TUT WURI HANDAYANI

HIMPUNAN MATEMATIKA INDONESIA 1976

IICMA 2013

Booklet : PROGRAMS & ABSTRACTS

IndoMS International Conference on Mathematics and Its Applications 2013

Department of Mathematics, UGM, 6-7 November 2013

Proceedings of IICMA 2013
Research Topic, pp. xx–xx.

SUPER ANTIMAGICNESS OF TRIANGULAR BOOK AND DIAMON LADDER GRAPHS

Dafik[1], Slamin[2], Fitriana Eka R[3], Laelatus Sya'diyah[4]

[1] **Math Edu. Depart., FKIP-University of Jember, d.dafik@gmail.com**
[2]**System Information Depart, University of Jember, slamin@gmail.com**
[3]**Math Edu. Depart., FKIP-University of Jember, fitriana.eka@gmail.com**
[4]**Math Edu. Depart., FKIP-University of Jember, laelatus@gmail.com**

Abstract. A graph G of order p and size q is called an (a, d)-*edge-antimagic total* if there exists a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that the edge-weights, $w(uv) = f(u) + f(v) + f(uv), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a graph G is called *super* if the smallest possible labels appear on the vertices. In this paper we study super (a, d)-edge-antimagic total properties of Triangular Book and Diamond Ladder graphs. The result shows that there are a super (a, d)-edge-antimagic total labeling of graph Bt_n and Dl_n, if $n \geq 1$ with $d \in \{0, 1, 2\}$.

Key Words: (a, d)-*edge-antimagic total labeling, super* (a, d)-*edge-antimagic total labeling, Triangular Book, Diamond Ladder.*

1. Introduction

Mathematics consists of several branches of science. Branch of current mathematics associated with a computer science is a graph theory. One of the interesting topics in graph theory is a graph labeling. Several applications of graph labeling can be found in [3]. There are various types of graph labeling, one is a super (a, d)-edge antimagic total labeling (SEATL for short). Assigning a label on each vertex and each edge such that the edge-weights form an arithmetic sequence is considered to be an NP-complete problem.

By a *labeling* we mean any mapping that carries a set of graph elements onto a set of numbers, called *labels*. In this paper, we deal with labelings with domain the set of all vertices and edges. This type of labeling belongs to the class of *total* labelings. We define the *edge-weight* of an edge $uv \in E(G)$ under a total labeling to be the sum of the vertex labels corresponding to vertices u, v and edge label corresponding to edge uv.

These labelings, introduced by Simanjuntak *at al.* in [16], are natural extensions of the concept of magic valuation, studied by Kotzig and Rosa [14] (see also [2],[11],[12],[15],[18]), and the concept of super edge-magic labeling, defined by

Enomoto *et al.* in [10]. Many other researchers investigated different forms of antimagic graphs. For example, see Bodendiek and Walther [4] and [5], and Hartsfield and Ringel [13].

In this paper we investigate the existence of super (a,d)-edge-antimagic total labelings for connected graphs. Some constructions of super (a,d)-edge-antimagic total labelings for $m\pounds_n$ and $m\pounds_{i,j,k}$ have been shown by Dafik, Slamin, Fuad and Rahmad in [6] and super (a,d)-edge-antimagic total labelings for disjoint union of caterpillars have been described by Bača in [1]. Dafik *et al* also found some families of graph which admits super (a,d)-edge-antimagic total labelings, namely $mC_n, mP_n, mK\underbrace{n,n,\dots,n}_{s}$ and m caterpilars in [7, 8, 9].

All graphs in this paper are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A (p,q)-graph G is a graph such that $|V(G)| = p$ and $|E(G)| = q$. We will now concentrate on Triangular Book and Diamond Ladder graphs, denoted by Bt_n and Dl_n.

2. Three useful Lemmas

We start this section with a necessary condition for a graph to be a super (a,d)-edge-antimagic total, which will provide a least upper bound, for a feasible value d.

Lemma 1. [17] *If a (p,q)-graph is super (a,d)-edge-antimagic total then* $d \leq \frac{2p+q-5}{q-1}$.

