Graphs and Combinatorics (2017) 33:233–243
DOI 10.1007/s00373-016-1748-1

ORIGINAL PAPER

On Ramsey (mK_2, H)-Minimal Graphs

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Received: 12 July 2015 / Revised: 7 March 2016 / Published online: 9 December 2016

Abstract Let $\mathcal{R}(G, H)$ denote the set of all graphs F satisfying $F \to (G, H)$ and for every $e \in E(F)$, $(F - e) \nrightarrow (G, H)$. In this paper, we derive the necessary and sufficient conditions for graphs belonging to $\mathcal{R}(mK_2, H)$ for any graph H and each positive integer m. We give all disconnected graphs in $\mathcal{R}(mK_2, H)$, for any connected graph H. Furthermore, we prove that if $F \in \mathcal{R}(mK_2, P_3)$, then any graph obtained by subdividing one non-pendant edge in F will be in $\mathcal{R}((m + 1)K_2, P_3)$.

Keywords Ramsey minimal graph · Edge coloring · Matching · Path · Subdivision

Mathematics Subject Classification 05D10 · 05C55

1 Introduction

The problem of finding Ramsey minimal graphs is one of the problems developed from the classical Ramsey theory. Let F, G, and H be nonempty graphs without isolated vertices. We write $F \to (G, H)$ if whenever each edge of F is colored either red or

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blue, then the red subgraph of F, denoted F_r, induced by all red edges contains a graph G or the blue subgraph of F, denoted F_b, induced by all blue edges contains a graph H. A graph F is *Ramsey graph* for a pair of graphs (G, H) if $F \rightarrow (G, H)$. If $F = K_n$, then the problem of determining the smallest n such that $K_n \rightarrow (G, H)$ has been studied extensively, extensively [1,9,13,17]. Such an integer $n = r(G, H)$ is usually called (graph) Ramsey number of a pair (G, H).

A red-blue coloring of edges of F so that F contains neither a red G nor a blue H is a (G, H)-*coloring*. A graph F is *Ramsey* (G, H)-*minimal* if $F \rightarrow (G, H)$ and for each $e \in E(F)$, $(F - e) \nrightarrow (G, H)$. The set of all Ramsey (G, H)-minimal graphs will be denoted by $\mathcal{R}(G, H)$. The pair (G, H) is called *Ramsey-finite* if $\mathcal{R}(G, H)$ is finite and *Ramsey-infinite* otherwise.

The main problem of Ramsey (G, H)-minimal graphs is to characterize all graphs F in $\mathcal{R}(G, H)$, for given graphs G and H. Numerous papers have studied the problem of Ramsey (G, H)-minimal graphs. Burr et al. [12] showed that the set $\mathcal{R}(G, H)$ is Ramsey infinite when both G and H are forest, with at least one of G or H having a non-star component. Łuczak [14] showed that the set $\mathcal{R}(G, H)$ is infinite for every forest G other than a matching and every graph H containing a cycle. Moreover, Borowiecki et al. [6] characterized the graphs in $\mathcal{R}(K_{1,2}, K_{1,m})$ for $m \geq 3$. Several papers discussed characterizing infinite families of Ramsey $(K_{1,2}, C_4)$-minimal graphs (see [2,5,18]). Yulianti et al. [23] gave constructions of some infinite classes Ramsey $(K_{1,2}, P_4)$-minimal graphs. Next, Borowiecki *et al.* [7] determined the graphs in $\mathcal{R}(K_{1,2}, K_3)$. Borowiecka-Olszewska and Hałuszczak [8] presented a procedure to generate an infinite family of Ramsey $(K_{1,m}, \mathcal{G})$-minimal graphs, where $m \geq 2$ and $\mathcal{G}$ is a family of 2-connected graphs.

In this paper, we focus on Ramsey-finite. Burr et al. [10] proved that $\mathcal{R}(mK_2, H)$ is Ramsey finite for any graph H and positive integer m. They showed that $\mathcal{R}(K_2, H) = \{H\}$ for any graph H, $\mathcal{R}(2K_2, 2K_2) = \{C_5, 3K_2\}$, and $\mathcal{R}(2K_2, K_3) = \{K_5, 2K_3, G_1\}$, where G_1 is the graph with the vertex set $V(G_1) = \{v_1, v_2, \ldots, v_7\}$ and the edge set $E(G_1) = \{v_1v_2, v_1v_3, v_2v_3\} \cup \{v_iv_7 \ |i = 1, 2, \ldots, 6\} \cup \{v_1v_4, v_2v_5, v_3v_6\}$. In the same paper, they described a collection of $\frac{n+1}{2}$ non-isomorphic graphs in $\mathcal{R}(2K_2, K_n)$, for $n \geq 4$ and $n - 2$ non-isomorphic graphs in $\mathcal{R}(2K_2, K_{1,n})$, for $n \geq 3$. Later, Burr et al. [11] investigated $\mathcal{R}(G, H)$ for the special case of $G = 2K_2$ and $H = tK_2$. Furthermore, Mengersen and Oeckermann [15] presented a characterization of graphs belonging to $\mathcal{R}(2K_2, K_{1,n})$, for $n \geq 3$. Baskoro and Yulianti [4] characterized all graphs in $\mathcal{R}(2K_2, P_n)$ for $n = 4, 5$. Mushi and Baskoro [16] derived the properties of graphs belonging to the class $\mathcal{R}(3K_2, P_3)$ and gave a proof to all members of the set $\mathcal{R}(3K_2, P_3)$ claimed in [10]. Recently, Baskoro and Wijaya [3] derived the necessary and sufficient conditions for graphs to be in $\mathcal{R}(2K_2, H)$ for any connected graph H. Moreover, Wijaya et al. [22] gave all graphs belonging to $\mathcal{R}(2K_2, C_4)$. Most recently, Wijaya et al. characterized all graphs belonging to $\mathcal{R}(2K_2, K_4)$ in [19], and all unicyclic graphs belonging to $\mathcal{R}(mK_2, P_3)$ in [20].

