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On Ramsey $(4K_2, P_3)$-minimal graphs

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Abstract

Let F, G, and H be simple graphs. We write $F \to (G, H)$ to mean that any red–blue coloring of all edges of F will contain either a red copy of G or a blue copy of H. A graph F (without isolated vertices) satisfying $F \to (G, H)$ and for each $e \in E(F)$, $(F - e) \nrightarrow (G, H)$ is called a Ramsey (G, H)-minimal graph. The set of all Ramsey (G, H)-minimal graphs is denoted by $\mathscr{R}(G, H)$. In this paper, we derive the necessary and sufficient condition of graphs belonging to $\mathscr{R}(4K_2, H)$, for any connected graph H. Moreover, we give a relation between Ramsey $(4K_2, P_3)$- and $(3K_2, P_3)$-minimal graphs, and Ramsey $(4K_2, P_3)$- and $(2K_2, P_3)$-minimal graphs. Furthermore, we determine all graphs in $\mathscr{R}(4K_2, P_3)$.

Keywords: Ramsey minimal graph; Edge coloring; Matching; Path

1. Introduction

Let F be a graph with n vertices and m edges. If $v \in V(F)$ and $e \in E(F)$, then $F - \{v\}$ is a graph on $n - 1$ vertices obtained by deleting the vertex v together with all edges incident with v, and $F - e$ is a graph on $m - 1$ edges obtained by deleting the edge e from F. A complete graph, cycle, and path with n vertices are denoted by K_n, C_n, and P_n, respectively. mK_2 will denote a graph consisting of m disjoint copies of a K_2.

Let F, G, and H be graphs without isolated vertices. We write $F \to (G, H)$ to mean that any red–blue coloring of the edges of F will contain either a red copy of G or a blue copy of H. A red–blue coloring of F such that neither a red G nor a blue H occurs is called a (G, H)-coloring. A graph F will be called a Ramsey (G, H)-minimal if $F \to (G, H)$ but for each $e \in E(F)$, $(F - e) \nrightarrow (G, H)$. The set of all Ramsey (G, H)-minimal graphs will be denoted by $\mathscr{R}(G, H)$.

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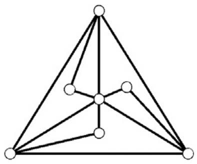

Fig. 1. Graph in $\mathscr{R}(2K_2, K_3)$.

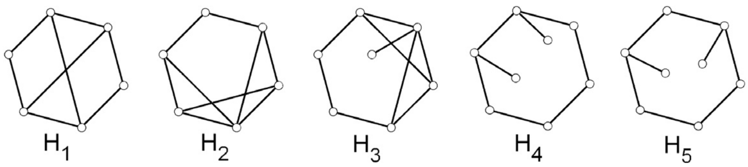

Fig. 2. All connected graphs with circumference 6 in $\mathscr{R}(3K_2, P_3)$.

The problem of characterizing all graphs F in $\mathscr{R}(G, H)$ for a fixed pair of graphs G and H is very interesting but it is also a difficult problem, even for small graphs G and H. Burr [1] showed that deciding whether $F \nrightarrow (G, H)$ is an NP-complete problem if G and H are fixed 3-connected graphs (or triangles).

Numerous papers discuss the problem of determining the set $\mathscr{R}(G, H)$. In particular, Burr et al. [2] proved that the set $\mathscr{R}(mK_2, H)$ is finite for any graph H. In particular, they proved that $\mathscr{R}(2K_2, 2K_2) = \{3K_2, C_5\}$, $\mathscr{R}(2K_2, K_3) = \{2K_3, K_5, G\}$, where G is the graph in Fig. 1. They also gave the maximal number of edges of graph F belonging to $\mathscr{R}(mK_2, H)$, that is $|E(F)| \leq \sum_{i=1}^{b} n^i$ where $n = |V(H)|$ and $b = (m-1)\left(\binom{2m-1}{2}+1\right)+1$ for m a positive integer. Burr et al. [3] gave some characterizations of all graphs in $\mathscr{R}(tK_2, 2K_2)$ for any $t \geq 2$. Mengersen and Oeckermann [4] proved that $\mathscr{R}(2K_2, P_3) = \{2P_3, C_4, C_5\}$. In the same paper, they also determined all graphs in $\mathscr{R}(2K_2, K_{1,3})$. Baskoro and Yulianti [5] determined all graphs in $\mathscr{R}(2K_2, P_n)$ for $n = 4, 5$. The characterization of all graphs which belong to $\mathscr{R}(2K_2, 2P_n)$ for $n = 4, 5$ was given by Tatanto and Baskoro [6]. Mushi and Baskoro [7] derived the properties of graphs belonging to the class $\mathscr{R}(3K_2, P_3)$ and determined all graphs in this class. They proved that $\mathscr{R}(3K_2, P_3) = \{3P_3, C_4 \cup P_3, C_5 \cup P_3, C_7, C_8, H_1, H_2, H_3, H_4, H_5\}$ where H_1, H_2, H_3, H_4, and H_5 are the graphs in Fig. 2.

