THE UNIVERSITY OF
NEWCASTLE
AUSTRALIA

Workshop Proceedings

International Workshop on Combinatorial Algorithms

5-9 November 2007 Newcastle · Australia

18th INTERNATIONAL WORKSHOP ON COMBINATORIAL ALGORITHMS 2007

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On diregularity of digraphs of defect two

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Abstract: Since Moore digraphs do not exist for $k \neq 1$ and $d \neq 1$, the problem of finding the existence of digraph of out-degree $d \geq 2$ and diameter $k \geq 2$ and order close to the Moore bound becomes an interesting problem. To prove the non-existence of such digraphs, we first may wish to establish their diregularity. It is easy to show that any digraph with out-degree at most $d \geq 2$, diameter $k \geq 2$ and order $n = d + d^2 + \ldots + d^k - 1$, that is, two less than Moore bound must have all vertices of out-degree d. However, establishing the regularity or otherwise of the in-degree of such a digraph is not easy. In this paper we prove that all digraphs of defect two are out-regular and almost in-regular.

Key Words: *Diregularity, digraph of defect two, degree-diameter problem.*

1 Introduction

By a *directed graph* or a *digraph* we mean a structure $G = (V(G), A(G))$, where $V(G)$ is a finite nonempty set of distinct elements called *vertices*, and $A(G)$ is a set of ordered pair (u, v) of distinct vertices $u, v \in V(G)$ called *arcs*.

[1] This research was supported by the Australian Research Council (ARC) Discovery Project grant DP04502994.

The *order* of the digraph G is the number of vertices in G. An *in-neighbour* (respectively, *out-neighbour*) of a vertex v in G is a vertex u (respectively, w) such that $(u, v) \in A(G)$ (respectively, $(v, w) \in A(G)$). The set of all in-neighbours (respectively, out-neighbours) of a vertex v is called the *in-neighbourhood* (respectively, the *out-neighbourhood*) of v and denoted by $N^-(v)$ (respectively, $N^+(v)$). The *in-degree* (respectively, *out-degree*) of a vertex v is the number of all its in-neighbours (respectively, out-neighbours). If every vertex of a digraph G has the same in-degree (respectively, out-degree) then G is said to be *in-regular* (respectively, *out-regular*). A digraph G is called a *diregular* digraph of degree d if G is in-regular of in-degree d and out-regular of out-degree d.

An alternating sequence $v_0a_1v_1a_2...a_lv_l$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$ for each i is called a *walk* of length l in G. A walk is *closed* if $v_0 = v_l$. If all the vertices of a $v_0 - v_l$ walk are distinct, then such a walk is called a *path*. A *cycle* is a closed path. A *digon* is a cycle of length 2.

The *distance* from vertex u to vertex v, denoted by $\delta(u, v)$, is the length of a shortest path from u to v, if any; otherwise, $\delta(u, v) = \infty$. Note that, in general, $\delta(u, v)$ is not necessarily equal to $\delta(v, u)$. The *in-eccentricity* of v, denoted by $e^-(v)$, is defined as $e^-(v) = max\{\delta(u, v) : u \in V\}$ and *out-eccentricity* of v, denoted by $e^+(v)$, is defined as $e^+(v) = max\{\delta(v, u) : u \in V\}$. The radius of G, denoted by rad(G), is defined as rad(G)= $min\{e^-(v) : v \in V\}$. The *diameter* of G, denoted by diam(G), is defined as diam(G)= $max\{e^-(v) : v \in V\}$. Note that if G is a strongly connected digraph then, equivalently, we could have defined the radius and the diameter of G in terms of out-eccentricity instead of in-eccentricity. The *girth* of a digraph G is the length of a shortest cycle in G.