Based on the above results, the aim of this paper is to derive the necessary and sufficient conditions for graphs in $\mathcal{R}(mK_2, H)$, for any graph H and integer $m > 1$. Some specific properties of these graphs are also obtained. Moreover, we determine all disconnected graphs in $\mathcal{R}(mK_2, H)$ for any connected graph H. Finally, we prove that if $F \in \mathcal{R}(mK_2, P_3)$, then any graph obtained by subdividing one non-pendant edge in F will be in $\mathcal{R}((m+1)K_2, P_3)$.

2 Main Results

The main results of this paper are given by three theorems. The first theorem (Theorem 1) gives the necessary and sufficient conditions for Ramsey (mK_2, H)-minimal graphs for any graph H. The second theorem (Theorem 4) shows that any disconnected graph in $\mathcal{R}(mK_2, H)$ is obtained from a disjoint union of graphs in $\mathcal{R}(sK_2, H)$ and $\mathcal{R}(tK_2, H)$, where $s+t=m$, for any connected graph H. In the last theorem (Theorem 6), we prove that if $F \in \mathcal{R}(mK_2, P_3)$, then every graph obtained by subdividing one non-pendant edge in F will be in $\mathcal{R}((m+1)K_2, P_3)$.

Before we discuss these theorems, some definitions and notations will be introduced. A complete graph and a path on n vertices are denoted by K_n and P_n, respectively. A union of m disjoint copies of K_2 is denoted by mK_2. Let F be a graph. For a k-subset $S_k \subseteq V(F)$, $k \geq 0$, $F[S_k]$ denotes the subgraph of F induced by all vertices in S_k. For odd k, we call *odd induced subgraph* $F[S_k]$. The notation $F(S_k)$ means that the subgraph of F induced by all edges incident with some vertices in S_k. For a nonnegative integer α, a disjoint union of α (not necessary isomorphic) induced subgraphs $F[S_k]$ will be denoted by $\alpha F(k)$. It means that

$$\alpha F(k) = F[S_k^1] \cup F[S_k^2] \cup \ldots \cup F[S_k^\alpha],$$

where $S_k^i \cap S_k^j = \emptyset$ for every $i \neq j$. Note that, $\alpha = 0$ in $\alpha F(k)$ means that an induced subgraph of order k is not considered. If $\alpha = 1$ then $F(k) = F[S_k]$.

Lemma 1 *Let F be a nonempty graph and $t > 1$ be an integer. The graph F has at most t independent edges if and only if there exists a k-subset $S_k \subseteq V(F)$ and α_i odd induced subgraphs $F[S_{2i+1}]$ of F, where $k + \sum_{i=1}^{t} i\alpha_i = t$ and $k, \alpha_i \in [0, t]$, such that F can be decomposed into*

$$F = F(S_k) \oplus \left(\bigcup_{i=1}^{t} \alpha_i F(2i+1) \right).$$

Proof Suppose that F has order n. It suffices to assume that F is a connected graph. Since for a disconnected graph, we can consider each of its components. Suppose that F has at most t independent edges. So, $t \leq \lfloor \frac{n}{2} \rfloor$. For $t = \lfloor \frac{n}{2} \rfloor$ and odd n, choose a 0-subset $S_0 = \emptyset \subseteq V(F)$ and the induced subgraph on $2t+1$ vertices $F[S_{2t+1}]$. Then $F = F[S_{2t+1}]$. For $t = \lfloor \frac{n}{2} \rfloor$ and even n, choose any 1-subset $S_1 \subseteq V(F)$ and the induced subgraph on the remaining vertices $F[S_{2t-1}]$. Then $F = F(S_1) \oplus F[S_{2t-1}]$. For $t < \lfloor \frac{n}{2} \rfloor$, set t independent edges in F, say, $M = \{e_1, e_2, \ldots, e_t\}$, where M is a maximum matching in F. Suppose $e_i = v_i v_{i+t}$. Define $S_t = \{u \mid u = v_i \text{ or } u = v_{i+t}\}$. If $E(F(S_t)) = E(F)$, then choose $S_k = S_t$. Thus, $F = F(S_k)$. Otherwise, suppose to the contrary that for each k-subset $S_k \subseteq V(F)$ and α_i odd induced subgraphs $F[S_{2i+1}]$ of F, where $k + \sum_{i=1}^{t} i\alpha_i = t$ and $k, \alpha_i \in [0, t]$, $F \neq F(S_k) \oplus \mathcal{F}$, where $\mathcal{F} = (\bigcup_{i=1}^{t} \alpha_i F(2i+1))$. It means that there is an $e = uv \in E(F)$, such that neither

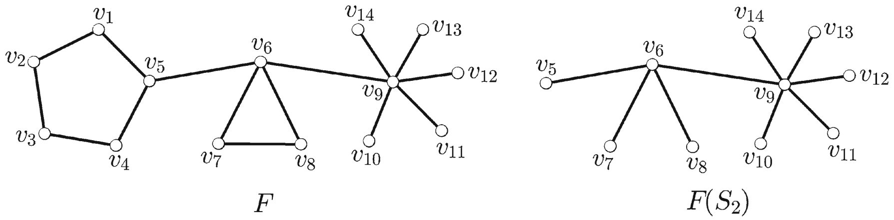

Fig. 1 The graph F with 5 independent *edges* and the graph $F(S_2)$ where $S_2 = \{v_6, v_9\}$

$e \in F(S_k)$ nor $e \in E(\mathcal{F})$. Now, define $S_{k'} = S_k \cup \{v\}$. Then the edge $e \in F(S_{k'})$. In this case, $F(S_{k'})$ has $k+1$ independent edges. Therefore, F has $t+1$ independent edges, a contradiction with the maximum matching M in F.