The following lemma about the necessary conditions of graphs in $\mathscr{R}(3K_2, P_3)$ given by Mushi and Baskoro [7].

Lemma 1.1 *([7]). Let $F \in \mathscr{R}(3K_2, P_3)$. Then*

(i) *$F - \{u, v\} \supseteq P_3$ for every $u, v \in V(F)$;*
(ii) *$F - \{u\} - E(C_3) \supseteq P_3$ for every $u \in V(F)$ and $C_3 \subseteq F$;*
(iii) *$F - E(2C_3) \supseteq P_3$ for every $2C_3 \subseteq F$;*
(iv) *$F - E(F_m^*) \supseteq P_3$ for every $F_m^* \subseteq F$ where F_m^* is an induced connected subgraph with m vertices ($m = 4$ or 5);*
(v) *Every vertex in F is contained in some P_3 in F.*

In this paper, we derive the necessary and sufficient condition for graphs belonging to $\mathscr{R}(4K_2, H)$. We give a relation between Ramsey $(4K_2, P_3)$-minimal graphs and Ramsey $(3K_2, P_3)$-minimal graphs as well as Ramsey $(2K_2, P_3)$-minimal graphs. Finally, we give characterizations of all graphs in $\mathscr{R}(4K_2, P_3)$.

2. Main results

2.1. Necessary and sufficient conditions

In this section, we derive the necessary and sufficient conditions for graphs in $\mathscr{R}(4K_2, H)$ for any connected graph H. Let F be a graph in $\mathscr{R}(4K_2, H)$. Let X, Y, and Z be induced subgraphs of F by 3, 5, and 7 vertices, respectively. If there are two different induced subgraphs on 3 vertices of F, then we will use the notation X_1 and X_2, where $V(X_1) \neq V(X_2)$. Then, we have the following theorem.

Theorem 2.1. *Let H be a connected graph. $F \in \mathscr{R}(4K_2, H)$ if and only if the following conditions are satisfied:*

(i) *For every $u, v, w \in V(F)$, $F - \{u, v, w\} \supseteq H$;*
(ii) *For every $u, v \in V(F)$ and X in F, $F - \{u, v\} - E(X) \supseteq H$;*
(iii) *For every $u \in V(F)$ and X_1, X_2 in F, $F - \{u\} - E(X_1 \cup X_2) \supseteq H$;*
(iv) *For every X_1, X_2, X_3 in F, $F - E(X_1 \cup X_2 \cup X_3) \supseteq H$;*
(v) *For every $u \in V(F)$ and Y in F, $F - \{u\} - E(Y) \supseteq H$;*
(vi) *For every X and Y in F, $F - E(X \cup Y) \supseteq H$;*
(vii) *For every Z in F, $F - E(Z) \supseteq H$;*
(viii) *For every edge $e \in E(F)$, at least one of seven conditions below is satisfied.*

(a) *There exists $u, v, w \in V(F)$ such that $(F - e) - \{u, v, w\} \not\supseteq H$;*
(b) *There exists $u, v \in V(F)$ and X in F such that $(F - e) - \{u, v\} - E(X) \not\supseteq H$;*
(c) *There exists $u \in V(F)$ and X_1, X_2 in F such that $(F - e) - \{u\} - E(X_1 \cup X_2) \not\supseteq H$;*
(d) *There exists X_1, X_2, X_3 in F such that $(F - e) - E(X_1 \cup X_2 \cup X_3) \not\supseteq H$;*
(e) *There exists $u \in V(F)$ and Y in F such that $(F - e) - \{u\} - E(Y) \not\supseteq H$;*
(f) *There exists X and Y in F such that $(F - e) - E(X \cup Y) \not\supseteq H$;*
(g) *There exists Z in F such that $(F - e) - E(Z) \not\supseteq H$.*

