Vol.14 No.1 Juni 2012

ISSN: 1411 – 5433

Saintifika

Jurnal Ilmu Pendidikan MIPA dan MIPA

Diterbitkan oleh: P MIPA FKIP Universitas Jember

Super Antimagicness of a Well-defined Graph

Dafik[23)], **Alfin Fajriatin**[24)], **Kunti Miladiyah F**[25)] [26)]

Abstract: A graph G of order p and size q is called an (a, d)-*edge-antimagic total* if there exist a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that the edge-weights, $w(uv) = f(u) + f(v) + f(uv), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a graph G is called *super* if the smallest possible labels appear on the vertices. In this paper we study super (a, d)-edge-antimagic total properties of connected and disconnected of a well-defined *mountain graph* and also show a new concept of a permutation of an arithmetic sequence.

Key Words *: SEATL, Permutation, Arithmetic Sequence, Mountain Graph.*

Introduction

The labeling of graph is the one of graph theory branch which is widely studied by a research group in combinatoric. Graph labelings provide useful mathematical models for a wide range of applications, such as radar and communication network addressing systems and circuit design, bioinformatics, various coding theory problems, automata, x-ray crystallography and data security. More detailed discussions about applications of graph labelings can be found in Bloom and Golomb's papers [4] and [5].

An (a, d)-*edge-antimagic total labeling* on a graph G is a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v), uv \in E(G)$, form an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a, d)-*edge-antimagic total graph.* Such a graph G is called *super* if the smallest possible labels appear on the vertices. Thus, a *super* (a, d)-*edge-antimagic total graph* is a graph that admits a super (a, d)-edge-antimagic total labeling.

The concept of (a, d)-edge-antimagic total labeling, introduced by Simanjuntak *at al.* in [11], is natural extension of the notion of *edge-magic* labeling

[23] Dafik adalah staf pengajar Prodi Pend. Matematika FKIP Universitas Jember
[24] Alfin Fajriatin adalah alumni Prodi Pend. MIPA FKIP Universitas Jember
[25] Alfin Fajriatin adalah alumni Prodi Pend. MIPA FKIP Universitas Jember
[26] Graph Research Group 2011, supported by DP2M Project

defined by Kotzig and Rosa [9] (see also [1], [7], [10] and [14]). The super (a, d)-edge-antimagic total labeling is natural extension of the notion of *super edge-magic* labeling which was defined by Enomoto *et al.* in [6].

In this paper we investigate the existence of super (a, d)-edge-antimagic total labelings for connected and disconnected graphs. We will now concentrate on a well-defined graph, namely the connected mountain graph and disjoint union of m copies mountain graph, denoted by M_{2n} and mM_{2n}. This research also show a new concept of a permutation of a consecutive number which is very useful especially for finding a super $(a, 1)$-edge-antimagic total labeling.

Some Useful Lemmas

We start this section by a necessary condition for a graph to be super (a, d)-edge-antimagic total, providing a least upper bound for feasible values of d.

Lemma 1 *If a (p, q)-graph is super (a, d)-edge-antimagic total then* $d \leq \frac{2p+q-5}{q-1}$.

Proof. Assume that a (p, q)-graph has a super (a, d)-edge-antimagic total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$. The edge-weights $w(uv) = f(u) + f(v)$, form an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$. The minimum possible edge weight in the labeling f is at least $1 + 2 + p + 1 = p + 4$. Thus, $a \geq p + 4$. On the other hand, the maximum possible edge weight is at most $(p - 1) + p + (p + q) = 3p + q - 1$. Hence $a + (q - 1)d \leq 3p + q - 1$. From the last inequality, we obtain the desired upper bound for the difference d. □

The following lemma, proved by Figueroa-Centeno *et al.* in [7], gives a necessary and sufficient condition for a graph to be super $(a, 0)$-edge-antimagic total or super edge-magic total.

Lemma 2 [13] *A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p + q + s$, where $s = min(S)$ and $S = \{a - (p + 1), a - (p + 2), \ldots, a - (p + q)\}$.*

In our terminology, the previous lemma states that a (p, q)-graph G is super $(a,0)$-edge-antimagic total if and only if there exists an $(a-p-q,1)$-edge-antimagic vertex labeling.

Research Method

There are three step of studies. Each study uses a different method.