Conversely, suppose there is a k-subset $S_k \subseteq V(F)$, and α_i odd induced subgraphs $F[S_{2i+1}]$ of F, where $k + \sum_{i=1}^{t} i\alpha_i = t$, and $k, \alpha_i \in [0, t]$, such that F can be decomposed into $F = F(S_k) \oplus \mathcal{F}$. We observe that each vertex in S_k can be viewed as the center of some star in $F(S_k)$. So, there is at most k independent edges of $F(S_k)$. On the other hand, the subgraph $F[S_{2i+1}]$ of F contains at most i independent edges. So, there are at most $\sum_{i=1}^{t} i\alpha_i$ independent edges of $\mathcal{F}$. Hence, F has at most $k + \sum_{i=1}^{t} i\alpha_i = t$ independent edges. □

As an illustration, consider the graph F of Fig. 1 having 5 independent edges, $v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}$. Set $S_2 = \{v_6, v_9\}$ and $S_7 = \{v_1, v_2, v_3, v_4, v_5, v_7, v_8\}$. We obtain $F = F(S_2) \oplus F[S_7]$. Another decomposition, we can set $S_3 = \{v_6, v_7, v_9\}$ and $S_5 = \{v_1, v_2, v_3, v_4, v_5\}$ such that $F = F(S_3) \oplus F[S_5]$. We also can set $S_4 = \{v_1, v_3, v_5, v_9\}$ and $S_3 = \{v_6, v_7, v_8\}$ such that $F = F(S_4) \oplus F[S_3]$. But, there is no S_5 such that $F = F(S_5)$.

Observe that if F contains at most t independent edges, then F contains a subgraph mK_2, for some $m \leq t$. Let F be a graph where every edge in F has either a red or blue color. Clearly, F can be decomposed into the red and blue subgraph, $F = F_r \oplus F_b$. Now, we apply Lemma 1 to obtain a Ramsey (mK_2, H)-minimal graph, namely how to color a graph F by red and blue such that the red subgraph of F contain at most $(m-1)$ independent edges. Let F be a graph of order n. Suppose that ϕ is a red-blue coloring of edges of F such that the red subgraph F_r has the maximal number of edges containing at most t independent edges, where $1 < t < \lfloor \frac{n}{2} \rfloor$. Then the red subgraph F_r can be decomposed into graphs as in Lemma 1. Furthermore, if we remove all red edges of F, then we obtain all blue edges of F. Removing the edges in $F(S_k)$ can be done by deleting all vertices in S_k. Note that, $F - S_k - E(\mathcal{F}) = F_b \cup N$, where N is an empty graph and $\mathcal{F} = \bigcup_{i=1}^{t} \alpha_i F(2i+1)$. Hence, to check whether the blue subgraph F_b contains a graph H or not, we can check whether the subgraph $F - S_k - E(\mathcal{F})$ contains a graph H or not.

2.1 Necessary and Sufficient Conditions for Graphs in $\mathcal{R}(mK_2, H)$

In this section, we discuss how to characterize all graphs F satisfying $F \rightarrow (mK_2, H)$ and for each $e \in E(F)$, $F - e \nrightarrow (mK_2, H)$. The following result gives the necessary and sufficient conditions for such graphs F.

Theorem 1 *Let H be a graph and $m > 1$ be an integer. A graph $F \in \mathcal{R}(mK_2, H)$ if and only if the following two conditions hold*:

(i) *for each k-subset $S_k \subseteq V(F)$ and α_i odd induced subgraphs $F[S_{2i+1}]$ of F where $k + \sum_{i=1}^{m-1} i\alpha_i = m-1$ and $k, \alpha_i \in [0, m-1]$ we have*

$$F - S_k - E\left(\bigcup_{i=1}^{m-1} \alpha_i F(2i+1)\right) \supseteq H,$$

(ii) *for each $e \in E(F)$, there exists a k-subset $S_k \subseteq V(F)$ and α_i odd induced subgraphs $F[S_{2i+1}]$ of F, where $k + \sum_{i=1}^{m-1} i\alpha_i = m-1$ and $k, \alpha_i \in [0, m-1]$, such that*

$$(F - e) - S_k - E\left(\bigcup_{i=1}^{m-1} \alpha_i F(2i+1)\right) \not\supseteq H.$$