Proof. Let H be a connected graph. Let $F \in \mathscr{R}(4K_2, H)$. So, $F \to (4K_2, H)$ and for each edge $e \in E(F)$, $(F - e) \nrightarrow (4K_2, H)$. We first consider $F \to (4K_2, H)$. We will prove that cases (i)–(vii) are satisfied. Suppose to the contrary that at least one of cases (i)–(vii) is violated. Then, color by red all edges incident to u, v, or w in cases (i)–(iii); all edges of all X in cases (ii), (iii), and (iv); all edges of Y in cases (v) and (vi); or all edges of Z in case (vii). Next, the remaining edges are colored by blue. Then, in any case we obtain a $(4K_2, H)$-coloring of F, a contradiction.

We now consider for each edge $e \in E(F)$, $(F - e) \nrightarrow (4K_2, H)$. Then, there exists a $(4K_2, H)$-coloring of $F - e$. In such a coloring, the subgraph of $F - e$ induced by all red edges does not contain a $4K_2$ and the subgraph of $F - e$ induced by all blue edges does not contain an H. Thus, the subgraph of $F - e$ induced by all blue edges is one of subgraphs: $(F - e) - \{u, v, w\}$, $(F - e) - \{u, v\} - E(X)$, $(F - e) - \{u\} - E(X_1 \cup X_2)$, $(F - e) - E(X_1 \cup X_2 \cup X_3)$, $(F - e) - \{u\} - E(Y)$, $(F - e) - E(X \cup Y)$, or $(F - e) - E(Z)$, for some $u, v, w \in V(F)$ and the induced subgraphs X, X_1, X_2, X_3, Y, Z of F whose order are $3, 3, 3, 3, 5, 7$, respectively. Thus, we obtain case (viii).

Conversely, suppose that all cases (i)–(viii) are satisfied. Consider any red–blue coloring of F not containing a red $4K_2$. Then, we have either all blue edges or the subgraph of F induced by all red edges contains at most 3 independent edges. Now, remove all red edges. This removal can be done by one of the cases (i)–(vii), namely deleting three vertices in case (i), deleting two vertices and all edges of the induced subgraph with 3 vertices in case (ii), deleting one vertex and all edges of two induced subgraphs with 3 vertices in case (iii), deleting all edges of three induced subgraphs with 3 vertices in case (iv), deleting one vertex and all edges of the induced subgraph with 5 vertices in case (v), deleting all edges of the induced subgraphs with 3 and 5 vertices in case (vi), or deleting all edges of the induced subgraph with 7 vertices in case (vii). In all cases, the existence of a blue H occurs. Thus, by cases (i)–(vii), we obtain $F \to (4K_2, H)$.

We now consider case (viii). We define a red–blue coloring ϕ of all edges of $F - e$ such that $\phi(x) =$ blue for every edge x in one of subgraphs $(F - e) - \{u, v, w\}$, $(F - e) - \{u, v\} - E(X)$, $(F - e) - \{u\} - E(X_1 \cup X_2)$, $(F - e) - E(X_1 \cup X_2 \cup X_3)$, $(F - e) - \{u\} - E(Y)$, $(F - e) - E(X) - E(Y)$, or $(F - e) - E(Z)$, for some $u, v, w \in V(F)$ and the induced subgraphs X, X_1, X_2, X_3, Y, Z of F whose order are $3, 3, 3, 3, 5, 7$, respectively and $\phi(x) =$ red otherwise. We obtain a $(4K_2, H)$-coloring ϕ of $F - e$. Thus, for each edge $e \in E(F)$, $(F - e) \nrightarrow (4K_2, H)$. ■

We now give some relations between a graph in $\mathscr{R}(4K_2, P_3)$ and graph in $\mathscr{R}(tK_2, P_3)$ for $t = 3$ and $t = 2$ in the following lemmas.

