pi}} \cos\theta\,(0) \right\}$$

$$- i\hbar^2 \left\{ -\sqrt{\frac{3}{8\pi}} cos\theta e^{-i\varphi} \left(ie^{i\varphi} \right) - \sqrt{\frac{3}{8\pi}} e^{-i\varphi} i \cos\theta \left(e^{i\varphi} \right) \right\}$$

$$[\hat{L}_z, \hat{L}_-]Y_{11} = -2\hbar^2 cos\theta e^{-i\varphi} \sqrt{\frac{3}{8\pi}} e^{i\varphi}$$

i. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2]Y_{11} = (\hat{L}_x \hat{L}^2 - \hat{L}^2 \hat{L}_x)Y_{11}$$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{11}$$

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$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{11}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right\} Y_{11}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left.+ \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\} Y_{11}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial Y_{11}}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{11}}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{11}}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial Y_{11}}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{11}}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{11}}{\partial\varphi^2}\right\}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{11}}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{11}}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial Y_{11}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial Y_{11}}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial Y_{11}}{\partial\theta}$$

$$\left.+ \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial Y_{11}}{\partial\varphi}\right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right.$$

$$+ \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$+ \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right)$$

$$\left. + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi} \right) \right\}$$

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$$= -i\hbar^3\left\{-\sqrt{\frac{3}{8\pi}}e^{i\varphi}\,sin\varphi\left(-\frac{1}{sin^2\theta}cos\,\theta - cos\,\theta\right)\right.$$

$$+\sqrt{\frac{3}{8\pi}}e^{i\varphi}\,sin\varphi(cos\,\theta) - \sqrt{\frac{3}{8\pi}}sin\varphi\frac{1}{sin^2\theta}cos\theta e^{i\varphi}$$

$$-\sqrt{\frac{3}{8\pi}}ie^{i\varphi}\,cot\theta cos\varphi\,cot\,\theta\,cos\theta + \sqrt{\frac{3}{8\pi}}cos\theta cos\varphi ie^{i\varphi}$$

$$\left.+\sqrt{\frac{3}{8\pi}}cos\theta cos\varphi\frac{1}{sin^2\theta}ie^{i\varphi}\right\}$$

$$+\,i\hbar^3\left\{\sqrt{\frac{3}{8\pi}}cos\,\theta\,sin\varphi e^{i\varphi}\right.$$

$$-\sqrt{\frac{3}{8\pi}}cot\,\theta cos\varphi ie^{i\varphi}\left(-\frac{1}{sin\theta} + cot\,\theta\,cos\theta\right)$$

$$+\sqrt{\frac{3}{8\pi}}sin\varphi cos\theta e^{i\varphi} + \sqrt{\frac{3}{8\pi}}cos\theta cos\varphi ie^{i\varphi}$$

$$-\sqrt{\frac{3}{8\pi}}cos\theta\frac{1}{sin^2\theta}\left(-2sin\varphi e^{i\varphi} + 2icos\varphi e^{i\varphi}\right)$$

$$\left.-\sqrt{\frac{3}{8\pi}}cos\theta\frac{1}{sin^2\theta}\left(-2icos\varphi e^{i\varphi} - 2sin\varphi e^{i\varphi}\right)\right\}$$

$[\hat{L}_x, \hat{L}^2]Y_{11} = 0$

j. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$[\hat{L}_y, \hat{L}^2]Y_{11} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{11}$

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$$= \left[\left\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{11}$$

$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + \cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + \cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}$$

$$\left.- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} Y_{11}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$\left.- \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\} Y_{11}$$

$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial Y_{11}}{\partial\theta} + \cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{11}}{\partial\theta^2} + \cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{11}}{\partial\varphi^2}\right.$$

$$- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial Y_{11}}{\partial\theta}$$

$$\left.- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{11}}{\partial\theta^2} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{11}}{\partial\varphi^2}\right\}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial Y_{11}}{\partial\theta} - \cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial Y_{11}}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial Y_{11}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial Y_{11}}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial Y_{11}}{\partial\theta}$$

$$\left.- \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial Y_{11}}{\partial\varphi}\right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right) \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right) \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right) \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$- \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right)$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \right) \right\}$$

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$$= i\hbar^3 \left\{ -\sqrt{\frac{3}{8\pi}} e^{i\varphi} cos\, \varphi \left(-\frac{1}{sin^2\, \theta}\; cos\theta - cos\, \theta \right) \right.$$

$$+ \sqrt{\frac{3}{8\pi}} e^{i\varphi} cos\, \varphi\, cos\theta - \sqrt{\frac{3}{8\pi}} \frac{1}{sin^2\, \theta} cos\, \varphi\, cos\theta e^{i\varphi}$$

$$+ i \sqrt{\frac{3}{8\pi}} sin\, \varphi\, cot\, \theta\, cot\, \theta\, cos\theta e^{i\varphi}$$

$$\left. - i \sqrt{\frac{3}{8\pi}} sin\, \varphi\, cos\, \theta\, e^{i\varphi} - \sqrt{\frac{3}{8\pi}} sin\, \varphi\, cos\, \theta \frac{1}{sin^2\, \theta} \left(ie^{i\varphi} \right) \right\}$$

$$- i\hbar^3 \left\{ \sqrt{\frac{3}{8\pi}} cos\, \theta\, cos\, \varphi\, e^{i\varphi} \right.$$

$$+ \sqrt{\frac{3}{8\pi}} cot\, \theta\, ie^{i\varphi} sin\, \varphi \left(-\frac{1}{sin\theta} + cot\, \theta\, cos\, \theta \right)$$

$$+ \sqrt{\frac{3}{8\pi}} e^{i\varphi}\; cos\, \varphi\, cos\theta - \sqrt{\frac{3}{8\pi}} sin\, \varphi\, cos\, \theta \left(ie^{i\varphi} \right)$$

$$- \sqrt{\frac{3}{8\pi}} \frac{1}{sin^2\, \theta} cos\theta \left(-2\, cos\, \varphi e^{i\varphi}\; - 2isin\varphi\; e^{-i\varphi} \right)$$

$$\left. + \sqrt{\frac{3}{8\pi}} \frac{1}{sin^2\, \theta} cos\, \theta \left(-2cos\varphi\; e^{-i\varphi}\; - 2\; isin\, \varphi\; e^{-i\varphi} \right) \right\}$$

$[\hat{L}_y, \hat{L}^2] Y_{11} = 0$

k. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$[\hat{L}_z, \hat{L}^2] Y_{11} = (\hat{L}_z \hat{L}^2 - \hat{L}^2 \hat{L}_z) Y_{11}$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial \varphi} \times -\hbar^2 \left(\cot\theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar^2 \left(\cot\theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right) \times -i\hbar \frac{\partial}{\partial \varphi} \right\} \right] Y_{11}$$

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$$= i\hbar^3 \left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} Y_{11}$$

$$- i\hbar^3 \left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\right\} Y_{11}$$

$$= i\hbar^3 \left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial Y_{11}}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{11}}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{11}}{\partial\varphi^2}\right\}$$

$$- i\hbar^3 \left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial Y_{11}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial Y_{11}}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial Y_{11}}{\partial\varphi}\right\}$$

$$= i\hbar^3 \left\{\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}\left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi}\right) + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}\left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi}\right)\right.$$

$$\left. + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi}\right)\right\}$$

$$- i\hbar^3 \left\{\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi}\right)\right.$$

$$\left. + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{3}{8\pi}} sin\theta e^{i\varphi}\right)\right\}$$

$$= i\hbar^3 \left\{-\sqrt{\frac{3}{8\pi}}\cot\theta\, cos\theta\left(ie^{i\varphi}\right) + \sqrt{\frac{3}{8\pi}} sin\theta\left(ie^{i\varphi}\right)\right.$$

$$\left. + \sqrt{\frac{3}{8\pi}}\frac{1}{\sin^2\theta} sin\theta\left(ie^{i\varphi}\right)\right\}$$

$$- i\hbar^3 \left\{-\sqrt{\frac{3}{8\pi}} cos\theta\cot\theta\left(ie^{i\varphi}\right) + \sqrt{\frac{3}{8\pi}} sin\theta\left(ie^{i\varphi}\right)\right.$$

$$\left. + \sqrt{\frac{3}{8\pi}}\frac{1}{\sin^2\theta} sin\theta\left(ie^{i\varphi}\right)\right\}$$

$$[\hat{L}_z, \hat{L}^2] Y_{11} = 0$$

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I.5 Harmonik Bola Y_{20}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y]Y_{20} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)Y_{20}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$- \left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{20}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial Y_{20}}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial Y_{20}}{\partial\varphi}\right.$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial Y_{20}}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial Y_{20}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{20}}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{20}}{\partial\varphi}\right.$$

$$\left.- cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{20}}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{20}}{\partial\varphi}\right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

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$$= \hbar^2 \left\{ -6\sqrt{\frac{5}{16\pi}} sin\varphi cos\varphi(-sin^2\theta + cos^2\theta) \right.$$

$$- \sqrt{\frac{5}{16\pi}} sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi(0)$$

$$+ \sqrt{\frac{5}{16\pi}} cot\theta cos\varphi sin\varphi(-6cos\theta sin\theta)$$

$$\left. - \sqrt{\frac{5}{16\pi}} cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi(0) \right\}$$

$$- \hbar^2 \left\{ -6\sqrt{\frac{5}{16\pi}} cos\varphi sin\varphi(-sin^2\theta + cos^2\theta) \right.$$

$$+ \sqrt{\frac{5}{16\pi}} cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi}(0)$$

$$- \sqrt{\frac{5}{16\pi}} cot\theta sin\varphi cos\varphi(-6cos\theta sin\theta)$$

$$\left. - \sqrt{\frac{5}{16\pi}} cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi}(0) \right\}$$

$$[\hat{L}_x, \hat{L}_y]Y_{20} = 0$$

b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z]Y_{20} = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)Y_{20}$$

$$= \left[\left\{ i\hbar \left(-\cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right.$$

$$\left. - \left\{ -i\hbar \frac{\partial}{\partial\varphi} \times i\hbar \left(-\cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{20}$$

$$= \hbar^2 \left\{ -\cos\varphi \frac{\partial}{\partial\theta} \frac{\partial Y_{20}}{\partial\varphi} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

$$- \hbar^2 \left\{ -\frac{\partial}{\partial\varphi} \cos\varphi \frac{\partial Y_{20}}{\partial\theta} + \frac{\partial}{\partial\varphi} \sin\varphi \cot\theta \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

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$$= \hbar^2 \left\{ -\cos\varphi \frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{5}{16\pi}}(3cos^2\theta - 1) \right.$$

$$\left. + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{5}{16\pi}}(3cos^2\theta - 1) \right\}$$

$$- \hbar^2 \left\{ -\frac{\partial}{\partial\varphi}\cos\varphi \frac{\partial}{\partial\theta}\sqrt{\frac{5}{16\pi}}(3cos^2\theta - 1) \right.$$

$$\left. + \frac{\partial}{\partial\varphi}\sin\varphi \cot\theta \frac{\partial}{\partial\varphi}\sqrt{\frac{5}{16\pi}}(3cos^2\theta - 1) \right\}$$

$$= \hbar^2 \left\{ -\sqrt{\frac{5}{16\pi}}\cos\varphi \frac{\partial}{\partial\theta}(0) + \sqrt{\frac{5}{16\pi}}\sin\varphi \cot\theta\,(0) \right\}$$

$$- \hbar^2 \left\{ -\sqrt{\frac{5}{16\pi}}\sin\varphi\,(6cos\theta sin\theta) + \sqrt{\frac{5}{16\pi}}\frac{\partial}{\partial\varphi}\sin\varphi \cot\theta\,(0) \right\}$$

$$[\hat{L}_y, \hat{L}_z]Y_{20} = \hbar^2 \sqrt{\frac{5}{16\pi}}\sin\varphi\,(6cos\theta sin\theta)$$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x]Y_{20} = (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)Y_{20}$$

$$= \left[\left\{ -i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi} \right) \times -i\hbar\frac{\partial}{\partial\varphi} \right\} \right] Y_{20}$$

$$= \hbar^2 \left\{ \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial Y_{20}}{\partial\theta} + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta}\frac{\partial Y_{20}}{\partial\varphi} + cot\theta cos\varphi \frac{\partial}{\partial\varphi}\frac{\partial Y_{20}}{\partial\varphi} \right\}$$

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$$= \hbar^2 \left\{ \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$\left. + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{5}{16\pi}} cos\varphi(-6cos\theta sin\theta) + \sqrt{\frac{5}{16\pi}} \frac{\partial}{\partial\varphi} cot\theta cos\varphi(0) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{5}{16\pi}} sin\varphi \frac{\partial}{\partial\theta}(0) + \sqrt{\frac{5}{16\pi}} cot\theta cos\varphi \frac{\partial}{\partial\varphi}(0) \right\}$$

$$[\hat{L}_z, \hat{L}_x] Y_{20} = -\hbar^2 \sqrt{\frac{5}{16\pi}} cos\varphi(6cos\theta sin\theta)$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+] Y_{20} = (\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z) Y_{20}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{20}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{20}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \right\} Y_{20}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial Y_{20}}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial Y_{20}}{\partial\varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

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$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$\left. - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$\left. - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$= i\hbar^2 \left\{ \sqrt{\frac{5}{16\pi}} ie^{i\varphi} (6cos\theta sin\theta) - \sqrt{\frac{5}{16\pi}} \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta\, (0) \right\}$$

$$- i\hbar^2 \left\{ -\sqrt{\frac{5}{16\pi}} e^{i\varphi} \frac{\partial}{\partial\theta} (0) - \sqrt{\frac{5}{16\pi}} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} (0) \right\}$$

$$[\hat{L}_z, \hat{L}_+] Y_{20} = i\hbar^2 \sqrt{\frac{5}{16\pi}} ie^{i\varphi} (6cos\theta sin\theta)$$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-] Y_{20} = (\hat{L}_z \hat{L}_- - \hat{L}_- \hat{L}_z) Y_{20}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{20}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{20}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \right\} Y_{20}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial Y_{20}}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i \cot\theta \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial Y_{20}}{\partial\varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

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$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$\left. - \frac{\partial}{\partial\varphi} e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$\left. - e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$= i\hbar^2 \left\{ \sqrt{\frac{5}{16\pi}} ie^{-i\varphi} (6cos\theta sin\theta) - \sqrt{\frac{5}{16\pi}} \frac{\partial}{\partial\varphi} e^{-i\varphi} i\,cot\,\theta\,(0) \right\}$$

$$- i\hbar^2 \left\{ \sqrt{\frac{5}{16\pi}} e^{-i\varphi} \frac{\partial}{\partial\theta} (0) - \sqrt{\frac{5}{16\pi}} e^{-i\varphi} i\,cot\,\theta\,(0) \right\}$$

$$[\hat{L}_z, \hat{L}_-]Y_{20} = i\hbar^2 \sqrt{\frac{5}{16\pi}} ie^{-i\varphi} (6cos\theta sin\theta)$$

f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2]Y_{20} = (\hat{L}_x\hat{L}^2 - \hat{L}^2\hat{L}_x)Y_{20}$$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{20}$$

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$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left. \times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left. \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{20}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{20}$$

$$+ i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left. + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\}Y_{20}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial Y_{20}}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{20}}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{20}}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial Y_{20}}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{20}}{\partial\theta^2}$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{20}}{\partial\varphi^2}\right\}$$

$$+ i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{20}}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{20}}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial Y_{20}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial Y_{20}}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial Y_{20}}{\partial\theta}$$

$$\left. + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial Y_{20}}{\partial\varphi}\right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right.$$

$$+ \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1) \right\}$$

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$$= -i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} sin\varphi(12cos\theta sin\theta) + \sqrt{\frac{5}{16\pi}} sin\varphi(8cos\theta sin\theta) \right.$$

$$+ \sqrt{\frac{5}{16\pi}} sin\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta}(0) + \sqrt{\frac{5}{16\pi}} cot\theta cos\varphi(0)$$

$$\left. + \sqrt{\frac{5}{16\pi}} cot\theta cos\varphi(0) + \sqrt{\frac{5}{16\pi}} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta}(0) \right\}$$

$$+ i\hbar^3 \left\{ -6\sqrt{\frac{5}{16\pi}} \cot\theta sin\varphi \left(-\sin^2\theta + cos^2\theta\right) \right.$$

$$+ \sqrt{\frac{5}{16\pi}} \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi(0) + \sqrt{\frac{5}{16\pi}} sin\varphi(8cos\theta sin\theta)$$

$$+ \sqrt{\frac{5}{16\pi}} \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi(0) + \sqrt{\frac{5}{16\pi}} \frac{1}{\sin^2\theta} sin\varphi(6cos\theta sin\theta)$$

$$\left. + \sqrt{\frac{5}{16\pi}} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi(0) \right\}$$

$$= -i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} sin\varphi(12cos\theta sin\theta) \right\}$$

$$+ i\hbar^3 \left\{ -6\sqrt{\frac{5}{16\pi}} \cot\theta sin\varphi \left(-\sin^2\theta + cos^2\theta\right) \right.$$

$$\left. + \sqrt{\frac{5}{16\pi}} \frac{1}{\sin^2\theta} sin\varphi(6cos\theta sin\theta) \right\}$$

$$= -i\hbar^3 \left\{ 6\sqrt{\frac{5}{16\pi}} sin\varphi(cos\theta sin\theta) \right\}$$

$$+ i\hbar^3 \left\{ 6\sqrt{\frac{5}{16\pi}} \cot\theta sin\varphi \left(1 - cos^2\theta\right) \right\}$$