Proof We refer the notation in Lemma 1 that $\mathcal{F} = \bigcup_{i=1}^{m-1} \alpha_i F(2i+1)$. Suppose to the contrary that $F \in \mathcal{R}(mK_2, H)$, but for some k-subset $S_k \subseteq V(F)$ and α_i odd induced subgraphs $F[S_{2i+1}]$ of F where $k+\sum_{i=1}^{m-1} i\alpha_i = m-1$ and $k, \alpha_i \in [0, m-1]$, we have $F - S_k - E(\mathcal{F}) \not\supseteq H$. Define a red-blue coloring of edges of F as follows. Color all edges of $F - S_k - E(\mathcal{F})$ with blue and the remaining edges with red. It is noticed easily that under this coloring, the blue subgraph F_b of F does not contain a blue H. While the red subgraph F_r of F is a subgraph $F(S_k) \oplus \mathcal{F}$. By Lemma 1, the red subgraph of F contains at most $(m-1)$ independent edges. So, we obtain an (mK_2, H)-coloring of edges of F, a contradiction. Next, by the minimality of F, for each $e \in E(F)$, there exists an (mK_2, H)-coloring ϕ of $F - e$. In such the coloring ϕ, the red subgraph F_r of $F - e$ contains at most $m - 1$ independent edges, while the blue subgraph F_b of $F - e$ does not contain a blue H. By Lemma 1, there is a k-subset $S_k \subseteq V(F)$ and α_i odd induced subgraphs $F[S_{2i+1}]$ of $F - e$, where $k + \sum_{i=1}^{m-1} i\alpha_i = m - 1$ and $k, \alpha_i \in [0, t]$, such that $F_r = F(S_k) \oplus \mathcal{F}$. Hence, $(F - e) - S_k - E(\mathcal{F}) \not\supseteq H$.

Conversely, let both conditions (i) and (ii) be satisfied. Consider any red-blue coloring of edges of F not containing a red mK_2. So, we have either all blue edges of F or the red subgraph F_r of F contains at most $(m-1)$ independent edges. Hence, by Lemma 1, $F_r = F(S_k) \oplus \mathcal{F}$. By condition (i), the blue subgraph F_b of F contains a blue H. Hence, $F \rightarrow (mK_2, H)$. Next, for each $e \in E(F)$, we color all edges of $(F - e) - S_k - E(\mathcal{F})$ with blue and the remaining edges with red. By condition (ii), under this coloring, $F - e$ does not contain a blue H. By Lemma 1, $F - e$ contains at most $(m-1)$ independent red edges. So, we obtain an (mK_2, H)-coloring of edges of $F - e$. Hence, $(F - e) \nrightarrow (mK_2, H)$. Therefore, $F \in \mathcal{R}(mK_2, H)$. □

The first condition of Theorem 1 means that $F \rightarrow (mK_2, H)$, while the second condition of Theorem 1 means that for each $e \in E(F)$, $F - e \nrightarrow (mK_2, H)$ and it is called the minimality property of a graph in $\mathcal{R}(mK_2, H)$. Although we have obtained the necessary and sufficient conditions for graphs belonging to $\mathcal{R}(mK_2, H)$, characterizing all graphs in $\mathcal{R}(mK_2, H)$ for a given graph H is difficult. The following result provides another property of a graph F satisfying $F \rightarrow (mK_2, H)$ based on a Ramsey $((m-1)K_2, H)$-minimal graph.

Lemma 2 *Let H be a graph and $m > 1$ be an integer. $F \rightarrow (mK_2, H)$ if and only if the following three conditions hold:*

(i) *for every $v \in V(F)$, $F - \{v\} \rightarrow ((m-1)K_2, H)$,*
(ii) *for every $K_3 \subseteq F$, $F - E(K_3) \rightarrow ((m-1)K_2, H)$,*
(iii) *for every $F[S_{2m-1}]$ of F, $F - E(F[S_{2m-1}])$ contains a graph H.*

Proof Suppose to the contrary that $F \rightarrow (mK_2, H)$, but at least one of three conditions is violated. Suppose that there exists an $((m-1)K_2, H)$-coloring ϕ_1 of edges of $F - \{v\}$. Let us define a red-blue coloring ϕ of edges of F such that

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \in E(F - \{v\}), \\ \text{red} & \text{if } x \text{ incident with } v. \end{cases}$$

Thus, ϕ is an (mK_2, H)-coloring of edges of F, a contradiction. A similar argument also leads to a contradiction when there exists an $((m-1)K_2, H)$-coloring of edges of $F - E(K_3)$. Finally, suppose that for some $F[S_{2m-1}]$ of F, $F - E(F[S_{2m-1}])$ does not contain a graph H. Color all edges of $F[S_{2m-1}]$ with red and otherwise with blue. We obtain an (mK_2, H)-coloring of edges of F, a contradiction.

Conversely, suppose that all conditions (i), (ii), and (iii) are satisfied. By applying Theorem 1(i), we obtain $F \rightarrow (mK_2, H)$. □

Theorem 1 may not be easy to apply to a given graph H and an integer m, since there are many candidates of graphs satisfying the first condition. The following theorem gives a relationship between graphs in $\mathcal{R}(mK_2, H)$ and the ones in $\mathcal{R}((m-1)K_2, H)$.

Theorem 2 *Let H be a graph and $m > 1$ be an integer. If $F \in \mathcal{R}(mK_2, H)$, then for any $v \in V(F)$ and $K_3 \subseteq F$, both graphs $F - \{v\}$ and $F - E(K_3)$ contain a Ramsey $((m-1)K_2, H)$-minimal graph.*

Proof Suppose to the contrary that for some $v \in V(F)$, $F - \{v\}$ contains no $G \in \mathcal{R}((m-1)K_2, H)$. This implies the existence of an $((m-1)K_2, H)$-coloring ϕ_1 of $F - \{v\}$. It means that $F - \{v\} \nrightarrow ((m-1)K_2, H)$. By Lemma 2, $F \nrightarrow (mK_2, H)$, a contradiction.

