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On r-Dynamic Coloring Of Some Graph Operations

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On r-Dynamic Coloring of Some Graph Operations

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Abstract

Let G be a simple, connected and undirected graph. Let r, k be natural number. By a proper k-coloring of a graph G, we mean a map $c : V(G) \rightarrow S$, where $|S| = k$, such that any two adjacent vertices receive different colors. An r-dynamic k-coloring is a proper k-coloring c of G such that $|c(N(v))| \geq min\{r, d(v)\}$ for each vertex v in $V(G)$, where $N(v)$ is the neighborhood of v and $c(S) = \{c(v) : v \in S\}$ for a vertex subset S. The r-dynamic chromatic number, written as $\chi_r(G)$, is the minimum k such that G has an r-dynamic k-coloring. The 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number of graph has been studied under the name a dynamic chromatic number, denoted by $\chi_d(G)$. By simple observation it is easy to see that $\chi_r(G) \leq \chi_{r+1}(G)$, for example $\chi(C_6) = 2 but \chi_d(C_6) = 3$. In this paper we will show the exact values of some graph operation of special graphs.

Keywords: r-dynamic coloring, r-dynamic chromatic number, graph operations.

Introduction

Salah satu topik dalam bidang graf yang menarik untuk dikaji adalah masalah pewarnaan. Salah satu macam pewarnaan yaitu pewarnaan titik (*vertex coloring*). Pewarnaan titik (*vertex coloring*) adalah pemberian warna pada titik-titik graf dimana dua titik yang bertetangga diberi warna yang berbeda. Jumlah warna paling sedikit yang digunakan untuk mewarnai titik pada graf G disebut bilangan kromatik yang dilambangkan dengan χ_r. Terdapat topik yang masih dalam ruang lingkup pewarnaan titik yaitu *r-Dynamic Coloring* atau yang biasa disebut pewarnaan dinamis. r-Dynamic Coloring dinotasikan dengan χ_r. Diberikan $G = (V, E)$ adalah graf dengan V sebagai titik dan E sebagai sisi. Derajat minimum dinotasikan $\delta(G)$ sedangkan derajat maksimum dinotasikan dengan $\chi(G)$. *r-Dynamic Coloring* dengan k-pewarnaan dari graf G adalah $|c(N(v))| \geq min\{r, d(v)\}$ [5].

Hasil penelitian mengenai *r-Dynamic Coloring* pada tahun 2015 dikaji oleh Mohanapriya yang mengkaji pewarnaan dinamis pada graf *4-Regular with Girth-3*[10], kemudian peneliti Lai dan Montgomery pada tahun 2002 melakukan penelitian pada graf *Particular*[8]. Hasil penelitian terkait ini terdapat pada [1, 2].

A Useful Lemma

The following lemmas are useful for determining the dynamic coloring of graphs. This lemma characterize the upper bound in term of the diameter of graph.

Theorem 1 *[9] If* $diam(G) = 2$*, then* $\chi_2(G) \leq \chi(G) + 2$*, with equality only when* G *is a complete bipartite graph or* C_5*.*

Theorem 2 *[9] If* G *is a* k*-chromatic graph with diameter at most 3, then* $\chi_2(G) \leq 3k$*, and this bound is sharp when* $k \geq 2$*.*

In term of the maximum degree of graph, the r-dynamic of graph satisfies as follows

Observation 1 *[9]* $\chi_r(G) \geq min\{\Delta(G), r\} + 1$*, and this is sharp. If* $\Delta(G) \leq r$ *then* $\chi_r(G) = min\{\Delta(G), r\}$*.*

Theorem 3 *[9]* $\chi_r(G) \leq r\Delta(G) + 1$*, with equality for* $r \geq 2$ *if and only if* G *is* r*-regular with diameter* 2 *and girth* 5*.*

Let G^2 denote the graph obtained from G by adding edges joining nonadjacent vertices that have a common neighbor, Jahanbekam *et. al* [9] proved the following.

