Heliyon 6 (2020) e03843

Contents lists available at ScienceDirect

Heliyon

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Research article

Subdivision of graphs in $\mathcal{R}(mK_2, P_4)$

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ARTICLE INFO

Keywords:
Mathematics
Ramsey minimal graphs
Red-blue coloring
Matching
Path

ABSTRACT

For any graphs F, G, and H, the notation $F \to (G, H)$ means that any red-blue coloring of all edges of F will contain either a red copy of G or a blue copy of H. The set $\mathcal{R}(G, H)$ consists of all Ramsey (G, H)-minimal graphs, namely all graphs F satisfying $F \to (G, H)$ but for each $e \in E(F)$, $(F - e) \nrightarrow (G, H)$. In this paper, we propose a simple construction for creating new Ramsey minimal graphs from the previous known Ramsey minimal graphs (by subdivision operation). In particular, suppose $F \in \mathcal{R}(mK_2, P_4)$ and let $e \in E(F)$ be an edge contained in a cycle of F, we construct a new Ramsey minimal graph in $\mathcal{R}((m+1)K_2, P_4)$ from graph F by subdividing the edge e four times.

1. Introduction

Let F, G, and H be simple graphs. Write $F \to (G, H)$ to mean that for any red-blue coloring of all edges of F there exists a red copy of G or a blue copy of H as a subgraph of F. A (G, H)-*coloring* of F is a red-blue coloring of F such that neither a red G nor a blue H occurs. A graph F will be called a *Ramsey* (G, H)-*minimal* if $F \to (G, H)$ but for each $e \in E(F)$, there exists a (G, H)-coloring of a graph $F - e$. The set of all Ramsey (G, H)-minimal graphs will be denoted by $\mathcal{R}(G, H)$.

The characterization of all graphs F in $\mathcal{R}(G, H)$ for a fixed pair of graphs G and H is an interesting but difficult problem. Even, it is for small graphs G and H. Burr et al. [1] showed that the problem of deciding whether a graph F is a Ramsey (G, H)-minimal graph is NP-complete for any fixed 3-connected graphs G and H. Numerous papers discuss the problem of determining the members of the set $\mathcal{R}(G, H)$. In particular, Burr et al. [2] proved that if G is a matching ($G = mK_2$), then the set $\mathcal{R}(mK_2, H)$ is finite for any graph H. One of the problems of Ramsey minimal graphs is characterizing graphs belonging to the set $\mathcal{R}(mK_2, H)$ for some classes of a graph H. For instance, the characterization of Ramsey minimal graphs belonging to $\mathcal{R}(3K_2, K_3)$ can be seen in [3]; $\mathcal{R}(2K_2, K_4)$ can be seen in [4, 5]. The set $\mathcal{R}(2K_2, P_3)$ is given by Mengersen and Oeckermann [6]. Furthermore, the set $\mathcal{R}(3K_2, P_3)$ is given by Burr et al. [2] (without proof) and by Mushi and Baskoro [7] (with a proof). Next, Wijaya et al. [8] determined all graphs in $\mathcal{R}(4K_2, P_3)$. Moreover, Baskoro and Yulianti [9] characterized all graphs in $\mathcal{R}(2K_2, P_4)$ and $\mathcal{R}(2K_2, P_5)$.

In 2016, Wijaya and Baskoro [10] constructed some Ramsey $(2K_2, 2H)$-minimal graphs by using some operations over graphs in $\mathcal{R}(2K_2, H)$ for H is a cycle, path, or star. Recently, Wijaya et al. [11] determined all unicyclic graphs in $\mathcal{R}(mK_2, P_3)$ for each integer $m > 1$. Most recently, Wijaya et al. [12] derived the necessary and sufficient conditions for all graphs belonging to $\mathcal{R}(mK_2, H)$, for any integer $m > 1$. They also proved that any graph obtained by subdividing one non-pendant edge in $F(\in \mathcal{R}(mK_2, P_3))$ will be in $\mathcal{R}((m+1)K_2, P_3)$. They also showed the following lemma.

Lemma 1. *Let H be a connected graph and m be a positive integer. Suppose $F \in \mathcal{R}(mK_2, H)$. For each $e \in E(F)$, let τ be an (mK_2, H)-coloring of edges of $F - e$. Then, there exists a red $(m-1)K_2$ in $F - e$.*

Motivated by subdividing one non-pendant edge of a Ramsey (mK_2, P_3)-minimal graph by Wijaya et al. [12], in this paper, our aim is to prove that if $F \in \mathcal{R}(mK_2, P_4)$, then any graph obtained by subdividing one edge contained in a cycle of F (four times) will be in $\mathcal{R}((m+1)K_2, P_4)$.

