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# Open Problems in the Construction of Large Directed Graphs

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**Abstract:** In this paper we consider the problem of how to construct directed graphs with given maximum out-degree and diameter. To deal with this problem, we describe several construction techniques. These fall into three broad categories, namely, algebraic specification, expansion and reduction methods.

**Key Words:** *Construction techniques, large order of directed graphs, degree, diameter.*

## 1. Introduction

One of the best known problems in extremal directed graphs is the so-called degree/diameter problem: For given numbers  $d$  and  $k$ , construct a digraph of maximum out-degree  $d$  and diameter  $\leq k$ , with the largest possible number of vertices  $n_{d,k}$ . In other words, find

$$N(d, k) = \max\{n : G(n, d, k) \neq \emptyset\}.$$

A straightforward general upper bound on the maximum order  $n_{d,k}$  of a directed graphs (digraph) of maximum degree  $d$  and diameter  $k$  is the *Moore bound*  $M_{d,k}$  for directed graphs.

$$\begin{aligned} n_{d,k} &\leq M_{d,k} = 1 + d + d^2 + \dots + d^k \\ &= \begin{cases} \frac{d^{k+1}-1}{d-1} & \text{if } d > 1 \\ k + 1 & \text{if } d = 1 \end{cases} \end{aligned}$$

The equality  $n_{d,k} = M_{d,k}$  holds only when  $d = 1$  or  $k = 1$  [19][17], and hence in all other cases the upper bound can be lowered by 1.

Case  $k = 2$ . It is well known that the bound  $M_{d,2} - 1$  can be achieved for all  $d \geq 2$  by line digraphs of complete digraphs of order  $d + 1$ . The existence of digraphs of  $M_{d,2} - 1$  vertices, other than the line digraphs, has been studied in [8][3][5].

Case  $k \geq 3$ . In general, it is not known whether or not  $M_{d,k} - 1$  is attainable. The existence of digraphs of degree  $d$ , diameter  $k \geq 3$  and order  $M_{d,k} - 1$  has been studied and several necessary conditions have been given in [3]–[5]. For degree  $d = 2$ , it was shown in [13] that  $M_{2,k} - 1$  is not attainable. For degree  $d = 3$ , it was proved that digraphs with  $M_{3,k} - 1$  nodes do not exist [7]. Moreover, it was shown in [15] that  $M_{2,k} - 2$  cannot be attained for  $k \geq 3$ .

Hence in many cases (depending on  $k$  and  $d$ ) the upper bound on  $n_{d,k}$  is actually 2 or 3 less than the Moore bound. But apart from that, no other upper bounds on  $n_{d,k}$  are known.

A general *lower* bound on the *largest order*  $n_{d,k}$  for the degree/diameter problem is given by Kautz digraphs  $K(d, k)$  [20] of order  $d^k + d^{k-1}$ ; these digraphs can be obtained by  $(k - 1)$ -fold iteration of the line digraph construction applied to the complete digraph of order  $d + 1$ . It is also known that  $n_{2,4} = 25$  (which implies that  $n_{2,j} \geq 25 \times 2^{j-4}$  for  $j \geq 4$  by the iterated line digraph construction); the corresponding digraph of order 25 was found by Alegre.

In view of the huge gap between the Moore bound and the best lower bound, much effort has been spent in generating large digraphs of given degree and diameter. In the next section we give an overview of construction techniques for large digraphs. Finally, in the last section we list open problems in the degree/diameter problem area for digraphs.

## 2. Construction Techniques for Large Directed Graphs

In general, Slamin [18] classified the techniques of constructing large digraphs into three classes.