Observation 2 *[9]* $\chi(G) \leq \chi_d(G) \leq \chi_3(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi(G^2)$*.*

The last for graph operations of cartesian product, we have the following

Theorem 4 *[9] If* $\delta(G) \geq r$ *then* $\chi_r(G \square H) = max\{\chi(G), \chi(H)\}$*.*

The Results

Now, we are ready to show our results on r-dynamic coloring for some special graph operations. Apart from showing the r-dynamic chromatic number we also show the colors $c(v \in V(G))$ for clarity. Some graph operations found in this paper are $W_n + C_m, W_n \square C_m, W_n \otimes C_m, W_n[C_m], W_n \odot C_m, shack(W_n + C_m, v, s), amal(S_n + P_m, v, s)$

Theorem 5 *Misal* G *adalah joint dari graf* W_n *dan graf* C_m*. For* $n \geq 3$ *dan* $m \geq 3$*, the* r*-dynamic chromatic number of* G *is*

$$\chi(G) = \chi_d(G) = \chi_3(G) = \chi_4(G) = \begin{cases} 5, & untuk\ n\ ganjil\ saat\ m\ genap \\ 6, & untuk\ n\ ganjil\ saat\ m\ genap \\ 6, & untuk\ n\ genap\ saat\ m\ ganjil \\ 7, & untuk\ n\ ganjil\ saat\ m\ ganjil \end{cases}$$

Proof. Graf W_n+C_m adalah sebuah graf terhubung dengan himpunan titik $V(W_n+C_m)=\{A, x_i, y_j; 1 \le i \le n; 1 \le j \le m\}$ and $E(W_n+C_m)=\{Ax_i; 1 \le i \le n\}$ $\cup\{x_ix_{i+1}; 1 \le i \le n-1\}$ $\cup\{x_1x_n\}$ $\cup\{Ay_j; 1 \le j \le m\}$ $\cup\{x_iy_j; 1 \le i \le n-1; 1 \le j \le m\}$ $\cup\{x_ny_j; 1 \le j \le m\}$ $\cup\{y_jy_{j+1}; 1 \le j \le m-1\}$ $\cup\{y_1y_n\}$. Thus $p = |V(W_n+C_m)| = n+m+1, q = |E(G)| = nm+2n+2m-1$ and $\Delta(W_n+C_m) = m+n$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(W_n+C_m) \ge min\{\Delta(W_n+C_m), r\}+1 = \{m+n, r\}+1$. Define the vertex coloring $c : V(W_n+C_m) \to \{1, 2, \ldots, k\}$ for $n \ge 3$ and $m \ge 2$ as follows: $f(A) = 3$
Untuk m ganjil

$$c(y_i) = \begin{cases} 5, & 1 \le j \le m-1, \ j \text{ odd} \\ 6, & 1 \le i \le m-1, \ j \text{ even} \\ 7, & j = m \end{cases}$$

Untuk n ganjil saat m ganjil

$$c(x_i) = \begin{cases} 1, & 1 \le i \le n-1, \ i \text{ even} \\ 2, & 1 \le i \le n-1, \ i \text{ odd} \\ 4, & i = n \end{cases}$$

Untuk n genap dan m ganjil

$$c(x_i) = \begin{cases} 1, & 1 \le i \le n, \ i \text{ even} \\ 2, & 1 \le i \le n, \ i \text{ odd} \end{cases}$$

Untuk m genap

$$c(y_j) = \begin{cases} 4, & 1 \le j \le m, \ j \text{ odd} \\ 5, & 1 \le j \le m, \ j \text{ even} \end{cases}$$

Untuk n ganjil dan m genap

$$c(x_i) = \begin{cases} 1, & 1 \le i \le n-1, \ i \text{ even} \\ 2, & 1 \le i \le n-1, \ i \text{ odd} \\ 6, & i = n \end{cases}$$

Untuk n genap dan m genap

$$c(x_i) = \begin{cases} 1, & 1 \le i \le n, \ i \text{ even} \\ 2, & 1 \le i \le n, \ i \text{ odd} \end{cases}$$