2. Subdivision graphs

The *subdivision* (k *vertices*) of a graph G on the edge $e = uv$ in $E(G)$, denoted by $SG(e, k)$, is a graph obtained from the graph G by removing the edge e and adding k new vertices $w_1, w_2, \ldots, w_k$ and $(k+1)$ new edges $uw_1, w_1w_2, w_2w_3, \ldots,$ $w_{k-1}w_k, w_kv$. Therefore, $SG(e, k)$ has

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https://doi.org/10.1016/j.heliyon.2020.e03843
Received 26 November 2019; Received in revised form 30 January 2020; Accepted 20 April 2020

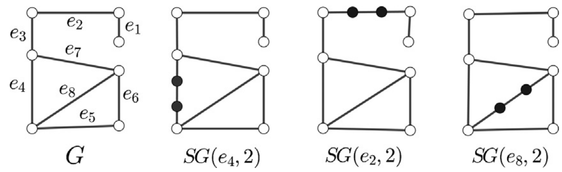

Fig. 1. The graphs $G, SG(e_4, 2), SG(e_2, 2)$, and $SG(e_8, 2)$, respectively.

the vertex set $V(SG(e,k)) = V(G) \cup \{w_1, w_2, \ldots, w_k\}$ and the edge set $E(SG(e,k)) = E(G-e) \cup \{uw_1, w_1w_2, \ldots, w_{k-1}w_k, w_kv\}$ Henceforth, the edge e in the notation $SG(e,k)$ will be called the *subdivision edge*. For example, consider a graph G as depicted in Fig. 1. Some subdivision (2 vertices, black vertex) of the graph G on the edge e_4 or e_2 or e_8 can be seen, respectively, in Fig. 1. We can see that the subdivision graphs $SG(e_1, 2)$, $SG(e_2, 2)$, and $SG(e_3, 2)$ are isomorphic.

Let F be a Ramsey (mK_2, P_4)-minimal graph for the pair matching mK_2 and a path on 4 vertices P_4. Let e be an edge in $E(F)$. From now on, we use the notation τ_e as an (mK_2, P_4)-coloring of $F-e$, namely the red-blue coloring of edges of a graph $F-e$ such that there is neither a red mK_2 nor a blue P_4. According to Lemma 1, under the coloring τ_e, there exists a red $(m-1)K_2$ in a graph $F-e$. Since $F \in \mathcal{R}(mK_2, P_4)$, if we return the edge e to a graph F, then e can have either a red or a blue color. If the edge e has a red color, then clearly there exists a red mK_2 on a graph F, while if it has a blue color, then there exists a blue path P_4 on a graph F. The next lemma discusses the property of the existence of a blue path P_4 in a graph $F \in \mathcal{R}(mK_2, P_4)$.

Lemma 2. *Let $m \geq 2$ be an integer and $F \in \mathcal{R}(mK_2, P_4)$. Then, for any $e \in E(F)$, there exists a red-blue coloring of F having no red mK_2 and the edge e satisfies one of the following four conditions:*

(i) *e is any edge of exactly one blue path P_4,*
(ii) *e is the middle edge of more than one blue path P_4 (there is no blue path P_5 in this case)*
(iii) *e is one of the middle edges of one or more than one blue path P_5 (there is no blue path P_6 in this case), or*
(iv) *e is the middle edge of one or more than one blue path P_6.*

Note: more than one blue path P_t for $t \in [4,6]$ in this Lemma are not independent; they have one or more than one edge together.

Proof. Let F be a Ramsey (mK_2, P_4)-minimal graph. Suppose $e \in E(F)$. Then, there exists an (mK_2, P_4)-coloring τ_e of $F-e$. Under the coloring τ_e, there does not exist a red mK_2 of a graph $F-e$. Now, define a new coloring τ of a graph F such that

$$\tau(x) = \begin{cases} \text{blue}, & \text{for } x = e, \\ \tau_e(x), & \text{for else.} \end{cases}$$

Then, under the coloring τ, there does not exist a red mK_2 of a graph F. Meanwhile, the edge e is contained in a blue path P_4, otherwise $F \nrightarrow (mK_2, P_4)$. Furthermore, we prove that the edge e is contained in a blue path P_t for some $t \in [4,6]$. If we assume that the edge e is contained in a blue path P_t, for each $t \geq 7$, then deleting the edge e from this path yields a blue P_4 in $F-e$ (under the coloring τ_e). So, $(F-e) \rightarrow (mK_2, P_4)$, a contradiction. Therefore, the edge e must be contained in a blue path P_t, for some $t \in [4,6]$. Next, since the path P_4 is a subgraph of both P_5 and P_6, it is easily verified the edge e satisfies one of the four conditions above. □

As an illustration, the four conditions of the edge e can be depicted in Fig. 2. All graphs in Fig. 2 are the only blue subgraphs of F containing a blue P_4, under coloring τ. Deleting the edge e of a graph F remains an (mK_2, P_4)-coloring τ_e of all edges of $F-e$.

Let F be a connected graph and e be an edge of F. We can see that there are two conditions about the edge e, as below.