The well known *degree/diameter* problem for digraphs is to determine the largest possible order $n_{d,k}$ of a digraph, given out-degree at most $d \geq 1$ and diameter $k \geq 1$. There is a natural upper bound on the order of digraphs given out-degree at most d and diameter k. For any given vertex v of a digraph G, we can count the number of vertices at a particular distance from that vertex. Let n_i , for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i \leq d^i$, for $0 \leq i \leq k$, and consequently,

$$n_{d,k} = \sum_{i=0}^{k} n_i \leq 1 + d + d^2 + \ldots + d^k. \qquad (1)$$

The right-hand side of (1), denoted by $M_{d,k}$, is called the *Moore bound*. If the equality sign holds in (1) then the digraph is called a *Moore digraph*. It is well known that Moore digraphs exist only in the cases when $d = 1$ (directed cycles of length $k+1$, C_{k+1} , for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d+1$, K_{d+1}, for any $d \geq 1$) [2, 11].

Note that every Moore digraph is diregular (of degree one in the case of C_{k+1} and of degree d in the case of K_{d+1}). Since for $d > 1$ and $k > 1$ there are no Moore digraphs, we are next interested in digraphs of order n 'close' to Moore bound.

It is easy to show that a digraph of order n, $M_{d,k} - M_{d,k-1} + 1 \leq n \leq M_{d,k} - 1$, with out-degree at most $d \geq 2$ and diameter $k \geq 2$ must have all vertices of out-degree d. In other words, the out-degree of such a digraph is constant ($= d$). This can be easily seen because if there were a vertex in the digraph with out-degree $d_1 < d$ (i.e., $d_1 \leq d - 1$), then the order of the digraph,

$$\begin{aligned} n &\leq 1 + d_1 + d_1 d + \ldots + d_1 d^{k-1} \\ &= 1 + d_1(1 + d + \ldots + d^{k-1}) \\ &\leq 1 + (d-1)(1 + d + \ldots + d^{k-1}) \\ &= (1 + d + \ldots + d^k) - (1 + d + \ldots + d^{k-1}) \\ &= M_{d,k} - M_{d,k-1} \\ &< M_{d,k} - M_{d,k-1} + 1, \end{aligned}$$

However, establishing the regularity or otherwise of in-degree for an *almost* Moore digraph is not easy. It is well known that there exist digraphs of out-degree d and diameter k whose order is just two or three less than the Moore bound and in which *not all* vertices have the same in-degree. In Fig. 1 we give two examples of digraphs of diameter 2, out-degree $d = 2, 3$, respectively, and order $M_{d,2} - d$, with vertices not all of the same in-degree.

Miller, Gimbert, Širáň and Slamin [7] considered the diregularity of digraphs of defect one, that is, $n = M_{d,k} - 1$, and proved that such digraphs are diregular. For defect two, diameter $k = 2$ and any out-degree $d \geq 2$, non-diregular digraphs always exist. One such family of digraphs can be generated from Kautz digraphs which contain vertices with identical out-neighbourhoods and so we can apply vertex deletion scheme, see [8], to obtain non-diregular digraphs of defect two, diameter $k = 2$, and any out-degree $d \geq 2$. Fig. 2(a) shows an example of Kautz digraph G of order $n = M_{3,2} - 1$ which we will use to illustrate the vertex deletion scheme. Note

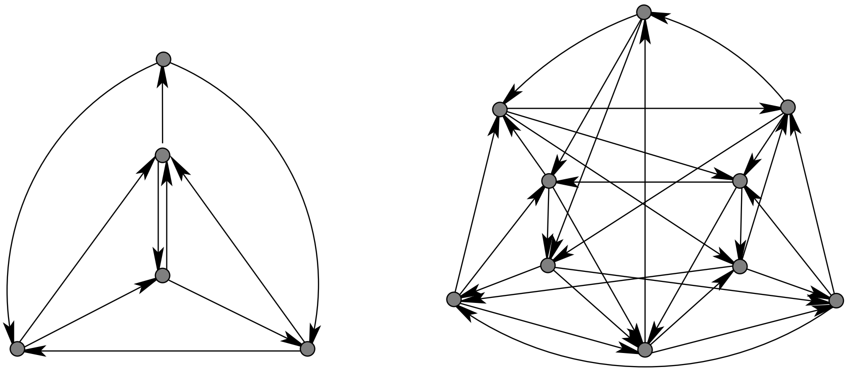

Fig. 1. Two examples of non-diregular digraphs.

the existence of identical out-neighbourhoods, for example, $N^+(v_{11}) = N^+(v_{12})$. Deleting vertex v_{12}, together with its outgoing arcs, and then reconnecting its incoming arcs to vertex 11, we obtain a new digraph G_1 of order $n = M_{3,2} - 2$, as shown in Fig. 2(b).