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$$= -i\hbar^3\left\{6\sqrt{\frac{5}{16\pi}}sin\varphi(cos\theta sin\theta)\right\}$$

$$+ i\hbar^3\left\{6\sqrt{\frac{5}{16\pi}}\cot\theta sin\varphi\,(sin^2\theta)\right\}$$

$$= -i\hbar^3\left\{6\sqrt{\frac{5}{16\pi}}sin\varphi(cos\theta sin\theta)\right\}$$

$$+ i\hbar^3\left\{6\sqrt{\frac{5}{16\pi}}\cos\theta sin\varphi\,(sin\theta)\right\}$$

$[\hat{L}_x, \hat{L}^2]Y_{20} = 0$

g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$[\hat{L}_y, \hat{L}^2]Y_{20} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{20}$

$$= \left[\left\{-i\hbar\left(cos\,\varphi\frac{\partial}{\partial\theta} - sin\,\varphi\,cot\,\theta\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times -i\hbar\left(cos\,\varphi\frac{\partial}{\partial\theta} - sin\,\varphi\,cot\,\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{20}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} + \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right.$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} Y_{20}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{20}$$

$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial Y_{20}}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{20}}{\partial\theta^2} + \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{20}}{\partial\varphi^2} \right.$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{20}}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{20}}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{20}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial Y_{00}}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial Y_{20}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial Y_{20}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial Y_{20}}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial Y_{20}}{\partial\theta}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta$$

$$\left. - 1) - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$- \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} cos\,\varphi\,(12cos\theta sin\theta) + \sqrt{\frac{5}{16\pi}} cos\,\varphi\,(8cos\theta sin\theta) \right.$$

$$+ \sqrt{\frac{5}{16\pi}} cos\,\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\,\theta}(0) - \sqrt{\frac{5}{16\pi}} sin\,\varphi\,cot\,\theta\,(0)$$

$$\left. - \sqrt{\frac{5}{16\pi}} sin\,\varphi\,cot\,\theta\,(0) - \sqrt{\frac{5}{16\pi}} sin\,\varphi\,cot\,\theta \frac{\partial}{\partial\varphi} \frac{1}{sin^2\,\theta}(0) \right\}$$

$$- i\hbar^3 \left\{ -6 \sqrt{\frac{5}{16\pi}} cot\,\theta\,cos\,\varphi\,(-\sin^2\theta + cos^2\theta) \right.$$

$$- \sqrt{\frac{5}{16\pi}} cot\,\theta \frac{\partial}{\partial\theta} sin\,\varphi\,cot\,\theta\,(0) + \sqrt{\frac{5}{16\pi}} cos\,\varphi\,(8cos\theta sin\theta)$$

$$- \sqrt{\frac{5}{16\pi}} \frac{\partial^2}{\partial\theta^2} sin\,\varphi\,cot\,\theta\,(0) + 6 \sqrt{\frac{5}{16\pi}} \frac{1}{sin^2\,\theta} cos\,\varphi\,(cos\theta sin\theta)$$

$$\left. - \sqrt{\frac{5}{16\pi}} \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} sin\,\varphi\,cot\,\theta\,(0) \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} cos\,\varphi\,(12cos\theta sin\theta) \right\}$$

$$- i\hbar^3 \left\{ -6 \sqrt{\frac{5}{16\pi}} cot\,\theta\,cos\,\varphi\,(-\sin^2\theta + cos^2\theta) \right.$$

$$\left. + 6 \sqrt{\frac{5}{16\pi}} \frac{1}{sin^2\,\theta} cos\,\varphi\,(cos\theta sin\theta) \right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} cos\,\varphi\,(12 cos\theta sin\theta) \right\}$$

$$- i\hbar^3 \left\{ 6\sqrt{\frac{5}{16\pi}} cot\,\theta\, cos\,\varphi \sin^2\theta - 6\sqrt{\frac{5}{16\pi}} cot\,\theta\, cos\,\varphi\, cos^2\theta \right.$$

$$\left. + 6\sqrt{\frac{5}{16\pi}} cot\theta\, cos\,\varphi \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} cos\,\varphi\,(6 cos\theta sin\theta) \right\}$$

$$- i\hbar^3 \left\{ 6\sqrt{\frac{5}{16\pi}} cot\theta\, cos\,\varphi\,(1 - cos^2\theta) \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} cos\,\varphi\,(6 cos\theta sin\theta) \right\}$$

$$- i\hbar^3 \left\{ 6\sqrt{\frac{5}{16\pi}} cot\theta\, cos\,\varphi\,(sin^2\theta) \right\}$$

$$[\hat{L}_y, \hat{L}^2]Y_{20} = 0$$

h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z, \hat{L}^2]Y_{20} = (\hat{L}_z\hat{L}^2 - \hat{L}^2\hat{L}_z)Y_{20}$$

$$= \left[\left\{ -i\hbar\frac{\partial}{\partial\varphi} \times -\hbar^2 \left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar^2 \left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2} \right) \times -i\hbar\frac{\partial}{\partial\varphi} \right\} \right] Y_{20}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2} \right\} Y_{20}$$

$$- i\hbar^3 \left\{ cot\,\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi} \right\} Y_{20}$$

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$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{20}}{\partial\theta} + \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{20}}{\partial\theta^2} + \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{20}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \frac{\partial Y_{20}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} \frac{\partial Y_{20}}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial Y_{20}}{\partial\varphi} \right\}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right.$$

$$+ \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$\left. + \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial}{\partial\varphi} \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} (0) + \sqrt{\frac{5}{16\pi}} (0) + \sqrt{\frac{5}{16\pi}} \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} (0) \right\}$$

$$- i\hbar^3 \left\{ \sqrt{\frac{5}{16\pi}} \cot\theta \frac{\partial}{\partial\theta} (0) + \sqrt{\frac{5}{16\pi}} \frac{\partial^2}{\partial\theta^2} (0) + \sqrt{\frac{5}{16\pi}} \frac{1}{\sin^2\theta} (0) \right\}$$

$$[\hat{L}_z, \hat{L}^2] Y_{20} = 0$$

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I.6 Harmonik Bola Y_{2-1}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y]Y_{2-1} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)Y_{2-1}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}$$

$$-\left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{2-1}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial Y_{2-1}}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial Y_{2-1}}{\partial\varphi}\right.$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial Y_{2-1}}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{2-1}}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{2-1}}{\partial\varphi}\right.$$

$$\left.- cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{2-1}}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi} \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi}$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi} \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi} \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi}$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi}$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \;\; e^{-i\varphi} \right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} e^{-i\varphi} sin\varphi cos\varphi(-4 \sin\theta \cos\theta) \right.$$

$$- 2\sqrt{\frac{15}{8\pi}} sin\varphi sin\varphi cos\theta \sin\theta \left(ie^{-i\varphi}\right)$$

$$+ \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi(-\sin^2\theta + \cos^2\theta)\left(-sin\varphi\, e^{-i\varphi} - icos\varphi\, e^{-i\varphi}\right)$$

$$\left. - \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi cos\theta \cos\theta \left(-icos\varphi\, e^{-i\varphi} - sin\varphi\, e^{-i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi cos\varphi e^{-i\varphi} (-4 \sin\theta \cos\theta) \right.$$

$$+ 2\sqrt{\frac{15}{8\pi}} cos\varphi cos\varphi \left(ie^{-i\varphi}\right) cos\theta \sin\theta$$

$$- \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi\left(cos\varphi\, e^{-i\varphi} - isin\varphi e^{-i\varphi}\right)(-\sin^2\theta + \cos^2\theta)$$

$$\left. - \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi cos\theta \cos\theta \left(isin\varphi\, e^{-i\varphi} - cos\varphi\, e^{-i\varphi}\right) \right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} e^{-i\varphi} sin\varphi cos\varphi(-4 \sin\theta \cos\theta) \right.$$

$$- 2\sqrt{\frac{15}{8\pi}} sin\varphi sin\varphi cos\theta \sin\theta \left(ie^{-i\varphi}\right)$$

$$+ \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi(sin^2\theta\, sin\varphi + sin^2\theta\, icos\varphi - cos^2\theta\, sin\varphi$$

$$- cos^2\theta\, icos\varphi)e^{-i\varphi}$$

$$\left. - \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi cos\theta \cos\theta \left(-icos\varphi\, e^{-i\varphi} - sin\varphi\, e^{-i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi cos\varphi e^{-i\varphi} (-4 \sin\theta \cos\theta) \right.$$

$$+ 2\sqrt{\frac{15}{8\pi}} cos\varphi cos\varphi \left(ie^{-i\varphi}\right) cos\theta \sin\theta$$

$$- \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi(- sin^2\theta\, cos\varphi + sin^2\theta\, isin\varphi + cos^2\theta\, cos\varphi$$

$$- cos^2\theta\, isin\varphi)e^{-i\varphi}$$

$$\left. - \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi cos\theta \cos\theta \left(isin\varphi\, e^{-i\varphi} - cos\varphi\, e^{-i\varphi}\right) \right\}$$

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$$= \hbar^2 \left\{ -2\sqrt{\frac{15}{8\pi}} sin^2\, \varphi\, cos\theta\, sin\,\theta\ \left(ie^{-i\varphi}\right) \right.$$

$$\left. + i\sqrt{\frac{15}{8\pi}} cot\theta cos^2\varphi (sin^2\,\theta) e^{-i\varphi} \right\}$$

$$- \hbar^2 \left\{ 2\sqrt{\frac{15}{8\pi}} cos^2\varphi \left(ie^{-i\varphi}\right) cos\theta\, sin\,\theta \right.$$

$$\left. - i\sqrt{\frac{15}{8\pi}} cot\theta\, sin^2\,\varphi\, (sin^2\,\theta) e^{-i\varphi} \right\}$$

$$= \hbar^2 \left\{ -2\sqrt{\frac{15}{8\pi}} cos\theta\, sin\,\theta\, ie^{-i\varphi}\ (sin^2\,\varphi + cos^2\varphi) \right.$$

$$\left. + i\sqrt{\frac{15}{8\pi}} cot\theta\, sin^2\,\theta\, (cos^2\varphi + sin^2\,\varphi) e^{-i\varphi} \right\}$$

$$= \hbar^2 \left\{ -2\sqrt{\frac{15}{8\pi}} cos\theta\, sin\,\theta\, ie^{-i\varphi}\ + i\sqrt{\frac{15}{8\pi}} cos\theta\, sin\,\theta\, e^{-i\varphi} \right\}$$

$$= \hbar^2 \left\{ -i\sqrt{\frac{15}{8\pi}} cos\theta\, sin\,\theta\, e^{-i\varphi} \right\}$$

$$= -i\hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} cos\theta\, sin\,\theta\, e^{-i\varphi} \right\}$$

$[\hat{L}_x, \hat{L}_y] Y_{2-1} = -i\hbar^2 Y_{2-1}$

$[\hat{L}_x, \hat{L}_y] = -i\hbar^2$

b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$[\hat{L}_y, \hat{L}_z] Y_{2-1} = (\hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y) Y_{2-1}$

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$$= \left[\left\{i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right.$$

$$\left.-\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{2-1}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{2-1}}{\partial\varphi} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

$$-\hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial Y_{2-1}}{\partial\theta} + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\ e^{-i\varphi}\right.$$

$$\left.+\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\ e^{-i\varphi}\right\}$$

$$-\hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\theta}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\ e^{-i\varphi}\right.$$

$$\left.+\frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\ e^{-i\varphi}\right\}$$

$$= \hbar^2\left\{\sqrt{\frac{15}{8\pi}}\cos\varphi\,(-\sin^2\theta + \cos^2\theta)ie^{-i\varphi}\right.$$

$$\left.-\sqrt{\frac{15}{8\pi}}\sin\varphi\cot\theta\sin\theta\cos\theta\ e^{-i\varphi}\right\}$$

$$-\hbar^2\left\{-\sqrt{\frac{15}{8\pi}}\left(-\sin\varphi\, e^{-i\varphi} - i\cos\varphi e^{-i\varphi}\right)(-\sin^2\theta + \cos^2\theta)\right.$$

$$\left.+\sqrt{\frac{15}{8\pi}}\cot\theta\sin\theta\cos\theta\ \left(-i\cos\varphi\ e^{-i\varphi} - \sin\varphi\, e^{-i\varphi}\right)\right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} cos\,\varphi\,(-sin^2\theta + cos^2\theta) ie^{-i\varphi} \right.$$

$$\left. - \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot\,\theta\, sin\,\theta\, cos\,\theta\;\; e^{-i\varphi} \right\}$$

$$- \hbar^2 \left\{ - \sqrt{\frac{15}{8\pi}} (sin^2\theta\, sin\varphi + sin^2\theta\, cos\,\varphi - cos^2\theta\, sin\varphi \right.$$

$$- cos^2\theta\, i\, cos\,\varphi) e^{-i\varphi}$$

$$\left. + \sqrt{\frac{15}{8\pi}} cot\,\theta\, sin\,\theta\, cos\,\theta\;\left(-icos\varphi\, e^{-i\varphi} - sin\,\varphi\, e^{-i\varphi}\right) \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} cos\,\varphi\,(cos^2\theta) ie^{-i\varphi} \right.$$

$$\left. + \sqrt{\frac{15}{8\pi}} sin\varphi (sin^2\theta - cos^2\theta) e^{-i\varphi} \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} cos^2\theta\,(i\,cos\,\varphi - sin\varphi) e^{-i\varphi} \right.$$

$$\left. + \sqrt{\frac{15}{8\pi}} sin\varphi (sin^2\theta) e^{-i\varphi} \right\} \frac{i}{i}$$

$$= \frac{\hbar^2}{i} \left\{ - \sqrt{\frac{15}{8\pi}} cos^2\theta\,(cos\,\varphi + isin\varphi) e^{-i\varphi} \right.$$

$$\left. + i \sqrt{\frac{15}{8\pi}} sin\varphi (sin^2\theta) e^{-i\varphi} \right\}$$

$$[\hat{L}_y, \hat{L}_z] Y_{2-1} = \frac{\hbar^2}{i} \{ - cos^2\theta\,\left(e^{i\varphi}\right) + isin\varphi(sin^2\theta) \} \sqrt{\frac{15}{8\pi}} e^{-i\varphi}$$

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c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z,\hat{L}_x]Y_{2-1} = (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)Y_{2-1}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]Y_{2-1}$$

$$= \hbar^2\left\{\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{2-1}}{\partial\theta} + \frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{2-1}}{\partial\varphi} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial}{\partial\theta}\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{-i\varphi}\right.$$

$$\left. + \frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{-i\varphi}\right\}$$

$$- \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{-i\varphi}\right.$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{-i\varphi}\right\}$$

$$= \hbar^2\left\{\sqrt{\frac{15}{8\pi}}\left(cos\varphi\;e^{-i\varphi} - isin\varphi e^{-i\varphi}\right)(-sin^2\theta + cos^2\theta)\right.$$

$$\left. + \sqrt{\frac{15}{8\pi}}cos^2\theta\left(isin\varphi\;e^{-i\varphi} - cos\varphi e^{-i\varphi}\right)\right\}$$

$$- \hbar^2\left\{-i\sqrt{\frac{15}{8\pi}}sin\varphi e^{-i\varphi}(-sin^2\theta + cos^2\theta)\right.$$

$$\left. - \sqrt{\frac{15}{8\pi}}cos^2\theta\;cos\varphi e^{-i\varphi}\right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} (-cos\varphi \ \ sin^2\theta + cos\varphi \, cos^2\theta + isin\varphi \, sin^2\theta \right.$$

$$\left. - isin\varphi \, cos^2\theta) e^{-i\varphi} + \sqrt{\frac{15}{8\pi}} cos^2\theta \, (isin\varphi \ - cos\varphi) e^{-i\varphi} \right\}$$

$$- \hbar^2 \left\{ -i \sqrt{\frac{15}{8\pi}} sin\varphi e^{-i\varphi} (-sin^2\theta + cos^2\theta) \right.$$

$$\left. - \sqrt{\frac{15}{8\pi}} cos^2\theta \, cos\varphi e^{-i\varphi} \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} (-cos\varphi \ \ sin^2\theta + cos\varphi \, cos^2\theta) e^{-i\varphi} \right.$$

$$\left. + \sqrt{\frac{15}{8\pi}} cos^2\theta \, (isin\varphi \,) e^{-i\varphi} \right\}$$

$$= \hbar^2 \left\{ - \sqrt{\frac{15}{8\pi}} cos\varphi \ \ sin^2\theta \, e^{-i\varphi} \right.$$

$$\left. + \sqrt{\frac{15}{8\pi}} cos^2\theta \, (cos\varphi + isin\varphi \,) e^{-i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_x] Y_{2-1} = \hbar^2 \left\{ -cos\varphi \ \ sin^2\theta + \sqrt{\frac{15}{8\pi}} cos^2\theta \left(e^{i\varphi} \right) \right\} \sqrt{\frac{15}{8\pi}} e^{-i\varphi}$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+] Y_{2-1} = (\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z) Y_{2-1}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{2-1}$$