It is easy to see that $c : V(W_n+C_m) \to \{1, 2, \ldots, 4\}$ for m even and $c : V(W_n+C_m) \to \{1, 2, \ldots, 5\}$, for m odd respectively, is proper coloring. Thus, $\chi(W_n+C_m) = 4$ for m odd and $\chi(W_n+C_m) = 5$ for m even respectively. By definition, since $min\{|c(N(v))|, \text{for every } v \in V(W_n+C_m)\} = 4$, it implies $\chi(W_n+C_m) = \chi_d(W_n+C_m) = \chi_3(W_n+C_m) = \chi_4(W_n+C_m) = \chi_5(W_n+C_m)$. It completes the proof. □

Open Problem 1 *Misal G adalah joint dari graf W_n and graf C_m. For $n \geq 3$ and $m \geq 3$, determine the r-dynamic chromatic number of G when $r \geq 6$.*

Theorem 6 *Let G be a cartesian product of W_n and C_m. For $n \geq 3$ dan $m \geq 3$, the r-dynamic chromatic number of G is*

$$\chi(G) = \chi_d(G) = \begin{cases} 3, & untuk\ n\ genap \\ 4, & untuk\ n\ ganjil\ saat\ m\ genap \end{cases}$$

$$\chi(G) = \chi_d(G) = \chi_3(G) = 4,\ untuk\ n\ ganjil\ saat\ m\ ganjil$$

Proof. The graph $W_n \square C_m$ is a connected graph with vertex set $V(W_n \square C_m)=$ $\{A_j, x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ dan $E(W_n \square C_m)=$ $\{A_j A_{j+1}; 1 \leq i \leq m-1\}$ $\cup\{A_m A_1\}$ $\cup\{A_j x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ $\cup\{x_{i,j} x_{i+1,j}; 1 \leq i \leq n-1; 1 \leq j \leq m\}$ $\cup\{x_{1,j} x_{n,j}; 1 \leq j \leq m\}$ $\cup\{x_{i,j} x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m-1\}$ $\cup\{x_{i,m} x_{i,1}; 1 \leq i \leq n\}$. Thus $|V(W_n \square C_m)| = nm + 2n + m$ and $|E(W_n \square C_m)| = 3nm + m$ and $\Delta(W_n \square C_m) = 6$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(W_n \square C_m) \geq min\{\Delta(W_n \square C_m), r\}+1 = \{6, r\}+1$. Define the vertex colouring $c : V(W_n \square C_m) \rightarrow \{1, 2, \dots, k\}$ for $n \geq 3$ and $m \geq 3$ as follows:

Untuk n genap dan n ganjil pada saat m ganjil

$$c(A_j) = \begin{cases} 1, & 1 \leq j \leq m-1,\ j \text{ ganjil} \\ 2, & 1 \leq j \leq m,\ j \text{ genap} \\ 3, & j = m \end{cases}$$

Untuk n genap saat m ganjil

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq j \leq m-1,\ j \text{ genap},\ 1 \leq i \leq n,\ i \text{ ganjil}, \jmath = m \\ 2, & 1 \leq j \leq m-1,\ j \text{ genap},\ 1 \leq i \leq n,\ i \text{ genap}, \jmath = m \\ 3, & 1 \leq i \leq n, \jmath = m \end{cases}$$

It is easy to see that $c : V(W_n \square C_m) \rightarrow \{1, 2\}$ and $c : V(W_n \square C_m) \rightarrow \{1, 2, 3, 4\}$, for n even and odd respectively, is proper coloring. By definition, since $min\{|c(N(v))|$, for every $v \in V(W_n \square C_m)\} = 1$, thus we only have $\chi(W_n \square C_m) = 3$ and $\chi(W_n \square C_m) = 4$, for n even and odd respectively. □

Open Problem 2 *Misal G adalah operasi cartesian product dari graf W_n dan C_m. Untuk $n \geq 3$ dan $m \geq 3$, determine the r-dynamic chromatic number of G when $r \geq 4$.*