- **Obtaining a network topology model**. By web-searching technique, we choose a Mountain Graph as well-defined family of graph.
- **Determining an algorithm of SEATL**. To find a SEATL bijective function, we firstly utilize an EAVL strategy.
- **Deriving a new Lemma, Theorem and Corollaries.** Deductive approach is the one of very popular way to prove mathematical truth.

Research Results

- The Mountain Graph

A connected Mountain Graph denoted by M_{2n} is a graph with vertex set $|V| = \{x_i, y_j; 1 \leq i \leq 2n$ dan $1 \leq j \leq 6n+2, n \epsilon N\}$ and *edge* set, $|E| = \{x_iy_{3i-2}, x_iy_{3i+3}$ if i is odd, x_iy_{3i-3}, x_iy_{3i+2} if i is even, $x_iy_{3i-1}, x_iy_{3i}, x_iy_{3i+1}$ if i is any, $1 \leq i \leq 2n$ and $y_jy_{j+1}, 1 \leq j \leq 6n+1\}$. Then $|V(M_{2n})| = p = 8n+2$ and $|E(M_{2n})| = q = 16n+1$. If mountain graph, has a super (a,d)-edge-antimagic total labeling then, for $p = 8n+2$ and $q = 16n+1$, it follows from Lemma 1 that the upper bound of d is $d \leq 2$ or $d \in \{0,1,2\}$.

The following new lemma describes an $(a,1)$-edge-antimagic vertex labeling for mountain graph.

Lemma 3 *If $n \geq 1$, then the mountain graph connected M_{2n} has an $(3,1)$-edge-antimagic vertex labeling.*

Proof. Define the vertex labeling $\alpha_1 : V(M_{2n}) \to \{1,2,\ldots,8n+2\}$ in the following way, for $1 \leq i \leq 2n$ and $1 \leq j \leq 6n+2$.

$$\alpha_1(x_i) = 4i - \tfrac{((-1)^{i+1}+1)}{2},$$

$$\alpha_1(y_j) = \begin{cases} \frac{4j-1}{3}, & \text{for } j = 1(\text{mod } 3), \\ \frac{4j-2}{3}, & \text{for } j = 2(\text{mod } 3), \\ \frac{4j+\frac{8j((-1)^j+1)}{2}}{3} - 1 - \frac{7((-1)^j+1)}{2}, & \text{for } j = 3(\text{mod } 3), \end{cases}$$

The vertex labeling α_1 is a bijective function. The edge-weights of M_{2n}, under the labeling α_1, constitute the following sets.

$$\begin{array}{lcll}
w^1_{\alpha_1}(x_iy_{3i-2}) &=& 8i-4; & \text{for } i \text{ is odd} \\
w^2_{\alpha_1}(x_iy_{3i-3}) &=& 8i-4; & \text{for } i \text{ is even} \\
w^3_{\alpha_1}(x_iy_{3i-1}) &=& 8i-2-\frac{((-1)^{i+1}+1)}{2}; & \text{for } i \text{ is any} \\
w^4_{\alpha_1}(x_iy_{3i}) &=& 8i-1; & \text{for } i \text{ is any} \\
w^5_{\alpha_1}(x_iy_{3i+1}) &=& 8i+\frac{((-1)^{i}+1)}{2}; & \text{for } i \text{ is any} \\
w^6_{\alpha_1}(x_iy_{3i+2}) &=& 8i+2; & \text{for } i \text{ is even} \\
w^7_{\alpha_1}(x_iy_{3i+3}) &=& 8i+2; & \text{for } i \text{ is odd} \\
w^8_{\alpha_1}(y_jy_{j+1}) &=& \frac{8j+1}{3}; & \text{for } j = 1(\text{mod } 3) \\
w^9_{\alpha_1}(y_jy_{j+1}) &=& \frac{8j-1+\frac{3((-1)^j+1)}{2}}{3}; & \text{for } j = 2(\text{mod } 3) \\
w^{10}_{\alpha_1}(y_jy_{j+1}) &=& \frac{8j}{3}+\frac{(-1)^{j+1}+1}{2}; & \text{for } j = 3(\text{mod } 3)
\end{array}$$

It is not difficult to see that the set $\bigcup_{t=1}^{10} w^t_{\alpha_1} = \{3, 4, 5, \ldots, \frac{8j+1}{3}\}$ consists of consecutive integers. Thus α_1 is a $(3,1)$-edge antimagic vertex labeling. □

We utilize the vertex labeling α_1 from the proof of Lemma 3 to prove the following theorem.