Proof. Assume that a (p,q)-graph has a super (a,d)-edge-antimagic total labeling $f : V(G) \cup E(G) \to \{1,2,\dots,p+q\}$ with the edge-weight set $W = \{w(uv) : uv \in E(G)\} = \{a, a+1, a+2, \dots, a+(q-1)d\}$. The minimum possible edge weight in the labeling f is at least $1+2+p+1 = p+4$. Thus, $a \geq p+4$. On the other hand, the maximum possible edge weight is at most $(p-1)+p+(p+q) = 3p+q-1$. Hence $a+(q-1)d \leq 3p+q-1$. From the last inequality, we obtain the desired upper bound for the difference d. □

The following lemma, proved by Figueroa-Centeno *et al.* in [11], proves a necessary and sufficient condition for a graph to be super edge-magic (super $(a,0)$-edge-antimagic total).

Lemma 2. [11] *A (p,q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1,2,\dots,p\}$, such that the set $S = \{f(u)+f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p+q+s$, where $s = min(S)$ and $S = \{a-(p+1), a-(p+2), \dots, a-(p+q)\}$.*

A similar lemma with Lemma 2, Bača, Lin, Miller and Simanjuntak, see [2], stated that a (p,q)-graph G is super $(a,0)$-edge-antimagic total if and only if there exists $(a-p-q,1)$-edge-antimagic vertex labeling. They extended the study with the following lemma.

Lemma 3. [2] *If (p, q)-graph G has an (a, d)-edge antimagic vertex labeling then G has a super$(a+p+q, d-1)$-edge antimagic total labeling and a super$(a+p+1, d+1)$-edge antimagic total labeling.*

In this paper, we will use the last lemma to prove the existence of a super $(a, 0)$-edge-antimagic total labeling and super $(a, 2)$-edge-antimagic total labeling for triangular book Bt_n and diamond ladder Dl_n.

3. Triangular Book Bt_n

Triangular Book graph denoted by Bt_n is a connected graph with vertex set $V(Bt_n) = \{x_i; i = 1, 2\} \bigcup \{y_j, 1 \leq j \leq n\}$ and edge set $E(Bt_n) = \{x_1x_2\} \bigcup \{x_iy_j; i = 1, 2, 1 \leq j \leq n\}$. Thus $|V(Bt_n)| = p = n + 2$ and $|E(Bt_n)| = q = 2n + 1$.

If Triangular Book graph has a super (a, d)-edge-antimagic total labeling then it follows from Lemma 1 that the upper bound of d is $d \leq 2$ or $d \in \{0, 1, 2\}$. The following theorem describes an $(a, 1)$-edge-antimagic vertex labeling for Triangular Book graph.

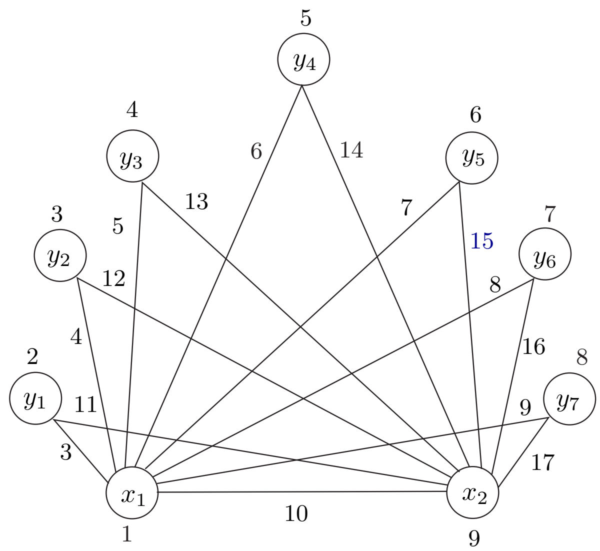

Figure 1. Example of $(3, 1)$-edge antimagic vertex labeling Bt_7 with its edge weight

Theorem 1. *A triangular book Bt_n has an $(a, 1)$-edge-antimagic vertex labeling if $n \geq 1$.*

Proof. Define the vertex labeling $\alpha_1 : V(Bt_n) \to \{1, 2, \ldots, n + 2\}$ in the following way:

$\alpha_1(x_i) = (i - 1)n + i,$ for $i = 1,\ 2$

$\alpha_1(y_j) = j + 1,\;$ for $1 \le j \le n$

The vertex labeling α_1 is a bijective function.