Theorem 7 *Let G be a tensor product of W_n and P_m. For $n \geq 3$ dan $m \geq 2$, the r-dynamic chromatic number of G is $\chi(W_n \otimes P_m) = 2$*

$$\chi(G) = 2,\ untuk\ m\ genap$$

$$\chi(G) = \chi_2(G) = 3, \ untuk\ m\ ganjil$$

Proof. The graph $W_n \otimes C_m$ is a connected graph with vertex set V= $\{A_j, x_{i,j}; 1 \le i \le n; 1 \le j \le m\}$ dan E= $\{A_j x_{i,j+1}; 1 \le i \le n; 1 \le j \le m-1\}$ $\cup\{A_j x_{i,j-1}; 1 \le i \le n; 2 \le j \le m-1\}$ $\cup\{A_1 x_{i,m}; 1 \le i \le n\}$ $\{A_m x_{i,1}; 1 \le i \le n\}$ $\cup\{x_{i,j} x_{i+1,j+1}; 1 \le i \le n-1; 1 \le j \le m-1\}$ $\cup\{x_{i,j} x_{i-1,j+1}; 2 \le i \le n-1; 1 \le j \le m-1\}$ $\cup\{x_{n,j} x_{1,j+1}; 1 \le j \le m-1\}$ $\cup\{x_{n,j+1} x_{1,j}; 1 \le j \le m-1\}$ $\cup\{x_{i,m} x_{i+1,1}; 1 \le i \le n-1\}$ $\cup\{x_{i,m} x_{i+1,1}; 1 \le i \le n-1\}$ $\cup\{x_{1,m} x_{n,1}\}$ $\cup\{x_{n,m} x_{1,1}\}$ $\cup\{x_{1,m} x_{n,1}\}$ $\cup\{x_{i,m} x_{i-1,1}; 2 \le i \le n\}$. Thus $|V(W_n \otimes C_m)| = nm + m$ dan $|E(W_n \otimes C_m)| = 4nm$ dan $\Delta(W_n \otimes C_m) = 2n$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(W_n \otimes C_m) \ge min\{\Delta(W_n \otimes C_m), r\} + 1 = \{2n, r\} + 1$. Define the vertex colouring $c : V(W_n \otimes C_m) \to \{1, 2, \ldots, k\}$ for $n \ge 3$ and $m \ge 3$ as follows:

Untuk n genap n ganjil saat m ganjil

$$c(A_j) = \begin{cases} 1, \ 1 \le j \le m-1, \ j \text{ ganjil} \\ 2, \ 1 \le j \le m, \ j \text{ genap} \\ 3, \ j = m \end{cases}$$

$$c(x_{i,j}) = \begin{cases} 1, \ 1 \le i \le n, \ 1 \le j \le m-1, \ j \text{ ganjil} \\ 2, \ 1 \le j \le n, \ 1 \le j \le m, \ j \text{ genap} \\ 3, \ 1 \le j \le n, \ j = m \end{cases}$$

Untuk n genap dan n ganjil saat m genap

$$c(x_{A,j}) = \begin{cases} 1, \ 1 \le j \le m-1, \ j \text{ ganjil} \\ 2, \ 1 \le j \le m, \ j \text{ genap} \end{cases}$$

It is easy to see that $c : V(W_n \otimes C_m) \to \{1, 2\}$ is proper coloring. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(W_n \otimes C_m)\} = 1$, thus we only have $\chi(W_n \otimes C_m) = 2$. □

Open Problem 3 *Misal G adalah operasi tensor product dari graf W_n dan graf C_m. Untuk $n \ge 3$ and $m \ge 3$, determine the r-dynamic chromatic number of G when $r \ge 2$ untuk m genap dan $r \ge 3$ untuk m ganjil.*

Theorem 8 *Misal G adalah operasi composition dari graf W_n dan C_m. Untuk $n \ge 3$ dan $m \ge 2$, bilangan kromatik r-dynamic dari G adalah*

$$\chi(W_n[C_m]) = \begin{cases} 6, \ \textit{for } n \textit{ even} \\ 8, \ \textit{for } n \textit{ odd} \end{cases}$$

Proof. The graph $W_n[P_m]$ is a connected graph with vertex set $V(W_n[P_m])$= $\{A_j, x_{i,j}; 1 \le i \le n; 1 \le j \le m\}$ and $E(W_n[P_m])$= $\{A_j x_{i,j+1}; 1 \le i \le n; 1 \le j \le m-1\}$ $\cup\{A_j x_{i,j-1}; 1 \le i \le n; 1 \le j \le m-1\}$ $\cup\{x_{i,j} x_{i+1,j+1}; 1 \le i \le n-1; 1 \le j \le m-1\}$ $\cup\{x_{i,j} x_{i-1,j+1}; 2 \le i \le n-1; 1 \le j \le m-1\}$ $\cup\{x_{n,j} x_{1,j+1}; 1 \le j \le m-1\}$