Lemma 2.2. *$F \to (4K_2, P_3)$ if and only if the following conditions are satisfied:*

(i) *for every $v \in V(F)$, $F - \{v\} \to (3K_2, P_3)$;*
(ii) *for every $C_3 \subset F$, $F - E(C_3) \to (3K_2, P_3)$;*
(iii) *for every induced subgraph on 7 vertices Z of F, $F - E(Z) \supseteq P_3$.*

Proof. Suppose to the contrary that at least one of cases (i)–(iii) is violated. Then, there exists a $(3K_2, P_3)$-coloring ϕ_1 of the edges of either $F - \{v\}$ in case (i) or $F - E(C_3)$ in case (ii). Now, let us define a new coloring ϕ of the edges of F such that $\phi(e) = \phi_1(e)$ for $e \in F - \{v\}$ in case (i) or $e \in F - E(C_3)$ in case (ii), and $\phi(e) =$ red for all edges e incident to v in case (i) or all edges $e \in E(C_3)$ in case (ii). In both cases, we obtain a $(4K_2, P_3)$-coloring of F, a contradiction. Next, suppose that $F - E(Z)$ does not contain a P_3 for an induced subgraph Z of F on 7 vertices. Color all edges $e \in E(Z)$ by red and otherwise by blue. We obtain a $(4K_2, P_3)$-coloring of F, a contradiction.

Conversely, let all cases (i)–(iii) be satisfied. By applying Lemma 1.1, the cases (i)–(vii) in Theorem 2.1 are satisfied. We obtain $F \to (4K_2, P_3)$. ■

Corollary 2.3. *Let $F \in \mathscr{R}(4K_2, P_3)$. For every $v \in V(F)$ and C_3 in F, then graphs $F - \{v\}$ and $F - E(C_3)$ contain a Ramsey $(3K_2, P_3)$-minimal graph.*

Proof. Suppose one of $F - \{v\}$ or $F - E(C_3)$ does not contain $G \in \mathscr{R}(3K_2, P_3)$ for some $v \in V(F)$ or C_3 in F. Then $F - \{v\} \nrightarrow (3K_2, P_3)$ or $F - E(C_3) \nrightarrow (3K_2, P_3)$. By Lemma 2.2, $F \nrightarrow (4K_2, P_3)$. This contradicts $F \in \mathscr{R}(4K_2, P_3)$. ■

Lemma 2.4. *If $F \to (4K_2, P_3)$, then*

(i) *for every $u, v \in V(F)$, $F - \{u, v\} \to (2K_2, P_3)$;*
(ii) *for every $u \in V(F)$ and C_3 in F, $F - \{u\} - E(C_3) \to (2K_2, P_3)$;*
(iii) *for every $2C_3$ in F, $F - E(2C_3) \to (2K_2, P_3)$.*

Proof. Suppose that at least one of cases (i)–(iii) is violated for some $u, v \in V(F)$, C_3 in F, or $2C_3$ in F. Then, there exists a $(2K_2, P_3)$-coloring ϕ_1 of all edges of either $F - \{u, v\}$, $F - \{u\} - E(C_3)$, or $F - E(2C_3)$. We now define a red–blue coloring ϕ of F such that $\phi(e) = \phi_1(e)$ for all edges $e \in F - \{u, v\}$, $e \in F - \{u\} - E(C_3)$, or $e \in F - E(2C_3)$ and $\phi(e) =$ red otherwise. In any case, we obtain a $(4K_2, P_3)$-coloring of F, a contradiction. ■

Corollary 2.5. *If $F \in \mathscr{R}(4K_2, P_3)$, then for every $u, v \in V(F)$ and tC_3 in F with $t = 1, 2$, all graphs $F - \{u, v\}$, $F - \{u\} - E(C_3)$, and $F - E(2C_3)$ contain a Ramsey $(2K_2, P_3)$-minimal graph.*

Proof. It follows directly from Lemma 2.4. ■

2.2. $\mathscr{R}(4K_2, P_3)$

In this section, we determine all graphs in $\mathscr{R}(4K_2, P_3)$. These graphs are constructed by applying Theorem 2.1. These graphs can be connected or disconnected. We first prove that all disconnected graphs in $\mathscr{R}(mK_2, P_3)$ are a disjoint union of graphs in $\mathscr{R}(sK_2, P_3)$ and $\mathscr{R}((m - s)K_2, P_3)$ for any positive integers $s, m \geq 1$ and $s < m$. We then give all disconnected graphs in $\mathscr{R}(4K_2, P_3)$.