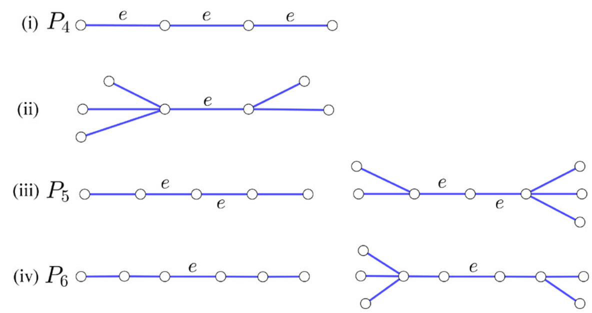

Fig. 2. The four conditions of the edge e in Lemma 2.

(i) The edge e is not contained in any cycle of F.
Then $SF(e,4) \supseteq (F \cup P_4)$. If $F \in \mathcal{R}(mK_2, P_4)$ then $F \cup P_4 \in \mathcal{R}((m+1)K_2, P_4)$ [12]. Hence, for each e is not contained in any cycle of F, if $F \in \mathcal{R}(mK_2, P_4)$ then $SF(e,4) \notin \mathcal{R}((m+1)K_2, P_4)$.
(ii) The edge e is contained in a cycle of F.
The graph $SF(e,4)$ is not contained $F \cup P_4$. That is why, for this case, we shall prove that if $F \in \mathcal{R}(mK_2, P_4)$ then $SF(e,4) \in \mathcal{R}((m+1)K_2, P_4)$ for each edge e is contained in a cycle of F, in theorem below.

Before doing this, we define the set $SF(4)$. Let F be a connected graph and e be an edge in a cycle of F. Let $SF(4) = \{SF(e,4) \mid e \in E(F)$ and e is an edge contained in a cycle of $F\}$ be the set of all graphs $SF(e,4)$ for all edges contained in a cycle of F. For example, $SG(4) = \{SG(e_2, 2), SG(e_4, 2), SG(e_5, 2), SG(e_8, 2)\}$ of a graph G as depicted in Fig. 1.

Theorem 3. *Let F be a connected graph and $m \geq 2$ be an integer. Suppose α is an edge contained in a cycle of F. If $F \in \mathcal{R}(mK_2, P_4)$, then $SF(\alpha, 4) \in \mathcal{R}((m+1)K_2, P_4)$. Consequently, $SF(4) \subseteq \mathcal{R}((m+1)K_2, P_4)$.*

Proof. Let $F \in \mathcal{R}(mK_2, P_4)$ be a connected graph and $\alpha \in E(F)$ be an edge contained in a cycle of F. We shall prove that $SF(\alpha, 4) \in \mathcal{R}((m+1)K_2, P_4)$. Let $E(SF(\alpha,4)) = E(F-\alpha) \cup \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be the edge set of $SF(\alpha,4)$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the five new consecutive edges of the *subdivision* (4 *vertices*) of the graph F on the edge α, $SF(\alpha, 4)$.

First, suppose to the contrary, that $SF(\alpha,4) \nrightarrow ((m+1)K_2, P_4)$. It means that there exists an $((m+1)K_2, P_4)$-coloring τ of $SF(\alpha,4)$. Under coloring τ, the graph $SF(\alpha,4)$ contains at most m independent red edges, where one or two red edges originated from the five new edges $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$.

- If one of the edges $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and α_5 provides one red independent edge, then the number of the disjoint red edges of $F-\alpha$ is exactly $m-1$. We now replace the edges $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ with the edge α and color α by blue. Then, we obtain a graph isomorphic to F containing a red $(m-1)K_2$ but no blue P_4. It means that F has an (mK_2, P_4)-coloring. The last statement contradicts the fact that $F \rightarrow (mK_2, P_4)$.
- If the edges $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and α_5 provide two independent red edges (red $2K_2$), then the number of the independent red edges of $F-\alpha$ is exactly $m-2$. Now, we replace the edges $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and α_5 by the edge α and color α by red. Then, we obtain a graph isomorphic to F containing a red $(m-1)K_2$ but no blue P_4, which contradicts $F \rightarrow (mK_2, P_4)$.

Therefore, from two cases above, we conclude that $SF(\alpha,4) \rightarrow ((m+1)K_2, P_4)$.

Next, we show that $SF(\alpha,4)$ is minimal. It means that for every $e \in E(SF(\alpha, 4))$, there exists an $((m+1)K_2, P_4)$-coloring of $SF(\alpha,4) - e$. We consider two cases, namely (i) $e \in E(F)$ and $e \neq \alpha$ and (ii) $e \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. First, for every $e \in E(F)$, there exists an (mK_2, P_4)-

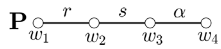

Fig. 3. A path **P** of length 3 in F containing the edge α.

coloring τ_e of $F-e$. Let $\alpha \in E(F-e)$ be the subdivision edge. So, the edge α becomes the edges $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ in $E(SF(\alpha,4))$. Under the coloring τ_e, the color of edge α can have either a red or blue color. Now, define a coloring τ of $SF(\alpha,4)-e$ such that $\tau(x)=\tau_e(x)$ for each $x \in E(F-\{e,\alpha\})$, and assign color to the five new edges $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, and α_5 depending the color of α under τ_e of the graph $F-e$ as follows.