$\cup\{x_{n,j+1}x_{1,j}; 1 \leq j \leq m-1\}$ $\{A_jA_{j+1}; 1 \leq i \leq m-1\}$ $\cup\{A_jx_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ $\cup\{x_{i,j}x_{i+1,j}; 1 \leq i \leq n-1; 1 \leq j \leq m\}$ $\cup\{x_{1,j}x_{n,j}; 1 \leq j \leq m\}$ $\cup\{x_{i,j}x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m-1\}$. Thus $|V(W_n[P_m])| = nm+m$ and $|E(W_n[P_m])| = 7nm-5n+m-1$ and $\Delta(W_n[P_m]) = 2n+2$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(W_n[P_m]) \geq min\{\Delta(W_n[P_m]), r\}+1 = \{2n+2, r\}+1$. Define the vertex coloring $c : V(W_n[P_m]) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows: For n even and n odd

$$c(A_j) = \begin{cases} 1, & 1 \leq j \leq m,\ j \text{ odd} \\ 4, & 1 \leq j \leq m,\ j \text{ even} \end{cases}$$

For n even

$$c(x_{i,j}) = \begin{cases} 2, & 1 \leq i \leq n-1,\ i \text{ odd};\ 1 \leq j \leq m,\ j \text{ odd} \\ 3, & 1 \leq i \leq n,\ i \text{ even};\ 1 \leq j \leq m,\ j \text{ odd} \\ 5, & 1 \leq i \leq n,\ i \text{ even};\ 1 \leq j \leq m,\ j \text{ even} \\ 6, & 1 \leq i \leq n,\ i \text{ even};\ 1 \leq j \leq m,\ j \text{ even} \end{cases}$$

For n odd

$$c(x_{i,j}) = \begin{cases} 2, & 1 \leq i \leq n-1,\ i \text{ odd};\ 1 \leq j \leq m,\ j \text{ odd} \\ 3, & 1 \leq i \leq n,\ i \text{ even};\ 1 \leq j \leq m,\ j \text{ odd} \\ 5, & 1 \leq i \leq n-1,\ i \text{ odd};\ 1 \leq j \leq m,\ j \text{ even} \\ 6, & 1 \leq i \leq n,\ i \text{ even};\ 1 \leq j \leq m,\ j \text{ even} \\ 7, & i = n,\ 1 \leq j \leq m,\ j \text{ odd} \\ 8, & i = n,\ 1 \leq j \leq m,\ j \text{ even} \end{cases}$$

It is easy to see that $c : V(W_n[P_m]) \rightarrow \{1, 2, \ldots, 6\}$ and $c : V(W_n[P_m]) \rightarrow \{1, 2, \ldots, 8\}$, for n even and odd respectively, is proper coloring. Thus, $\chi(W_n[P_m]) = 6$ and $\chi(W_n[P_m]) = 8$, for n even and odd respectively. By definition, since $min\{|c(N(v))|,$
for every $v \in V(W_n[P_m])\} = 5$, it implies $\chi(W_n[P_m]) = \chi_d(W_n[P_m]) = \chi_3(W_n[P_m]) = \chi_4(W_n[S_m]) = \chi_5(W_n[P_m])$. It completes the proof. □

Open Problem 4 *Let G be a composition of W_n and P_m. For $n \geq 3$ and $m \geq 2$, determine the r-dynamic chromatic number of G when $r \geq 6$.*

Theorem 9 *Let G be a crown product of W_n on P_m. For $n \geq 3$ dan $m \geq 2$, the r-dynamic chromatic number of G is*

$$\chi(W_n \odot P_m) = \chi_d(W_n \odot P_m) = \begin{cases} 3, & \text{for } n \text{ even} \\ 4, & \text{for } n \text{ odd} \end{cases}$$