Next, the proof for the case of $F - E(K_3)$ containing a Ramsey $((m-1)K_2, H)$-minimal graph for any $K_3 \subseteq F$ is similar. □

Note that, Theorem 2 can be used to construct a graph F satisfying $F \rightarrow (mK_2, H)$ based on a Ramsey $((m-1)K_2, H)$-minimal graph. For example, the construction can be seen in Wijaya et al. [21], where they use it to construct all graphs belonging to $\mathcal{R}(3K_2, K_3)$.

The next two results are similar to Lemma 2 and Theorem 2. We present the property of a graph F satisfying $F \rightarrow (mK_2, H)$ based on a Ramsey $((m-2)K_2, H)$-minimal graph and the relationship between graphs in $\mathcal{R}(mK_2, H)$ and the ones in $\mathcal{R}((m-2)K_2, H)$.

Lemma 3 *Let H be a graph and $m > 2$ be an integer. If $F \rightarrow (mK_2, H)$, then the following three conditions hold:*

(i) *for every $u, v \in V(F)$, $F - \{u, v\} \rightarrow ((m-2)K_2, H)$,*
(ii) *for every $u \in V(F)$ and K_3 in F, $F - \{u\} - E(K_3) \rightarrow ((m-2)K_2, H)$,*
(iii) *for every $2K_3$ in F, $F - E(2K_3) \rightarrow ((m-2)K_2, H)$.*

Proof Suppose to the contrary that at least one of three conditions is violated. Suppose first for some $u, v \in V(F)$, there exists an $((m-2)K_2, H)$-coloring ϕ_1 of edges of $F - \{u, v\}$. We now define a red-blue coloring ϕ of F such that $\phi(x) = \phi_1(x)$ for all $x \in E(F - \{u, v\})$ and $\phi(x) =$ red otherwise. Thus, ϕ is an (mK_2, H)-coloring of edges of F, a contradiction. A similar argument works for the conditions (ii) and (iii). □

Theorem 3 *Let H be a graph and $m > 2$ be an integer. If $F \in \mathcal{R}(mK_2, H)$, then for any $u, v \in V(F)$ and tK_3 in F with $t = 1, 2$, each of the graphs $F - \{u, v\}$, $F - \{u\} - E(K_3)$, and $F - E(2K_3)$ contains a Ramsey $((m-2)K_2, H)$-minimal graph.*

Proof It follows directly from Lemma 3. □

2.2 Disconnected Graphs in $\mathcal{R}(mK_2, H)$

In this section, we show that all disconnected graphs in $\mathcal{R}(mK_2, H)$ are obtained from a disjoint union of graphs in $\mathcal{R}(sK_2, H)$ and in $\mathcal{R}(tK_2, H)$, for any connected graph H and for every positive integer s, t, and m, where $s + t = m$. Moreover, we show a class of disconnected Ramsey $(mK_2, \bigcup_{i=1}^{t} H_i)$-minimal graphs for any connected graph H_i, for each $i \in [1, t]$.

Theorem 4 *Let F and G be graphs and H be a connected graph. The graph $F \cup G \in \mathcal{R}(mK_2, H)$ if and only if $F \in \mathcal{R}(sK_2, H)$ and $G \in \mathcal{R}((m-s)K_2, H)$ for every positive integer $s < m$.*

Proof Before the details of the proof is given, we begin with some colorings. Let ϕ_1 be an (sK_2, H)-coloring of edges of $F - e$ and ϕ_2 be a red-blue coloring of edges of G such that G contains a red $(m-s)K_2$ but it has no blue H.

Suppose to the contrary that $F \rightarrow (sK_2, H)$ and $G \rightarrow ((m-s)K_2, H)$ but $F \cup G \nrightarrow (mK_2, H)$. Then there is an (mK_2, H)-coloring ϕ of edges of $F \cup G$, namely $\phi(x) = \varphi(x)$ for all $x \in E(F)$ and $\phi(x) = \phi_2(x)$ for all $x \in E(G)$. Therefore, φ must be an (sK_2, H)-coloring of edges of F. This leads to a contradiction with $F \rightarrow (sK_2, H)$. To prove the minimality, suppose $e \in E(F \cup G)$. It suffices to consider $e \in E(F)$. Now, define ϕ as a red-blue coloring of edges of $(F \cup G) - e$ such that $\phi(x) = \phi_1(x)$ for all $x \in E(F - e)$ and $\phi(x) = \phi_2(x)$ for all $x \in E(G)$. We obtain an (mK_2, H)-coloring of edges of $(F \cup G) - e$.

Conversely, suppose to the contrary that $F \cup G \in \mathcal{R}(mK_2, H)$, $F \notin \mathcal{R}(sK_2, H)$ but $G \in \mathcal{R}((m-s)K_2, H)$ for some positive integer $s < m$. If $F \nrightarrow (sK_2, H)$, then define a red-blue coloring ϕ of $F \cup G$ such that $\phi(x) = \varphi(x)$ for all $x \in E(F)$ and

$\phi(x) = \phi_2(x)$ for all $x \in E(G)$. Then ϕ is an (mK_2, H)-coloring of edges of $F \cup G$, a contradiction. If $F \to (sK_2, H)$ but F is not minimal, then there exists a Ramsey (sK_2, H)-minimal graph $F^* \subseteq F$. By the first case, we have $F^* \cup G$ is a Ramsey (mK_2, H)-minimal graph. This contradicts the minimality of $F \cup G$. □

For a connected graph H, Theorem 4 shows that the characterization of disconnected graphs belonging to $\mathcal{R}(mK_2, H)$ has completely done. But, characterizing all connected graphs in $\mathcal{R}(mK_2, H)$ is still open. For a disconnected graph H, the following theorem provides a disconnected graph belonging to $\mathcal{R}(mK_2, H)$.