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$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi}\, i \cot\theta \frac{\partial}{\partial \varphi} \right\} Y_{2-1}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} - e^{i\varphi}\, i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right\} Y_{2-1}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial Y_{2-1}}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi}\, i \cot\theta \frac{\partial Y_{2-1}}{\partial \varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial Y_{2-1}}{\partial \varphi} - e^{i\varphi}\, i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial Y_{2-1}}{\partial \varphi} \right\}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{-i\varphi} \right.$$

$$\left. - \frac{\partial}{\partial \varphi} e^{i\varphi}\, i \cot\theta \frac{\partial}{\partial \varphi} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{-i\varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{-i\varphi} \right.$$

$$\left. - e^{i\varphi}\, i \cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{-i\varphi} \right\}$$

$$= i\hbar^2 \left\{ -\sqrt{\frac{15}{8\pi}} (0) - \sqrt{\frac{15}{8\pi}} (0) \right\}$$

$$- i\hbar^2 \left\{ i \sqrt{\frac{15}{8\pi}} e^{i\varphi} (-\sin^2\theta + \cos^2\theta) e^{-i\varphi} \right.$$

$$\left. + \sqrt{\frac{15}{8\pi}} e^{i\varphi}\, i \cos^2\theta\, e^{-i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_+] Y_{2-1} = \hbar^2 e^{i\varphi} (-\sin^2\theta + 2\cos^2\theta) \sqrt{\frac{15}{8\pi}} e^{-i\varphi}$$

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e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-]Y_{2-1} = (\hat{L}_z\hat{L}_- - \hat{L}_-\hat{L}_z)Y_{2-1}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right] Y_{2-1}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right\}Y_{2-1}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}Y_{2-1}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial Y_{2-1}}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial Y_{2-1}}{\partial\varphi} - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{2-1}}{\partial\varphi}\right\}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\varphi}\right.$$

$$\left. - \frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\varphi}\right\}$$

$$- i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\varphi}\right.$$

$$\left. - e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\varphi}\right\}$$

$$= i\hbar^2\left\{-2i\sqrt{\frac{15}{8\pi}}e^{-2i\varphi}(-\sin^2\theta + \cos^2\theta)\right.$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}}\frac{\partial}{\partial\varphi}e^{-2i\varphi}\cos^2\theta\right\}$$

$$- i\hbar^2\left\{-i\sqrt{\frac{15}{8\pi}}e^{-2i\varphi}(-\sin^2\theta + \cos^2\theta) + \sqrt{\frac{15}{8\pi}}e^{-2i\varphi}i\cos^2\theta\right\}$$

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$$= i\hbar^2 \left\{ i\sqrt{\frac{15}{8\pi}} e^{-2i\varphi} (sin^2\theta) \right\}$$

$$[\hat{L}_z, \hat{L}_-] Y_{2-1} = -\hbar^2 e^{-i\varphi} \, sin^2\theta \sqrt{\frac{15}{8\pi}} e^{-i\varphi}$$

f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2] Y_{2-1} = (\hat{L}_x \hat{L}^2 - \hat{L}^2 \hat{L}_x) Y_{2-1}$$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\frac{1}{sin\,\theta} \frac{\partial}{\partial\theta} \left(sin\,\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\frac{1}{sin\,\theta} \frac{\partial}{\partial\theta} \left(sin\,\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times\, i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{2-1}$$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(cot\,\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(cot\,\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times\, i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{2-1}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} cot\,\theta \frac{\partial}{\partial\theta} + sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} + sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right.$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} Y_{2-1}$$

$$+ i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} + cot\,\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta}$$

$$\left. + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right\} Y_{2-1}$$

$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} cot\,\theta \frac{\partial Y_{2-1}}{\partial\theta} + sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{2-1}}{\partial\theta^2} \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\theta} \frac{\partial^2 Y_{2-1}}{\partial\varphi^2} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial Y_{2-1}}{\partial\theta}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{2-1}}{\partial\theta^2} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\theta} \frac{\partial^2 Y_{2-1}}{\partial\varphi^2} \right\}$$

$$+ i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial Y_{2-1}}{\partial\theta} + cot\,\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial Y_{2-1}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial Y_{2-1}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial Y_{2-1}}{\partial\varphi}$$

$$\left. + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial Y_{2-1}}{\partial\theta} + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial Y_{2-1}}{\partial\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} cot\,\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi} \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi} \right.$$

$$+ cot\,\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$+ \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi}$$

$$\left. + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} sin\theta\; cos\theta e^{-i\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} sin\varphi \left(\frac{1}{sin^2\,\theta}(cos^2\theta - sin^2\theta) \right. \right.$$

$$\left. + 4sin\theta cos\theta cot\theta \right) e^{-i\varphi} - 4\sqrt{\frac{15}{8\pi}} sin\varphi(-sin^2\theta + \ cos^2\theta)e^{-i\varphi}$$

$$+ \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\,\theta} e^{-i\varphi}$$

$$- i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi(-sin^2\theta + \ cos^2\theta)e^{-i\varphi}$$

$$\left. + 4i\sqrt{\frac{15}{8\pi}} cot\theta cos\varphi sin\theta\ cos\theta e^{-i\varphi} + i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi e^{-i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ -4\sqrt{\frac{15}{8\pi}} cot\,\theta\ sin\varphi sin\theta\ cos\theta e^{-i\varphi} \right.$$

$$+ 2i\sqrt{\frac{15}{8\pi}} cot\,\theta\ cos\varphi sin\theta\ cos\theta e^{-i\varphi}$$

$$- 4\sqrt{\frac{15}{8\pi}} sin\varphi(-sin^2\theta + \ cos^2\theta)e^{-i\varphi}$$

$$+ 2i\sqrt{\frac{15}{8\pi}} cos\varphi(-sin^2\theta + \ cos^2\theta)$$

$$+ \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\,\theta}(-2cos^2\theta sin\varphi - 2icos^2\theta cos\varphi + 2sin^2\theta sin\varphi$$

$$\left. + 2isin^2\theta cos\varphi)e^{-i\varphi} + \sqrt{\frac{15}{8\pi}} cot^2\theta(2icos\varphi + 2sin\varphi)e^{-i\varphi} \right\}$$

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$$= -i\hbar^3\left\{-\sqrt{\frac{15}{8\pi}}sin\varphi\left(\frac{1}{sin^2\theta}(cos^2\theta)\right)e^{-i\varphi}\right.$$

$$+\sqrt{\frac{15}{8\pi}}sin\varphi\frac{1}{sin^2\theta}e^{-i\varphi}$$

$$-i\sqrt{\frac{15}{8\pi}}cot^2\theta cos\varphi(-sin^2\theta + cos^2\theta)e^{-i\varphi}$$

$$\left.+i\sqrt{\frac{15}{8\pi}}cot^2\theta cos\varphi e^{-i\varphi}\right\}$$

$$+i\hbar^3\left\{2i\sqrt{\frac{15}{8\pi}}cos\varphi(-sin^2\theta)e^{-i\varphi} + 2\sqrt{\frac{15}{8\pi}}sin\varphi e^{-i\varphi}\right.$$

$$\left.+\sqrt{\frac{15}{8\pi}}\frac{1}{sin^2\theta}(2isin^2cos\varphi)e^{-i\varphi}\right\}$$

$$= i\hbar^3\left\{\sqrt{\frac{15}{8\pi}}sin\varphi\left(\frac{1}{sin^2\theta}(cos^2\theta)\right)e^{-i\varphi} - \sqrt{\frac{15}{8\pi}}sin\varphi\frac{1}{sin^2\theta}e^{-i\varphi}\right.$$

$$-i\sqrt{\frac{15}{8\pi}}cot^2\theta cos\varphi sin^2\theta e^{-i\varphi} + i\sqrt{\frac{15}{8\pi}}cot^2\theta cos\varphi cos^2\theta e^{-i\varphi}$$

$$-i\sqrt{\frac{15}{8\pi}}cot^2\theta cos\varphi e^{-i\varphi} + 2i\sqrt{\frac{15}{8\pi}}cos\varphi(-sin^2\theta)e^{-i\varphi}$$

$$\left.+2\sqrt{\frac{15}{8\pi}}sin\varphi e^{-i\varphi} + \sqrt{\frac{15}{8\pi}}\frac{1}{sin^2\theta}(2isin^2\theta cos\varphi)e^{-i\varphi}\right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi \left(\frac{1}{sin^2\theta} (cos^2\theta) \right) e^{-i\varphi} - \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} e^{-i\varphi} \right.$$

$$- i\sqrt{\frac{15}{8\pi}} cos^2\theta cos\varphi e^{-i\varphi} - i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi e^{-i\varphi}$$

$$+ i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi cos^2\theta e^{-i\varphi} + 2i\sqrt{\frac{15}{8\pi}} cos\varphi(-sin^2\theta) e^{-i\varphi}$$

$$\left. + 2\sqrt{\frac{15}{8\pi}} sin\varphi e^{-i\varphi} + \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\theta} (2isin^2\theta cos\varphi) e^{-i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} (cos^2\theta - 1) e^{-i\varphi} + 2\sqrt{\frac{15}{8\pi}} sin\varphi e^{-i\varphi} \right.$$

$$- i\sqrt{\frac{15}{8\pi}} cos^2\theta cos\varphi e^{-i\varphi} + i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi (cos^2\theta - 1) e^{-i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} cos\varphi(-sin^2\theta + 1) \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} (-sin^2\theta) e^{-i\varphi} + 2\sqrt{\frac{15}{8\pi}} sin\varphi e^{-i\varphi} \right.$$

$$- i\sqrt{\frac{15}{8\pi}} cos^2\theta cos\varphi e^{-i\varphi} + i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi (-sin^2\theta) e^{-i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} cos\varphi (cos^2\theta) \right\}$$

$$[\hat{L}_x, \hat{L}^2] Y_{2-1} = 0$$

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g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$$[\hat{L}_y,\hat{L}^2]Y_{2-1} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{2-1}$$

$$= \left[\left\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$- \left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{2-1}$$

$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + \cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + \cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}$$

$$\left.- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} Y_{2-1}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$\left.- \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\} Y_{2-1}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial Y_{2-1}}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{2-1}}{\partial\theta^2} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{2-1}}{\partial\varphi^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{2-1}}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{2-1}}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{2-1}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial Y_{2-1}}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial Y_{2-1}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial Y_{2-1}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial Y_{2-1}}{\partial\varphi}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial Y_{2-1}}{\partial\theta} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial Y_{2-1}}{\partial\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi} \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$- \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{-i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} \cos\varphi \left(\frac{1}{\sin^2\theta} (\cos^2\theta - \sin^2\theta) \right. \right.$$

$$\left. + 4 \sin\theta \cos\theta \cot\theta \right) e^{-i\varphi} - 4\sqrt{\frac{15}{8\pi}} \cos\varphi \, (-\sin^2\theta + \cos^2\theta) e^{-i\varphi}$$

$$+ \sqrt{\frac{15}{8\pi}} \cos\varphi \frac{1}{\sin^2\theta} e^{-i\varphi}$$

$$+ i\sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta \, (-\sin^2\theta + \cos^2\theta) e^{-i\varphi}$$

$$\left. - 4i\sqrt{\frac{15}{8\pi}} \sin\varphi \cot\theta \sin\theta \cos\theta e^{-i\varphi} - i\sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta \, e^{-i\varphi} \right\}$$

$$- i\hbar^3 \left\{ -4\sqrt{\frac{15}{8\pi}} \cot\theta \cos\varphi \sin\theta \cos\theta e^{-i\varphi} \right.$$

$$- 2i\sqrt{\frac{15}{8\pi}} \sin\varphi \cot\theta \sin\theta \cos\theta e^{-i\varphi}$$

$$- 4\sqrt{\frac{15}{8\pi}} \cos\varphi \, (-\sin^2\theta + \cos^2\theta) e^{-i\varphi}$$

$$- 2i\sqrt{\frac{15}{8\pi}} \sin\varphi \, (\cos^2\theta - \sin^2\theta) e^{-i\varphi}$$

$$+ \sqrt{\frac{15}{8\pi}} \frac{1}{\sin^2\theta} (2i\cos^2\theta \sin\varphi - 2\cos^2\theta \cos\varphi - 2i\sin^2\theta \sin\varphi$$

$$\left. + 2\sin^2\theta \cos\varphi) e^{-i\varphi} + \sqrt{\frac{15}{8\pi}} \cot^2\theta \, (-2i\sin\varphi + 2\cos\varphi) e^{-i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} cos\,\varphi \left(\frac{1}{sin^2\,\theta} (cos^2\theta) \right) e^{-i\varphi} \right.$$

$$+ \sqrt{\frac{15}{8\pi}} cos\,\varphi \frac{1}{sin^2\,\theta} e^{-i\varphi}$$

$$+ i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\,\theta\, (-sin^2\theta$$

$$\left. +\; cos^2\theta) e^{-i\varphi} - i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\,\theta\; e^{-i\varphi} \right\}$$

$$- i\hbar^3 \left\{ -2i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, (-sin^2\theta) e^{-i\varphi} \right.$$

$$\left. + \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\,\theta} (-2isin^2\theta sin\varphi + sin^2\theta cos\varphi) e^{-i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} cos\,\varphi \left(\frac{1}{sin^2\,\theta} (cos^2\theta) \right) e^{-i\varphi} \right.$$

$$+ \sqrt{\frac{15}{8\pi}} cos\,\varphi \frac{1}{sin^2\,\theta} e^{-i\varphi} - i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\,\theta\, sin^2\theta e^{-i\varphi}$$

$$+ i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\,\theta\, cos^2\theta e^{-i\varphi} - i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\,\theta\; e^{-i\varphi}$$

$$- 2i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, (sin^2\theta) e^{-i\varphi} + \sqrt{\frac{15}{8\pi}} 2sin\varphi e^{-i\varphi}$$

$$\left. - \sqrt{\frac{15}{8\pi}} cos\varphi e^{-i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} \cos\varphi \left(\frac{1}{\sin^2\theta} (\cos^2\theta) \right) e^{-i\varphi} \right.$$

$$+ \sqrt{\frac{15}{8\pi}} \cos\varphi \frac{1}{\sin^2\theta} e^{-i\varphi} - \sqrt{\frac{15}{8\pi}} \cos\varphi e^{-i\varphi}$$

$$- i \sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta \, e^{-i\varphi}$$

$$+ i \sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta \cos^2\theta e^{-i\varphi} - i \sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta \, e^{-i\varphi}$$

$$\left. - 2i \sqrt{\frac{15}{8\pi}} \sin\varphi (\sin^2\theta) e^{-i\varphi} + \sqrt{\frac{15}{8\pi}} 2\sin\varphi e^{-i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} \cos\varphi \frac{1}{\sin^2\theta} (\cos^2\theta - 1) e^{-i\varphi} - \sqrt{\frac{15}{8\pi}} \cos\varphi e^{-i\varphi} \right.$$

$$- i \sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta \, e^{-i\varphi} + i \sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta \, (\cos^2\theta - 1) e^{-i\varphi}$$

$$\left. + 2i \sqrt{\frac{15}{8\pi}} \sin\varphi (1 - \sin^2\theta) e^{-i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} \cos\varphi \frac{1}{\sin^2\theta} (-\sin^2\theta) e^{-i\varphi} - \sqrt{\frac{15}{8\pi}} \cos\varphi e^{-i\varphi} \right.$$

$$- i \sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta \, e^{-i\varphi} + i \sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta \, (-\sin^2\theta) e^{-i\varphi}$$

$$\left. + 2i \sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta \, e^{-i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} cos\,\varphi\, e^{-i\varphi} - \sqrt{\frac{15}{8\pi}} cos\varphi e^{-i\varphi} \right.$$

$$\left. - 2i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cos^2\,\theta\, e^{-i\varphi} + 2i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cos^2\,\theta\, e^{-i\varphi} \right\}$$