- **Algebraic specification.** By algebraic specification we mean that a digraph is obtained by using a construction technique specified by some algebraic formula. Construction techniques that can be classified as algebraic specifications include *generalised de Bruijn digraphs* and *generalised Kautz digraphs*.
- **Expansion method.** By expansion method we mean that a new digraph is obtained from another digraph of smaller order according to some specified rules. In this way, we start from a base digraph, then follow the procedure to obtain a new digraph with order larger than that of the original digraph. The construction techniques that can be classified as expansion methods are *line digraphs* and *partial line digraphs*, *de Bruijn* and *Kautz digraphs on alphabets* and *voltage assignments*.
- **Reduction method.** Using a reduction method, we start from a digraph then follow some procedure to obtain a new digraph with order smaller than that of the original digraph. The construction techniques that can be classified as reduction methods are *digon reduction* and *vertex deletion scheme*.

### 2.1 Generalised de Bruijn and Kautz Digraphs

Imase and Itoh [11] constructed digraphs for given arbitrary order  $n$  and out-degree  $d$ ,  $1 < d < n$ , by the following procedure. Let the vertices of digraphs be labeled by  $0, 1, \dots, n - 1$ . A vertex  $u$  is adjacent to  $v$ , if

$$v \equiv du + i \pmod{n}, i = 0, 1, \dots, d - 1.$$

For example, Figure 1 shows the digraph of order  $n = 9$ , out-degree  $d = 3$  and diameter  $k = 2$ , obtained from this construction. Note that when  $n = d^k$ , the digraphs obtained from this construction are isomorphic to the de Bruijn digraphs of degree  $d$  and diameter  $k$ .

Miller [12] gave a construction technique which is generalised Kautz digraph for given arbitrary order  $n$  and out-degree  $d$ , for  $1 < d < n$ . The technique is using the following procedure. Let the vertices of digraphs be labeled by  $0, 1, \dots, n - 1$ . A vertex  $u$  is adjacent to  $v$ , if

$$v \equiv -du + i \pmod{n}, i = 0, 1, \dots, d - 1.$$

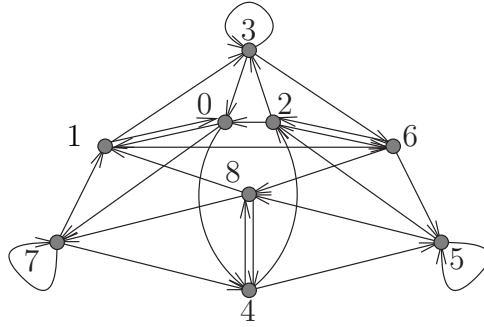


Figure 1: Generalised de Bruijn digraphs  $G \in \mathcal{G}(9, 3, 2)$ .

The digraph in Figure 2 is obtained from the construction for  $n = 9$  and  $d = 2$ . Note that when  $n = d^k + d^{k-1}$ , the digraphs obtained from this construction are isomorphic to the Kautz digraphs of degree  $d$  and diameter  $k$ .

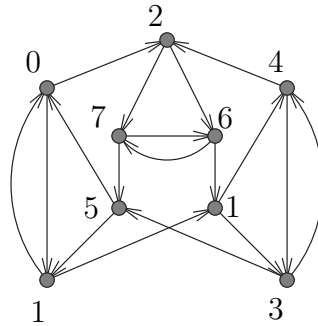


Figure 2: Generalised Kautz digraphs  $G \in \mathcal{G}(9, 2, 3)$ .

## 2.2 Line Digraphs and Partial Line Digraphs

Let  $G = (V, A)$  and let  $N$  be the multiset of all walks of length two in  $G$ . The *line digraph* of a digraph  $G$ ,  $L(G) = (A, N)$ , that is, the set of vertices of  $L(G)$  is equal to the set of arcs of  $G$  and the set of arcs of  $L(G)$  is equal to the set of walks of length two in  $G$ . This means that a vertex  $uv$  of  $L(G)$  is adjacent to a vertex  $wx$  if and only if  $v = w$ .

The order of the line digraph  $L(G)$  is equal to the number of arcs in the digraph  $G$ . For a diregular digraph  $G$  of out-degree  $d \geq 2$ , the sequence of line digraph iterations  $L(G), L^2(G) = L(L(G)), \dots, L^i(G) = L(L^{i-1}(G)), \dots$  is an infinite sequence of diregular digraphs of degree  $d$ . Figure 3 shows an example of a digraph and its line digraph.