Theorem 5 *Let m and t be positive integers and H_i be a connected graph for $i \in [1, t]$. If $H_i \neq H_j$ and $H_i \nsubseteq H_j$ for every $i \neq j$, and $i, j \in [1, t]$, then $m\mathcal{H} \in \mathcal{R}(mK_2, \mathcal{H})$, where $\mathcal{H} = \bigcup_{i=1}^{t} H_i$.*

Proof Observe that $m\mathcal{H} \to (mK_2, \mathcal{H})$. We now prove that for each $e \in m\mathcal{H}$, $m\mathcal{H} - e \nrightarrow (mK_2, \mathcal{H})$. Observe that $m\mathcal{H} - e = (m-1)\mathcal{H} \cup (\mathcal{H} - e)$. We can only consider when $e \in H_1$. Hence, $\mathcal{H} - e = (H_1 - e) \cup \left(\bigcup_{i=2}^{t} H_i\right)$. Let us define a red-blue coloring ϕ of edges of $m\mathcal{H} - e$ such that every edge of H_1 in $(m-1)(\mathcal{H})$ is colored by red and the remaining edges are colored by blue. Under the coloring ϕ, the red subgraph of $m\mathcal{H} - e$ is a graph $(m-1)K_2$ and the blue subgraph of $m\mathcal{H} - e$ is a graph $m\left(\bigcup_{i=2}^{t} H_i\right) \cup m(H_1 - e)$. Since $H_i \neq H_j$ and $H_i \nsubseteq H_j$ for every $i \neq j$, and $i, j \in [1, t]$, the graph $m\left(\bigcup_{i=2}^{t} H_i\right) \cup m(H_1 - e)$ does not contain a graph H_1. So, ϕ is an $(mK_2, \mathcal{H})$-coloring of edges of $m\mathcal{H} - e$. □

2.3 Subdivision of Graphs in $\mathcal{R}(mK_2, P_3)$

In this section, we discuss how to obtain a graph in $\mathcal{R}((m+1)K_2, P_3)$, namely, by subdividing one non-pendant edge of a graph in $\mathcal{R}(mK_2, P_3)$. We begin with some lemmas.

Lemma 4 *Let H be a connected graph and m be a positive integer. Suppose $F \in \mathcal{R}(mK_2, H)$. For each $e \in E(F)$, let ϕ be an (mK_2, H)-coloring of edges of $F - e$. Then there exists a red $(m-1)K_2$ in $F - e$.*

Proof Let $F \in \mathcal{R}(mK_2, H)$, $e \in E(F)$, and ϕ be an (mK_2, H)-coloring of edges of $F - e$. Clearly, a red $(m-1)K_2$ in $F - e$ exists for $H = K_2$, since $\mathcal{R}(mK_2, K_2) = \{mK_2\}$. We now consider $H \neq K_2$. Suppose to the contrary that the red subgraph of $F - e$ contains a red $(m-2)K_2$ under coloring ϕ. Define ϕ_1 as a red-blue coloring of edges of F such that $\phi_1(x) = \phi(x)$ for all $x \in E(F - e)$ and $\phi_1(e) =$ red. Hence, ϕ_1 is an (mK_2, H)-coloring of edges of F, a contradiction. □

Lemma 5 *Let m be a positive integer, $F \in \mathcal{R}(mK_2, P_3)$, and $e = uv$ be an edge in F for some $u, v \in V(F)$. Let ϕ be an (mK_2, P_3)-coloring of edges of $F - e$. Then u or v is incident with a blue edge in $F - e$.*

Proof Suppose to the contrary that ϕ is an (mK_2, P_3)-coloring of edges of $F - e$ but both vertices u and v are incident with the red edges in $F - e$. Define ϕ_1 as a red-blue coloring of edges of F such that $\phi_1(x) = \phi(x)$ for all $x \in E(F - e)$ and $\phi_1(e) =$ blue. Hence, ϕ_1 is an (mK_2, P_3)-coloring of edges of F, a contradiction. □

Lemma 6 *Let m be a positive integer and $F \in \mathcal{R}(mK_2, P_3)$. Let $e = uv$ be an edge of F for some $u, v \in V(F)$. The following three statements are equivalent.*

(i) *There exists an (mK_2, P_3)-coloring of edges of $F - e$.*
(ii) *There exists a red-blue coloring of edges of F such that F contains a red $(m-1)K_2$ and a unique blue P_3 or P_4.*
(iii) *There exists a red-blue coloring of edges of F such that F contains a red mK_2, where one independent red edge is represented by a K_2 but F does not contain a blue P_3.*

Proof Let ϕ_1 be an (mK_2, P_3)-coloring of edges of $F - e$. Under the coloring ϕ_1, by Lemma 4, $F - e$ contains a red $(m-1)K_2$, and by Lemma 5, u or v is incident with a blue edge in $F - e$. Now, let ϕ be a red-blue coloring of edges of F such that $\phi(x) = \phi_1(x)$ for all $x \in E(F - e)$ and $\phi(e) =$ blue. Hence, F contains a red $(m-1)K_2$ and a unique blue P_3 or P_4. Next, we change the one of a blue edge in P_3 or the middle blue edge in P_4 to red ones. Then F contain a red mK_2 where one independent red edge is represented by a K_2 but F does not contain a blue P_3. Finally, by deleting the one independent red edge e represented by a K_2, we obtain an (mK_2, P_3)-coloring of edges of $F - e$. □

Our final theorem shows that if $F \in \mathcal{R}(mK_2, P_3)$, then any graph obtained by subdividing on one non-pendant edge e of F, for each $e \in E(F)$, will be in $\mathcal{R}((m+1)K_2, P_3)$. To do this, we begin with the following definition.