$[\hat{L}_y, \hat{L}^2] Y_{2-1} = 0$

h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z, \hat{L}^2] Y_{2-1} = (\hat{L}_z \hat{L}^2 - \hat{L}^2 \hat{L}_z) Y_{2-1}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times -\hbar^2 \left(cot\,\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar^2 \left(cot\,\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{2-1}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi} \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \right\} Y_{2-1}$$

$$- i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} \frac{\partial}{\partial\varphi} + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial}{\partial\varphi} \right\} Y_{2-1}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial Y_{2-1}}{\partial\theta} + \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{2-1}}{\partial\theta^2} + \frac{\partial}{\partial\varphi} \frac{1}{sin^2\,\theta} \frac{\partial^2 Y_{2-1}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} \frac{\partial Y_{2-1}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} \frac{\partial Y_{2-1}}{\partial\varphi} + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial Y_{2-1}}{\partial\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \frac{\partial}{\partial \varphi} cot\,\theta \frac{\partial}{\partial \theta} \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi} \right.$$

$$+ \frac{\partial}{\partial \varphi} \frac{\partial^2}{\partial \theta^2} \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi}$$

$$\left. + \frac{\partial}{\partial \varphi} \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial \varphi^2} \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi} \right\}$$

$$- i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi} \right.$$

$$+ \frac{\partial^2}{\partial \theta^2} \frac{\partial}{\partial \varphi} \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi}$$

$$\left. + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial \varphi^2} \frac{\partial}{\partial \varphi} \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -i \sqrt{\frac{15}{8\pi}} cot\,\theta\,(-\,sin^2\,\theta + cos^2\,\theta) e^{-i\varphi} \right.$$

$$\left. + 4i \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi} + i \sqrt{\frac{15}{8\pi}} \frac{cos\,\theta}{sin\,\theta}\;\; e^{-i\varphi} \right\}$$

$$- i\hbar^3 \left\{ -i \sqrt{\frac{15}{8\pi}} cot\,\theta\,(-\,sin^2\,\theta + cos^2\,\theta) e^{-i\varphi} \right.$$

$$\left. + 4i \sqrt{\frac{15}{8\pi}} sin\,\theta\,cos\,\theta\;\; e^{-i\varphi} + i \sqrt{\frac{15}{8\pi}} \frac{cos\,\theta}{sin\,\theta} e^{-i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}^2] Y_{2-1} = 0$$

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I.7 Harmonik Bola Y_{21}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x,\hat{L}_y]Y_{21} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)Y_{21}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$- \left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{21}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial Y_{21}}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial Y_{21}}{\partial\varphi}\right.$$

$$\left.+\, cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial Y_{21}}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial Y_{21}}{\partial\varphi}\right\}$$

$$-\, \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{21}}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{21}}{\partial\varphi}\right.$$

$$\left.-\, cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{21}}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{21}}{\partial\varphi}\right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right) \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right)$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right) \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right)$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right)$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \ e^{i\varphi} \right) \right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} e^{i\varphi} sin\varphi cos\varphi (4 \sin\theta \; cos\, \theta \,) \right.$$

$$- 2\sqrt{\frac{15}{8\pi}} sin\varphi sin\varphi cos\theta \sin\theta \; \left(ie^{i\varphi}\right)$$

$$- \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi (-sin^2 \theta + cos^2 \theta)\left(-sin\varphi \; e^{i\varphi} + icos\varphi \; e^{i\varphi}\right)$$

$$\left. + \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi cos\theta \cos\theta \left(icos\varphi \; e^{i\varphi} - sin\varphi \; e^{i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi cos\varphi e^{i\varphi} (4 \sin\theta \; cos\, \theta \,) \right.$$

$$+ 2\sqrt{\frac{15}{8\pi}} cos\varphi cos\varphi \left(ie^{i\varphi}\right) cos\theta \sin\theta$$

$$+ \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi \left(cos\varphi \; e^{i\varphi} + isin\varphi e^{i\varphi}\right)(-sin^2 \theta + cos^2 \theta)$$

$$\left. - \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi cos\theta \cos\theta \left(isin\varphi \; e^{i\varphi} + cos\varphi \; e^{i\varphi}\right) \right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} e^{i\varphi} sin\varphi cos\varphi (4 \sin\theta \cos\theta) \right.$$

$$- 2\sqrt{\frac{15}{8\pi}} sin\varphi sin\varphi cos\theta \sin\theta \left(ie^{i\varphi}\right)$$

$$- \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi (sin^2\theta \, sin\varphi - sin^2\theta \, icos\varphi - cos^2\theta \, sin\varphi$$

$$+ cos^2\theta \, icos\varphi) e^{i\varphi}$$

$$\left. + \sqrt{\frac{15}{8\pi}} cot\theta cos\varphi cos\theta \cos\theta \left(icos\varphi \, e^{i\varphi} - sin\varphi \, e^{i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi cos\varphi e^{i\varphi} (4 \sin\theta \cos\theta) \right.$$

$$+ 2\sqrt{\frac{15}{8\pi}} cos\varphi cos\varphi \left(ie^{i\varphi}\right) cos\theta \sin\theta$$

$$+ \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi (- sin^2\theta \, cos\varphi - sin^2\theta \, isin\varphi + cos^2\theta \, cos\varphi$$

$$+ cos^2\theta \, isin\varphi) e^{i\varphi}$$

$$\left. - \sqrt{\frac{15}{8\pi}} cot\theta sin\varphi cos\theta \cos\theta \left(isin\varphi \, e^{i\varphi} + cos\varphi \, e^{i\varphi}\right) \right\}$$

$$= \hbar^2 \left\{ -\sqrt{\frac{15}{8\pi}} sin^2\varphi \, cos\theta \sin\theta \left(ie^{i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} cos^2\varphi \left(ie^{i\varphi}\right) cos\theta \sin\theta \right\}$$

$$= -\hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} cos\theta \sin\theta \, (sin^2\varphi + cos^2\varphi) ie^{i\varphi} \right\}$$

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$$= -i\hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} cos\theta \sin\theta\, e^{i\varphi} \right\}$$

$$[\hat{L}_x, \hat{L}_y] Y_{21} = i\hbar^2 Y_{21}$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar^2$$

b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z] Y_{21} = (\hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y) Y_{21}$$

$$= \left[\left\{ i\hbar \left(-\cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right.$$

$$\left. - \left\{ -i\hbar \frac{\partial}{\partial\varphi} \times i\hbar \left(-\cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{21}$$

$$= \hbar^2 \left\{ -\cos\varphi \frac{\partial}{\partial\theta} \frac{\partial Y_{21}}{\partial\varphi} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{21}}{\partial\varphi} \right\}$$

$$- \hbar^2 \left\{ -\frac{\partial}{\partial\varphi} \cos\varphi \frac{\partial Y_{21}}{\partial\theta} + \frac{\partial}{\partial\varphi} \sin\varphi \cot\theta \frac{\partial Y_{21}}{\partial\varphi} \right\}$$

$$= \hbar^2 \left\{ -\cos\varphi \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\ e^{i\varphi} \right) \right.$$

$$\left. + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\ e^{i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ -\frac{\partial}{\partial\varphi} \cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\ e^{i\varphi} \right) \right.$$

$$\left. + \frac{\partial}{\partial\varphi} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\ e^{i\varphi} \right) \right\}$$

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$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} \cos\varphi \,(-\sin^2\theta + \cos^2\theta) i e^{i\varphi} + \sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta \, e^{i\varphi} \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} (-\sin^2\theta + \cos^2\theta)\left(-\sin\varphi \, e^{i\varphi} + i\cos\varphi \, e^{i\varphi}\right) \right.$$

$$\left. - \sqrt{\frac{15}{8\pi}} \cos^2\theta \left(i\cos\varphi e^{i\varphi} - \sin\varphi \, e^{i\varphi}\right) \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} \cos\varphi \,(-\sin^2\theta + \cos^2\theta) i e^{i\varphi} + \sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta \, e^{i\varphi} \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} (\sin^2\theta \sin\varphi - \sin^2\theta \, i\cos\varphi - \cos^2\theta \sin\varphi \right.$$

$$\left. + \cos^2\theta \, i\cos\varphi) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} \cos^2\theta \left(i\cos\varphi e^{i\varphi} - \sin\varphi \, e^{i\varphi}\right) \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} \cos\varphi \,(\cos^2\theta) i e^{i\varphi} + \sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta \, e^{i\varphi} \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} (\sin^2\theta \sin\varphi) e^{i\varphi} \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} \cos^2\theta \,(i\cos\varphi + \sin\varphi) e^{i\varphi} \right\}$$

$$- \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} (\sin^2\theta \sin\varphi) e^{i\varphi} \right\}$$

$$= \frac{\hbar^2}{i} \left\{ -\sqrt{\frac{15}{8\pi}} \cos^2\theta \,(\cos\varphi - i\sin\varphi) e^{i\varphi} \right\}$$

$$- \frac{\hbar^2}{i} \left\{ i\sqrt{\frac{15}{8\pi}} (\sin^2\theta \sin\varphi) e^{i\varphi} \right\}$$

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$$= \frac{\hbar^2}{i}\left\{-\sqrt{\frac{15}{8\pi}} cos^2\,\theta\,(e^{-i\varphi})e^{i\varphi}\right\} - \frac{\hbar^2}{i}\left\{i\sqrt{\frac{15}{8\pi}}(sin^2\,\theta\, sin\varphi)e^{i\varphi}\right\}$$

$$[\hat{L}_y, \hat{L}_z]Y_{21} = \frac{\hbar^2}{i}\{-\,cos^2\,\theta\,(e^{-i\varphi}) - i\,sin^2\,\theta\, sin\varphi\}\sqrt{\frac{15}{8\pi}}e^{i\varphi}$$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x]Y_{21} = (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)Y_{21}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]Y_{21}$$

$$= \hbar^2\left\{\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{21}}{\partial\theta} + \frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{21}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{21}}{\partial\varphi} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial Y_{21}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial}{\partial\theta}\left(-\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{i\varphi}\right)\right.$$

$$\left. + \frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{i\varphi}\right)\right\}$$

$$- \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{i\varphi}\right)\right.$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{15}{8\pi}}sin\,\theta\,cos\,\theta\;\;e^{i\varphi}\right)\right\}$$

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$$= \hbar^2 \left\{ -\sqrt{\frac{15}{8\pi}} (cos\varphi\ e^{i\varphi} + isin\varphi e^{i\varphi})(-sin^2\theta + cos^2\theta) \right.$$

$$\left. - \sqrt{\frac{15}{8\pi}} cos^2\theta \left(-isin\varphi\ e^{i\varphi} - cos\varphi e^{i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ -\sqrt{\frac{15}{8\pi}} sin\varphi(-sin^2\theta + cos^2\theta) ie^{i\varphi} \right.$$

$$\left. + \sqrt{\frac{15}{8\pi}} cos^2\theta\ cos\varphi e^{i\varphi} \right\}$$

$$= \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} (cos\varphi\ \ sin^2\theta - isin\varphi\ cos^2\theta) e^{i\varphi} \right\}$$

$$- \hbar^2 \left\{ + \sqrt{\frac{15}{8\pi}} cos^2\theta\ cos\varphi e^{i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_x] Y_{21} = \hbar^2 \{cos\varphi\ \ sin^2\theta - cos^2\theta\ (cos\varphi + isin\varphi)\} \sqrt{\frac{15}{8\pi}} e^{i\varphi}$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+] Y_{21} = (\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z) Y_{21}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial \varphi} \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i\cot\theta \frac{\partial}{\partial \varphi} \right) \right\} \right.$$

$$\left. - \left\{ \hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i\cot\theta \frac{\partial}{\partial \varphi} \right) \times -i\hbar \frac{\partial}{\partial \varphi} \right\} \right] Y_{21}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi} i\cot\theta \frac{\partial}{\partial \varphi} \right\} Y_{21}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} - e^{i\varphi} i\cot\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right\} Y_{21}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial \varphi} e^{i\varphi} \frac{\partial Y_{21}}{\partial \theta} - \frac{\partial}{\partial \varphi} e^{i\varphi} i\cot\theta \frac{\partial Y_{21}}{\partial \varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial \theta} \frac{\partial Y_{21}}{\partial \varphi} - e^{i\varphi} i\cot\theta \frac{\partial}{\partial \varphi} \frac{\partial Y_{21}}{\partial \varphi} \right\}$$

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$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\, cos\theta e^{i\varphi} \right) \right.$$

$$\left. - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\, cos\theta e^{i\varphi} \right) \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\, cos\theta e^{i\varphi} \right) \right.$$

$$\left. - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\, cos\theta e^{i\varphi} \right) \right\}$$

$$= i\hbar^2 \left\{ 2i \sqrt{\frac{15}{8\pi}} e^{i\varphi} (-\sin^2\theta + \cos^2\theta) e^{i\varphi} - 2i \sqrt{\frac{15}{8\pi}} e^{i\varphi} \cos^2\theta\, e^{i\varphi} \right\}$$

$$- i\hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} e^{i\varphi} (-\sin^2\theta + \cos^2\theta) i e^{i\varphi} - \sqrt{\frac{15}{8\pi}} e^{i\varphi} i \cos^2\theta\, e^{i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_+] Y_{21} = \hbar^2 \left\{ \sqrt{\frac{15}{8\pi}} e^{i\varphi} (\sin^2\theta) e^{i\varphi} \right\}$$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-] Y_{21} = (\hat{L}_z \hat{L}_- - \hat{L}_- \hat{L}_z) Y_{21}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{21}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{21}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \right\} Y_{21}$$

$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial Y_{21}}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i \cot\theta \frac{\partial Y_{21}}{\partial\varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial Y_{21}}{\partial\varphi} - e^{-i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{21}}{\partial\varphi} \right\}$$

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$$= i\hbar^2 \left\{ \frac{\partial}{\partial \varphi} e^{-i\varphi} \frac{\partial}{\partial \theta} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right.$$

$$\left. - \frac{\partial}{\partial \varphi} e^{-i\varphi} i\, cot\, \theta \frac{\partial}{\partial \varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right.$$

$$\left. - e^{-i\varphi} i\, cot\, \theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right\}$$

$$= i\hbar^2 \left\{ -\sqrt{\frac{15}{8\pi}} (0) + \sqrt{\frac{15}{8\pi}} (0) \right\}$$

$$- i\hbar^2 \left\{ -i \sqrt{\frac{15}{8\pi}} e^{-i\varphi} (-\, sin^2\, \theta + cos^2\, \theta) e^{i\varphi} \right.$$

$$\left. - \sqrt{\frac{15}{8\pi}} e^{-i\varphi} i\, cos^2\, \theta\, e^{i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_-] Y_{21} = \hbar^2 e^{-i\varphi} \{ sin\theta - 2cos^2\theta \}$$

f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2] Y_{21} = (\hat{L}_x \hat{L}^2 - \hat{L}^2 \hat{L}_x) Y_{21}$$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial \theta} + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\frac{1}{sin\, \theta} \frac{\partial}{\partial \theta} \left(sin\, \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{sin^2\, \theta} \frac{\partial^2}{\partial \varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\frac{1}{sin\, \theta} \frac{\partial}{\partial \theta} \left(sin\, \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{sin^2\, \theta} \frac{\partial^2}{\partial \varphi^2} \right) \right.$$

$$\left. \left. \times i\hbar \left(sin\varphi \frac{\partial}{\partial \theta} + cot\theta cos\varphi \frac{\partial}{\partial \varphi} \right) \right\} \right] Y_{21}$$

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$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{21}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right\} Y_{21}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left.+ \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\} Y_{21}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial Y_{21}}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{21}}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{21}}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial Y_{21}}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{21}}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{21}}{\partial\varphi^2}\right\}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{21}}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{21}}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial Y_{21}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial Y_{21}}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial Y_{21}}{\partial\theta}$$

$$\left.+ \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial Y_{21}}{\partial\varphi}\right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right\}$$

$$+ i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right.$$

$$+ \cot\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$+ \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right)$$

$$\left. + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} sin\theta\ cos\theta e^{i\varphi} \right) \right\}$$