Line digraphs do not have expandability properties since the order of resulting digraph comes out with a composite number. To resolve this drawback of the line digraph technique, Fiol and Llado [10] presented a revised technique, namely, the partial line digraph.

Let  $G$  be a digraph with vertex-set  $V$  and arc-set  $A$ . Let  $A' \subset A$  be a subset of arcs which are incident to all vertices of  $G$ . A digraph  $\mathcal{L}G$  is said to be a *partial line digraph* of  $G$  if its vertices represent the arcs of  $A'$  and a vertex  $uv$  is adjacent to the vertices  $v'w$ , for each

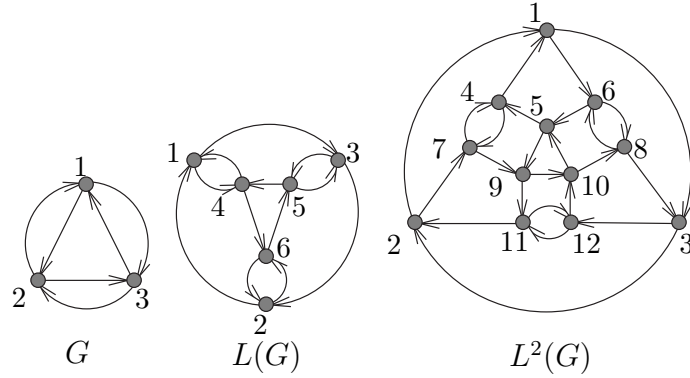


Figure 3: Digraphs  $G \in \mathcal{G}(3, 2, 1)$  with its line digraphs  $L(G) \in \mathcal{G}(6, 2, 2)$  and  $L^2(G) \in \mathcal{G}(12, 2, 3)$ .

$w \in N^+(v)$  in  $G$ , where

$$v' = \begin{cases} v, & \text{if } vw \in V(\mathcal{L}G) \\ v'', \text{ for any } v'' \in N^-(w) \text{ in } G \\ \text{such that } v''w \in V(\mathcal{L}G) & \text{otherwise} \end{cases}$$

For example, Figure 4(a) shows the digraph  $G$  with a vertex-set  $V = \{0, 1, 2, 3, 4, 5\}$  and an arc-set

$$A = \{01, 03, 12, 15, 20, 23, 34, 35, 40, 41, 52, 54\}.$$

Let  $A' = \{01, 03, 12, 15, 20, 23, 34, 35, 41, 52, 54\}$  (shown as solid lines in Figure 4(a)) is a subset of  $A$ . Then a partial line digraph  $\mathcal{L}G$  of the digraph  $G$  has the vertex-set

$$V(\mathcal{L}G) = \{01, 03, 12, 15, 20, 23, 34, 35, 41, 52, 54\}$$

as shown in Figure 4(b). In this example, vertices 34 and 54 are adjacent to vertex 20 because the arc  $40 \notin A'$ .

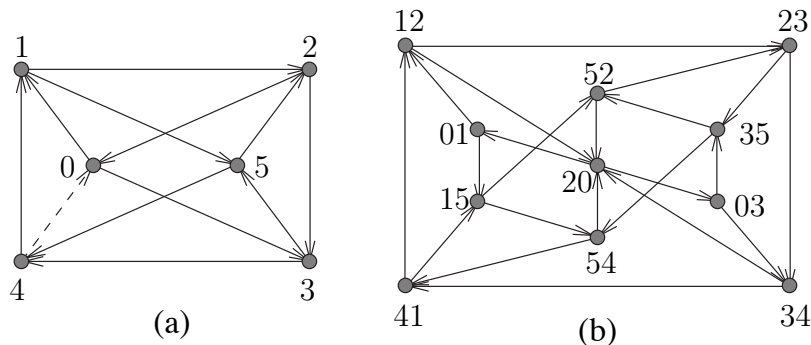


Figure 4: Digraph  $G$  and one of its partial line digraphs.