Proof. The graph $W_n \odot P_m$ is a connected graph with vertex set $V(W_n \odot P_m)$= $\{A, x_i, x_{i,j}, y_j; 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(W_n \odot P_m)$= $\{Ax_i; 1 \leq i \leq n\}$ $\cup\{x_ix_{i+1}; 1 \leq i \leq n-1\}$ $\cup\{Ay_j; 1 \leq j \leq m\}$ $\cup\{y_jy_{j+1}; 1 \leq j \leq m-1\}$ $\cup\{x_1x_n\}$

$\cup\{x_ix_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ $\cup\{x_{i,j}x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m-1\}$. Thus $|V(W_n[P_m])| = nm + n + m + 1$ and $|E(W_n \odot P_m)| = 2nm + n + 2m - 1$ and $\Delta(W_n \odot P_m) = n + m$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(W_n \odot P_m) \geq min\{\Delta(W_n \odot P_m), r\} + 1 = \{n+m, r\} + 1$. Define the vertex coloring $c : V(W_n \odot P_m) \to \{1, 2, \dots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows: $A = 4$ and

$$c(y_j) = \begin{cases} 1, & 1 \leq j \leq m,\ j \text{ even} \\ 3, & 1 \leq j \leq m,\ j \text{ odd} \end{cases}$$

For n even

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n,\ i \text{ odd};\ 1 \leq j \leq m,\ j \text{ even} \\ 2, & 1 \leq i \leq n,\ i \text{ even};\ 1 \leq j \leq m,\ j \text{ even} \\ 3, & 1 \leq j \leq m,\ j \text{ odd};\ 1 \leq i \leq n \end{cases}$$

$$c(x_i) = \begin{cases} 1, & 1 \leq i \leq n,\ i \text{ even} \\ 2, & 1 \leq i \leq n,\ i \text{ odd} \end{cases}$$

For n odd

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n,\ i \text{ odd};\ 1 \leq j \leq m,\ j \text{ even} \\ 2, & 1 \leq i \leq n,\ i \text{ even}, 1 \leq j \leq m,\ i \text{ even} \\ 3, & 1 \leq j \leq m-1,\ j \text{ even};\ 1 \leq i \leq n-1 \\ 4, & 1 \leq j \leq m,\ j \text{ odd};\ i = n \end{cases}$$

$$c(x_i) = \begin{cases} 1, & 1 \leq i \leq n-1,\ i \text{ even} \\ 2, & 1 \leq i \leq n-1,\ i \text{ odd} \\ 3, & i = n \end{cases}$$

It is easy to see that $c : V(W_n \odot P_m) \to \{1, 2, \dots, 3\}$ and $c : V(W_n \odot P_m) \to \{1, 2, \dots, 4\}$, for n even and odd respectively, is proper coloring. Thus, $\chi(W_n \odot P_m) = 3$ and $\chi(W_n \odot P_m) = 4$, for n even and odd respectively. By definition, since $min\{|c(N(v))|, \text{for every } v \in V(W_n \odot P_m)\} = 2$, it implies $\chi(W_n \odot P_m) = \chi_d(W_n \odot P_m)$. It completes the proof. □

Open Problem 5 *Let G be a crown product of W_n on P_m. For $n \geq 3$ dan $m \geq 2$, determine the r-dynamic chromatic number of G when $r \geq 3$.*

Theorem 10 *Let G be a shackle of joint S_n and P_m. For $n \geq 3$ and $m \geq 2$, the r-dynamic chromatic number of G is*

$$\chi(shack(S_n + P_m, v, s)) = \chi_d(shack(S_n + P_m, v, s)) = \chi_3(shack(S_n + P_m, v, s)) = 4$$

Proof. The shackle of joint S_n and P_m, denoted by $shack(S_n + P_m, v, s)$, is a connected graph with vertex set $V = \{A_k, x_1^k, x_i^k, y_j^k, p; 1 \leq i \leq n; 1 \leq j \leq m; 1 \leq k \leq s\}$ and $E = \{A_kx_i^k; 1 \leq i \leq n-1; 1 \leq k \leq s\}$ $\cup\{A_kx_i^{k+1}; 1 \leq k \leq s\}$ $\cup\{A_sp\}$ $\cup\{y_j^ky_{j+1}^k; 1 \leq j \leq m-1; 1 \leq k \leq s\}$ $\cup\{A^ky_j^k; 1 \leq j \leq m; 1 \leq k \leq s\}$ $\cup\{x_i^ky_j^k; 1 \leq i \leq$