The *subdivision* (k vertices) of a graph G on the edge $e = uv$, denoted by $SG(e, k)$, is a graph obtained from the graph G by removing the edge e and adding k new vertices $w_1, w_2, \ldots, w_k$ and $(k+1)$ new edges uw_1, w_1w_2, w_2w_3, $\ldots$, $w_{k-1}w_k$, w_kv. Therefore, $SG(e, k)$ has the vertex set

$$V(SG(e,k)) = V(G) \cup \{w_1, w_2, \ldots, w_k\}$$

and the edge set

$$E(SG(e,k)) = E(G - e) \cup \{uw_1, w_1w_2, \ldots, w_{k-1}w_k, w_kv\}.$$

Let $F \in \mathcal{R}(mK_2, P_3)$ and e be a non-pendant edge of F. Suppose that $SF(e, 3)$ is the subdivision (3 vertices) of a graph F on the edge e. Let $SF(3) = \{SF(e, 3) \mid e \in E(F)$ and e is a non-pendant edge$\}$. Then we have the following theorem.

Theorem 6 *If $F \in \mathcal{R}(mK_2, P_3)$, then $SF(3) \subseteq \mathcal{R}((m+1)K_2, P_3)$.*

Proof Let $F^* \in SF(3)$. Then $F^* = SF(a, 3)$ for some the subdivided edge a in F. We will prove that $F^* \in \mathcal{R}((m+1)K_2, P_3)$. Suppose first to the contrary that $F \in \mathcal{R}(mK_2, P_3)$ but $F^* \nrightarrow ((m+1)K_2, P_3)$. It means that, there exists an $((m+1)K_2, P_3)$-coloring ϕ of F^*. By Lemma 4, there exists a red mK_2 in F^*. Then the edges a_1, a_2, a_3, a_4 can contribute to either a red K_2 or a red $2K_2$. If the edges a_1, a_2, a_3, a_4 contribute to a red K_2, then both edges a_1 and a_4 are not adjacent to each blue edge in F. Next, replace the edges a_1, a_2, a_3, a_4 with the edge a and color it with blue. Then F contains a red $(m-1)K_2$ but F does not contain a blue P_3, a contradiction. While, if the edges a_1, a_2, a_3, a_4 contribute to a red $2K_2$, and replace them with the edge a and color it by red, then F contains a red $(m-1)K_2$ but F does not contain a blue P_3, a contradiction. Hence, $F^* \rightarrow ((m+1)K_2, P_3)$.

It remains to show the minimality of F^*. Let $e \in E(F^*)$. There are two cases: either $e \in E(F)$ or $e \notin E(F)$ (it means that e is a_1, a_2, a_3, or a_4). We first consider $e \in E(F)$. Then by Lemma 6(i), there exists an (mK_2, P_3)-coloring ϕ_1 of $F - e$. Under the coloring ϕ_1, the subdivided edge a can have either a red or a blue color, namely either $\phi_1(a) =$ red or $\phi_1(a) =$ blue. Let us define ϕ be a red-blue coloring of edges of $F^* - e$ as follows. When $\phi_1(a) =$ red, color the edges a_1, a_3, and a_4 with red and a_2 with blue. When $\phi_1(a) =$ blue, color the edges a_2 and a_3 with red and a_1 and a_4 with blue. Otherwise $\phi(x) = \phi_1(x)$. We obtain an $((m+1)K_2, P_3)$-coloring ϕ of edges of $F^* - e$.

Next, we consider $e \notin E(F)$. Then e can be either a_1 or a_2, since a similar argument works for a_3 and a_4. We consider $e = a_1$. By Lemma 6(i), there exists an (mK_2, P_3)-coloring ψ_1 of $F - a$. If a_1 is deleted from F^*, then a_2 is a pendant edge of $F^* - a_1$. We define ψ as a red-blue coloring of edges of $F^* - a_1$ such that $\psi(x) = \psi_1(x)$ for all $x \in E(F - a)$, $\psi(a_3) = \psi(a_4) =$ red, and $\psi(a_2) =$ blue. By Lemma 6, under the coloring ψ, there exists neither a red $(m+1)K_2$ nor a blue P_3 in $F^* - a_1$. Hence, ψ is an $((m+1)K_2, P_3)$-coloring of edges of $F^* - e$. We now consider $e = a_2$. If a_2 is deleted from F^*, then both a_1 and a_3 are pendant edges of $F^* - a_2$. Let b be an edge of F which is adjacent to a_4. By Lemma 6(i), there is an (mK_2, P_3)-coloring φ_1 of $F - b$. Let us define φ be a red-blue coloring of edges of $F^* - a_2$, such that $\varphi(a_1) = \varphi(a_3) =$ blue, $\varphi(a_4) = \varphi(b) =$ red, and otherwise $\varphi(x) = \varphi_1(x)$. By Lemma 6, the edge a_1 is not adjacent to a blue edge. Thus, φ is an $((m+1)K_2, P_3)$-coloring of edges of $F^* - a_2$. Hence, for each $e \in E(F^*)$, there exists an $((m+1)K_2, P_3)$-coloring of edges of $F^* - e$. □

Acknowledgements This research was supported by Research Grant "Program Hibah Riset Unggulan ITB-DIKTI", Ministry of Research, Technology and Higher Education, Indonesia. The first author would like to acknowledge Prof. Zdeněk Ryjáček for providing support during her research visit of two months durations in October - November 2014 at University of West Bohemia in Pilsen.