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$$= -i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi \left(\frac{1}{sin^2\theta}(cos^2\theta - sin^2\theta) + 4sin\theta cos\theta cot\theta \right) e^{i\varphi} \right.$$

$$+ 4\sqrt{\frac{15}{8\pi}} sin\varphi(-sin^2\theta + cos^2\theta)e^{i\varphi} - \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} e^{i\varphi}$$

$$- i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi(-sin^2\theta + cos^2\theta)e^{i\varphi}$$

$$\left. + 4i\sqrt{\frac{15}{8\pi}} cot\theta cos\varphi sin\theta\, cos\theta e^{i\varphi} + i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi e^{i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ 4\sqrt{\frac{15}{8\pi}} cot\,\theta\, sin\varphi sin\theta\, cos\theta e^{i\varphi} \right.$$

$$+ 2i\sqrt{\frac{15}{8\pi}} cot\,\theta\, cos\varphi sin\theta\, cos\theta e^{i\varphi}$$

$$+ 4\sqrt{\frac{15}{8\pi}} sin\varphi(-sin^2\theta + cos^2\theta)e^{i\varphi}$$

$$+ 2i\sqrt{\frac{15}{8\pi}} cos\varphi(-sin^2\theta + cos^2\theta)e^{i\varphi}$$

$$- \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\theta}(-2cos^2\theta sin\varphi + 2icos^2\theta cos\varphi + 2sin^2\theta sin\varphi$$

$$\left. - 2isin^2\theta cos\varphi)e^{i\varphi} - \sqrt{\frac{15}{8\pi}} cot^2\theta(-2icos\varphi + 2sin\varphi)e^{i\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi \left(\frac{1}{sin^2\theta} (cos^2\theta) \right) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} e^{i\varphi} \right.$$

$$- i \sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi(-sin^2\theta + cos^2\theta) e^{i\varphi}$$

$$\left. + i \sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi e^{i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ 2i \sqrt{\frac{15}{8\pi}} cos\varphi(-sin^2\theta) e^{i\varphi} \right.$$

$$\left. - \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\theta} (sin^2\theta sin\varphi - 2isin^2\theta cos\varphi) e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ - \sqrt{\frac{15}{8\pi}} sin\varphi \left(\frac{1}{sin^2\theta} (cos^2\theta) \right) e^{i\varphi} + \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} e^{i\varphi} \right.$$

$$- i \sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi sin^2\theta e^{i\varphi} + i \sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi cos^2\theta e^{i\varphi}$$

$$- i \sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi e^{i\varphi} - 2i \sqrt{\frac{15}{8\pi}} cos\varphi(sin^2\theta) e^{i\varphi}$$

$$\left. - \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\theta} (sin^2\theta sin\varphi - 2isin^2\theta cos\varphi) e^{i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} sin\varphi \left(\frac{1}{sin^2\theta} (cos^2\theta) \right) e^{i\varphi} + \sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} e^{i\varphi} \right.$$

$$- \sqrt{\frac{15}{8\pi}} sin\varphi - i\sqrt{\frac{15}{8\pi}} cos^2\theta cos\varphi e^{i\varphi} + i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi cos^2\theta e^{i\varphi}$$

$$- i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi e^{i\varphi} - 2i\sqrt{\frac{15}{8\pi}} cos\varphi (sin^2\theta) e^{i\varphi}$$

$$\left. + \sqrt{\frac{15}{8\pi}} (2icos\varphi) e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} (cos^2\theta - 1) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} sin\varphi \right.$$

$$- i\sqrt{\frac{15}{8\pi}} cos^2\theta cos\varphi e^{i\varphi} + i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi (cos^2\theta - 1) e^{i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} cos\varphi (1 - sin^2\theta) e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} sin\varphi \frac{1}{sin^2\theta} (-sin^2\theta) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} sin\varphi \right.$$

$$- i\sqrt{\frac{15}{8\pi}} cos^2\theta cos\varphi e^{i\varphi} + i\sqrt{\frac{15}{8\pi}} cot^2\theta cos\varphi (-sin^2\theta) e^{i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} cos\varphi (cos^2\theta) e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} sin\varphi e^{i\varphi} - \sqrt{\frac{15}{8\pi}} sin\varphi - 2i\sqrt{\frac{15}{8\pi}} cos^2\theta cos\varphi e^{i\varphi} \right.$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} cos\varphi (cos^2\theta) e^{i\varphi} \right\}$$

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$[\hat{L}_x, \hat{L}^2]Y_{21} = 0$

g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$$[\hat{L}_y, \hat{L}^2]Y_{21} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{21}$$

$$= \left[\left\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{21}$$

$$= i\hbar^3\left\{\cos\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + \cos\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + \cos\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta}$$

$$\left.- \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{21}$$

$$- i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\theta}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}\cos\varphi\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\cos\varphi\frac{\partial}{\partial\theta}$$

$$\left.- \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right\}Y_{21}$$

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$$= i\hbar^3 \left\{ cos\,\varphi \frac{\partial}{\partial\theta} cot\,\theta \frac{\partial Y_{21}}{\partial\theta} + cos\,\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{21}}{\partial\theta^2} + cos\,\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\,\theta} \frac{\partial^2 Y_{21}}{\partial\varphi^2} \right.$$

$$- sin\,\varphi\, cot\,\theta \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial Y_{21}}{\partial\theta}$$

$$\left. - sin\,\varphi\, cot\,\theta \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{21}}{\partial\theta^2} - sin\,\varphi\, cot\,\theta \frac{\partial}{\partial\varphi} \frac{1}{sin^2\,\theta} \frac{\partial^2 Y_{21}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} cos\,\varphi \frac{\partial Y_{21}}{\partial\theta} - cot\,\theta \frac{\partial}{\partial\theta} sin\,\varphi\, cot\,\theta \frac{\partial Y_{21}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} cos\,\varphi \frac{\partial Y_{21}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} sin\,\varphi\, cot\,\theta \frac{\partial Y_{21}}{\partial\varphi} + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} cos\,\varphi \frac{\partial Y_{21}}{\partial\theta}$$

$$\left. - \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} sin\,\varphi\, cot\,\theta \frac{\partial Y_{21}}{\partial\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right) \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right) \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right) \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$- \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right)$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi} \right) \right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} cos\,\varphi \left(\frac{1}{sin^2\,\theta} (cos^2\theta - sin^2\theta) + 4sin\theta cos\theta cot\theta \right) e^{i\varphi} \right.$$

$$+ 4\sqrt{\frac{15}{8\pi}} cos\,\varphi\,(-sin^2\theta + \ cos^2\theta) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} cos\,\varphi \frac{1}{sin^2\,\theta} e^{i\varphi}$$

$$+ i\sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta (cos^2\theta - sin^2\theta) e^{i\varphi}$$

$$\left. - 4i\sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot\,\theta\,(sin\theta\ cos\theta) e^{i\varphi} - i\sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta e^{i\varphi} \right\}$$

$$- i\hbar^3 \left\{ 4\sqrt{\frac{15}{8\pi}} cot\,\theta\, cos\,\varphi\,(sin\theta\ cos\theta) e^{i\varphi} \right.$$

$$- 2i\sqrt{\frac{15}{8\pi}} cot\,\theta\, sin\,\varphi \frac{\partial}{\partial\theta} (sin\theta\ cos\theta) e^{i\varphi}$$

$$+ 4\sqrt{\frac{15}{8\pi}} cos\,\varphi\,(cos^2\theta - sin^2\theta) e^{i\varphi}$$

$$- 2i\sqrt{\frac{15}{8\pi}} sin\,\varphi\,(cos^2\theta - sin^2\theta) e^{i\varphi}$$

$$- \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\,\theta} (-2icos^2\theta sin\varphi - 2cos^2\theta cos\varphi + 2isin^2\theta sin\varphi$$

$$\left. + 2sin^2\theta cos\varphi) e^{i\varphi} + \sqrt{\frac{15}{8\pi}} cot^2\theta (-2i\,sin\,\varphi - 2cos\varphi) e^{i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} cos\,\varphi \left(\frac{1}{sin^2\,\theta} (cos^2\theta) \right) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} cos\,\varphi \frac{1}{sin^2\,\theta} e^{i\varphi} \right.$$

$$\left. + i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta (cos^2\theta - sin^2\theta) e^{i\varphi} - i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta e^{i\varphi} \right\}$$

$$- i\hbar^3 \left\{ -2i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, (-sin^2\theta) e^{i\varphi} \right.$$

$$\left. - \sqrt{\frac{15}{8\pi}} \frac{1}{sin^2\,\theta} (2isin^2\theta sin\varphi + sin^2\theta cos\varphi) e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} cos\,\varphi \left(\frac{1}{sin^2\,\theta} (cos^2\theta) \right) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} cos\,\varphi \frac{1}{sin^2\,\theta} e^{i\varphi} \right.$$

$$+ i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta cos^2\theta e^{i\varphi}$$

$$- i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta sin^2\theta e^{i\varphi} - i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta e^{i\varphi}$$

$$\left. - 2i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, (sin^2\theta) e^{i\varphi} + \sqrt{\frac{15}{8\pi}} 2isin\varphi e^{i\varphi} + \sqrt{\frac{15}{8\pi}} cos\varphi e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} cos\,\varphi \frac{1}{sin^2\,\theta} (cos^2\theta) e^{i\varphi} - \sqrt{\frac{15}{8\pi}} cos\,\varphi \frac{1}{sin^2\,\theta} e^{i\varphi} \right.$$

$$+ \sqrt{\frac{15}{8\pi}} cos\varphi e^{i\varphi} - i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cos^2\theta e^{i\varphi}$$

$$+ i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta cos^2\theta e^{i\varphi} - i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, cot^2\theta e^{i\varphi}$$

$$\left. - 2i \sqrt{\frac{15}{8\pi}} sin\,\varphi\, (sin^2\theta) e^{i\varphi} + \sqrt{\frac{15}{8\pi}} 2isin\varphi e^{i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} \cos\varphi \frac{1}{\sin^2\theta}(\cos^2\theta - 1)e^{i\varphi} + \sqrt{\frac{15}{8\pi}} \cos\varphi e^{i\varphi} \right.$$

$$- i\sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta e^{i\varphi} + i\sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta(\cos^2\theta - 1)e^{i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} \sin\varphi (1 - \sin^2\theta)e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \sqrt{\frac{15}{8\pi}} \cos\varphi \frac{1}{\sin^2\theta}(-\sin^2\theta)e^{i\varphi} + \sqrt{\frac{15}{8\pi}} \cos\varphi e^{i\varphi} \right.$$

$$- i\sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta e^{i\varphi} + i\sqrt{\frac{15}{8\pi}} \sin\varphi \cot^2\theta(-\sin^2\theta)e^{i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} \sin\varphi (\cos^2\theta)e^{i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{8\pi}} \cos\varphi\, e^{i\varphi} + \sqrt{\frac{15}{8\pi}} \cos\varphi e^{i\varphi} - 2i\sqrt{\frac{15}{8\pi}} \sin\varphi \cos^2\theta e^{i\varphi} \right.$$

$$\left. + 2i\sqrt{\frac{15}{8\pi}} \sin\varphi (\cos^2\theta)e^{i\varphi} \right\}$$

$$[\hat{L}_y, \hat{L}^2]Y_{21} = 0$$

h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z, \hat{L}^2]Y_{21} = (\hat{L}_z\hat{L}^2 - \hat{L}^2\hat{L}_z)Y_{21}$$

$$= \left[\left\{ -i\hbar\frac{\partial}{\partial\varphi} \times -\hbar^2 \left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) \right\} \right.$$

$$\left. - \left\{ -\hbar^2 \left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) \times -i\hbar\frac{\partial}{\partial\varphi} \right\} \right] Y_{21}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right\} Y_{21}$$

$$- i\hbar^3 \left\{ \cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi} \right\} Y_{21}$$

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$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial Y_{21}}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{21}}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{21}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta}\frac{\partial Y_{21}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial Y_{21}}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial Y_{21}}{\partial\varphi} \right\}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial}{\partial\theta}\left(-\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} \right) \right.$$

$$+ \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}\left(-\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} \right)$$

$$\left. + \frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\left(-\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} \right) \right\}$$

$$- i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} \right) \right.$$

$$+ \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} \right)$$

$$\left. + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\left(-\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} \right) \right\}$$

$$= i\hbar^3 \left\{ -i\sqrt{\frac{15}{8\pi}} cot\,\theta\,(-sin^2\,\theta + cos^2\,\theta) e^{i\varphi} \right.$$

$$\left. + 4i\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} + i\sqrt{\frac{15}{8\pi}\frac{cos\,\theta}{sin\,\theta}}\; e^{i\varphi} \right\}$$

$$- i\hbar^3 \left\{ -i\sqrt{\frac{15}{8\pi}} cot\,\theta\,(-sin^2\,\theta + cos^2\,\theta) e^{i\varphi} \right.$$

$$\left. + 4i\sqrt{\frac{15}{8\pi}} sin\,\theta\, cos\,\theta\;\, e^{i\varphi} + i\sqrt{\frac{15}{8\pi}\frac{cos\,\theta}{sin\,\theta}} e^{i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}^2] Y_{21} = 0$$

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I.8 Harmonik Bola $Y_{2\text{-}2}$

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$$[\hat{L}_x, \hat{L}_y]Y_{2-2} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)Y_{2-2}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}$$

$$-\left\{-i\hbar\left(cos\varphi\frac{\partial}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right] Y_{2-2}$$

$$= \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}cos\varphi\frac{\partial Y_{2-2}}{\partial\theta} - sin\varphi\frac{\partial}{\partial\theta}cot\theta sin\varphi\frac{\partial Y_{2-2}}{\partial\varphi}\right.$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}cos\varphi\frac{\partial Y_{2-2}}{\partial\theta} - cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\theta sin\varphi\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{cos\varphi\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{2-2}}{\partial\theta} + cos\varphi\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{2-2}}{\partial\varphi}\right.$$

$$\left. - cot\theta sin\varphi\frac{\partial}{\partial\varphi}sin\varphi\frac{\partial Y_{2-2}}{\partial\theta} - cot\theta sin\varphi\frac{\partial}{\partial\varphi}cot\theta cos\varphi\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi} \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi}$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi} \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi} \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi}$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi}$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{-2i\varphi} \right\}$$

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$$= \hbar^2 \left\{ 2\sqrt{\frac{15}{32\pi}} sin\varphi cos\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi} \right.$$

$$+ 2i\sqrt{\frac{15}{32\pi}} sin\varphi sin\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi cos\theta sin\theta \left(-sin\varphi\, e^{-2i\varphi} - 2icos\varphi e^{-2i\varphi} \right)$$

$$\left. - \sqrt{\frac{15}{32\pi}} cot\theta cos\varphi cos\theta sin\theta \left(-2icos\varphi\, e^{-2i\varphi} - 4sin\varphi e^{-2i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ 2\sqrt{\frac{15}{32\pi}} cos\varphi sin\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi} \right.$$

$$- 2i\sqrt{\frac{15}{32\pi}} cos\varphi cos\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi}$$

$$- 2\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi cos\theta sin\theta \left(cos\varphi\, e^{-2i\varphi} - 2isin\varphi e^{-2i\varphi} \right)$$

$$\left. - \sqrt{\frac{15}{32\pi}} cot\theta sin\varphi cos\theta sin\theta \left(2isin\varphi\, e^{-2i\varphi} - 4cos\varphi e^{-2i\varphi} \right) \right\}$$

$$= \hbar^2 \left\{ 2i\sqrt{\frac{15}{32\pi}} sin^2\varphi (-sin^2\theta) e^{-2i\varphi} \right\}$$

$$- \hbar^2 \left\{ -2i\sqrt{\frac{15}{32\pi}} cos^2\varphi (-sin^2\theta) e^{-2i\varphi} \right\}$$

$$= \hbar^2 \left\{ 2i\sqrt{\frac{15}{32\pi}} sin^2\varphi (-sin^2\theta) e^{-2i\varphi} \right.$$

$$\left. - 2i\sqrt{\frac{15}{32\pi}} cos^2\varphi (sin^2\theta) e^{-2i\varphi} \right\}$$

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$$= -2i\hbar^2(sin^2\varphi + cos^2\varphi)\sqrt{\frac{15}{32\pi}}sin^2\theta\, e^{-2i\varphi}$$

$$[\hat{L}_x, \hat{L}_y]Y_{2-2} = -2i\hbar^2 Y_{2-2}$$

$$[\hat{L}_x, \hat{L}_y] = -2i\hbar^2$$

b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y, \hat{L}_z]Y_{2-2} = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)Y_{2-2}$$

$$= \left[\left\{i\hbar\left(-cos\,\varphi\frac{\partial}{\partial\theta} + sin\,\varphi\,cot\,\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right.$$

$$\left.-\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(-cos\,\varphi\frac{\partial}{\partial\theta} + sin\,\varphi\,cot\,\theta\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{2-2}$$

$$= \hbar^2\left\{-cos\,\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{2-2}}{\partial\varphi} + sin\,\varphi\,cot\,\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

$$-\hbar^2\left\{-\frac{\partial}{\partial\varphi}cos\,\varphi\frac{\partial Y_{2-2}}{\partial\theta} + \frac{\partial}{\partial\varphi}sin\,\varphi\,cot\,\theta\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{-cos\,\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\theta\;\; e^{-2i\varphi}\right.$$

$$\left.+ sin\,\varphi\,cot\,\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\theta\;\; e^{-2i\varphi}\right\}$$

$$-\hbar^2\left\{-\frac{\partial}{\partial\varphi}cos\,\varphi\frac{\partial}{\partial\theta}\sqrt{\frac{15}{32\pi}}sin^2\theta\;\; e^{-2i\varphi}\right.$$

$$\left.+\frac{\partial}{\partial\varphi}sin\,\varphi\,cot\,\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\theta\;\; e^{-2i\varphi}\right\}$$

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$$= \hbar^2 \left\{ 4i\sqrt{\frac{15}{32\pi}} \cos\varphi \cos\theta \sin\theta e^{-2i\varphi} \right.$$

$$\left. - 4\sqrt{\frac{15}{32\pi}} \sin\varphi \cos\theta \sin\theta e^{-2i\varphi} \right\}$$

$$- \hbar^2 \left\{ -2\sqrt{\frac{15}{32\pi}} \cos\theta \sin\theta \left(-\sin\varphi\, e^{-2i\varphi} - 2i \cos\varphi\, e^{-2i\varphi}\right) \right.$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} \cos\theta \sin\theta \left(i\cos\varphi\, e^{-2i\varphi} + 2 \sin\varphi\, e^{-2i\varphi}\right) \right\}$$

$$= \hbar^2 \left\{ -2\sqrt{\frac{15}{32\pi}} \sin\varphi \cos\theta \sin\theta e^{-2i\varphi} \right.$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} \cos\theta \sin\theta \left(i\cos\varphi\, e^{-2i\varphi}\right) \right\}$$

$$= -2\hbar^2 \cos\theta (\sin\varphi - i\cos\varphi) \sqrt{\frac{15}{32\pi}} \sin\theta e^{-2i\varphi}$$

$$= -2\frac{\hbar^2}{i} \cos\theta (i\sin\varphi + \cos\varphi) \sqrt{\frac{15}{32\pi}} \sin\theta e^{-2i\varphi}$$

$$[\hat{L}_y, \hat{L}_z] Y_{2-2} = -2\frac{\hbar^2}{i} \cos\theta \left(e^{i\varphi}\right) \sqrt{\frac{15}{32\pi}} \sin\theta e^{-2i\varphi}$$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x] Y_{2-2} = (\hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z) Y_{2-2}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times i\hbar \left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ i\hbar \left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \cos\varphi \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{2-2}$$