- If $\tau_e(\alpha)=$ red, then color the edges $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ by either red, blue, blue, red, red, respectively, if α_1 is adjacent to a red edge of F, or red, red, blue, blue, red, respectively, if α_5 is adjacent to a red edge of F. Otherwise, if both α_1 and α_5 are adjacent to a blue edge of F, then color the edges $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ by red, blue, blue, red, red, respectively. In this case, the red edge α_1 displaces the red edge α. That is why the coloring of the five new edges donates one independent red edge. So, τ is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-e$.
- If $\tau_e(\alpha)=$ blue, then the only one vertex, which is incident with the edge α, will be incident with a blue edge. Otherwise, F will not be minimal. Furthermore, color the edges $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ by blue, red, red, blue, blue, respectively if α_5 is adjacent to a red edge of F, and color by blue, blue, red, red, blue, respectively, if α_1 is adjacent to a red edge of F. In this case, the five new edges only contribute to one independent red edge. Hence, τ is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-e$.

Now, consider the case if $e \in \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$. By symmetry, it is enough to consider if e is either α_1,α_2, or α_3.

(1) Case of $e=\alpha_1$. Then, α_2 is a pendant edge of $SF(\alpha,4)-\alpha_1$. Let τ_α be an (mK_2,P_4)-coloring of $F-\alpha$. Now, define τ_{α_1} as a red-blue coloring of edges of $SF(\alpha,4)-\alpha_1$ such that

$$\tau_{\alpha_1}(x)=\begin{cases}\text{red}, & \text{for } x=\alpha_4,\alpha_5,\\ \text{blue}, & \text{for } x=\alpha_2,\alpha_3,\\ \tau_\alpha(x), & \text{for else.}\end{cases}$$

It is easy to see that under coloring τ_{α_1}, there is neither a red $(m+1)K_2$ nor a blue P_4 in $SF(\alpha,4)-\alpha_1$. Hence, τ_{α_1} is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-\alpha_1$.

(2) Case of $e=\alpha_2$. Then, both α_1 and α_3 are pendant edges of $SF(\alpha,4)-\alpha_2$. Consider the edge of F adjacent to α_5, say b. Then there exists an (mK_2,P_4)-coloring τ_b of $F-b$. Now, define a red-blue coloring τ_{α_2} of $SF(\alpha,4)-\alpha_2$ such that

$$\tau_{\alpha_2}(x)=\begin{cases}\text{red}, & \text{for } x=\alpha_5,b,\\ \text{blue}, & \text{for } x=\alpha_3,\alpha_4,\\ \tau_b(\alpha), & \text{for } x=\alpha_1,\\ \tau_b(x), & \text{otherwise.}\end{cases}$$

We can easily see that τ_{α_2} is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-\alpha_2$.

(3) Next, we consider $e=\alpha_3$. Let **P** be a path of length 3 in F containing the subdivision edge α with the vertex-set $V(\mathbf{P})=\{w_1,w_2,w_3,w_4\}$ and the edge-set $E(\mathbf{P})=\{r,s,\alpha\}$, where $r=w_1w_2, s=w_2w_3$, and $\alpha=w_3w_4$ (see Fig. 3). In this case, the edge α_1 is incident with the vertex w_3. We now consider an (mK_2,P_4)-coloring τ_r of $F-r$ and an (mK_2,P_4)-coloring τ_s of $F-s$. Without loss of generality, we consider the subdivision edge α is contained in a path P_4,P_5, or P_6 as referred to in Lemma 2. It means that under both coloring τ_r and τ_s and according to Lemma 2, the subdivision edge α has a blue color. Therefore, there are 3 possibilities about the path **P**, where from all possibilities, we consider either the coloring τ_r or τ_s.

(a) A path **P** is contained in a blue path P_4 (but **P** is not contained in a blue path P_5). Then $\mathbf{P}=P_4$. Under the coloring τ_r, the blue edge incident with w_2 is the only edge s. Let τ_{α_3} be a red-blue coloring of edges of $SF(\alpha,4)-\alpha_3$ such that

$$\tau_{\alpha_3}(x)=\begin{cases}\text{red}, & \text{for } x=\alpha_1,s,\\ \text{blue}, & \text{for } x=\alpha_2, \text{ and } x=\alpha_4,\alpha_5,r,\\ \tau_r(x), & \text{otherwise.}\end{cases}$$

So, τ_{α_3} is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-\alpha_3$.