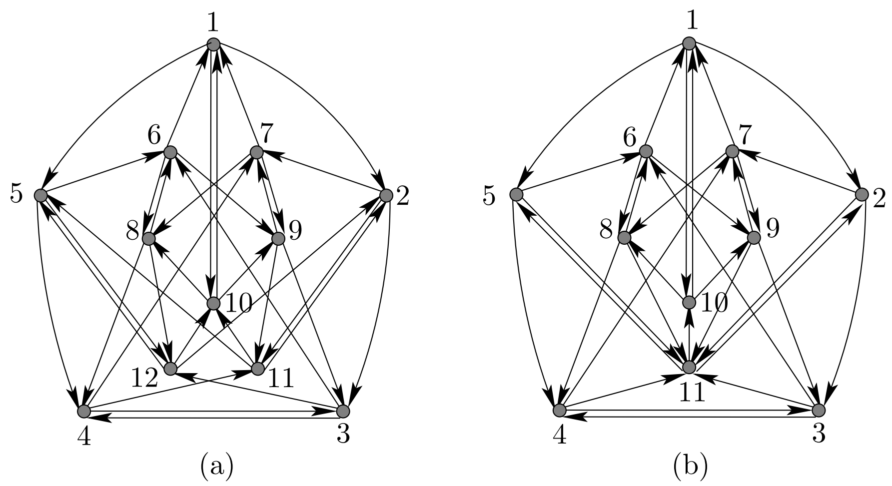

Fig. 2. Digraphs G of order 12 and G_1 of order 11.

We now introduce the notion of 'almost diregularity'. Throughout this paper, let S be the set of all vertices of G whose in-degree is less than d. Let S' be the set of all vertices of G whose in-degree is greater than d; and let σ^- be the *in-excess*, $\sigma^- = \sigma^-(G) = \sum_{w \in S'}(d^-(w) - d) = \sum_{v \in S}(d - d^-(v))$. Similarly, let R be the set of all vertices of G whose out-degree is less than d. Let R' be the set of all vertices of G whose out-degree is greater than d. We define the *out-excess*, $\sigma^+ = \sigma^+(G) = \sum_{w \in R'}(d^+(w) - d) = \sum_{v \in R}(d - d^+(v))$. A digraph of average in-degree d is called *almost in-regular* if the in-excess is at most equal to d. Similarly, a digraph of average out-degree d is called *almost out-regular* if the out-excess is at most equal to d. A digraph is *almost diregular* if it is almost in-regular and almost out-regular. Note that if $\sigma^- = 0$ (respectively, $\sigma^+ = 0$) then G is in-regular (respectively, out-regular). In this paper we prove that all digraphs of defect two, diameter $k \geq 3$ and out-degree $d \geq 2$ are out-regular and almost in-regular.

2 Results

Let G be a digraph of out-degree $d \geq 2$, diameter $k \geq 3$ and order $M_{d,k} - 2$. Since the order of G is $M_{d,k} - 2$, using a counting argument, it is easy to show that for each vertex u of G there exist exactly two vertices $r_1(u)$ and $r_2(u)$ (not necessarily distinct) in G with the property that there are two $u \to r_i(u)$ walks, for $i = 1, 2$, in G of length not exceeding k. The vertex $r_i(u)$, for each $i = 1, 2$, is called the *repeat* of u; this concept was introduced in [5].

We will use the following notation throughout. For each vertex u of a digraph G described above, and for $1 \leq s \leq k$, let $T_s^+(u)$ be the multiset of all endvertices of directed paths in G of length at most s which start at u. Similarly, by $T_s^-(u)$ we denote the multiset of all starting vertices of directed paths of length at most s in G which terminate at u. Observe that the vertex u is in both $T_s^+(u)$ and $T_s^-(u)$, as it corresponds to a path of zero length. Let $N_s^+(u)$ be the set of all endvertices of directed paths in G of length exactly s which start at u. Similarly, by $N_s^-(u)$ we denote the set of all starting vertices of directed paths of length exactly s in G which terminate at u. If $s = 1$, the sets $T_1^+(u) \setminus \{u\}$ and $T_1^-(u) \setminus \{u\}$ represent the out- and in-neighbourhoods of the vertex u in the digraph G; we denote these neighbourhoods simply by $N^+(u)$ and $N^-(u)$, respectively. We illustrate the notations $T_s^+(u)$ and $N_s^+(u)$ in Fig. 3.