### 2.3 Generalised de Bruijn and Kautz Digraphs on Alphabets

Fiol, Llado and Villar [9] constructed generalisations of de Bruijn digraphs using their representation as digraphs on alphabets, that is, digraphs whose vertices are represented by words from a given alphabet and whose arcs are defined by an adjacency rule that relates pairs of words. Let  $Br(d, k)$  be a de Bruijn digraph of degree  $d$  and diameter  $k$ , then  $Br(d, k)$  has the set of vertices  $V(Br) = \Omega^k$  where  $\Omega = \{x | x \in \text{words}\}$  and  $|\Omega| = d$  and the adjacency: a vertex  $x_1x_2 \dots x_k$  adjacent to the vertices  $x_2 \dots x_kx_{k+1}$  for  $x_{k+1} \in \Omega$ .

The generalisation of Kautz digraphs was defined as follows. Consider the vertex-set  $V(Ka(d, k))$  of Kautz sequences or words of length  $k$  without consecutive letters on an alphabet  $X$ ,  $|X| = d + 1$ . Let  $u = x_1x_2 \dots x_k \in V(Ka(d, k))$  and  $\bar{u}$  be a Kautz sequence obtained from  $u$  by deleting the first letter of  $u$ , that is,  $\bar{u} = x_2x_3 \dots x_k \in V(Ka(d, k - 1))$ . Let  $n$  be any integer such that  $d^{k-1} + d^{k-2} < n \leq d^k + d^{k-1}$ . A digraph  $H(d, k, n)$  has vertex-set  $V \subset V(Ka(d, k))$ , such that  $\{\bar{u} | u \in V\} = V(Ka(d, k - 1))$  and a vertex  $u = x_1x_2 \dots x_k$  is adjacent to the vertices  $v = x'_2x_3 \dots x_k\alpha$  for every  $\alpha \in X$ ,  $\alpha \neq x_k$ , where

$$x'_2 = \begin{cases} x_2, & \text{if } x_2x_3 \dots x_k\alpha \in V \\ x''_2, & \text{for any } x''_2 \text{ such that} \\ & x''_2x_3 \dots x_k\alpha \in V, & \text{otherwise} \end{cases}$$

Figure 5 shows the digraph  $H(2, 3, 10)$  with vertex-set

$$\{010, 012, 020, 021, 101, 102, 120, 121, 201, 210\}.$$

In this example, vertices 020 and 120 are adjacent to vertex 102, vertices 021 and 121 are adjacent to vertex 012 because vertices 202, 212  $\notin V$ .

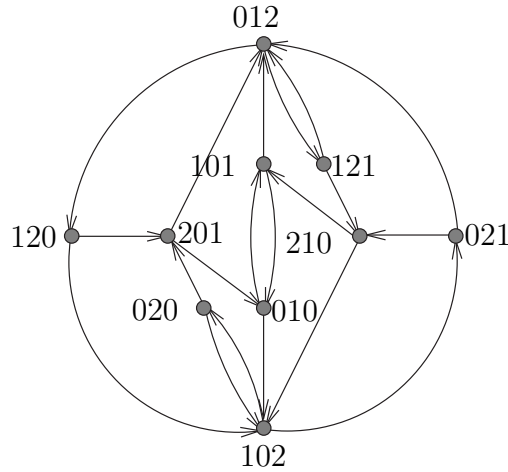


Figure 5: Example of generalised Kautz digraph on alphabet.