(b) A path **P** is contained in a blue path P_5 (but **P** is not contained in a blue path P_6). Under the coloring τ_r, then (i) there exists at least one edge incident with the vertex w_1 has a blue color, and (ii) the blue edge which is incident with the vertex w_4 is the only α. We now define τ_{α_3} as a red-blue coloring of edges of $SF(\alpha,4)-\alpha_3$ such that

$$\tau_{\alpha_3}(x)=\begin{cases}\text{red}, & \text{for } x=\alpha_1,s,\\ \text{blue}, & \text{for } x=\alpha_2,\alpha_4,\alpha_5,r,\\ \tau_r(x), & \text{otherwise.}\end{cases}$$

So, τ_{α_3} is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-\alpha_3$.

(c) A path **P** is contained in a blue path P_6. Under the coloring τ_r, then (i) there is exactly one blue edge incident with the vertex w_1, say $p=vw_1$, and (ii) at least one edge incident with the vertex v also having a blue color. Now, consider two cases below.

- If s is the only blue edge adjacent to α, then define τ_{α_3} as a red-blue coloring of edges of $SF(\alpha,4)-\alpha_3$ such that

$$\tau_{\alpha_3}(x)=\begin{cases}\text{red}, & \text{for } x=r,s,\\ \text{blue}, & \text{for } x=\alpha_1,\alpha_2, \text{ and } x=\alpha_4,\alpha_5,\\ \tau_r(x), & \text{otherwise.}\end{cases}$$

  So, τ_{α_3} is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-\alpha_3$.
- If the blue edge adjacent to α is not only s, then there exists at least one blue edge, say s_1, which is incident with the vertex w_3. For this case, we consider an $((m+1)K_2,P_4)$-coloring τ_s of edges of $F-s$. Under the coloring τ_s, the blue edge incident with the vertex v is only the edge p. Now, define τ_{α_3} as a red-blue coloring of edges of $SF(\alpha,4)-\alpha_3$ such that

$$\tau_{\alpha_3}(x)=\begin{cases}\text{red}, & \text{for } x=\alpha_1,s,\\ \text{blue}, & \text{for } x=\alpha_2, \text{ and } x=\alpha_4,\alpha_5,\\ \tau_r(x), & \text{for } x \text{ is incident with } w_4,\\ \tau_s(x), & \text{for else.}\end{cases}$$

  So, τ_{α_3} is an $((m+1)K_2,P_4)$-coloring of $SF(\alpha,4)-\alpha_3$.

Therefore, $SF(\alpha,4)\in\mathcal{R}((m+1)K_2,P_4)$. □

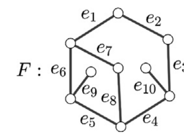

Fig. 4. The graph $F \in \mathcal{R}(3K_2,P_4)$.

As an illustration, consider the graph F in Fig. 4. We can prove that the graph F in Fig. 4 is in $\mathcal{R}(3K_2,P_4)$. The graph F satisfies the following conditions (see [3, 12]):

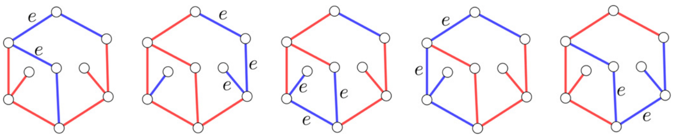

Fig. 5. Some red-blue colorings of F such that removing a blue edge e satisfying Lemma 2 results a $(3K_2, P_4)$-coloring of $F - e$.

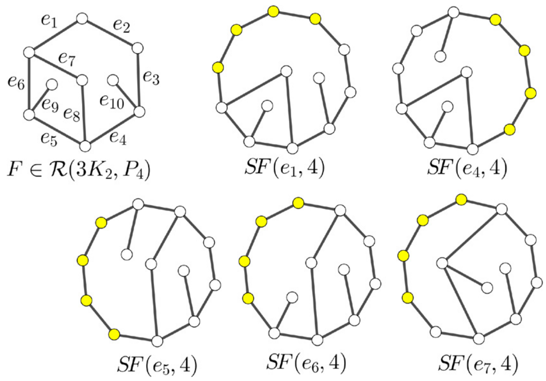

Fig. 6. Five non-isomorphism graphs belonging to $\mathcal{R}(4K_2, P_4)$ which is obtained by subdividing four times (4 yellow vertices) an edge in a cycle of $F \in \mathcal{R}(3K_2, P_4)$.

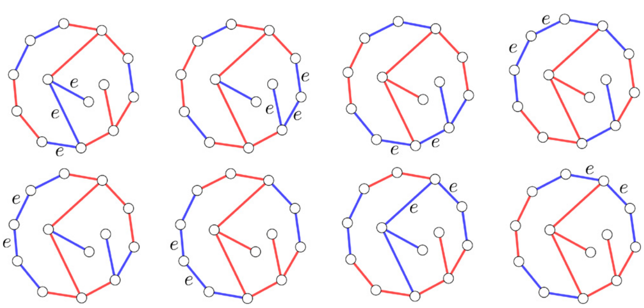

Fig. 7. Some red-blue colorings of $SF(e_5, 4)$ such that removing the blue edge e satisfying Lemma 2 results a $(4K_2, P_4)$-coloring of $SF(e_5, 4) - e$.