Theorem 1 *If $n \geq 1$ then the graph M_{2n} has a super $(24n+6,0)$-edge-antimagic total labeling and a super $(8n+6,2)$-edge-antimagic total labeling.*

Proof.

Case 1. $d = 0$

Label the vertices of M_{2n} with $\alpha_2(x_i) = \alpha_1(x_i)$ and $\alpha_2(y_j) = \alpha_1(y_j)$, for $1 \leq i \leq 2n$ and $1 \leq j \leq 6n+2$; and label the edges with $\alpha_2(x_i)$, $\alpha_2(y_j)$, $\alpha_2(x_iy_{3i-2})$, $\alpha_2(x_iy_{3i-3})$, $\alpha_2(x_iy_{3i-1})$, $\alpha_2(x_iy_{3i})$, $\alpha_2(x_iy_{3i+1})$, $\alpha_2(x_iy_{3i+2})$, $\alpha_2(x_iy_{3i+3})$ and $\alpha_2(y_jy_{j+1})$. It follows from Lemma 2 that the labeling α_2 can be extended, by completing the edge label $p+1, p+2, \ldots, p+q$, to a super $(a,0)$-edge antimagic total labeling, where, in the case $p = 8mn+2m$ and $q = 16mn+m$.

We can find the total labeling W_{α_2} with summing $w_{\alpha_1} = w_{\alpha_2}$ with edge label α_2. It is not difficult to see that the set $\bigcup_{t=1}^{14} W_{\alpha_2}^t = \{24n+6, 24n+6, \ldots, 24n+6\}$ contains an arithmetic sequence with the first term $24n+6$ and common difference 0. Thus α_2 is a super $(24n+6, 0)$-edge-antimagic total labeling. This concludes the proof. □

Case 2. $d = 2$

Label the vertices of M_{2n} with $\alpha_3(x_i) = \alpha_1(x_i)$ and $\alpha_3(y_j) = \alpha_1(y_j)$, for $1 \leq i \leq 2n$ and $1 \leq j \leq 6n+2$; and label the edges with $\alpha_3(x_i)$, $\alpha_3(y_j)$, $\alpha_3(x_iy_{3i-2})$, $\alpha_3(x_iy_{3i-3})$, $\alpha_3(x_iy_{3i-1})$, $\alpha_3(x_iy_{3i})$, $\alpha_3(x_iy_{3i+1})$, $\alpha_3(x_iy_{3i+2})$, $\alpha_3(x_iy_{3i+3})$ and $\alpha_3(y_jy_{j+1})$. The total labeling α_1 is a bijective function from $V(M_{2n}) \bigcup E(M_{2n})$ onto the set $\{1, 2, 3, \ldots, 24n+3\}$. For the edge weight of M_{2n}, under the total labeling α_1 we have:

$$
\begin{aligned}
W_{\alpha_3}^1 &= \{w_{\alpha_3}^1 + \alpha_3(x_iy_{3i-2}); \text{ if } i \text{ is odd}\} \\
&= (8i-4) + (8n+8i-4) \\
W_{\alpha_3}^2 &= \{w_{\alpha_3}^2 + \alpha_3(x_iy_{3i-3}); \text{ if } i \text{ is even}\} \\
&= (8i-4) + (8n+8i-4) \\
W_{\alpha_3}^3 &= \{w_{\alpha_3}^3 + \alpha_3(x_iy_{3i-1}); \text{ if } i \text{ is odd}\} \\
&= (8i-3) + (8n+8i-3) \\
W_{\alpha_3}^4 &= \{w_{\alpha_3}^4 + \alpha_3(x_iy_{3i-1}); \text{ if } i \text{ is even}\} \\
&= (8i-2) + (8n+8i-2) \\
W_{\alpha_3}^5 &= \{w_{\alpha_3}^5 + \alpha_3(x_iy_{3i}); \text{ if } i \text{ is any}\} \\
&= (8i-1) + (8n+8i-1) \\
W_{\alpha_3}^6 &= \{w_{\alpha_3}^6 + \alpha_3(x_iy_{3i+1}); \text{ if } i \text{ is odd}\} \\
&= (8i) + (8n+8i) \\
W_{\alpha_3}^7 &= \{w_{\alpha_3}^7 + \alpha_3 x_iy_{3i+1}); \text{ if } i \text{ is even}\} \\
&= (8i+1) + (8n+8i+1)
\end{aligned}
$$