Theorem 2.6. *Let G and H be connected graphs. The graph $G \cup H \in \mathscr{R}(mK_2, P_3)$ if and only if $G \in \mathscr{R}(sK_2, P_3)$ and $H \in \mathscr{R}((m - s)K_2, P_3)$ for any integers $s, m \geq 1$ and $s < m$.*

Proof. First, we prove that $G \cup H \to (mK_2, P_3)$ if $G \in \mathscr{R}(sK_2, P_3)$ and $H \in \mathscr{R}((m - s)K_2, P_3)$ for any integers $s, m \geq 1$ and $s < m$. Let φ_1 be a red–blue coloring of G such that G contains at most $(s - 1)$ independent red edges (form a red $(s - 1)K_2$) and a blue P_3. Let φ_2 be a red–blue coloring of H such that H contains a red $(m - s)K_2$ and no blue P_3. Use such a coloring in $G \cup H$. Thus, the red–blue coloring of $G \cup H$ implies $G \cup H$ containing at most $(m - 1)$ independent red edges (form a red $(m - 1)K_2$) and a blue P_3. Now, we show that for every $e \in E(G \cup H)$, $(G \cup H) - e \nrightarrow (mK_2, P_3)$. Without loss of generality, we need only to consider when $e \in E(G)$. So, $G - e \nrightarrow (sK_2, P_3)$. Thus, there exists an (sK_2, P_3)-coloring ϕ_1 of all edges of $G - e$. We define a red–blue coloring ϕ of all edges of $(G \cup H) - e$ such that $\phi(a) = \phi_1(a)$ for all edges $a \in E(G - e)$ and $\phi(a) = \varphi_2(a)$ for all edges $a \in E(H)$. Thus, we obtain an (mK_2, P_3)-coloring ϕ of $(G \cup H) - e$.

Conversely, let $G \cup H \in \mathscr{R}(mK_2, P_3)$. Suppose that $G \notin \mathscr{R}(sK_2, P_3)$ for a positive integer s. If $G \nrightarrow (sK_2, P_3)$, then there exists an (sK_2, P_3)-coloring ϕ_2 of all edges of G. Now, let us define a red–blue coloring ϕ of $G \cup H$ such that $\phi(e) = \phi_2(e)$ for all edges $e \in E(G)$ and $\phi(e) = \varphi_2(e)$ for all edges $e \in E(H)$. Then, ϕ is an (mK_2, P_3)-coloring of $G \cup H$. It means that $G \cup H \nrightarrow (mK_2, P_3)$, a contradiction. If $G \rightarrow (sK_2, P_3)$ but G is not minimal, then there exists a Ramsey (sK_2, P_3)-minimal graph $G^* \subseteq G$. By the first case, we have $G^* \cup H$ is a Ramsey (mK_2, P_3)-minimal graph, a contradiction to the minimality of $G \cup H$. ■

Corollary 2.7. *The only disconnected graphs in* $\mathscr{R}(4K_2, P_3)$ *are* $4P_3$, $C_4 \cup 2P_3$, $C_5 \cup 2P_3$, $2C_4$, $2C_5$, $C_4 \cup C_5$, $C_7 \cup P_3$, $C_8 \cup P_3$, $H_1 \cup P_3$, $H_2 \cup P_3$, $H_3 \cup P_3$, $H_4 \cup P_3$, $H_5 \cup P_3$, *where* H_i *for* $i \in [1, 5]$ *is the graph depicted in Fig. 2.*

Proof. We know that $\mathscr{R}(K_2, P_3) = \{P_3\}$ in [2], $\mathscr{R}(2K_2, P_3) = \{2P_3, C_4, C_5\}$ in [4] and $\mathscr{R}(3K_2, P_3) = \{3P_3, C_4 \cup P_3, C_5 \cup P_3, C_7, C_8, H_1, H_2, H_3, H_4, H_5\}$ in [7]. We then apply Theorem 2.6. ■

Start now, we will investigate all connected graphs in $\mathscr{R}(4K_2, P_3)$. We will prove that all connected graphs in $\mathscr{R}(4K_2, P_3)$ must contain a cycle. For $u, v \in V$, we will use the notation $u \sim v$ to denote u adjacent to v. Observe that if $F, G \in \mathscr{R}(4K_2, P_3)$, then by the minimality property $F \nsubseteq G$ and $G \nsubseteq F$. In the following, we will use this fact to eliminate graphs not belonging to $\mathscr{R}(4K_2, P_3)$.