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$$= \hbar^2 \left\{ \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial Y_{2-2}}{\partial\theta} + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial Y_{2-2}}{\partial\varphi} \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial Y_{2-2}}{\partial\varphi} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial Y_{2-2}}{\partial\varphi} \right\}$$

$$= \hbar^2 \left\{ \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta \;\; e^{-2i\varphi} \right.$$

$$\left. + \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta \;\; e^{-2i\varphi} \right\}$$

$$- \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta \;\; e^{-2i\varphi} \right.$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta \;\; e^{-2i\varphi} \right\}$$

$$= \hbar^2 \left\{ 2 \sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(cos\varphi \; e^{-2i\varphi} - 2isin\varphi e^{-2i\varphi} \right) \right.$$

$$\left. + \sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(2isin\varphi \; e^{-2i\varphi} - 4cos\varphi \; e^{-2i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ -4i \sqrt{\frac{15}{32\pi}} sin\varphi cos\theta sin\theta e^{-2i\varphi} \right.$$

$$\left. - 4 \sqrt{\frac{15}{32\pi}} cos\theta sin\theta cos\varphi e^{-2i\varphi} \right\}$$

$$= \hbar^2 \left\{ 2 \sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(cos\varphi \; e^{-2i\varphi} \right) \right.$$

$$\left. + \sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(2isin\varphi \; e^{-2i\varphi} \right) \right\}$$

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$$= 2\hbar^2 cos\theta(cos\varphi + isin\varphi)\sqrt{\frac{15}{32\pi}} sin\theta e^{-2i\varphi}$$

$$[\hat{L}_z, \hat{L}_x]Y_{2-2} = 2\hbar^2 cos\theta(e^{i\varphi})\sqrt{\frac{15}{32\pi}} sin\theta e^{-2i\varphi}$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+]Y_{2-2} = (\hat{L}_z\hat{L}_+ - \hat{L}_+\hat{L}_z)Y_{2-2}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right] Y_{2-2}$$

$$= i\hbar^2\left\{-\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right\}Y_{2-2}$$

$$- i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}Y_{2-2}$$

$$= i\hbar^2\left\{-\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial Y_{2-2}}{\partial\theta} - \frac{\partial}{\partial\varphi}e^{i\varphi}i\cot\theta\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

$$- i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial Y_{2-2}}{\partial\varphi} - e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

$$= i\hbar^2\left\{-\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta}\sqrt{\frac{15}{32\pi}}\sin^2\theta\ e^{-2i\varphi}\right.$$

$$\left. - \frac{\partial}{\partial\varphi}e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}\sin^2\theta\ e^{-2i\varphi}\right\}$$

$$- i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}\sin^2\theta\ e^{-2i\varphi}\right.$$

$$\left. - e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}\sin^2\theta\ e^{-2i\varphi}\right\}$$

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$$= i\hbar^2 \left\{ 2i\sqrt{\frac{15}{32\pi}} cos\theta sin\theta e^{-i\varphi} + 2\sqrt{\frac{15}{32\pi}} icos\theta sin\theta e^{-i\varphi} \right\}$$

$$- i\hbar^2 \left\{ 4i\sqrt{\frac{15}{32\pi}} cos\theta sin\theta e^{-i\varphi} + 4\sqrt{\frac{15}{32\pi}} icos\theta sin\theta e^{-i\varphi} \right\}$$

$$= -i\hbar^2 \left\{ 2i\sqrt{\frac{15}{32\pi}} cos\theta sin\theta e^{-i\varphi} + 2\sqrt{\frac{15}{32\pi}} icos\theta sin\theta e^{-i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_+]Y_{2-2} = 4\hbar^2 e^{-i\varphi} cos\theta sin\theta \sqrt{\frac{15}{32\pi}}$$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z, \hat{L}_-]Y_{2-2} = (\hat{L}_z\hat{L}_- - \hat{L}_-\hat{L}_z)Y_{2-2}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right] Y_{2-2}$$

$$= i\hbar^2 \left\{\frac{\partial}{\partial\varphi} e^{-i\varphi}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i\cot\theta\frac{\partial}{\partial\varphi}\right\} Y_{2-2}$$

$$- i\hbar^2 \left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - e^{-i\varphi} i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\} Y_{2-2}$$

$$= i\hbar^2 \left\{\frac{\partial}{\partial\varphi} e^{-i\varphi}\frac{\partial Y_{2-2}}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{-i\varphi} i\cot\theta\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

$$- i\hbar^2 \left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial Y_{2-2}}{\partial\varphi} - e^{-i\varphi} i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{2-2}}{\partial\varphi}\right\}$$

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$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\ \ e^{-2i\varphi} \right.$$

$$\left. - \frac{\partial}{\partial\varphi} e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\ \ e^{-2i\varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\ \ e^{-2i\varphi} \right.$$

$$\left. - e^{-i\varphi} i\,cot\,\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\ \ e^{-2i\varphi} \right\}$$

$$= i\hbar^2 \left\{ -6i \sqrt{\frac{15}{32\pi}} e^{-i\varphi} cos\theta sin\theta e^{-2i\varphi} \right.$$

$$\left. + 6 \sqrt{\frac{15}{32\pi}} e^{-i\varphi} icos\theta sin\theta e^{-2i\varphi} \right\}$$

$$- i\hbar^2 \left\{ -4i \sqrt{\frac{15}{32\pi}} e^{-i\varphi} cos\theta sin\theta e^{-2i\varphi} \right.$$

$$\left. + 4 \sqrt{\frac{15}{32\pi}} e^{-i\varphi} icos\theta sin\theta e^{-2i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_-] Y_{2-2} = 0$$

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f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2]Y_{2-2} = (\hat{L}_x\hat{L}^2 - \hat{L}^2\hat{L}_x)Y_{2-2}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$- \left\{-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{2-2}$$

$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$- \left\{-\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{2-2}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\cot\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{2-2}$$

$$+ i\hbar^3\left\{\cot\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left.+ \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\}Y_{2-2}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} cot\,\theta \frac{\partial Y_{2-2}}{\partial\theta} + sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{2-2}}{\partial\theta^2} \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\theta} \frac{\partial^2 Y_{2-2}}{\partial\varphi^2} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial Y_{2-2}}{\partial\theta}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{2-2}}{\partial\theta^2} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\theta} \frac{\partial^2 Y_{2-2}}{\partial\varphi^2} \right\}$$

$$+ i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial Y_{2-2}}{\partial\theta} + cot\,\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial Y_{2-2}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial Y_{2-2}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial Y_{2-2}}{\partial\varphi}$$

$$\left. + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial Y_{2-2}}{\partial\theta} + \frac{1}{sin^2\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial Y_{2-2}}{\partial\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} cot\,\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi} \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi} \right.$$

$$+ cot\,\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$+ \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi}$$

$$\left. + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{-2i\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ -4\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.$$

$$- \sqrt{\frac{15}{32\pi}} 8 sin\varphi cos\theta \, sin\theta e^{-2i\varphi} + \sqrt{\frac{15}{32\pi}} sin\varphi(0)$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot^2\theta cos\varphi cos\theta \, sin\theta e^{-2i\varphi}$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi}$$

$$\left. + 8i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, sin\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi} \right.$$

$$- 2i\sqrt{\frac{15}{32\pi}} \cot\theta \, cos\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi}$$

$$- 8\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} + 8i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta \, sin\theta \, e^{-2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (-5 sin\varphi - 4i cos\varphi) e^{-2i\varphi}$$

$$\left. + \sqrt{\frac{15}{32\pi}} cot\theta (8 sin\varphi + 10i cos\varphi) e^{-2i\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot^2\theta cos\varphi cos\theta \, sin\theta e^{-2i\varphi}$$

$$\left. - 2i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{-2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, sin\varphi (cos^2\theta) e^{-2i\varphi} \right.$$

$$\left. + 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta \, sin\theta \, e^{-2i\varphi} + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (-sin\varphi) e^{-2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{-2i\varphi}$$

$$+ 2i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, sin\varphi (cos^2\theta) e^{-2i\varphi}$$

$$\left. + 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta \, sin\theta \, e^{-2i\varphi} - 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (sin\varphi) e^{-2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} sin\varphi\, cos\,\theta\, sin\theta\, e^{-2i\varphi} \right.$$

$$+ 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, sin\varphi (cos^2\theta) e^{-2i\varphi}$$

$$\left. + 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta\, sin\theta\, e^{-2i\varphi} - 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, (sin\varphi) e^{-2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi\, cos\,\theta\, sin\theta\, e^{-2i\varphi} \right. \right.$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, sin\varphi (cos^2\theta) e^{-2i\varphi} - 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, (sin\varphi) e^{-2i\varphi} \right]$$

$$+ \left[6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi} \right.$$

$$\left. \left. + 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta\, sin\theta\, e^{-2i\varphi} \right] \right\}$$

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$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.\right.$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot^2\,\theta \, sin\varphi \;\; \cos\theta \, sin\theta e^{-2i\varphi}$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} \cot\theta \,(sin\varphi) e^{-2i\varphi} \right]$$

$$+ \left[6i\sqrt{\frac{15}{32\pi}} cot^2\,\theta \, cos\varphi \cos\theta \, sin\theta e^{-2i\varphi} \right.$$

$$\left.\left. - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta \; sin\theta \; e^{-2i\varphi} \right] \right\}$$

$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta (1 + cot^2\,\theta) \, e^{-2i\varphi} \right.\right.$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} \cot\theta \,(sin\varphi) e^{-2i\varphi} \right]$$

$$+ \left[6i\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta (cot^2\,\theta + 1) e^{-2i\varphi} \right.$$

$$\left.\left. - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi} \right] \right\}$$

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$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi\, cos\,\theta\, sin\theta \left(\frac{1}{sin^2\theta}\right) e^{-2i\varphi} \right.\right.$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} cot\,\theta\,(sin\varphi) e^{-2i\varphi} \right]$$

$$+ \left[6i\sqrt{\frac{15}{32\pi}} cos\varphi\, cos\,\theta\, sin\theta \left(\frac{1}{sin^2\theta}\right) e^{-2i\varphi} \right.$$

$$\left.\left. - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi} \right]\right\}$$

$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi\, cot\,\theta\;\; e^{-2i\varphi} - 2\sqrt{\frac{15}{32\pi}} cot\,\theta\,(sin\varphi) e^{-2i\varphi} \right] \right.$$

$$\left. + \left[6i\sqrt{\frac{15}{32\pi}} cos\varphi\, cot\,\theta\, e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{-2i\varphi} \right] \right\}$$

$[\hat{L}_x, \hat{L}^2]Y_{2-2} = 0$

g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$[\hat{L}_y, \hat{L}^2]Y_{2-2} = (\hat{L}_y\hat{L}^2 - \hat{L}^2\hat{L}_y)Y_{2-2}$

$$= \left[\left\{ -i\hbar \left(cos\,\varphi \frac{\partial}{\partial\theta} - sin\,\varphi\, cot\,\theta \frac{\partial}{\partial\varphi} \right) \right.\right.$$

$$\left. \times -\hbar^2 \left(cot\,\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(cot\,\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left.\left. \times -i\hbar \left(cos\,\varphi \frac{\partial}{\partial\theta} - sin\,\varphi\, cot\,\theta \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{2-2}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} + \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right.$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} Y_{2-2}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{2-2}$$

$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial Y_{2-2}}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{2-2}}{\partial\theta^2} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{2-2}}{\partial\varphi^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{2-2}}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{2-2}}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{2-2}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial Y_{2-2}}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial Y_{2-2}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial Y_{2-2}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial Y_{2-2}}{\partial\varphi}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial Y_{2-2}}{\partial\theta} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial Y_{2-2}}{\partial\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$- \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$- \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$\left. - \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi} \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$- \frac{\partial^2}{\partial\theta^2} \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta\ e^{-2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -4\sqrt{\frac{15}{32\pi}} cos\varphi\, cos\, \theta\, sin\theta\, e^{-2i\varphi} \right.$$

$$- \sqrt{\frac{15}{32\pi}} 8cos\varphi cos\theta\, sin\theta e^{-2i\varphi} + \sqrt{\frac{15}{32\pi}} sin\varphi(0)$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot^2\theta sin\varphi cos\theta\, sin\theta e^{-2i\varphi}$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi}$$

$$\left. - 8i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{-2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, cos\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi} \right.$$

$$+ 2i\sqrt{\frac{15}{32\pi}} cot\,\theta\, sin\varphi (cos^2\theta - sin^2\theta) e^{-2i\varphi}$$

$$- 8\sqrt{\frac{15}{32\pi}} cos\varphi\, cos\,\theta\, sin\theta\, e^{-2i\varphi} - 8i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta\, sin\theta\, e^{-2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, (-5cos\varphi + 4isin\varphi) e^{-2i\varphi}$$

$$\left. - \sqrt{\frac{15}{32\pi}} cot\theta (10isin\varphi - 8cos\varphi) e^{-2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot^2\theta sin\varphi cos\theta \, sin\theta e^{-2i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi (cos^2\theta) e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{-2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, cos\varphi (cos^2\theta) e^{-2i\varphi} \right.$$

$$\left. - 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{-2i\varphi} + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (-cos\varphi) e^{-2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot^2\theta sin\varphi cos\theta \, sin\theta e^{-2i\varphi}$$

$$+ 2i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi (cos^2\theta) e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{-2i\varphi}$$

$$- 2\sqrt{\frac{15}{32\pi}} \cot\theta \, cos\varphi (cos^2\theta) e^{-2i\varphi}$$

$$\left. + 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{-2i\varphi} + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{-2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{32\pi}} 2cos\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot\theta cos^2\theta sin\varphi e^{-2i\varphi}$$

$$+ 2i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi (cos^2\theta) e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{-2i\varphi}$$

$$- 2\sqrt{\frac{15}{32\pi}} \cot\theta \, cos\varphi (cos^2\theta) e^{-2i\varphi}$$

$$\left. + 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{-2i\varphi} + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{-2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, e^{-2i\varphi} \right.$$

$$+ 6i\sqrt{\frac{15}{32\pi}} cot\theta cos^2\theta sin\varphi e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{-2i\varphi}$$

$$- 2\sqrt{\frac{15}{32\pi}} \cot\theta \, cos\varphi (cos^2\theta) e^{-2i\varphi}$$

$$\left. + 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{-2i\varphi} + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{-2i\varphi} \right\}$$

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$$= i\hbar^3\left\{\left[-2\sqrt{\frac{15}{32\pi}}cos\varphi\, cos\,\theta\, sin\theta\, e^{-2i\varphi}\right.\right.$$

$$-\,2\sqrt{\frac{15}{32\pi}}cos\,\theta\, sin\theta cos\varphi(cot^2\theta)e^{-2i\varphi}$$

$$\left.+\,2\sqrt{\frac{15}{32\pi}}cot\,\theta\,(cos\varphi)e^{-2i\varphi}\right]$$

$$+\left[6i\sqrt{\frac{15}{32\pi}}cos\,\theta\, sin\theta cot^2\theta sin\varphi e^{-2i\varphi}\right.$$

$$\left.\left.+\,6i\sqrt{\frac{15}{32\pi}}sin\varphi cos\theta\, sin\theta\, e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}}cot\theta sin\varphi e^{-2i\varphi}\right]\right\}$$

$$= i\hbar^3\left\{\left[-2\sqrt{\frac{15}{32\pi}}cos\varphi\, cos\,\theta\, sin\theta\, e^{-2i\varphi}\right.\right.$$

$$-\,2\sqrt{\frac{15}{32\pi}}cos\,\theta\, sin\theta cos\varphi(cot^2\theta)e^{-2i\varphi}$$

$$\left.+\,2\sqrt{\frac{15}{32\pi}}cot\,\theta\,(cos\varphi)e^{-2i\varphi}\right]$$

$$+\left[6i\sqrt{\frac{15}{32\pi}}cos\,\theta\, sin\theta cot^2\theta sin\varphi e^{-2i\varphi} + 6i\sqrt{\frac{15}{32\pi}}sin\varphi cos\theta\, sin\theta\right.$$

$$\left.\left.-\,6i\sqrt{\frac{15}{32\pi}}cot\theta sin\varphi e^{-2i\varphi}\right]\right\}$$