$$\begin{aligned}
W^{8}_{\alpha_3} &= \{w^{8}_{\alpha_3} + \alpha_3(x_i y_{3i+2});\text{ if } i \text{ is even}\}\\
&= (8i+2) + (8n+8i+2)\\
W^{9}_{\alpha_3} &= \{w^{9}_{\alpha_3} + \alpha_3(x_i y_{3i+3});\text{ if } i \text{ is odd}\}\\
&= (8i+2) + (8n+8i+2)\\
W^{10}_{\alpha_3} &= \{w^{10}_{\alpha_3} + \alpha_3(y_j y_{j+1});\text{ if } j = 1(\text{mod } 3)\}\\
&= (\frac{8j+1}{3}) + (8n + \frac{8j+1}{3})\\
W^{11}_{\alpha_3} &= \{w^{11}_{\alpha_3} + \alpha_3(y_j y_{j+1});\text{ if } j = 2(\text{mod } 3), j \text{ is odd}\}\\
&= (\frac{8j-1}{3}) + (8n + \frac{8j-1}{3})\\
W^{12}_{\alpha_3} &= \{w^{12}_{\alpha_3} + \alpha_3(y_j y_{j+1});\text{ if } j = 2(\text{mod } 3), j \text{ is even}\}\\
&= (\frac{8j+2}{3}) + (\frac{8j+2}{3})\\
W^{13}_{\alpha_3} &= \{w^{13}_{\alpha_3} + \alpha_3(y_j y_{j+1});\text{ if } j = 3(\text{mod } 3), j \text{ is odd}\}\\
&= (\frac{8j}{3}+1) + (8n + \frac{8j+3}{3})\\
W^{14}_{\alpha_3} &= \{w^{14}_{\alpha_3} + \alpha_3(y_j y_{j+1});\text{ if } j = 3(\text{mod } 3), j \text{ is even}\}\\
&= (\frac{8j}{3}) + (8n + \frac{8j}{3})
\end{aligned}$$

It is not difficult to see that the set $\bigcup_{t=1}^{14} W^{t}_{\alpha_3} = \{8n+6,\, 8n+8,\, 8n+10 \ldots, 40n+6\}$ contains an arithmetic sequence with the first term $8n+6$ and common difference 0. Thus α_3 is a super $(8n+6, 2)$-edge-antimagic total labeling. This concludes the proof. □

Now, we will show our a progressive result for permutation lemma. This lemma is very useful especially for finding a super $(a, 1)$-edge-antimagic total labeling.

Lemma 4 *Let Υ be a sequence of consecutive number $\Upsilon = \{c, c+1, c+2, \ldots c+k\}$, k even. Then there exists a permutation $\Pi(\Upsilon)$ of the elements of Υ such that $\Upsilon + \Pi(\Upsilon) = \{2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, 2c + \frac{k}{2} + 3, \ldots, 2c + \frac{3k}{2}, 2c + \frac{3k}{2} + 1\}$ is also a sequence of consecutive number.*

Proof. Let Υ be a sequence $\Upsilon = \{a_i |\ a_i = c + (i-1),\ 1 \le i \le k+1\}$ and k be even. Define a permutation $\Pi(\Upsilon) = \{b_i |\ 1 \le i \le k+1\}$ of the elements of

Υ as follows:

$$b_i = \begin{cases} c+k+\frac{3-i}{2} & \text{if } i \text{ is odd}, 1 \le i \le k+1 \\ c+\frac{k}{2}+\frac{2-i}{2} & \text{if } i \text{ is even}, 2 \le i \le k. \end{cases}$$

By direct computation, we obtain that $\Upsilon + \Pi(\Upsilon) = \{a_i + b_i \mid 1 \le i \le k+1\} = \{2c+k+\frac{1+i}{2} \mid i \text{ odd}, 1 \le i \le k+1\} \cup \{2c+\frac{k}{2}+\frac{i}{2} \mid i \text{ even}, 2 \le i \le k\} = \{2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, 2c+\frac{k}{2}+3, \ldots, 2c+\frac{3k}{2}, 2c+\frac{3k}{2}+1\}$. □

Directly from Lemma 3, with respect to Lemma 4, it follows that *mountain graph* has a super $(a,1)$-edge-antimagic total labeling.