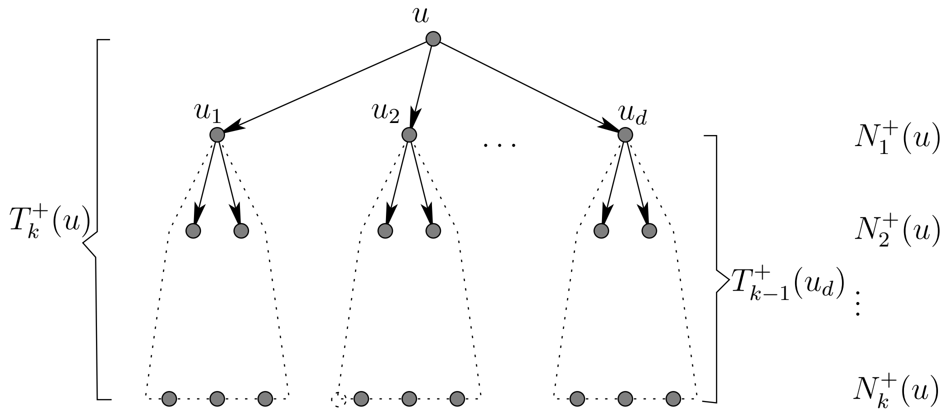

Fig. 3. Multiset $T_k^+(u)$

We will also use the following notation throughout.

Notation 1 *Let $\mathcal{G}(d,k,\delta)$ be the set of all digraphs of maximum out-degree d and diameter k and defect δ. The we refer to any digraph $G \in \mathcal{G}(d,k,\delta)$ as a (d,k,δ)-digraph.*

We will present our new results concerning the diregularity of digraphs of order close to Moore bound in the following sections.

2.1 Diregularity of $(d,k,2)$-digraphs

In this section we present a new result concerning the in-regularity of digraphs of defect two for any out-degree $d \geq 2$ and diameter $k \geq 3$. Let S be the set of all vertices of G whose in-degree is less than d. Let S' be the set of all vertices of G whose in-degree is greater than d; and let σ be the *in-excess*, $\sigma^- = \sum_{w \in S'}(d^-(w) - d) = \sum_{v \in S}(d - d^-(v))$.

Lemma 1 *Let $G \in \mathcal{G}(d,k,2)$. Let S be the set of all vertices of G whose in-degree is less than d. Then $S \subseteq N^+(r_1(u)) \cup N^+(r_2(u))$, for any vertex u.*

Proof. Let $v \in S$. Consider an arbitrary vertex $u \in V(G)$, $u \neq v$, and let $N^+(u) = \{u_1, u_2, ..., u_d\}$. Since the diameter of G is equal to k, the vertex v must occur in each of the sets $T_k^+(u_i)$, $i = 1, 2, ..., d$. It follows that for each i there exists a vertex $x_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $x_i v$ is an arc of G. Since the in-degree of v is less than d then the in-neighbours x_i of v are not all distinct. This implies that there exists some vertex which occurs at least twice in $T_k^+(u)$. Such a vertex must be a repeat of u. As G has defect 2, there are at most two vertices of G which are repeats of u, namely, $r_1(u)$ and $r_2(u)$. Therefore, $S \subseteq N^+(r_1(u)) \cup N^+(r_2(u))$. □

Combining Lemma 1 with the fact that every vertex in G has out-degree d gives

Corollary 1 $|S| \leq 2d$.

In principle, we might expect that the in-degree of $v \in S$ could attain any value between 1 and $d - 1$. However, the next lemma asserts that the in-degree cannot be less than $d - 1$.