The edge-weights of Bt_n, under the labeling α_1, constitute the following sets

$$w_{\alpha_1}(x_1x_2) = n + 3$$

$$w_{\alpha_1}(x_iy_j) = n(i-1) + j + (i+1),\;\text{ for } i = 1,\ 2 \text{ dan } 1 \le j \le n$$

It is not difficult to see that the union of the set w_{α_1} equals to $\{3, 4, 5, \ldots, n+3, \ldots, 2n+3\}$ and consists of consecutive integers. Thus α_1 is a $(3,1)$-edge antimagic vertex labeling. □

Figure 1 gives an example of $(a,1)$-edge-antimagic vertex labeling of Bt_n.

With Theorem 1 in hand and by using Lemma 3, we obtain the following result.

Theorem 2. *A triangular book Bt_n has a super $(3n+6,0)$-edge-antimagic total labeling and a super $(n+6,2)$-edge-antimagic total labeling for $n \ge 1$.*

Proof.

We have proved that the vertex labeling α_1 is a $(3,1)$-edge antimagic vertex labeling. With respect to Lemma 2, by completing the edge labels $p+1, p+2, \ldots, p+q$, we are able to extend labeling α_1 to a super $(a_1,0)$-edge-antimagic total labeling and a super $(a_2,2)$-edge-antimagic total labeling, where, for $p = n+2$ and $q = 2n+1$, the value $a_1 = 3n+6$ and the value $a_2 = n+6$. □

Theorem 3. *A triangular book Bt_n has a super $(2n+6,1)$-edge-antimagic total labeling.*

Proof. Label the vertices of Bt_n with $\alpha_2(x_i) = \alpha_1(x_i)$ and $\alpha_2(y_j) = \alpha_1(y_j)$, for $i = 1,2$ and $1 \le j \le n$; and label the edges with the following way.

$$\alpha_2(x_1x_2) = \begin{cases} \frac{5n+7}{2}; & \text{if } n \text{ is odd} \\ \frac{3n+6}{2}; & \text{if } n \text{ is even} \end{cases}$$

For n odd, any j, and $i = 1,2$

$$\alpha_2(x_iy_j) = \frac{(5-i)n - j + (8-i) + ((-1)^j + 1)n}{2} + \frac{(-1)^j + 1}{4}$$

For n even, any j and $i = 1,2$

$$\alpha_2(x_iy_j) = \frac{(5-i)n - j + (8-i) + ((-1)^{j+(i-1)} + 1)n}{2} + \frac{(-1)^{j+(i-1)} + 1}{4}$$

The total labeling α_2 is a bijective function from $V(Bt_n) \cup E(Bt_n)$ onto the set $\{1, 2, 3, \ldots, 3n+3\}$. The edge-weights of Bt_n, under the labeling α_2, constitute the following sets:

$$W_{\alpha_2}(x_1x_2) = \begin{cases} \frac{7n+13}{2}; & \text{if } n \text{ is odd} \\ \frac{5n+12}{2}; & \text{if } n \text{ is even} \end{cases}$$

For n odd, any j, and $i = 1, 2$, the edge-weights of x_iy_j

$W_{\alpha_2}(x_iy_j) = \frac{(3+i)n+(10+i)+j+((-1)^j+1)n}{2} + \frac{(-1)^j+1}{4}$;

For n even, any j, and $i = 1, 2$,

$W_{\alpha_2}(x_iy_j) = \frac{(3+i)n+(10+i)+j+((-1)^{j+(i-1)}+1)n}{2} + \frac{(-1)^{j+(i-1)}+1}{4}$;

It is not difficult to see that the union of the set w_{α_2} equals to $\{2n+6, 2n+7, 2n+8, \ldots, 4n+6\}$ and contains an arithmetic sequence with the first term $2n+6$ and common difference 1. Thus α_2 is a super $(2n+6, 1)$-edge-antimagic total labeling. This concludes the proof. □