Theorem 2 *If $n \ge 1$, then the graph M_{2n} has a super $(16n+6,1)$-edge-antimagic total labeling.*

Proof. From Lemma 3, the graph M_{2n} has a $(3,1)$-edge-antimagic vertex labeling. Let $\mathfrak{A} = \{c, c+1, c+2, \ldots, c+k\}$ be a set of the edge weights of the vertex labeling α_3, for $c = 3$ and $k = 16n$. In light of Lemma 4, there exists a permutation $\Pi(\Upsilon)$ of the elements of Υ such that $\Upsilon + \left[\Pi(\Upsilon) + \frac{k}{2} - 1\right] = \{2c+16n, 2c+16n+1, \ldots, 2c+24n\}$. If $\left[\Pi(\Upsilon) + \frac{k}{2} - 1\right]$ is an edge labeling of M_{2n} then $\Upsilon + [\Pi(\Upsilon) + \frac{k}{2} - 1]$ gives the set of the edge weights of M_{2n}, which implies that the total labeling is super $(a,1)$-edge-antimagic total, where $a = 2c + 16n = 2(3) + 16n = 16n + 6$. This concludes that the graph M_{2n} admid a super $(16n+6,1)$-edge antimagic totallabeling. □

- Disjoint Union of Mountain Graph

Disjoint union of m copies of mountain graph denoted by mM_{2n} is a disconnected graph with vertex set, $|V| = \{x_i^k, y_j^k; 1 \le i \le 2n \text{ and } 1 \le j \le 6n+2, n \in N\}$ and edge set, $|E| = \{x_i^k y_{3i-2}^k, x_i^k y_{3i+3}^k$ for i odd, $x_i^k y_{3i-3}^k, x_i^k y_{3i+2}^k$ for i even, $x_i^k y_{3i-1}^k, x_i^k y_{3i}^k, x_i^k y_{3i+1}^k$ for any i, $1 \le i \le 2n$ and $y_j^k y_{j+1}^k, 1 \le j \le 6n+1\}$. We bounded mM_{2n} for $1 \le k \le m$, $m \ge 2$ and $n \ge 1$. Thus $|V(mM_{2n})| = p = m(8n+2)$ and $|E(mM_{2n})| = q = m(16n+1)$.

If the disjoint union of m copies of a Mountain Graph mM_{2n}, has a super (a,d)-edge-antimagic total labeling then, for $p = m(8n+2)$ and $q = m(16n+1)$, it follows from Lemma 1 that the upper bound of d is $d \le 2 - \frac{3m-3}{16mn+m-1}$ or $d \in \{0,1,2\}$.

Lemma 5 *The graph mM_{2n} for $d = \{0, 2\}$ has a $(\frac{3m+3}{2}, 1)$-edge-antimagic vertex labeling if $m \geq 3$ is odd and $n \geq 1$.*

Proof. Define the vertex labeling $\alpha_4 : V(mM_{2n}) \rightarrow \{1, 2, \ldots, 8nm + 2m\}$ in the following way:

$$\alpha_4(y_j^k) = \begin{cases} \frac{(j-1)4m}{3} + \frac{k+1+(\frac{(-1)^k+1}{2})m}{2}; & \text{for } j \equiv 1(mod\ 9) \\ \frac{(j-2)4m}{3} + \frac{2m+k+(\frac{(-1)^{k+1}+1}{2})m}{2}; & \text{for } j \equiv 2(mod\ 9) \\ \frac{(j-3)4m}{3} + \frac{6m+k+1+(\frac{(-1)^k+1}{2})m}{2}; & \text{for } j \equiv 3(mod\ 9)\ \ , j \text{ is odd} \\ \frac{(j-12)4m}{3} + 15m - k + 1; & \text{for } j \equiv 3(mod\ 9)\ \ , j \text{ is even} \\ \frac{(j-4)4m}{3} + \frac{8m+k+(\frac{(-1)^{k+1}+1}{2})m}{2}; & \text{for } j \equiv 4(mod\ 9) \\ \frac{(j-5)4m}{3} + 6m - k + 1; & \text{for } j \equiv 5(mod\ 9) \\ \frac{(j-6)4m}{3} + \frac{12m+k+1+(\frac{(-1)^k+1}{2})m}{2}; & \text{for } j \equiv 6(mod\ 9)\ \ , j \text{ is even} \\ \frac{(j-15)4m}{3} + \frac{38m+k+(\frac{(-1)^{k+1}+1}{2})m}{2}; & \text{for } j \equiv 6(mod\ 9)\ \ , j \text{ is odd} \\ \frac{(j-7)4m}{3} + 9m - k + 1; & \text{for } j \equiv 7(mod\ 9) \\ \frac{(j-8)4m}{3} + \frac{18m+k+1+(\frac{(-1)^k+1}{2})m}{2}; & \text{for } j \equiv 8(mod\ 9) \\ \frac{(j-9)4m}{3} + 12m - k + 1; & \text{for } j \equiv 9(mod\ 9)\ \ , j \text{ is odd} \\ \frac{(j-18)4m}{3} + \frac{44m+k+(\frac{(-1)^{k+1}+1}{2})m}{2}; & \text{for } j \equiv 9(mod\ 9)\ \ , j \text{ is even} \end{cases}$$

$$\alpha_4(x_i^k) = \begin{cases} (i-1)4m + 3m - k + 1; & \text{for } i \equiv 1(mod\ 6) \\ (i-2)4m + \frac{14m+k+(\frac{(-1)^{k+1}+1}{2})m}{2}; & \text{for } i \equiv 2(mod\ 6) \\ (i-3)4m + \frac{20m+k+(\frac{(-1)^{k+1}+1}{2})m}{2}; & \text{for } i \equiv 3(mod\ 6) \\ (i-4)4m + \frac{30m+k+1+(\frac{(-1)^k+1}{2})m}{2}; & \text{for } i \equiv 4(mod\ 6) \\ (i-5)4m + \frac{36m+k+1+(\frac{(-1)^k+1}{2})m}{2}; & \text{for } i \equiv 5(mod\ 6) \\ (i-6)4m + 24m - k + 1; & \text{for } i \equiv 6(mod\ 6) \end{cases}$$

for $1 \leq i \leq 2n$ and $1 \leq j \leq 6n + 2$.

The vertex labeling α_4 is a bijective function. We have the same way with lemma 4 to determine the value of the edge-weights of mM_{2n}. It is not difficult to see that set $\bigcup_{t=1}^{45} w_{\alpha_4}^t = \{\frac{3m+3}{2}, \frac{3m+5}{2}, \frac{3m+7}{2}, \ldots, \frac{(16n-3)m+1}{2}\}$ consists of consecutive integers. Thus α_4 is a $(\frac{3m+3}{2}, 1)$-edge antimagic vertex labeling. □

Baĉa, Lin, Miller and simanjutak (see[9],Theorem 5) have proved that if (p, q)-graph G has an (a, d)-edge-antimagic vertex labeling then G has a super $(a+p+q, d-1)$-edge-antimagic total labeling and a super $(a+p+1, d+1)$-edge-antimagic total labeling. With the Theorem 3.3.1 in hand, and using Theorem 5 from [9], we obtain the following result(Dafik,2007:41).

Theorem 3 *If $m \geq 3$ is odd and $n \geq 1$ then the graph mM_{2n} has a super $(24mn + \frac{(9m+3)}{2}, 0)$-edge-antimagic total labeling and a super $(8mn + \frac{7m+5}{2}, 2)$-edge-antimagic total labeling.*

Proof.

Case 1. $d = 0$

Label the vertices of mM_{2n} with $\alpha_5(x_i^k) = \alpha_4(x_i^k)$ and $\alpha_5(y_j^k) = \alpha_4(y_j^k)$, for $1 \leq i \leq 2n$ and $1 \leq j \leq 6n+2$; and label the edges with $\alpha_5(y_j^k y_{j+1}^k)$, $\alpha_5(x_i^k y_{3i-2}^k)$, $\alpha_5(x_i^k y_{3i-1}^k)$, $\alpha_5(x_i^k y_{3i}^k)$, $\alpha_5(x_i^k y_{3i+1}^k)$, $\alpha_5(x_i^k y_{3i+3}^k)$, $\alpha_5(x_i^k y_{3i+2}^k)$ and $\alpha_5(x_i^k y_{3i-3}^k)$.