Lemma 2.8. $\mathscr{R}(4K_2, P_3)$ *contains no tree.*

Proof. Suppose to the contrary that $\mathscr{R}(4K_2, P_3)$ contains a tree T. Let L be the longest path in T, then $|V(L)| \leq 11$. Otherwise $T \supseteq 4P_3$, contradict to the minimality of F. Suppose $V(L) = \{v_1, v_2, \ldots, v_\ell\}$. We consider the vertex v_2. If $d(v_2) = 2$, then v_2 is only adjacent to two vertices in $V(L)$, namely v_1 and v_3. By Corollary 2.3, $T - \{v_3\}$ must contain a Ramsey minimal graph $G \in \mathscr{R}(3K_2, P_3)$. So, G is a acyclic graph. Hence, $G = 3P_3$. Clearly v_1, v_2, v_3 are not contained in G. It implies that $T \supseteq 4P_3$, a contradiction. If $d(v_2) \geq 3$, then there exists a vertex $u \notin V(L)$ such that $u \sim v_2$. Since L is the longest path in T, $d(u) = 1$. By Corollary 2.3, $T - \{v_2\} \supseteq G$ for some $G \in \mathscr{R}(3K_2, P_3)$. So, $G = 3P_3$. Clearly v_1, v_2, u are not contained in G. It implies that $T \supseteq 4P_3$, a contradiction. ■

Since there is no tree in $\mathscr{R}(4K_2, P_3)$, for every connected graph in $\mathscr{R}(4K_2, P_3)$ must contain a cycle. Therefore, we will construct all connected graphs in $\mathscr{R}(4K_2, P_3)$ based on the circumference. The *circumference* is the length of the longest cycle in a graph. In general, the construction graph in $\mathscr{R}(4K_2, P_3)$ is done by applying Theorem 2.1.

Lemma 2.9. *Let* F *be a connected graph in* $\mathscr{R}(mK_2, P_3)$. *If* t *is the circumference of* F, *then* $3 \leq t \leq 3m - 1$.

Proof. Let t be the circumference of F. It implies $t \geq 3$. Next, suppose that $t \geq 3m$, then F contains C_{3m}. Hence, $F \supseteq mP_3$. By Theorem 2.6, $mP_3 \in \mathscr{R}(mK_2, P_3)$. So, F is not minimal, a contradiction. ■

In the following lemmas, we show that the set $\mathscr{R}(4K_2, P_3)$ contains no connected graphs with circumferences 3, 4, or 5.

Lemma 2.10. $\mathscr{R}(4K_2, P_3)$ *contains no connected graph with circumference* 3.

Proof. Suppose to the contrary that there exists a connected graph F with circumference 3 such that $F \in \mathscr{R}(4K_2, P_3)$. Let C_3 be a cycle in F, where $V(C_3) = \{v_1, v_2, v_3\}$. By Corollary 2.5, $F - \{v_1, v_2\}$ contains a Ramsey minimal $G \in \mathscr{R}(2K_2, P_3)$. Since $2P_3$ is the only graph in $\mathscr{R}(2K_2, P_3)$ having the circumference less than 3, $G = 2P_3$. Thus, we obtain $E(F) \supseteq E(C_3) \cup E(2P_3)$ and $V(C_3) \cap V(2P_3) = \{v_3\}$. Next, by Corollary 2.3, there must be a P_3 in $F - E(C_3)$ containing no vertices of $2P_3$. Without loss of generality, we assume the vertex v_2 is contained in the P_3. We obtain $E(F) \supseteq E(C_3) \cup E(3P_3)$ and $V(C_3) \cap V(3P_3) = \{v_2, v_3\}$. Next, by Corollary 2.3, $F - \{v_2\}$ must contain a $3P_3$. But the $3P_3$ in $F - \{v_2\}$ implies that $F \supseteq 4P_3$, contradicts to the minimality of F. ■

Lemma 2.11. $\mathscr{R}(4K_2, P_3)$ *contains no connected graph with circumference* 4.