$n-1; 1 \le j \le m; 1 \le k \le s\} \cup \{x_1^{k+1} y_j^k; 1 \le j \le m; 1 \le k \le s-1\} \cup \{py_j^s; 1 \le j \le m\}$. Thus $|V(shack(S_n+P_m, v, s))| = nr+mr+1$ and $|E(shack(S_n+P_m, v, s))| = 2nms+ns+2ms-s$ and $\Delta(shack(S_n+P_m, v, s)) = 6$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(shack(S_n+P_m, v, s)) \ge min\{\Delta(shack(S_n+P_m, v, s)), r\}+1 = \{6, r\}+1$. Define the vertex coloring $c : V(shack(S_n+P_m, v, s)) \rightarrow \{1, 2, \dots, k\}$ for $n \ge 3$ and $m \ge 2$ as follows: $c(A^k) = 4$

$$c(x_i^k) = \{ \ 3, \ 1 \le i \le n-1; \ 1 \le k \le s$$

$$c(y_i^k) = \begin{cases} 1, \ 1 \le j \le m, \ j \text{ odd}; \ 1 \le k \le s \\ 2, \ 1 \le j \le m, \ j \text{ even}; \ 1 \le k \le s \end{cases}$$

It is easy to see that $c : V(shack(S_n+P_m, v, s)) \rightarrow \{1, 2, \dots, 4\}$ is proper coloring. Thus, $\chi(shack(S_n+P_m, v, s))$
$= 4$. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(shack(S_n+P_m, v, s)\} = 3$, it implies $\chi(shack(S_n+P_m)) = \chi_d(shack(S_n+P_m)) = \chi_3(shack(S_n+P_m))$. It completes the proof. □

Open Problem 6 *Let G be a shackle of joint S_n and P_m. For $n \ge 3$ and $m \ge 2$, determine the r-dynamic chromatic number of G when $r \ge 4$.*

Theorem 11 *Let G be an amalgamation of joint W_n and P_m. For $n \ge 2$ and $m \ge 3$, the r-dynamic chromatic number of G is*

$$\chi(Amal(W_n+P_m, v, s)) = \begin{cases} 5, \ \text{for } n \text{ even} \\ 6, \ \text{for } n \text{ odd} \end{cases}$$

Proof. Amalgamation of joint W_n and P_m, denoted by $amal(W_n+P_m, v, s)$, is a connected graph with vertex set $V(amal(W_n+P_m, v, s))= \{A_k, x_i^k, y_j^k, y_1; 1 \le i \le n; 2 \le j \le m; 1 \le k \le r\}$ and $E(amal(W_n+P_m, v, s))= \{A_k x_i^k; 1 \le i \le n; 1 \le k \le s\}$ $\cup \{A_k y_1; 1 \le k \le s\} \cup \{A_k y_j; 2 \le j \le m-1; 1 \le k \le s\} \cup \{x_{i^k} x_{i+1^k}; 1 \le i \le n-1; 1 \le k \le s\} \cup \{x_i^k y_j^k; 1 \le i \le n; 2 \le j \le m-1; 1 \le k \le s\} \cup \{x_i^k y_j + 1^k; 1 \le i \le n; 2 \le j \le m-1; 1 \le k \le s\} \cup \{y_1 x_i^k; 1 \le i \le n; 1 \le k \le s\} \cup \{y_j^k y_j + 1^k; 2 \le j \le m-1; 1 \le k \le s\} \cup \{y_1 y_2^k; 1 \le k \le s\}$. Thus $|V(amal(W_n+P_m, v, s))| = ns+ms+1$ and $|E(amal(W_n+P_m, v, s))| = 2nm+n+2m-2$ and $\Delta(amal(W_n+P_m, v, s)) = (n+m)r$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(amal(W_n+P_m, v, s)) \ge min\{\Delta(amal(W_n+P_m, v, s)), r\}+1 = \{3(n+m)s, r\}+1$. Define the vertex coloring $c : V(amal(W_n+P_m, v, s)) \rightarrow \{1, 2, \dots, k\}$ for $n \ge 3$ and $m \ge 2$ as follows: $c(A^k) = 1$ and $c(y_1) = 4$

$$c(y_j^k) = \begin{cases} 4, \ 2 \le j \le m, \ j \text{ odd}; \ 1 \le k \le s \\ 5, \ 2 \le j \le m, \ j \text{ even}; \ 1 \le k \le s \end{cases}$$