## 2.4 Voltage Assignments

Baskoro, Branković, Miller, Plesník, Ryan and Širáň [2] introduced the use of voltage assignments to construct large digraphs as described below. Let  $G$  be a digraph and  $A(G)$  be the set of arcs of  $G$ . Let  $\Gamma$  be an arbitrary group. Any mapping  $\alpha : A(G) \rightarrow \Gamma$  is called a *voltage assignment*. The *lift* of  $G$  by  $\alpha$ , denoted by  $G^\alpha$ , is the digraph defined as follows: the vertex-set  $V(G^\alpha) = V(G) \times \Gamma$ , the arc-set  $A(G^\alpha) = A(G) \times \Gamma$ , and there is an arc  $(a, f)$  in  $G^\alpha$  from  $(u, g)$  to  $(v, h)$  if and only if  $f = g$ ,  $a$  is an arc from  $u$  to  $v$ , and  $h = g\alpha(a)$ .

For example, Figure 6 shows a digraph  $G$  and its lift  $G^\alpha$  with  $\Gamma = Z_6$  and the voltage assignment  $\alpha(a) = \alpha(d) = 5, \alpha(b) = 0, \alpha(c) = \alpha(f) = 1$  and  $\alpha(e) = 2$ . Informally, voltage assignment technique enables us to “blow up” a given base digraph  $G$  in order to obtain a larger digraph (called a “lift”) whose incidence structure depends on both  $G$  and a mapping (“voltage assignment”) from the edge set of  $G$  into a finite group. Since the lift is completely determined in terms of the original *base digraph*  $G$  and the voltage assignment  $\alpha$ , this type of construction is suitable for handling large digraphs in terms of the properties of the base digraph and the assignment.

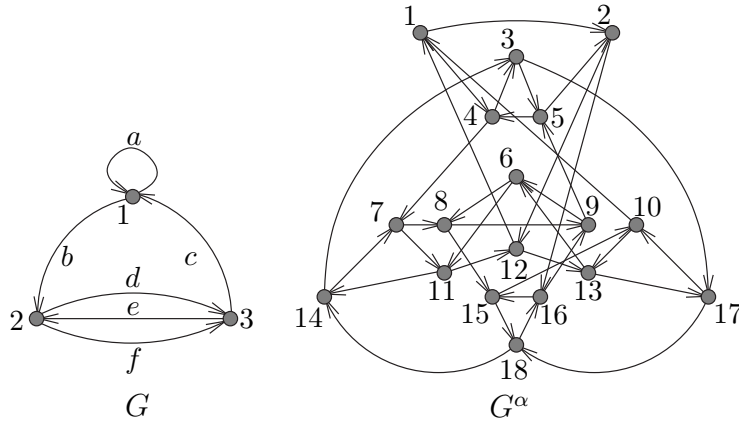


Figure 6: Digraph  $G$  and its lift  $G^\alpha$ .

## 2.5 Digon Reduction and Vertex Deletion Scheme

Miller and Fris [14] gave a construction technique for digraphs of degree 2 with minimum diameter using digon reduction scheme which they combined with line digraph iterations. Let  $G \in \mathcal{G}(n, d, k)$  be a digraph of order  $n$ , out-degree  $d = 2$  and diameter  $k$  which contains  $p$  digons. Then  $G' \in \mathcal{G}(n - 1, d, k')$ , for  $k' \leq k$ , can be obtained from  $G$  by ‘gluing’ two vertices which share a digon. Figure 7 shows the digraph  $G'$  which is obtained from digraph  $G$  by gluing two vertices  $y$  and  $z$ .

Miller and Slamin [16] also constructed digraphs using vertex deletion scheme. Suppose that  $N^+(u) = N^+(v)$  for any vertex  $u, v \in G$ . Let  $G_1$  be a digraph deduced from  $G$  by deleting vertex  $u$  together with its outgoing arcs and reconnecting the incoming arcs of  $u$  to the vertex  $v$ . Obviously, the new digraph  $G_1$  has maximum out-degree the same as the maximum out-degree of  $G$ . It can be proved that the diameter of  $G_1$  is at most  $k$ .