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$$= i\hbar^3\left\{\left[-2\sqrt{\frac{15}{32\pi}}cos\varphi\, cos\,\theta\, sin\theta\,(1+cot^2\theta)e^{-2i\varphi}\right.\right.$$

$$\left.+\,2\sqrt{\frac{15}{32\pi}}cot\,\theta\,(cos\varphi)e^{-2i\varphi}\right]$$

$$\left.+\left[6i\sqrt{\frac{15}{32\pi}}cos\,\theta\, sin\theta sin\varphi e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}}cot\theta sin\varphi e^{-2i\varphi}\right]\right\}$$

$$= i\hbar^3\left\{\left[-2\sqrt{\frac{15}{32\pi}}cos\varphi\, cos\,\theta\, sin\theta\left(\frac{1}{sin^2\theta}\right)e^{-2i\varphi}\right.\right.$$

$$\left.+\,2\sqrt{\frac{15}{32\pi}}cot\,\theta\,(cos\varphi)e^{-2i\varphi}\right]$$

$$+\left[6i\sqrt{\frac{15}{32\pi}}cos\,\theta\, sin\theta sin\varphi\left(\frac{1}{sin^2\theta}\right)e^{-2i\varphi}\right.$$

$$\left.\left.-\,6i\sqrt{\frac{15}{32\pi}}cot\theta sin\varphi e^{-2i\varphi}\right]\right\}$$

$$= i\hbar^3\left\{\left[-2\sqrt{\frac{15}{32\pi}}cos\varphi\, cot\,\theta\, e^{-2i\varphi} + 2\sqrt{\frac{15}{32\pi}}cot\,\theta\,(cos\varphi)e^{-2i\varphi}\right]\right.$$

$$\left.+\left[6i\sqrt{\frac{15}{32\pi}}cot\,\theta\, sin\varphi e^{-2i\varphi} - 6i\sqrt{\frac{15}{32\pi}}cot\theta sin\varphi e^{-2i\varphi}\right]\right\}$$

$$[\hat{L}_y,\hat{L}^2]Y_{2-2} = 0$$

h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z,\hat{L}^2]Y_{2-2} = (\hat{L}_z\hat{L}^2 - \hat{L}^2\hat{L}_z)Y_{2-2}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi}\times -\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}\right.$$

$$\left.-\left\{-\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]Y_{2-2}$$

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$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} Y_{2-2}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial}{\partial\varphi} \right\} Y_{2-2}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{2-2}}{\partial\theta} + \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{2-2}}{\partial\theta^2} + \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{2-2}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \frac{\partial Y_{2-2}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2} \frac{\partial Y_{2-2}}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial Y_{2-2}}{\partial\varphi} \right\}$$

$$= i\hbar^3 \left\{ \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{-2i\varphi} \right.$$

$$+ \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{-2i\varphi}$$

$$\left. + \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{-2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{-2i\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{-2i\varphi}$$

$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{-2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -4i\sqrt{\frac{15}{32\pi}} cos^2\theta e^{-2i\varphi} - 4i\sqrt{\frac{15}{32\pi}} (cos^2\theta - sin^2\theta) e^{-2i\varphi} \right.$$

$$\left. + 8i\sqrt{\frac{15}{32\pi}} e^{-2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ -4i\sqrt{\frac{15}{32\pi}} cos^2\theta e^{-2i\varphi} - 4i\sqrt{\frac{15}{32\pi}} (cos^2\theta - sin^2\theta) e^{-2i\varphi} \right.$$

$$\left. + 8i\sqrt{\frac{15}{32\pi}} e^{-2i\varphi} \right\}$$

$[\hat{L}_z, \hat{L}^2] Y_{2-2} = 0$

I.9 Harmonik Bola Y_{22}

a. Komutator operator $\hat{L}_x$ dan $\hat{L}_y$

$[\hat{L}_x, \hat{L}_y] Y_{22} = (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) Y_{22}$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \times -i\hbar \left(cos\varphi \frac{\partial}{\partial\theta} - cot\theta sin\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$- \left\{ -i\hbar \left(cos\varphi \frac{\partial}{\partial\theta} - cot\theta sin\varphi \frac{\partial}{\partial\varphi} \right) \right.$$

$$\left. \left. \times\, i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{22}$$

$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial Y_{22}}{\partial\theta} - sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial Y_{22}}{\partial\varphi} \right.$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial Y_{22}}{\partial\theta} - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial Y_{22}}{\partial\varphi} \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial Y_{22}}{\partial\theta} + cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial Y_{22}}{\partial\varphi} \right.$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial Y_{22}}{\partial\theta} - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial Y_{22}}{\partial\varphi} \right\}$$

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$$= \hbar^2 \left\{ sin\varphi \frac{\partial}{\partial\theta} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta \ e^{2i\varphi} \right.$$

$$- sin\varphi \frac{\partial}{\partial\theta} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta \ e^{2i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta \, e^{2i\varphi}$$

$$\left. - cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\theta sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta \ e^{2i\varphi} \right\}$$

$$- \hbar^2 \left\{ cos\varphi \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta \ e^{2i\varphi} \right.$$

$$+ cos\varphi \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta \ e^{2i\varphi}$$

$$- cot\theta sin\varphi \frac{\partial}{\partial\varphi} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta \ e^{2i\varphi}$$

$$\left. - cot\theta sin\varphi \frac{\partial}{\partial\varphi} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta \ e^{2i\varphi} \right\}$$

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$$= \hbar^2 \left\{ 2\sqrt{\frac{15}{32\pi}} sin\varphi cos\varphi (cos^2\theta - sin^2\theta) e^{2i\varphi} \right.$$

$$- 2i\sqrt{\frac{15}{32\pi}} sin\varphi sin\varphi (cos^2\theta - sin^2\theta) e^{2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi cos\theta sin\theta \left(-sin\varphi\, e^{2i\varphi} + 2icos\varphi e^{2i\varphi}\right)$$

$$\left. - \sqrt{\frac{15}{32\pi}} cot\theta cos\varphi cos\theta sin\theta \left(2icos\varphi\, e^{2i\varphi} - 4sin\varphi e^{2i\varphi}\right) \right\}$$

$$- \hbar^2 \left\{ 2\sqrt{\frac{15}{32\pi}} cos\varphi sin\varphi (cos^2\theta - sin^2\theta) e^{2i\varphi} \right.$$

$$+ 2i\sqrt{\frac{15}{32\pi}} cos\varphi cos\varphi (cos^2\theta - sin^2\theta) e^{2i\varphi}$$

$$- 2\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi cos\theta sin\theta \left(cos\varphi\, e^{2i\varphi} + 2isin\varphi e^{2i\varphi}\right)$$

$$\left. - \sqrt{\frac{15}{32\pi}} cot\theta sin\varphi cos\theta sin\theta \left(-2isin\varphi\, e^{2i\varphi} - 4cos\varphi e^{2i\varphi}\right) \right\}$$

$$= \hbar^2 \left\{ -2i\sqrt{\frac{15}{32\pi}} sin\varphi sin\varphi (-sin^2\theta) e^{2i\varphi} \right\}$$

$$- \hbar^2 \left\{ 2i\sqrt{\frac{15}{32\pi}} cos\varphi cos\varphi (-sin^2\theta) e^{2i\varphi} \right\}$$

$$= 2i\hbar^2 (sin^2\varphi + cos^2\varphi) \sqrt{\frac{15}{32\pi}} sin^2\theta\, e^{2i\varphi}$$

$$[\hat{L}_x, \hat{L}_y] Y_{22} = 2i\hbar^2 Y_{22}$$

$$[\hat{L}_x, \hat{L}_y] = 2i\hbar^2$$

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b. Komutator operator $\hat{L}_y$ dan $\hat{L}_z$

$$[\hat{L}_y,\hat{L}_z]Y_{22} = (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)Y_{22}$$

$$= \left[\left\{i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\} - i\hbar\frac{\partial}{\partial\varphi}\right.$$

$$\left. \times\, i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right\} Y_{22}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{22}}{\partial\varphi} + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial Y_{22}}{\partial\theta} + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{-\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\varphi}\right.$$

$$\left. + \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\varphi}\right\}$$

$$- \hbar^2\left\{-\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\theta}\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\varphi}\right.$$

$$\left. + \frac{\partial}{\partial\varphi}\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\varphi}\right\}$$

$$= \hbar^2\left\{-4i\sqrt{\frac{15}{32\pi}}\cos\varphi\sin\theta\cos\theta e^{2i\varphi} - 4\sqrt{\frac{15}{32\pi}}\sin\varphi\sin\theta\cos\theta e^{2i\varphi}\right\}$$

$$- \hbar^2\left\{-2\sqrt{\frac{15}{32\pi}}\sin\theta\cos\theta\left(-\sin\varphi\, e^{2i\varphi} + 2i\cos\varphi\, e^{2i\varphi}\right)\right.$$

$$\left. + \sqrt{\frac{15}{32\pi}}\sin\theta\cos\theta\left(2i\cos\varphi\, e^{2i\varphi} - 4\sin\varphi\, e^{2i\varphi}\right)\right\}$$

$$= -\frac{2\hbar^2}{i}\cos\theta\{i\cos\varphi + \sin\varphi\}\sqrt{\frac{15}{32\pi}}\sin\theta e^{2i\varphi}$$

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$$= \frac{2\hbar^2}{i} cos\theta\{cos\,\varphi - i\,sin\,\varphi\} \sqrt{\frac{15}{32\pi}} sin\theta e^{2i\varphi}$$

$$[\hat{L}_y, \hat{L}_z]Y_{22} = \frac{2\hbar^2}{i} cos\theta e^{-i\varphi} \sqrt{\frac{15}{32\pi}} sin\theta e^{2i\varphi}$$

c. Komutator operator $\hat{L}_z$ dan $\hat{L}_x$

$$[\hat{L}_z, \hat{L}_x]Y_{22} = (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)Y_{22}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left. - \left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right] Y_{22}$$

$$= \hbar^2\left\{\frac{\partial}{\partial\varphi} sin\varphi\frac{\partial Y_{22}}{\partial\theta} + \frac{\partial}{\partial\varphi} cot\theta cos\varphi\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

$$- \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}\frac{\partial Y_{22}}{\partial\varphi} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

$$= \hbar^2\left\{\frac{\partial}{\partial\varphi} sin\varphi\frac{\partial}{\partial\theta}\sqrt{\frac{15}{32\pi}} sin^2\theta\ \ e^{2i\varphi}\right.$$

$$\left. + \frac{\partial}{\partial\varphi} cot\theta cos\varphi\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}} sin^2\theta\ \ e^{2i\varphi}\right\}$$

$$- \hbar^2\left\{sin\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}} sin^2\theta\ \ e^{2i\varphi}\right.$$

$$\left. + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}} sin^2\theta\ \ e^{2i\varphi}\right\}$$

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$$= \hbar^2 \left\{ 2\sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(cos\varphi\ e^{2i\varphi} + 2isin\varphi e^{2i\varphi} \right) \right.$$

$$\left. + \sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(-2isin\varphi\ e^{2i\varphi} - 4cos\varphi\ e^{2i\varphi} \right) \right\}$$

$$- \hbar^2 \left\{ 4i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta sin\theta e^{2i\varphi} - 4\sqrt{\frac{15}{32\pi}} cos\theta sin\theta cos\varphi e^{2i\varphi} \right\}$$

$$= \hbar^2 \left\{ 2\sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(cos\varphi\ e^{2i\varphi} \right) \right.$$

$$\left. + \sqrt{\frac{15}{32\pi}} cos\theta sin\theta \left(-2isin\varphi\ e^{2i\varphi} \right) \right\}$$

$$= 2\hbar^2 cos\theta (cos\varphi\ - isin\varphi) \sqrt{\frac{15}{32\pi}} sin\theta e^{2i\varphi}$$

$$[\hat{L}_z, \hat{L}_x] Y_{22} = 2\hbar^2 cos\theta \left(e^{-i\varphi} \right) \sqrt{\frac{15}{32\pi}} sin\theta e^{2i\varphi}$$

d. Komutator operator $\hat{L}_z$ dan $\hat{L}_+$

$$[\hat{L}_z, \hat{L}_+] Y_{22} = \left(\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z \right) Y_{22}$$

$$= \left[\left\{ -i\hbar \frac{\partial}{\partial\varphi} \times \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right.$$

$$\left. - \left\{ \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \times -i\hbar \frac{\partial}{\partial\varphi} \right\} \right] Y_{22}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{22}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \right\} Y_{22}$$

$$= i\hbar^2 \left\{ -\frac{\partial}{\partial\varphi} e^{i\varphi} \frac{\partial Y_{22}}{\partial\theta} - \frac{\partial}{\partial\varphi} e^{i\varphi} i \cot\theta \frac{\partial Y_{22}}{\partial\varphi} \right\}$$

$$- i\hbar^2 \left\{ -e^{i\varphi} \frac{\partial}{\partial\theta} \frac{\partial Y_{22}}{\partial\varphi} - e^{i\varphi} i \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial Y_{22}}{\partial\varphi} \right\}$$

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$$= i\hbar^2\left\{-\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\;e^{2i\varphi}\right.$$

$$\left.-\frac{\partial}{\partial\varphi}e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\;e^{2i\varphi}\right\}$$

$$-i\hbar^2\left\{-e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\;e^{2i\varphi}\right.$$

$$\left.-e^{i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\;e^{2i\varphi}\right\}$$

$$= i\hbar^2\left\{-6i\sqrt{\frac{15}{32\pi}}cos\theta sin\theta e^{3i\varphi}+6\sqrt{\frac{15}{32\pi}}icos\theta sin\theta e^{3i\varphi}\right\}$$

$$-i\hbar^2\left\{-4i\sqrt{\frac{15}{32\pi}}cos\theta sin\theta e^{3i\varphi}+4\sqrt{\frac{15}{32\pi}}icos\theta sin\theta e^{3i\varphi}\right\}$$

$$[\hat{L}_z,\hat{L}_+]Y_{22}=0$$

e. Komutator operator $\hat{L}_z$ dan $\hat{L}_-$

$$[\hat{L}_z,\hat{L}_-]Y_{22}=(\hat{L}_z\hat{L}_- - \hat{L}_-\hat{L}_z)Y_{22}$$

$$=\left[\left\{-i\hbar\frac{\partial}{\partial\varphi}\times-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right)\right\}\right.$$

$$\left.-\left\{-\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right)\times-i\hbar\frac{\partial}{\partial\varphi}\right\}\right]Y_{22}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial}{\partial\theta}-\frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\right\}Y_{22}$$

$$-i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}-e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right\}Y_{22}$$

$$= i\hbar^2\left\{\frac{\partial}{\partial\varphi}e^{-i\varphi}\frac{\partial Y_{22}}{\partial\theta}-\frac{\partial}{\partial\varphi}e^{-i\varphi}i\cot\theta\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

$$-i\hbar^2\left\{e^{-i\varphi}\frac{\partial}{\partial\theta}\frac{\partial Y_{22}}{\partial\varphi}-e^{-i\varphi}i\cot\theta\frac{\partial}{\partial\varphi}\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

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$$= i\hbar^2 \left\{ \frac{\partial}{\partial\varphi} e^{-i\varphi} \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{2i\varphi} \right.$$

$$\left. - \frac{\partial}{\partial\varphi} e^{-i\varphi} i\cot\theta \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{2i\varphi} \right\}$$

$$- i\hbar^2 \left\{ e^{-i\varphi} \frac{\partial}{\partial\theta} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{2i\varphi} \right.$$

$$\left. - e^{-i\varphi} i\cot\theta \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\theta\ e^{2i\varphi} \right\}$$

$$= i\hbar^2 \left\{ 2i \sqrt{\frac{15}{32\pi}} e^{-i\varphi} cos\theta sin\theta e^{2i\varphi} + 2 \sqrt{\frac{15}{32\pi}} e^{-i\varphi} icos\theta sin\theta e^{2i\varphi} \right\}$$

$$- i\hbar^2 \left\{ 4i \sqrt{\frac{15}{32\pi}} e^{-i\varphi} cos\theta sin\theta e^{2i\varphi} \right.$$

$$\left. + 4 \sqrt{\frac{15}{32\pi}} e^{-i\varphi} icos\theta sin\theta e^{2i\varphi} \right\}$$

$$[\hat{L}_z, \hat{L}_-]Y_{22} = 4i\hbar^2 \left\{ \sqrt{\frac{15}{32\pi}} e^{-i\varphi} cos\theta sin\theta e^{2i\varphi} \right\}$$

f. Komutator operator $\hat{L}_x$ dan $\hat{L}^2$

$$[\hat{L}_x, \hat{L}^2]Y_{22} = (\hat{L}_x\hat{L}^2 - \hat{L}^2\hat{L}_x)Y_{22}$$

$$= \left[\left\{ i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times i\hbar \left(sin\varphi \frac{\partial}{\partial\theta} + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{22}$$