4. Diamond Ladder Dl_n

Diamond ladder graph denoted by Dl_n is a connected graph with a vertex set $V(Dl_n) = \{x_i, y_i, z_j; 1 \leq i \leq n, 1 \leq j \leq 2n\}$ and an edge set $E(Dl_n) = \{x_ix_{i+1}, y_iy_{i+1}; 1 \leq i \leq n-1\} \cup \{x_iy_i; 1 \leq i \leq n\} \cup \{z_jz_{j+1}; 2 \leq j \leq 2n-2 \text{ for } j \text{ even}\} \cup \{x_iz_{2i-1}, x_iz_{2i}, y_iz_{2i-1}, y_iz_{2i}; 1 \leq i \leq n\}$. Thus $|V(Dl_n)| = p = 4n$ and $|E(Dl_n)| = q = 8n-3$.

If diamond ladder graph has a super (a, d)-edge-antimagic total labeling then it follows from Lemma 1 that the upper bound of d is $d \leq 2$ or $d \in \{0, 1, 2\}$. The following lemma describes an $(a, 1)$-edge-antimagic vertex labeling for diamond ladder.

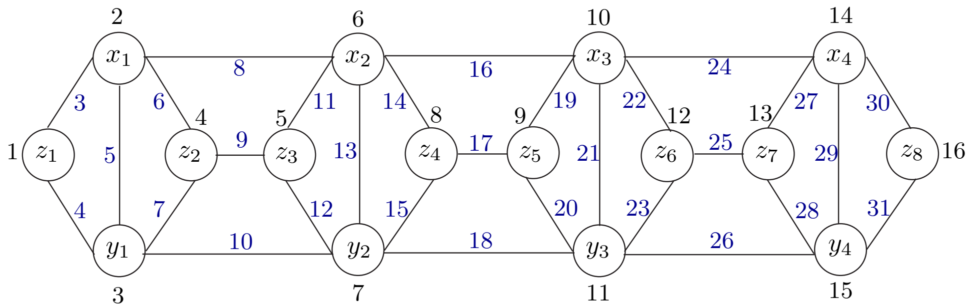

Figure 2. A (3,1)-edge antimagic vertex labeling of Dl_4

Theorem 4. *If $n \geq 2$ then the diamond ladder graph Dl_n has an $(a, 1)$-edge-antimagic vertex labeling.*

Proof. Define the vertex labeling $\beta_1 : V(Dl_n) \to \{1, 2, \ldots, 4n\}$ in the following way:

$$\beta_1(x_i) = 4i - 2, \text{ for } 1 \leq i \leq n$$

$$\beta_1(y_i) = 4i - 1, \text{ for } 1 \leq i \leq n$$

$$\beta_1(z_j) = 2j - \frac{((-1)^{j+1} + 1)}{2}, \text{ for } 1 \leq j \leq 2n$$

The vertex labeling β_1 is a bijective function. The edge-weights of Dl_n, under the labeling β_1, constitute the following sets

$$
\begin{aligned}
w_{\beta_1}(x_ix_{i+1}) &= 8i; \text{ for } 1 \le i \le n-1;\\
w_{\beta_1}(y_iy_{i+1}) &= 8i+2; \text{ for } 1 \le i \le n-1;\\
w_{\beta_1}(x_iy_i) &= 8i-3; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(z_jz_{j+1}) &= 4j+1; \text{ for } 2 \le j \le 2n-2 \ \ j \text{ even};\\
w_{\beta_1}(x_iz_{2i-1}) &= 8i-5; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(x_iz_{2i}) &= 8i-2; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(y_iz_{2i-1}) &= 8i-4; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(y_iz_{2i}) &= 8i-1; \text{ for } 1 \le i \le n;
\end{aligned}
$$

It is not difficult to see that the union of $w_{\beta_1} = \{3, 4, \dots, 8n-1\}$ and consists of consecutive integers. Thus β_1 is a $(3, 1)$-edge antimagic vertex labeling. □

Figure 2 gives an example of a $(3, 1)$-edge antimagic vertex labeling of Dl_4.

In similar way, with Theorem 4 in hand and by using Lemma 3, we obtain the following result.