Figure 8(a) shows an example of digraph  $G \in \mathcal{G}(12, 2, 3)$  with the property that some vertices have identical out-neighbourhoods. For example,  $N^+(7) = N^+(11)$ . Deleting vertex 12 together with its outgoing arcs and then reconnecting its incoming arcs to vertex 7 (since  $N^+(7) = N^+(11)$ ), we obtain a new digraph  $G_1 \in \mathcal{G}(11, 2, 2)$  as shown in Figure 8(b).

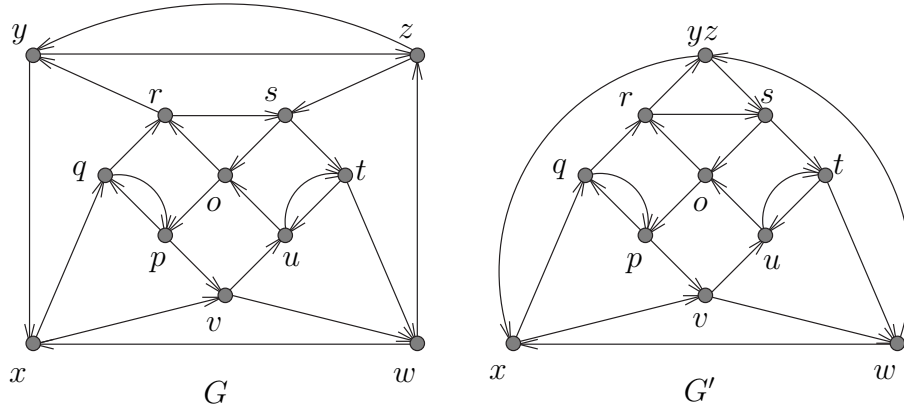


Figure 7: Digraphs  $G \in \mathcal{G}(12, 2, 3)$  and  $G' \in \mathcal{G}(11, 2, 3)$ .

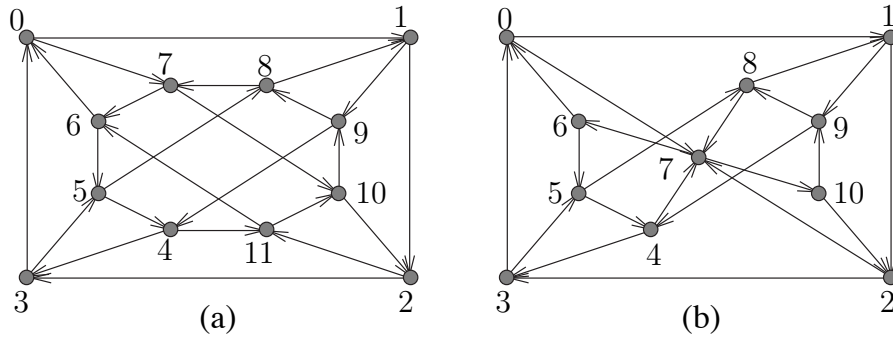


Figure 8: Digraph  $G \in \mathcal{G}(12, 2, 3)$  and  $G_1 \in \mathcal{G}(11, 2, 3)$  obtained from  $G$ .

### 3. Open Problems

A frequent goal in constructions of large digraphs with given properties is to keep the diameter as small as possible. The *voltage assignment technique* seems a very promising technique for finding large digraphs. Preliminary results regarding the voltage assignment technique are extremely encouraging, see [6] and [1]. In [6] the authors presented an upper bound on the diameter of lifted digraphs, applicable to a fairly general class of base digraphs and general groups.

Table 1-4 summarise our knowledge of best current lower bound  $N_l(d, k)$  and upper bound  $N_u(d, k)$  for  $2 \leq k \leq 10$  and  $d = 2, 3, 4$  and  $5$ . The tables also show the range of the potential larger orders  $n_{d,k}$ . The huge gap between the upper bound and the best lower bound strongly represent an open problem.

Using the voltage assignment technique or some other established technique, possibly with the aid of computers, there is a good chance that a new digraphs with larger order can be found.