We can found the total labeling W_{α_5} with summing edge weight $w_{\alpha_5} = w_{\alpha_4}$ with edge label α_5. It is not difficult to see that the set $\bigcup_{t=1}^{45} W_{\alpha_5}^t = \{24mn + \frac{(9m+3)}{2}, 24mn + \frac{(9m+3)}{2}, \ldots, 24mn + \frac{(9m+3)}{2}\}$ contains an arithmetic sequence with the first term $\{24mn + \frac{(9m+3)}{2}$ and common difference 0. Thus α_2 is a super $(24mn + \frac{(9m+3)}{2}, 0)$-edge-antimagic total labeling. This concludes the proof. □

Case 2. $d = 2$

Label the vertices of mM_{2n} with $\alpha_6(x_i^k) = \alpha_4(x_i^k)$ and $\alpha_6(y_j^k) = \alpha_4(y_j^k)$, for $1 \leq i \leq 2n$ and $1 \leq j \leq 6n+2$; and label the edges with $\alpha_6(y_j^k y_{j+1}^k)$, $\alpha_6(x_i^k y_{3i-2}^k)$, $\alpha_5(x_i^k y_{3i-1}^k)$, $\alpha_6(x_i^k y_{3i}^k)$, $\alpha_5(x_i^k y_{3i+1}^k)$, $\alpha_6(x_i^k y_{3i+3}^k)$, $\alpha_5(x_i^k y_{3i+2}^k)$ and $\alpha_6(x_i^k y_{3i-3}^k)$.

We can find the total labeling W_{α_6} with summing edge weight $w_{\alpha_6} = w_{\alpha_4}$ with edge label α_6. It is not difficult to see that the set $\bigcup_{t=1}^{45} W_{\alpha_6}^t = \{\frac{7m+5}{2} + 8mn, \frac{7m+7}{2} + 8mn, \frac{7m+9}{2} + 8mn \ldots, \frac{11m+1}{2} + 40mn\}$ contains an arithmetic sequence with the first term $8mn + \frac{7m+5}{2}$ and common difference 2. Thus α_6 is a super $(8mn + \frac{7m+5}{2}, 2)$-edge-antimagic total labeling. This concludes the proof. □

Here, we will present our new permutation lemma. This lemma is also very useful for proving a super $(a,1)$-edge-antimagic total labeling.

Lemma 6 *Let Ψ be a sequence of consecutive number $\Psi = \{c, c+1, c+2, \dots c+k\}$, k even. Then there exists a permutation $\Pi(\Psi)$ of the elements of Ψ such that $\Psi + \Pi(\Psi) = \{2c + \frac{k}{2}, 2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, \dots, 2c + \frac{3k}{2}\}$ is also a sequence of consecutive number.*

Proof. Let Ψ be a sequence $\Psi = \{a_i | \ a_i = c + (i-1),\ 1 \le i \le k+1\}$ and k be even. Define a permutation $\Pi(\Psi) = \{b_i | \ 1 \le i \le k+1\}$ of the elements of Ψ as follows:

$$b_i = \begin{cases} c + i + \frac{k}{2} & \text{if } 1 \le i \le \frac{k}{2} \\ c + i - (\frac{k}{2} + 1) & \text{if } \frac{k}{2} + 1 \le i \le k + 1. \end{cases}$$

By direct computation, we obtain that $\Psi + \Pi(\Psi) = \{a_i + b_i | \ 1 \le i \le k+1\} = \{c + i + \frac{k}{2} \text{if } 1 \le i \le \frac{k}{2}\} \cup \{c + i - (\frac{k}{2} + 1) \text{if, } \frac{k}{2} + 1 \le i \le k + 1.\} = \{2c + \frac{k}{2}, 2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, 2c + \frac{k}{2} + 3, \dots, 2c + \frac{3k}{2}, 2c + \frac{3k}{2}\}$. □

Directly from Lemma 3, with respect to Lemma 6, it follows that *mountain graph* has a super $(a,1)$-edge-antimagic total labeling.