References

1. Baskoro, E.T., Nababan, S.M., Miller, M.: On Ramsey numbers for trees versus wheels of five or six vertices. Graphs Comb. **18**(4), 717–721 (2002)
2. Baskoro, E.T., Vetrík, T., Yulianti, L.: A note on Ramsey $(K_{1,2}, C_4)$-minimal graphs of diameter 2. In: Proceedings of the International Conference 70 Years of FCE STU, pp. 1–4. Bratislava, Slovakia (2008)
3. Baskoro, E.T., Wijaya, K.: On Ramsey $(2K_2, K_4)$-minimal graphs. In: Mathematics in the 21st Century, Springer Proceedings in Mathematics & Statistics, vol. 98, pp. 11–17 (2015)
4. Baskoro, E.T., Yulianti, L.: On Ramsey minimal graphs for $2K_2$ versus P_n. Adv. Appl. Discrete Math. **8**(2), 83–90 (2011)
5. Baskoro, E.T., Yulianti, L., Assiyatun, H.: Ramsey $(K_{1,2}, C_4)$-minimal graphs. J. Combin. Math. Combin. Comput. **65**, 79–90 (2008)
6. Borowiecki, M., Hałuszczak, M., Sidorowicz, E.: On Ramsey minimal graphs. Discrete Math. **286**(1–2), 37–43 (2004)
7. Borowiecki, M., Schiermeyer, I., Sidorowicz, E.: Ramsey $(K_{1,2}, K_3)$-minimal graphs. Electron. J. Combin. **12**, R20 (2005)

8. Borowiecka-Olszewska, M., Hałuszczak, M.: On Ramsey $(K_{1,m}, \mathcal{G})$-minimal graphs. Discrete Math. **313**(19), 1843–1855 (2012)
9. Burr, S.A.: Generalized Ramsey Theory for Graphs–A Survey. Graphs and Combinatorics. Springer, Berlin (1973)
10. Burr, S.A., Erdös, P., Faudree, R.J., Schelp, R.H.: A class of Ramsey-finite graphs. In: Proceeding of the Ninth Southeastern Conference on Combinatorics, Graph Theory and Computing, pp. 171–180. Boca Raton (1978)
11. Burr, S.A., Erdös, P., Faudree, R.J., Schelp, R.H.: Ramsey minimal graphs for matchings. In: The Theory and Applications of Graphs (Kalamazoo, Mich.) (1980), pp. 159–168. Wiley, New York (1981)
12. Burr, S.A., Erdös, P., Faudree, R.J., Rousseau, C.C., Schelp, R.H.: Ramsey minimal graphs for forests. Discrete Math. **38**(1), 23–32 (1982)
13. Jayawardene, C.J., Rousseau, C.C.: The Ramsey number for a cycle of length five vs. a complete graph of order six. J. Graph Theory **35**, 99–108 (2000)
14. Łuczak, T.: On Ramsey minimal graphs. Electron. J. Combin. **1**, R4 (1994)
15. Mengersen, I., Oeckermann, J.: Matching-star Ramsey sets. Discrete Appl. Math. **95**, 417–424 (1999)
16. Muhshi, H., Baskoro, E.T.: On Ramsey $(3K_2, P_3)$-minimal graphs. AIP Conference Proceeding **1450**, 110–117 (2012)
17. Surahmat, Baskoro, E.T., Broersma, H.J.: The Ramsey numbers of large cycles versus small wheels. INTEGERS Electron. J. Combin. Number Theory 4(A#10), 1–9 (2004)
18. Vetrík, T., Yulianti, L., Baskoro, E.T.: On Ramsey $(K_{1,2}, C_4)$-minimal graphs. Discuss. Math. Graph Theory **30**, 637–649 (2010)
19. Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D.: The complete list of Ramsey $(2K_2, K_4)$-minimal graphs. Electron. J. Graph Theory Appl. **3**(2), 216–227 (2015)
20. Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D.: On unicyclic Ramsey (mK_2, P_3)-minimal graphs. Proc. Comput. Sci. **74**, 10–14 (2015)
21. Wijaya, K., Baskoro, E.T., Assiyatun, H., Suprijanto, D.: On Ramsey $(3K_2, K_3)$-minimal graphs. In: AIP Conference Proceedings, vol. 1707, pp. 020025 (2016). doi:10.1063/1.4940826
22. Wijaya, K., Yulianti, L., Baskoro, E.T., Assiyatun, H., Suprijanto, D.: All Ramsey $(2K_2, C_4)$-minimal graphs. J. Algorithms Comput. **46**, 9–25 (2015)
23. Yulianti, L., Assiyatun, H., Uttunggadewa, S., Baskoro, E.T.: On Ramsey $(K_{1,2}, P_4)$-minimal graphs. Far East J. Math. Sci. **40**(1), 23–36 (2010)