$k$	Lower bound $N_l(2, k)$	Upper bound $N_u(2, k)$	Potential Larger Order
2	6	6	—
3	12	12	—
4	25	28	26, 27, 28
5	50	60	$n_{2,k} = N_l(2, k) + i, i = 1 \dots 10$
6	100	124	$n_{2,k} = N_l(2, k) + i, i = 1 \dots 24$
7	200	252	$n_{2,k} = N_l(2, k) + i, i = 1 \dots 52$
8	400	508	$n_{2,k} = N_l(2, k) + i, i = 1 \dots 108$
9	800	1,020	$n_{2,k} = N_l(2, k) + i, i = 1 \dots 220$
10	1,600	2,044	$n_{2,k} = N_l(2, k) + i, i = 1 \dots 444$

Table 1: Potential orders  $n_{d,k}$  for out-degree  $d = 2$ .

$k$	Lower bound $N_l(3, k)$	Upper bound $N_u(3, k)$	Potential Larger Order
2	12	12	—
3	36	38	37, 38
4	108	119	$n_{3,k} = N_l(3, k) + i, i = 1 \dots 11$
5	324	362	$n_{3,k} = N_l(3, k) + i, i = 1 \dots 38$
6	972	1,091	$n_{3,k} = N_l(3, k) + i, i = 1 \dots 119$
7	2,916	3,278	$n_{3,k} = N_l(3, k) + i, i = 1 \dots 362$
8	8,748	9,839	$n_{3,k} = N_l(3, k) + i, i = 1 \dots 1,091$
9	26,244	29,522	$n_{3,k} = N_l(3, k) + i, i = 1 \dots 3,278$
10	78,732	88,571	$n_{3,k} = N_l(3, k) + i, i = 1 \dots 9,839$

Table 2: Potential orders  $n_{d,k}$  for out-degree  $d = 3$ .

$k$	Lower bound $N_l(4, k)$	Upper bound $N_u(4, k)$	Potential Larger Order
2	20	20	—
3	80	84	81, 82, 83, 84
4	320	340	$n_{4,k} = N_l(4, k) + i, i = 1 \dots 20$
5	1,280	1,364	$n_{4,k} = N_l(4, k) + i, i = 1 \dots 84$
6	5,120	5,460	$n_{4,k} = N_l(4, k) + i, i = 1 \dots 340$
7	20,480	21,844	$n_{4,k} = N_l(4, k) + i, i = 1 \dots 1,364$
8	81,920	87,380	$n_{4,k} = N_l(4, k) + i, i = 1 \dots 5,460$
9	327,680	349,524	$n_{4,k} = N_l(4, k) + i, i = 1 \dots 21,844$
10	1,310,720	1,398,100	$n_{4,k} = N_l(4, k) + i, i = 1 \dots 87,380$

Table 3: Potential orders  $n_{d,k}$  for out-degree  $d = 4$ .

$k$	Lower bound $N_l(5, k)$	Upper bound $N_u(5, k)$	Potential Larger Order
2	30	30	—
3	150	155	151, 152, 153, 154, 155
4	750	780	$n_{5,k} = N_l(5, k) + i, i = 1 \dots 30$
5	3,750	3,905	$n_{5,k} = N_l(5, k) + i, i = 1 \dots 155$
6	18,750	19,530	$n_{5,k} = N_l(5, k) + i, i = 1 \dots 780$
7	93,750	97,655	$n_{5,k} = N_l(5, k) + i, i = 1 \dots 3,905$
8	468,750	488,280	$n_{5,k} = N_l(5, k) + i, i = 1 \dots 19,530$
9	2,343,750	2,441,405	$n_{5,k} = N_l(5, k) + i, i = 1 \dots 97,655$
10	11,718,750	12,207,030	$n_{5,k} = N_l(5, k) + i, i = 1 \dots 488,280$

Table 4: Potential orders  $n_{d,k}$  for out-degree  $d = 5$ .

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