(i) for each $u, v \in V(F)$, $F - \{u, v\} \supseteq P_4$,
(ii) for each subset on 5 vertices $S_5 \subseteq V(F)$, $F - E(F[S_5]) \supseteq P_4$, where $F[S_5]$ is an induced subgraph of 5 vertices in S_5 of a graph F.

So, $F \to (3K_2, P_4)$. The minimality property of a graph F, that is for each $e \in E(F)$, there exists a $(3K_2, P_4)$-coloring of $F - e$, can be seen in Fig. 5. Removing one blue edge e satisfying Lemma 2 results a $(3K_2, P_4)$-coloring of $F - e$. By Theorem 3, up to isomorphism, if we subdivide an edge e_i ($i \in [1, 8]$), four times, of a graph F in Fig. 4, then we obtain five non-isomorphism subdivision graphs (4 vertices) belonging to $\mathcal{R}(4K_2, P_4)$, namely $SF(e_1, 4)$, $SF(e_4, 4)$, $SF(e_5, 4)$, $SF(e_6, 4)$, and $SF(e_7, 4)$ as depicted in Fig. 6. The proof of the minimality of a graph $SF(e_5, 4)$ can be seen in Fig. 7, while the minimality of the other graphs can be shown in the same fashion.

3. Some classes of Ramsey (mK_2, P_4) minimal graphs

In this section, we give some connected graphs other than cycle belonging to $\mathcal{R}(mK_2, P_4)$ for an integer m. We construct these graphs by subdivision (4 vertices) on the edge contained in a cycle of a graph F, where F is either in $\mathcal{R}(2K_2, P_4)$ or in $\mathcal{R}(3K_2, P_4)$. Baskoro and Yulianti [9], proved that $\mathcal{R}(2K_2, P_4) = \{2P_4, C_5, C_6, C_7, C_4^2(1)\}$, where $C_4^2(1)$ is a cycle on 4 vertices with two additional pendant vertices so that the two vertices of degree 3 are adjacent, as depicted in Fig. 8. In general, we define a graph $C_n^2(s)$ as a graph formed from a cycle on n vertices with two

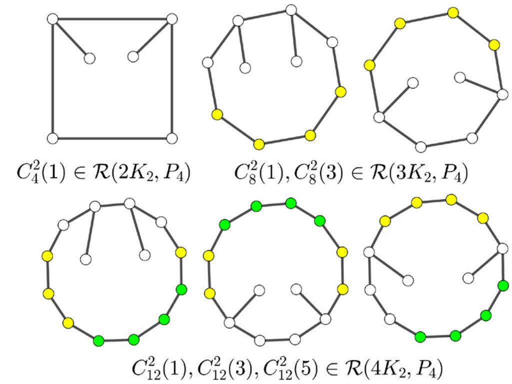

Fig. 8. Some graphs are in $\mathcal{R}(mK_2, P_4)$ for $m = 2, 3$, or 4.

additional pendant vertices such that the two vertices of degree 3 are at distance s. By Theorem 3, the subdivision (4 (yellow) vertices) on any edge contained in a cycle of the graph $C_4^2(1)$ yields $C_8^2(1)$ and $C_8^2(3)$, as depicted in Fig. 8; and both are in $\mathcal{R}(3K_2, P_4)$. Next, the subdivision (4 (green) vertices) on any edge contained in a cycle of a graph $C_8^2(1)$ will produce graphs in $\mathcal{R}(4K_2, P_4)$, namely $C_{12}^2(1)$ and $C_{12}^2(5)$. Meanwhile, the subdivision (4 (green) vertices) on any edge contained in a cycle of a graph $C_8^2(3)$ will yield graphs in $\mathcal{R}(4K_2, P_4)$, namely $C_{12}^2(3)$ and $C_{12}^2(5)$, as depicted in Fig. 8. By continuing this step recursively, we get corollary below.

Corollary 4. *Let $m \geq 2$ be a natural number. Then, the graph $C_{4(m-1)}^2(s)$ is in $\mathcal{R}(mK_2, P_4)$, for any odd integer $s \leq 2m - 3$.* □

We now define four special graphs formed by a cycle C_n with circumference n by adding two new edges connecting vertices of the cycle C_n. Suppose $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$ are the vertex-set and edge-set of C_n, respectively. Let i, j, k, l be four distinct integers, we denote by $C_n[(i, k), (j, l)]$ the graph formed from a cycle C_n by adding two new edges v_iv_k and v_jv_l. Now, consider graphs: $C_8[(1, 4), (2, 7)]$, $C_8[(1, 5), (3, 7)]$, $C_8[(1, 5), (2, 6)]$ and $C_8[(1, 4), (2, 6)]$ as depicted in Figs. 9, 10, 11, and 12, respectively. In the following lemma, we prove that these four graphs $C_8[(1, 4), (2, 7)]$, $C_8[(1, 5), (3, 7)]$, $C_8[(1, 5), (2, 6)]$ and $C_8[(1, 4), (2, 6)]$ are in $\mathcal{R}(3K_2, P_4)$.