Proof. Suppose to the contrary that there exists a connected graph $F \in \mathscr{R}(4K_2, P_3)$ with circumference 4. Then, F contains a C_4, where $V(C_4) = \{v_1, v_2, v_3, v_4\}$. By Corollary 2.3, $F - \{v_1\}$ contains a Ramsey minimal $G \in \mathscr{R}(3K_2, P_3)$. Since $3P_3$ and $C_4 \cup P_3$ are the only graphs in $\mathscr{R}(3K_2, P_3)$ having the circumference at most 4,

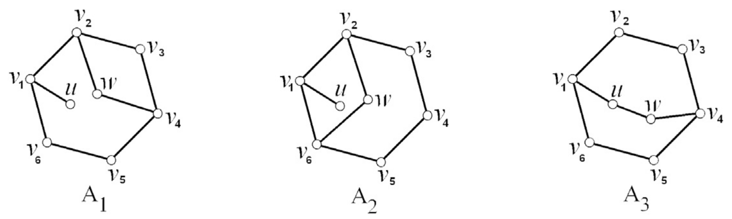

Fig. 3. Graphs A_1, A_2, and A_3.

$G = 3P_3$ or $G = C_4 \cup P_3$. We now consider $F - \{v_1\} \supseteq 3P_3$. Then, there is at most a P_3 containing no vertices in $V(C_4 \setminus v_1)$. Otherwise $F \supseteq C_4 \cup 2P_3$. Next, by Corollary 2.3, for every $i = 2, 3, 4$, $F - \{v_i\}$ must contain a $3P_3$. Otherwise F is not minimal or F has circumference greater than 4. But the $3P_3$ in $F - \{v_i\}$ forces that F is not minimal, a contradiction.

We next consider $F - \{v_1\} \supseteq C_4 \cup P_3$. Since $F \not\supseteq 2C_4$, F contains a C_4 containing either (i) one vertex in $V(C_4 \setminus v_1)$, say v_2 or (ii) all vertices in $V(C_4 \setminus v_1)$. Otherwise F has circumference greater than 4. For case (i), by Corollary 2.3, $F - \{v_2\} \supseteq 3P_3$, otherwise F is not minimal. But the $3P_3$ in $F - \{v_2\}$ implies that F is not minimal. For case (ii), there exists a vertex $u \in V(F \setminus C_4)$ such that $uv_2, uv_4 \in E(F)$. By Corollary 2.3, $F - \{v_2\}$ can contain a $3P_3$ or $C_4 \cup P_3$. But the $3P_3$ or $C_4 \cup P_3$ in $F - \{v_2\}$ implies that F is not minimal or F has circumference greater than 4, a contradiction. ■

Lemma 2.12. *$\mathscr{R}(4K_2, P_3)$ contains no connected graph with circumference 5.*

Proof. Suppose to the contrary that there exists a connected graph $F \in \mathscr{R}(4K_2, P_3)$ with circumference 5. Then, F contains a C_5, where $V(C_5) = \{v_1, v_2, \ldots, v_5\}$. By Theorem 2.1(vii), F has order at least 8. By Corollary 2.3, $F - \{v_1\}$ contains a $G \in \mathscr{R}(3K_2, P_3)$. Then, one of the following 3 cases must hold: (i) $G = 3P_3$, (ii) $G = C_4 \cup P_3$, or (iii) $G = C_5 \cup P_3$.

For case (i), $F - \{v_1\} \supseteq 3P_3$. Consider $3P_3 = 2P_3 \cup P_3$. Since there is a P_3 in C_5, say $v_3v_4v_5$, then v_2 is contained in $2P_3$. By Corollary 2.3, $F - \{v_2\}$ must contain a Ramsey minimal $G \in \mathscr{R}(3K_2, P_3)$. By the minimality of F, there is no G satisfying this condition.

For case (ii), $F - \{v_1\} \supseteq C_4 \cup P_3$. Then, one of the following 3 cases must hold: (a) one vertex in $V(C_5 \setminus v_1)$, say v_2 is contained in a C_4, (b) 3 vertices in $V(C_5 \setminus v_1)$, say v_2, v_3, v_4 are contained in a C_4, or (c) all vertices in $V(C_5 \setminus v_1)$ are contained in a C_4. Otherwise F has circumference greater than 5. Next, by Corollary 2.3, $F - \{v_2\}$ must contain a Ramsey minimal $G \in \mathscr{R}(3K_2, P_3)$. By the minimality of F, there is no G satisfying this condition.