Lemma 2 *Let $G \in \mathcal{G}(d, k, 2)$. If $v_1 \in S$ then $d^-(v_1) = d - 1$.*

Proof. Let $v_1 \in S$. Consider an arbitrary vertex $u \in V(G)$, $u \neq v_1$, and let $N^+(u) = \{u_1, u_2, ..., u_d\}$. Since the diameter of G is equal to k, the vertex v_1 must occur in each of the sets $T_k^+(u_i)$, $i = 1, 2, ..., d$. It follows that for each i there exists a vertex $x_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $x_i v_1$ is an arc of G. If $d^-(v_1) \leq d-3$ then there are at least three repeats of u, which is impossible. Suppose that $d^-(v_1) \leq d - 2$. By Lemma 1, the in-excess must satisfy

$$\sigma^- = \sum_{x \in S'} (d^-(x) - d) = \sum_{v_1 \in S} (d - d^-(v_1)) = |S| \leq 2d.$$

We now consider the number of vertices in the multiset $T_k^-(v_1)$. To reach v_1 from all the other vertices in G, the number of distinct vertices in $T_k^-(v_1)$ must be

$$|T_k^-(v_1)| \leq \sum_{t=0}^{k} |N_t^-(v)|. \tag{2}$$

To estimate the above sum we can observe the following inequality

$$|N_t^-(v)| \leq \sum_{u \in N_{t-1}^-(v)} d^-(u) = d|N_{t-1}^-(v)| + \varepsilon_t, \tag{3}$$

where $2 \leq t \leq k$ and $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma$. If $d^-(v_1) = d - 2$ then $|N^-(v_1)| = |N_1^-(v_1)| = d - 2$. It is not difficult to see that a safe upper bound on the sum of $|T_k^-(v_1)|$ is obtained from inequality (3) by setting $\varepsilon_2 = 2d$, and $\varepsilon_t = 0$ for $3 \le t \le k$. This gives

$$\begin{aligned}
|T_k^-(v_1)| &\le 1 + |N_1^-(v_1)| + |N_2^-(v_1)| + |N_3^-(v_1)| + \ldots + |N_k^-(v_1)| \\
&= 1 + (d-2) + (d(d-2) + \varepsilon_2) + (d(d(d-2) + \varepsilon_2) + \varepsilon_3) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&= 1 + (d-2) + (d(d-2) + 2d) + (d(d(d-2) + 2d) + 0) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&= 1 + d - 2 + d^2 + d^3(1 + d + \cdots + d^{k-3}) \\
&= M_{d,k} - 2.
\end{aligned}$$

Since $\varepsilon_2 = 2d$, $\varepsilon_t = 0$ for $3 \le t \le k$, and G contains a vertex of in-degree $d-2$ then $|S| = d$. Let $S = \{v_1, v_2, \ldots, v_d\}$. Every v_i, for $i = 2, 3, \ldots, d$, has to reach v_1 at distance at most k. Since v_1 and every v_i have exactly the same in-neighbourhood then v_1 is forced to be selfrepeat. This implies that v_1 occurs twice in the multiset $T_k^-(v_1)$. Hence $|T^-(v_1)| < M_{d,k} - 2$, which is a contradiction. Therefore $d^-(v_1) = d - 1$, for any $v_1 \in S$. □

Lemma 3 *If S is the set of all vertices of G whose in-degree is $d-1$ then $|S| \le d$.*

Proof. Suppose $|S| \ge d+1$. Then there exist $v_i \in S$ such that $d^-(v_i) = d-1$, for $i = 1, 2, \ldots, d+1$. The in-excess $\sigma^- = \sum_{v \in S}(d - d^-(v)) \ge d+1$. This implies that $|S'| \ge 1$. However, we cannot have $|S'| = 1$. Suppose, for a contradiction, $S' = \{x\}$. To reach v_1 (and v_i, $i = 2, 3, \ldots, d+1$) from all the other vertices in G, we must have $x \in \bigcap_{i=1}^{d+1} N^-(v_i)$, which is impossible as the out-degree of x is d. Hence $|S'| \ge 2$.