Lemma 5. *The graphs $C_8[(1, 4), (2, 7)]$, $C_8[(1, 5), (3, 7)]$, $C_8[(1, 5), (2, 6)]$, and $C_8[(1, 4), (2, 6)]$ are Ramsey $(3K_2, P_4)$-minimal graphs.*

Proof. Let F be one of the graphs $C_8[(1, 4), (2, 7)]$, $C_8[(1, 5), (3, 7)]$, $C_8[(1, 5), (2, 6)]$, or $C_8[(1, 4), (2, 6)]$. We can easily show the graph $F \to (3K_2, P_4)$, since it satisfies the following conditions (see [3, 12]):

(i) for any distinct two vertices $u, v \in V(F)$, $F - \{u, v\} \supseteq P_4$,
(ii) for any 5-subset $S_5 \subseteq V(F)$, $F - E(F[S_5]) \supseteq P_4$, where $F[S_5]$ is the induced subgraph of S_5 of F.

Next, the minimality of a graph F can be seen in Figs. 9, 10, 11, and 12, where removing one blue edge labeled (i, j) will result a $(3K_2, P_4)$-coloring of $F - v_iv_j$, for some distinct $i, j \in [1, 8]$. □

Now, we consider the graph $C_8[(1, 4), (2, 7)]$. Since every edge in $C_8[(1, 4), (2, 7)]$ is contained in a cycle then by Theorem 3, the subdivision (4 vertices) on any edge of $C_8[(1, 4), (2, 7)]$ will result some graphs in $\mathcal{R}(4K_2, P_4)$. By repeating the process to the resulting graph again and again, we obtain the following corollary.

Corollary 6. *Let $m \geq 3$ be an integer. Then, the graphs $C_{4(m-1)}[(1, 4), (2, 4m-5)]$, $C_{4(m-1)}[(1, 4m-8), (2, 4m-5)]$, and $C_{4(m-1)}[(1, 4m-8), (4m-10, 4m-6)]$ are in $\mathcal{R}(mK_2, P_4)$.*

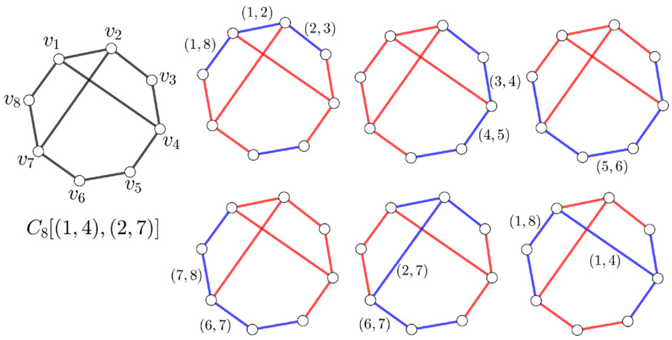

Fig. 9. Some red-blue colorings of $C_8[(1,4),(2,7)]$ such that removing one labeled blue edge (i,j) will result a $(3K_2,P_4)$-coloring of $C_8[(1,4),(2,7)]-v_iv_j$ for some distinct $i,j\in[1,8]$.

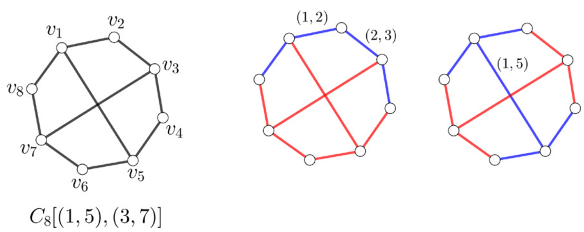

Fig. 10. Some red-blue colorings of $C_8[(1,5),(3,7)]$ such that removing one labeled blue edge (i,j) will result a $(3K_2,P_4)$-coloring of $C_8[(1,5),(3,7)]-v_iv_j$ for some distinct $i,j\in[1,8]$.

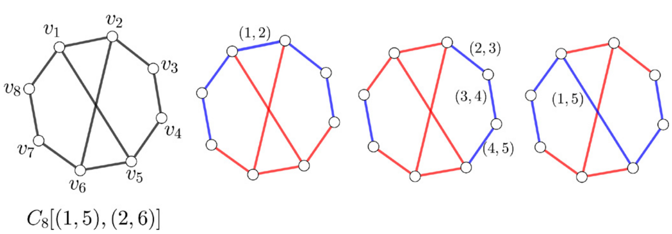

Fig. 11. Some red-blue colorings of $C_8[(1,5),(2,6)]$ such that removing one labeled blue edge (i,j) will result a $(3K_2,P_4)$-coloring of $C_8[(1,5),(2,6)]-v_iv_j$ for some distinct $i,j\in[1,8]$.