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$$= \left[\left\{i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right.\right.$$

$$\left.\times -\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

$$-\left\{-\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$

$$\left.\left.\times\, i\hbar\left(sin\varphi\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\right]Y_{22}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{22}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial}{\partial\theta}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial}{\partial\theta}$$

$$\left.+ \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial}{\partial\varphi}\right\}Y_{22}$$

$$= -i\hbar^3\left\{sin\varphi\frac{\partial}{\partial\theta}cot\,\theta\frac{\partial Y_{22}}{\partial\theta} + sin\varphi\frac{\partial}{\partial\theta}\frac{\partial^2 Y_{22}}{\partial\theta^2} + sin\varphi\frac{\partial}{\partial\theta}\frac{1}{sin^2\theta}\frac{\partial^2 Y_{22}}{\partial\varphi^2}\right.$$

$$+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial Y_{22}}{\partial\theta} + cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{22}}{\partial\theta^2}$$

$$\left.+ cot\theta cos\varphi\frac{\partial}{\partial\varphi}\frac{1}{sin^2\theta}\frac{\partial^2 Y_{22}}{\partial\varphi^2}\right\}$$

$$+ i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}sin\varphi\frac{\partial Y_{22}}{\partial\theta} + cot\,\theta\frac{\partial}{\partial\theta}cot\theta cos\varphi\frac{\partial Y_{22}}{\partial\varphi}\right.$$

$$+ \frac{\partial^2}{\partial\theta^2}sin\varphi\frac{\partial Y_{22}}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}cot\theta cos\varphi\frac{\partial Y_{22}}{\partial\varphi} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}sin\varphi\frac{\partial Y_{22}}{\partial\theta}$$

$$\left.+ \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}cot\theta cos\varphi\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

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$$= -i\hbar^3 \left\{ sin\varphi \frac{\partial}{\partial\theta} cot\,\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi} \right.$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$+ sin\varphi \frac{\partial}{\partial\theta} \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} cot\,\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$+ cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$\left. + cot\theta cos\varphi \frac{\partial}{\partial\varphi} \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ cot\,\theta \frac{\partial}{\partial\theta} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi} \right.$$

$$+ cot\,\theta \frac{\partial}{\partial\theta} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$+ \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} sin\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi}$$

$$\left. + \frac{1}{sin^2\,\theta} \frac{\partial^2}{\partial\varphi^2} cot\theta cos\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} sin^2\,\theta\;\; e^{2i\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ -4\sqrt{\frac{15}{32\pi}} sin\varphi\, cos\,\theta\, sin\theta\, e^{2i\varphi} \right.$$

$$- \sqrt{\frac{15}{32\pi}} 8 sin\varphi cos\theta\, sin\theta e^{2i\varphi} + \sqrt{\frac{15}{32\pi}} sin\varphi(0)$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot^2\theta cos\varphi cos\theta\, sin\theta e^{2i\varphi}$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi(cos^2\theta - sin^2\,\theta)e^{2i\varphi}$$

$$\left. - 8i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, sin\varphi(cos^2\theta - sin^2\,\theta)e^{2i\varphi} \right.$$

$$+ 2i\sqrt{\frac{15}{32\pi}} cot\,\theta\, cos\varphi(cos^2\theta - sin^2\,\theta)e^{2i\varphi}$$

$$- 8\sqrt{\frac{15}{32\pi}} sin\varphi\, cos\,\theta\, sin\theta\, e^{2i\varphi} - 8i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta\, sin\theta\, e^{2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot\,\theta\,(-5 sin\varphi + 4i cos\varphi)e^{2i\varphi}$$

$$\left. + \sqrt{\frac{15}{32\pi}} cot\theta(8 sin\varphi - 10i cos\varphi)e^{2i\varphi} \right\}$$

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$$= -i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{2i\varphi} \right.$$

$$+ 4i\sqrt{\frac{15}{32\pi}} cot^2\theta cos\varphi cos\theta \, sin\theta e^{2i\varphi}$$

$$\left. + 2i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi} \right\}$$

$$+ i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, sin\varphi (cos^2\theta) e^{2i\varphi} \right.$$

$$\left. - 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta \, sin\theta \, e^{2i\varphi} + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (-sin\varphi) e^{2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{2i\varphi} \right.$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{2i\varphi}$$

$$- 2i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi (cos^2\theta) e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, sin\varphi (cos^2\theta) e^{2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta \, sin\theta \, e^{2i\varphi}$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (sin\varphi) e^{2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} sin\varphi\, cos\,\theta\, sin\theta\; e^{2i\varphi} \right.$$

$$- 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi(cos^2\theta)e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, sin\varphi(cos^2\theta)e^{2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta\; sin\theta\; e^{2i\varphi}$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} cot\,\theta\,(sin\varphi)e^{2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi\, cos\,\theta\, sin\theta\; e^{2i\varphi} \right. \right.$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, sin\varphi(cos^2\theta)e^{2i\varphi} - 2\sqrt{\frac{15}{32\pi}} cot\,\theta\,(sin\varphi)e^{2i\varphi} \right]$$

$$+ \left[-6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi(cos^2\theta)e^{-2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi} \right.$$

$$\left. \left. - 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta\; sin\theta\; e^{2i\varphi} \right] \right\}$$

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$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \, e^{2i\varphi} \right.\right.$$

$$+ 2\sqrt{\frac{15}{32\pi}} cot^2\theta \, sin\varphi \, \cos\theta \, sin\theta e^{2i\varphi}$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (sin\varphi) e^{2i\varphi} \right]$$

$$+ \left[-6i\sqrt{\frac{15}{32\pi}} cot^2\theta \, cos\varphi \cos\theta \, sin\theta e^{2i\varphi} \right.$$

$$\left.\left. + 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cos\varphi cos\theta \, sin\theta \, e^{2i\varphi} \right] \right\}$$

$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta (1 + cot^2\theta) \, e^{2i\varphi} \right.\right.$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (sin\varphi) e^{2i\varphi} \right]$$

$$+ \left[-6i\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta (cot^2\theta + 1) e^{2i\varphi} \right.$$

$$\left.\left. + 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi} \right] \right\}$$

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$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi \cos\theta \, sin\theta \left(\frac{1}{sin^2\theta} \right) e^{2i\varphi} \right. \right.$$

$$\left. - 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (sin\varphi) e^{2i\varphi} \right]$$

$$+ \left[-6i\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \left(\frac{1}{sin^2\theta} \right) e^{2i\varphi} \right.$$

$$\left. \left. + 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi} \right] \right\}$$

$$= i\hbar^3 \left\{ \left[2\sqrt{\frac{15}{32\pi}} sin\varphi \cot\theta \;\, e^{2i\varphi} - 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (sin\varphi) e^{2i\varphi} \right] \right.$$

$$\left. + \left[-6i\sqrt{\frac{15}{32\pi}} cos\varphi \cot\theta \, e^{2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} cot\theta cos\varphi e^{2i\varphi} \right] \right\}$$

$$[\hat{L}_x, \hat{L}^2] Y_{22} = 0$$

g. Komutator operator $\hat{L}_y$ dan $\hat{L}^2$

$$[\hat{L}_y, \hat{L}^2] Y_{22} = (\hat{L}_y \hat{L}^2 - \hat{L}^2 \hat{L}_y) Y_{22}$$

$$= \left[\left\{ -i\hbar \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right. \right.$$

$$\left. \times -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right\}$$

$$- \left\{ -\hbar^2 \left(\cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right) \right.$$

$$\left. \left. \times -i\hbar \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right\} \right] Y_{22}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} + \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right.$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} Y_{22}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right\} Y_{22}$$

$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial Y_{22}}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2 Y_{22}}{\partial\theta^2} + \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{22}}{\partial\varphi^2} \right.$$

$$- \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial Y_{22}}{\partial\theta}$$

$$\left. - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{\partial^2 Y_{22}}{\partial\theta^2} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{22}}{\partial\varphi^2} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial Y_{22}}{\partial\theta} - \cot\theta \frac{\partial}{\partial\theta} \sin\varphi \cot\theta \frac{\partial Y_{22}}{\partial\varphi} \right.$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial Y_{22}}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \sin\varphi \cot\theta \frac{\partial Y_{22}}{\partial\varphi} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial Y_{22}}{\partial\theta}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sin\varphi \cot\theta \frac{\partial Y_{22}}{\partial\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \cos\varphi \frac{\partial}{\partial\theta} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi} \right.$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$+ \cos\varphi \frac{\partial}{\partial\theta} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$- \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \cot\theta \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$- \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \frac{\partial^2}{\partial\theta^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$\left. - \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ \cot\theta \frac{\partial}{\partial\theta} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi} \right.$$

$$- \cot\theta \frac{\partial}{\partial\theta} \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$+ \frac{\partial^2}{\partial\theta^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$- \frac{\partial^2}{\partial\theta^2} \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$+ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cos\varphi \frac{\partial}{\partial\theta} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi}$$

$$\left. - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \sqrt{\frac{15}{32\pi}} \sin^2\theta \;\, e^{2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -4\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, e^{2i\varphi} - \sqrt{\frac{15}{32\pi}} 8cos\varphi cos\theta \, sin\theta e^{2i\varphi} \right.$$

$$+ \sqrt{\frac{15}{32\pi}} sin\varphi(0) - 4i\sqrt{\frac{15}{32\pi}} cot^2\theta sin\varphi cos\theta \, sin\theta e^{2i\varphi}$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi(cos^2\theta - sin^2\theta)e^{2i\varphi}$$

$$\left. + 8i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, cos\varphi(cos^2\theta - sin^2\theta)e^{2i\varphi} \right.$$

$$- 2i\sqrt{\frac{15}{32\pi}} \cot\theta \, sin\varphi(cos^2\theta - sin^2\theta)e^{2i\varphi}$$

$$- 8\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, e^{2i\varphi} + 8i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{2i\varphi}$$

$$+ 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (-5cos\varphi - 4isin\varphi)e^{2i\varphi}$$

$$\left. - \sqrt{\frac{15}{32\pi}} cot\theta(-10isin\varphi + 8cos\varphi)e^{2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} cos\varphi\, cos\,\theta\, sin\theta\; e^{2i\varphi} \right.$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot^2\theta sin\varphi cos\theta\; sin\theta e^{2i\varphi}$$

$$\left. - 2i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi (cos^2\theta) e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, cos\varphi (cos^2\theta) e^{2i\varphi} \right.$$

$$\left. + 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta\; sin\theta\; e^{2i\varphi} + 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, (-cos\varphi) e^{2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} cos\varphi\, cos\,\theta\, sin\theta\; e^{2i\varphi} \right.$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot^2\theta sin\varphi cos\theta\; sin\theta e^{2i\varphi}$$

$$- 2i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi (cos^2\theta) e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi}$$

$$- 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, cos\varphi (cos^2\theta) e^{2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta\; sin\theta\; e^{2i\varphi}$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} cot\,\theta\, (cos\varphi) e^{2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ -\sqrt{\frac{15}{32\pi}} 2cos\varphi \, cos\,\theta \, sin\theta \, e^{2i\varphi} \right.$$

$$- 4i\sqrt{\frac{15}{32\pi}} cot\theta cos^2\theta sin\varphi e^{2i\varphi} - 2i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi (cos^2\theta) e^{2i\varphi}$$

$$+ 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} - 2\sqrt{\frac{15}{32\pi}} cot\,\theta \, cos\varphi (cos^2\theta) e^{2i\varphi}$$

$$\left. - 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{2i\varphi} + 2\sqrt{\frac{15}{32\pi}} cot\,\theta \, (cos\varphi) e^{2i\varphi} \right\}$$

$$= i\hbar^3 \left\{ -2\sqrt{\frac{15}{32\pi}} cos\varphi \, cos\,\theta \, sin\theta \, e^{2i\varphi} \right.$$

$$- 6i\sqrt{\frac{15}{32\pi}} cot\theta cos^2\theta sin\varphi e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi}$$

$$- 2\sqrt{\frac{15}{32\pi}} cot\,\theta \, cos\varphi (cos^2\theta) e^{2i\varphi} - 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{2i\varphi}$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} cot\,\theta \, (cos\varphi) e^{2i\varphi} \right\}$$

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$$= i\hbar^3 \left\{ \left[-2\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, e^{2i\varphi} \right. \right.$$

$$- 2\sqrt{\frac{15}{32\pi}} \cos\theta \, sin\theta cos\varphi (cot^2\theta) e^{2i\varphi}$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{2i\varphi} \right]$$

$$+ \left[-6i\sqrt{\frac{15}{32\pi}} \cos\theta \, sin\theta cot^2\theta sin\varphi e^{2i\varphi} \right.$$

$$\left. \left. - 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} \right] \right\}$$

$$= i\hbar^3 \left\{ \left[-2\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, e^{2i\varphi} \right. \right.$$

$$- 2\sqrt{\frac{15}{32\pi}} \cos\theta \, sin\theta cos\varphi (cot^2\theta) e^{2i\varphi}$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{2i\varphi} \right]$$

$$+ \left[-6i\sqrt{\frac{15}{32\pi}} \cos\theta \, sin\theta cot^2\theta sin\varphi e^{2i\varphi} \right.$$

$$\left. \left. - 6i\sqrt{\frac{15}{32\pi}} sin\varphi cos\theta \, sin\theta \, e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} \right] \right\}$$

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$$= i\hbar^3 \left\{ \left[-2\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \, (1 + cot^2\theta) e^{2i\varphi} \right.\right.$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{2i\varphi} \right]$$

$$+ \left[-6i \sqrt{\frac{15}{32\pi}} \cos\theta \, sin\theta sin\varphi (cot^2\theta + 1) e^{2i\varphi} \right.$$

$$\left.\left. + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} \right] \right\}$$

$$= i\hbar^3 \left\{ \left[-2\sqrt{\frac{15}{32\pi}} cos\varphi \cos\theta \, sin\theta \left(\frac{1}{sin^2\theta} \right) e^{2i\varphi} \right.\right.$$

$$\left. + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{2i\varphi} \right]$$

$$+ \left[-6i \sqrt{\frac{15}{32\pi}} \cos\theta \, sin\theta sin\varphi \left(\frac{1}{sin^2\theta} \right) e^{2i\varphi} \right.$$

$$\left.\left. + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} \right] \right\}$$

$$= i\hbar^3 \left\{ \left[-2\sqrt{\frac{15}{32\pi}} cos\varphi \cot\theta \, e^{2i\varphi} + 2\sqrt{\frac{15}{32\pi}} \cot\theta \, (cos\varphi) e^{2i\varphi} \right] \right.$$

$$\left. + \left[-6i \sqrt{\frac{15}{32\pi}} \cot\theta \, sin\varphi e^{2i\varphi} + 6i\sqrt{\frac{15}{32\pi}} cot\theta sin\varphi e^{2i\varphi} \right] \right\}$$

$$[\hat{L}_y, \hat{L}^2] Y_{22} = 0$$

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h. Komutator operator $\hat{L}_z$ dan $\hat{L}^2$

$$[\hat{L}_z, \hat{L}^2]Y_{22} = (\hat{L}_z\hat{L}^2 - \hat{L}^2\hat{L}_z)Y_{22}$$

$$= \left[\left\{-i\hbar\frac{\partial}{\partial\varphi} \times -\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}\right.$$

$$\left. - \left\{-\hbar^2\left(cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right) \times -i\hbar\frac{\partial}{\partial\varphi}\right\}\right]Y_{22}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{22}$$

$$- i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\right\}Y_{22}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial Y_{22}}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial^2 Y_{22}}{\partial\theta^2} + \frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2 Y_{22}}{\partial\varphi^2}\right\}$$

$$- i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}\frac{\partial Y_{22}}{\partial\varphi} + \frac{\partial^2}{\partial\theta^2}\frac{\partial Y_{22}}{\partial\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial Y_{22}}{\partial\varphi}\right\}$$

$$= i\hbar^3\left\{\frac{\partial}{\partial\varphi}cot\,\theta\frac{\partial}{\partial\theta}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\, e^{2i\varphi} + \frac{\partial}{\partial\varphi}\frac{\partial^2}{\partial\theta^2}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\, e^{2i\varphi}\right.$$

$$\left. + \frac{\partial}{\partial\varphi}\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\, e^{2i\varphi}\right\}$$

$$- i\hbar^3\left\{cot\,\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\, e^{2i\varphi}\right.$$

$$\left. + \frac{\partial^2}{\partial\theta^2}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\, e^{2i\varphi} + \frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\frac{\partial}{\partial\varphi}\sqrt{\frac{15}{32\pi}}sin^2\,\theta\;\, e^{2i\varphi}\right\}$$

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$$= i\hbar^3 \left\{ 4i\sqrt{\frac{15}{32\pi}} cos^2\theta e^{2i\varphi} + 4i\sqrt{\frac{15}{32\pi}} (cos^2\theta - sin^2\theta) e^{2i\varphi} \right.$$

$$\left. - 8i\sqrt{\frac{15}{32\pi}} e^{2i\varphi} \right\}$$

$$- i\hbar^3 \left\{ 4i\sqrt{\frac{15}{32\pi}} cos^2\theta e^{2i\varphi} + 4i\sqrt{\frac{15}{32\pi}} (cos^2\theta - sin^2\theta) e^{2i\varphi} \right.$$

$$\left. - 8i\sqrt{\frac{15}{32\pi}} e^{2i\varphi} \right\}$$

$[\hat{L}_z, \hat{L}^2] Y_{22} = 0$