Theorem 4 *If $m \ge 2$ and $n \ge 1$, then the graph mM_{2n} has a super $(16nm + 4m + 2, 1)$-edge-antimagic total labeling.*

Proof. From Lemma 5, the graph mM_{2n} has a $(\frac{3m+3}{2}, 1)$-edge-antimagic vertex labeling. Let $\mathfrak{A} = \{c, c+1, c+2, \dots, c+k\}$ be a set of the edge weights of the vertex labeling α_4, for $c = \frac{3m+3}{2}$ and $k = 16mn + m - 1$. In light of Lemma 6, there exists a permutation $\Pi(\Psi)$ of the elements of Ψ such that $\Psi + \left[\Pi(\Psi) + \frac{k}{2} - 1\right] = \{2c + 16mn + m - 1, 2c + 16mn + m, \dots, 2c + 32mn + 2m - 2\}$. If $\left[\Pi(\Upsilon) + \frac{k}{2} - 1\right]$ is an edge labeling of mM_{2n} then $\Upsilon + [\Pi(\Upsilon) + \frac{k}{2} - 1]$ gives the set of the edge weights of mM_{2n}, which implies that the total labeling is super $(a,1)$-edge-antimagic total, where $a = 2c + 16mn + m - 1 = 2(\frac{3m+3}{2}) + 16mn + m - 1 = 16mn + 4m + 2$. This concludes that the graph mM_{2n} admid a super $(16mn + 4m + 2, 1)$-edge antimagic totallabeling. □

Conclusion

We have proved that mountain graph M_{2n} and disjoint union of mountain graph mM_{2n} admit super (a, d)-edge-antimagic for $d \in \{0, 1, 2\}$ and for specific m, n. Apart from those cases, we have not found any super (a, d)-edge-antimagic total labeling. Therefore we propose the following open problems.

Open Problem 1 *For the graph mM_{2n}, $n \geq 1$; $1 \leq k \leq m$; m is even, determine if there is a super (a, d)-edge-antimagic total labeling with $d = 0$ dan $d = 2$.*

Bibliography

[1] M. Bača, Y. Lin, M. Miller and R. Simanjuntak, New constructions of magic and antimagic graph labelings, *Utilitas Math.* **60** (2001), 229–239.

[2] M. Bača and L. Brankovic, Edge-antimagicness for a class of disconnected graphs, *Ars Combin.*, in press.

[3] M. Bača, Dafik, M. Miller and J. Ryan, On super (a, d)-edge antimagic total labeling of caterpillars, *J. Combin. Math. Combin. Comput.*, **65** (2008), 61–70.

[4] G.S. Bloom and S.W. Golomb, Applications of numbered undirected graphs, *Proc. IEEE*, **65** (1977) 562–570.

[5] G.S. Bloom and S.W. Golomb, Numbered complete graphs, unusual rules and assorted applications, *In: Theory and Applications of Graphs, Lecture Notes in Math.*, **642**, Springer-Verlag, New York (1978) 53–65.

[6] H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[7] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (2001), 153–168.

[8] J. Ivančo and I. Lučkaničová, On edge-magic disconnected graphs, *SUT Journal of Math.* **38** (2002), 175–184.

[9] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[10] G. Ringel and A.S. Lladó, Another tree conjecture, *Bull. Inst. Combin. Appl.* **18** (1996), 83–85.

[11] R. Simanjuntak, F. Bertault and M. Miller, Two new (a, d)-antimagic graph labelings, *Proc. of Eleventh Australasian Workshop on Combinatorial Algorithms* (2000), 179–189.

[12] I.W. Sudarsana, D. Ismaimuza, E.T. Baskoro and H. Assiyatun, On super (a, d)-edge-antimagic total labeling of disconnected graphs, *JCMCC* **55** (2005), 149–158.

[13] K.A. Sugeng, M. Miller, Slamin and M. Bača, (a, d)-edge-antimagic total labelings of caterpillars, *Lecture Notes in Computer Science* 3330 (2005), 169–180.

[14] W. D. Wallis, E. T. Baskoro, M. Miller and Slamin , Edge-magic total labelings, *Austral. J. Combin.* **22** (2000), 177–190.

[15] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston - Basel - Berlin, 2001.