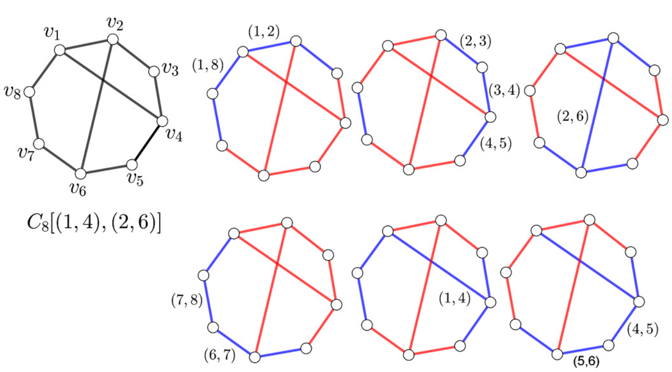

Fig. 12. Some red-blue colorings of $C_8[(1,4),(2,6)]$ such that removing one labeled blue edge (i,j) will result a $(3K_2,P_4)$-coloring of $C_8[(1,4),(2,6)]-v_iv_j$ for some distinct $i,j\in[1,8]$.

Proof. Consider the graph $C_8[(1,4),(2,7)]\in\mathcal{R}(3K_2,P_4)$. Let $\{v_1,v_2,\ldots,v_8\}$ be the vertex-set of $C_8[(1,4),(2,7)]$. The subdivision (4 vertices) on the edge $e=v_4v_5$ will result $C_{12}[(1,4),(2,11)]$. By Theorem 3, $C_{12}[(1,4),(2,11)]\in\mathcal{R}(4K_2,P_4)$. Furthermore, by considering the edge $e=v_4v_5$ of $C_{12}[(1,4),(2,11)]$ and subdivision (4 vertices) on this edge, we obtain the graph $C_{16}[(1,4),(2,15)]$. Again, by Theorem 3, $C_{16}[(1,4),(2,15)]\in\mathcal{R}(5K_2,P_4)$. If we continue this process and apply to the resulting graph, then we obtain the graph $C_{4(m-1)}[(1,4),(2,4m-5)]$. By Theorem 3, $C_{4(m-1)}[(1,4),(2,4m-5)]\in\mathcal{R}(mK_2,P_4)$.

Now, by subdivision (4 vertices) on the edge $e=v_2v_3$ of the graph $C_8[(1,4),(2,7)]$, repeatedly, and apply Theorem 3, we obtain $C_{4(m-1)}[(1,4m-8),(2,4m-5)]\in\mathcal{R}(mK_2,P_4)$. The last graph $C_{4(m-1)}[(1,4m-8),(4m-10,4m-6)]$ is in $\mathcal{R}(mK_2,P_4)$. If this above process is applied to the edge $e=v_1v_2$, then we obtain $C_{4(m-1)}[(1,4m-8),(2,4m-5)]\in\mathcal{R}(mK_2,P_4)$. □

In the same fashion, we can construct the other graphs which are in $\mathcal{R}(mK_2,P_4)$ from graphs $C_8[(1,5),(3,7)]$, $C_8[(1,5),(2,6)]$ and $C_8[(1,4),(2,6)]$. Therefore, we have the following corollary.

Corollary 7. *Let $m\geq 3$ be an integer. Then the graphs:*

(i) $C_{4(m-1)}[(1,5),(3,7)]$,
(ii) $C_{4(m-1)}[(1,4m-7),(2,4m-6)]$,
(iii) $C_{4(m-1)}[(1,4m-7),(4m-10,4m-6)]$,
(iv) $C_{4(m-1)}[(1,4m-8),(2,4m-6)]$, *and*
(v) $C_{4(m-1)}[(1,4m-8),(4m-10,4m-6)]$ *are in* $\mathcal{R}(mK_2,P_4)$. □

Declarations

Author contribution statement

Kristiana Wijaya: Conceived and designed experiments; Performed the experiments; Wrote the paper.
Edy Tri Baskoro: Conceived and designed experiments; Analyzed the data; Wrote the paper.
Hilda Assiyatun, Djoko Suprijanto: Analyzed and interpreted the data.

Funding statement

This research has been supported by the World Class Research (WCR) Program, Ministry of Research, Technology and Higher Education, Indonesia, Decree No. 7/E/KPT/2019 and Contract No. 127/SP2H/LT/DRPM/2019.
The research of the first author has been supported by DRPM, Directorate General of Strengthening for Research and Development, Ministry of Research, Technology and Higher Education, Indonesia through “Hibah Penelitian Dasar”, Decree No. 7/E/KPT/2019 and Contract No. 175/SP2H/LT/DRPM/2019.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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