Theorem 5. *If $n \ge 2$ then the graph Dl_n has a super $(12n, 0)$-edge-antimagic total labeling and a super $(4n+4, 2)$-edge-antimagic total labeling.*

Theorem 6. *If $n \ge 2$, then the graph Dl_n has a super $(8n+2, 1)$-edge-antimagic total labeling.*

Proof. Label the vertices of Dl_n with $\beta_2(x_i) = \beta_1(x_i)$, $\beta_2(y_i) = \beta_1(y_i)$ and $\beta_2(z_j) = \beta_1(z_j)$, for $1 \le i \le n$ and $1 \le j \le 2n$; and label the edges with the following way.

$$
\begin{aligned}
\beta_2(x_ix_{i+1}) &= 12n-4i-1; \text{ for } 1 \le i \le n-1\\
\beta_2(y_iy_{i+1}) &= 12n-4i-2; \text{ for } 1 \le i \le n-1\\
\beta_2(x_iy_i) &= 8n-4i+2; \text{ for } 1 \le i \le n\\
\beta_2(z_jz_{j+1}) &= 8n-2j; \text{ for } 2 \le j \le 2n-2 \text{ even}\\
\beta_2(x_iz_{2i-1}) &= 8n-4i+3; \text{ for } 1 \le i \le n\\
\beta_2(x_iz_{2i}) &= 12n-4i; \text{ for } 1 \le i \le n\\
\beta_2(y_iz_{2i-1}) &= 12n-4i+1; \text{ for } 1 \le i \le n\\
\beta_2(y_iz_{2i}) &= 8n-4i+1; \text{ for } 1 \le i \le n
\end{aligned}
$$

The total labeling β_2 is a bijective function from $V(Dl_n) \cup E(Dl_n)$ onto the set $\{1, 2, 3, \dots, 12n-3\}$. The edge-weights of Dl_n, under the labeling β_2, constitute the sets

$$
\begin{aligned}
W_{\beta_2}(x_i x_{i+1}) &= 12n+4i-1; \text{ for } 1 \le i \le n-1 \\
W_{\beta_2}(y_i y_{i+1}) &= 12n+4i; \text{ for } 1 \le i \le n-1 \\
W_{\beta_2}(x_i y_i) &= 8n+4i-1; \text{ for } 1 \le i \le n \\
W_{\beta_2}(z_j z_{j+1}) &= 8n+2j+1; \text{ for } 2 \le j \le 2n-2, \text{ for } j \text{ even} \\
W_{\beta_2}(x_i z_{2i-1}) &= 8n+4i-2; \text{ for } 1 \le i \le n \\
W_{\beta_2}(x_i z_{2i}) &= 12n+4i-2; \text{ for } 1 \le i \le n \\
W_{\beta_2}(y_i z_{2i-1}) &= 12n+4i-3; \text{ for } 1 \le i \le n \\
W_{\beta_2}(y_i z_{2i}) &= 8n+4i; \text{ for } 1 \le i \le n
\end{aligned}
$$

It is not difficult to see that the union of the the set $W_{\beta_2} = \{8n+2, 8n+3, \ldots, 16n-2\}$ contains an arithmetic sequence with the first term $8n+2$ and common difference 1. Thus β_2 is a super $(8n+2,1)$-edge-antimagic total labeling. This concludes the proof. □

Figure 3 gives an example of super (a,d)-edge antimagic total labeling of Dl_4 for $d=1$.

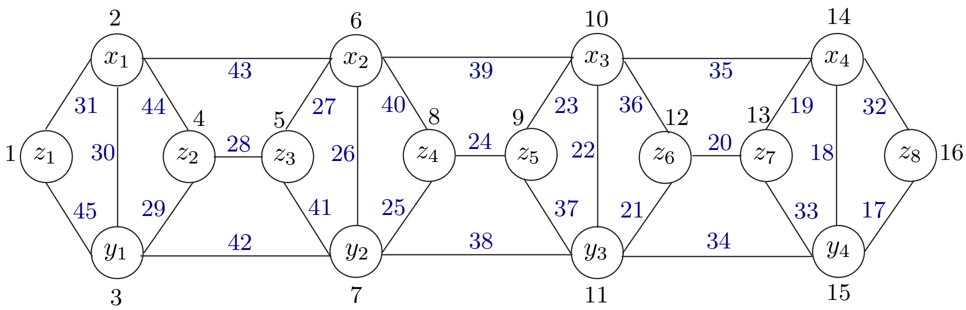

FIGURE 3. Super (34,1)-edge antimagic total labeling of Dl_4

5. Conclusion

In this paper, we have studied the existence of super antimagicness of two special families of graphs, namely triangular book and diamond ladder. The research shows the following results:

(1) The upper bound d of a super (a,d)-edge-antimagic total labeling at Bt_n and Dl_n is $d \le 2$

(2) There are a super (a,d)-edge-antimagic total labeling of graph Bt_n and Dl_n, if $n \ge 1$ with $d \in \{0,1,2\}$.

Further interested research is then to answer following problem: If a graph Bt_n and Dl_n are super (a,d)-edge-antimagic total, are the disjoint union of multiple copies of the graphs Bt_n and Dl_n super (a,d)-edge-antimagic total as well? Therefore, we propose the following open problem.

Open Problem 1. *For the graph mBt_n, $n \geq 1$ and $m \geq 2$, determine if there exists a super (a,d)-edge-antimagic total labeling with any feasible upper bound d.*

Open Problem 2. *For the graph mDl_n, $n \geq 1$ and $m \geq 1$, determine if there exists a super (a,d)-edge-antimagic total labeling with any feasible upper bound d.*

References

[1] M. Bača, Dafik, M. Miller and J. Ryan, On super (a,d)-edge antimagic total labeling of caterpillars, *J. Combin. Math. Combin. Comput.*, **65** (2008), 61–70.

[2] M. Bača, Y. Lin, M. Miller and R. Simanjuntak, New constructions of magic and antimagic graph labelings, *Utilitas Math.* **60** (2001), 229–239.

[3] G.S. Bloom and S.W. Golomb, Applications of numbered undirected graphs, *Proc. IEEE*, **65** (1977) 562–570.

[4] R. Bodendiek and G. Walther, On (a,d)-antimagic parachutes, Ars Combin. 42 (1996), 129–149.

[5] R. Bodendiek and G. Walther, (a,d)-antimagic parachutes II, Ars Combin. 46 (1997), 33–63.

[6] Dafik, Slamin, Fuad and Riris. 2009. *Super Edge-antimagic Total Labeling of Disjoint Union of Triangular Ladder and Lobster Graphs.*Yogyakarta: Proceeding of IndoMS International Conference of Mathematics and Applications (IICMA) 2009.

[7] Dafik, M. Miller, J. Ryan and M. Bača, Antimagic total labeling of disjoint union of complete s-partite graphs, *J. Combin. Math. Combin. Comput.*, **65** (2008), 41–49.

[8] Dafik, M. Miller, J. Ryan and M. Bača, On super (a,d)-edge antimagic total labeling of disconnected graphs, *Discrete Math.*, **309** (2009), 4909-4915.

[9] Dafik, M. Miller, J. Ryan and M. Bača, Super edge-antimagic total labelings of $mK_{n,n,n}$, *Ars Combinatoria* , **101** (2011), 97-107

[10] H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[11] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (2001), 153–168.

[12] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, On super edge-magic graph, Ars Combin. 64 (2002), 81–95.

[13] N. Hartsfield and G. Ringel, Pearls in Graph Theory, Academic Press, Boston - San Diego - New York - London, 1990.

[14] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[15] G. Ringel and A.S. Lladó, Another tree conjecture, *Bull. Inst. Combin. Appl.* **18** (1996), 83–85.

[16] R. Simanjuntak, F. Bertault and M. Miller, Two new (a,d)-antimagic graph labelings, *Proc. of Eleventh Australasian Workshop on Combinatorial Algorithms* (2000), 179–189.

[17] K.A. Sugeng, M. Miller and M. Bača, Super edge-antimagic total labelings, *Utilitas Math.*, **71** (2006) 131-141.

[18] W. D. Wallis, E. T. Baskoro, M. Miller and Slamin , Edge-magic total labelings, *Austral. J. Combin.* **22** (2000), 177–190.