For n even

$$c(x_i^k) = \begin{cases} 2, \ 1 \le i \le n-1, \ j \text{ odd}; \ 1 \le k \le s \\ 3, \ 1 \le i \le n-1, \ j \text{ even}; \ 1 \le k \le s \end{cases}$$

$$c(p) = 3$$

For n odd

$$c(x_i^k) = \begin{cases} 2, \; 1 \leq i \leq n-1, \; j \text{ odd}; \; 1 \leq k \leq s \\ 3, \; 1 \leq i \leq n-1, \; i \text{ even}; \; 1 \leq k \leq s \end{cases}$$

Clearly $c : V(amal(W_n{+}P_m, v, s)) \rightarrow \{1, \ldots, 5\}$ and $c : V(amal(W_n{+}P_m, v, s)) \rightarrow \{1, \ldots, 6\}$, for n even and odd respectively, are proper coloring. Thus, for n even, $\chi(amal(W_n + P_m, v, s)) = 5$ and for n odd, $\chi(amal(W_n + P_m, v, s)) = 6$. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(amal(W_n + P_m, v, s)\} = 4$, it implies $\chi(amal(W_n + P_m, v, s)) = \chi_d(amal(W_n + P_m, v, s)) = \chi_3(amal(W_n + P_m, v, s)) = \chi_4(amal(W_n + P_m, v, s))$. It completes the proof. □

Open Problem 7 *Let G be a shackle of joint W_n and P_m. For $n \geq 3$ and $m \geq 2$, determine the r-dynamic chromatic number of G when $r \geq 5$.*

Conclusions

We have studied the r-dynamic coloring of some graph operations. The results show for each graph operation we have not obtained completely all values of r, therefore we left as an open problem for the reader.

References

[1] Desy Tri Puspasari, Dafik Dafik, Slamin Slamin, Pewarnaan Titik pada Graf Khusus: Operasi dan Aplikasinya, **Prosiding Seminar Matematika dan Pendidikan Matematik**, **Vol. 2, Issue 1**, (2014), 50-58

[2] Harsya Alfian Yulia, Dafik Dafik, Ika Hesti Agustin, Bilangan Kromatik pada Pengoperasian Graf Lintasan dengan Graf Lingkaran, **Proceeding of International Workshop on Mathematics UAD**, (2014), 1-18

[3] J.L. Gross, J. Yellen and P. Zhang, *Handbook of Graph Theory*, Second Edition, CRC Press, Taylor and Francis Group, 2014

[4] S.J. Kim and W.J. Park, List dynamic coloring of sparse graphs, *Combinatorial optimization and applications. Lect. Notes Comput. Sci.* 6831 (Springer, 2011), 156 162.

[5] S.J. Kim, S. J. Lee, and W.J. Park, Dynamic coloring and list dynamic coloring of planar graphs. *Discrete Applied Math.* 161 (2013), 22072212.

[6] S. Akbari, M. Ghanbari, S. Jahanbekam, On the dynamic chromatic number of graphs, *Combinatorics and graphs. Contemp. Math.* 531 (Amer. Math. Soc. 2010), 118.

[7] B. Montgomery, Dynamic Coloring of Graphs. *Ph.D Dissertation*, West Virginia University, 2001.

[8] H.J. Lai, B. Montgomery, and H. Poon, Upper bounds of dynamic chromatic number. Ars Combin. 68 (2003), 193201.

[9] S Jahanbekam, J Kim, O Suil, D.B. West, On r-dynamic Coloring of Graphs, 2014, In Press

[10] R.L. Brooks, On colouring the nodes of a network. Proc. Cambridge Philos. Soc. 37 (1941), 194197.