Let $u \in V(G)$ and $u \ne v_i$. To reach v_i from u, we must have $\bigcup_{i=1}^{d+1} N^-(v_i) \subseteq \{r_1(u), r_2(u)\}$. Since the out-degree is d then $|\bigcup_{i=1}^{d+1} N^-(v_i)| = d$. Let $r_1(u) = x_1$ and $r_2(u) = x_2$. Without loss of generality, we suppose $x_1 \in \bigcup_{i=1}^{d} N^-(v_i)$ and $x_2 \in N^-(v_{d+1})$. Now consider the multiset $T_k^+(x_1)$. Since every v_i, for $i = 1, 2, \ldots, d$, respectively, must reach $\{v_{j \ne i}\}$, for $j = 1, 2, \ldots, d+1$, within distance at most k, then x_1 occurs three times in $T_k^+(x_1)$, otherwise x_1 will have at least three repeats, which is impossible. This implies that x_1 is a double selfrepeat. Since two of v_i, say v_k and v_l, for $k, l \in \{1, 2, \ldots, d+1\}$, occur in the walk joining two selfrepeats then v_k and v_l are selfrepeats. Then it is not possible for the d out-neighbours of x_1 to reach v_{d+1}. □

Theorem 1 *For $d \geq 2$ and $k \geq 3$, every $(d,k,2)$-digraph is out-regular and almost in-regular.*

Proof. Out-regularity of $(d,k,2)$-digraphs was explained in the Introduction. Hence we only need to proof that every $(d,k,2)$-digraph is almost in-regular. If $S = \emptyset$ then $(d,k,2)$-digraph is diregular. By Lemma 2, if $S \neq \emptyset$ then all vertices in S have in-degree $d-1$. This gives

$$\sigma = \sum_{x \in S'} (d^-(x) - d) = \sum_{v \in S} (d - d^-(v)) = |S| \leq 2d.$$

Take an arbitrary vertex $v \in S$; then $|N^-(v)| = |N_1^-(v)| = d-1$. By the diameter assumption, the union of all the sets $N_t^-(v)$ for $0 \leq t \leq k$ is the entire vertex set $V(G)$ of G, which implies that

$$|V(G)| \leq \sum_{t=0}^{k} |N_t^-(v)|. \tag{4}$$

To estimate the above sum we can observe the following inequality

$$|N_t^-(v)| \leq \sum_{u \in N_{t-1}^-(v)} d^-(u) = d|N_{t-1}^-(v)| + \varepsilon_t, \tag{5}$$

where $2 \leq t \leq k$ and $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma$.

It is not difficult to see that a safe upper bound on the sum of $|V(G)|$ is obtained from inequality (5) by setting $\varepsilon_2 = \sigma = |S|$, and $\varepsilon_t = 0$, for $3 \leq t \leq k$; note that the latter is equivalent to assuming that *all* vertices from $S \setminus \{v\}$ are contained in $N_k^-(v)$ and that all vertices of S' belong to $N_1^-(v)$. This way we successively obtain:

$$\begin{aligned}
|V(G)| &\leq 1 + |N_1^-(v)| + |N_2^-(v)| + |N_3^-(v)| + \ldots + |N_k^-(v)| \\
&\leq 1 + (d-1) + (d(d-1) + |S|)(1 + d + \cdots + d^{k-2}) \\
&= d + d^2 + \cdots + d^k + (|S| - d)(1 + d + \cdots + d^{k-2}) \\
&= M_{d,k} - 2 + (|S| - d)(1 + d + \cdots + d^{k-2}) + 1.
\end{aligned}$$

But G is a digraph of order $M_{d,k} - 2$; this implies that

$$\begin{aligned}
(|S| - d)(1 + d + \cdots + d^{k-2}) + 1 &\geq 0 \\
(|S| - d)\frac{d^{k-1} - 1}{d-1} + 1 &\geq 0 \\
|S| &\geq d - \frac{d-1}{d^{k-1} - 1}
\end{aligned}$$

As $0 < \frac{d-1}{d^{k-1}-1} < 1$, whenever $k \geq 3$ and $d \geq 2$, it follows that $|S| \geq d$. Since $1 \leq |S| \leq d$. This implies $|S| = d$. □

We conclude with a conjecture.

Conjecture 1 *All digraphs of defect 2 are diregular for maximum out-degree $d \geq 2$ and diameter $k \geq 3$.*

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