For case (iii), $F - \{v_1\} \supseteq C_5 \cup P_3$. Then, one of the following 2 cases must hold: (a) one vertex in $V(C_5 \setminus v_1)$, say v_2 is contained in a C_5 or (b) all vertices in $V(C_5 \setminus v_1)$ are contained in a C_5. Otherwise F has circumference greater than 5. Next, by Corollary 2.3, $F - \{v_2\}$ must contain a Ramsey minimal $G \in \mathscr{R}(3K_2, P_3)$. By the minimality of F, there is no G satisfying this condition. ■

We now construct graphs with circumference 6 in $\mathscr{R}(4K_2, P_3)$. Furthermore, we show that the set $\mathscr{R}(4K_2, P_3)$ contains no graph with circumference 7. We first consider graphs A_1, A_2, and A_3 as pictured in Fig. 3.

Lemma 2.13. *Let F be a connected graph with circumference 6. If $F \in \mathscr{R}(4K_2, P_3)$ then F contains A_3, where A_3 is the graph as depicted in Fig. 3.*

Proof. Let F be a connected graph with circumference 6. So, F contains a C_6, where $V(C_6) = \{v_1, v_2, \ldots, v_6\}$. If $F \in \mathscr{R}(4K_2, P_3)$, then $V(F) \geq 8$, by Theorem 2.1(vii). Now, suppose $u, w \in V(F \setminus C_6)$ and assume u adjacent to v_1. By Theorem 2.1(i) and (vii), $F - \{v_1, v_3, v_5\}$ and $F - E(Z)$ must contain a P_3 where $V(Z) = V(C_6) \cup \{u\}$. Since we cannot have a cycle of length greater than 6 in F, then one of the following 3 cases must hold: (i) both $w \sim v_2$ and $w \sim v_4$, (ii) both $w \sim v_2$ and $w \sim v_6$, or (iii) both $w \sim u$ and $w \sim v_4$ (the graphs A_1, A_2, A_3, respectively in Fig. 3). Therefore, if $F \in \mathscr{R}(4K_2, P_3)$, then F must contain either A_1, A_2, or A_3.

Fig. 4. All connected graphs with circumference 6 in $\mathscr{R}(4K_2, P_3)$.

We now prove that if $F \in \mathscr{R}(4K_2, P_3)$, then F contains neither A_1 nor A_2. Let us consider $F \supseteq A_1$. By Theorem 2.1(i), $F - \{v_1, v_2, v_4\}$ must contain a P_3. Then, one of the following 3 cases must hold: (i) $x_1 \sim v_5$, (ii) $x_1 \sim v_6$, or (iii) both $x_1 \sim v_3$ and $x_1 \sim x_2$ (up to isomorphism), where $x_1, x_2 \in V(F \setminus A_1)$. For all cases, by Theorem 2.1(i), $F - \{v_2, v_4, v_6\}$ and $F - \{v_1, v_3, v_4\}$ must contain a P_3. But the P_3 in both $F - \{v_2, v_4, v_6\}$ and $F - \{v_1, v_3, v_4\}$ forces F containing $C_4 \cup 2P_3$, $H_4 \cup P_3$, or $H_5 \cup P_3$, a contradiction to the minimality of F.

Let us consider $F \supseteq A_2$. By Theorem 2.1(i), $F - \{v_2, v_4, v_6\}$ must contain a P_3. Then one of the following 4 cases must hold: (i) $x_1 \sim v_1$, (ii) $x_1 \sim u$, (iii) both $v_3 \sim x_1$ and $x_1 \sim x_2$, or (iv) both $w \sim x_1$ and $x_1 \sim x_2$, where $x_1, x_2 \in V(F \setminus A_1)$. For cases (i) and (ii), by Corollary 2.3, $F - \{v_1\}$ must contain a Ramsey minimal $G \in \mathscr{R}(3K_2, P_3)$. Every $G \in \mathscr{R}(3K_2, P_3)$ yields F which is not minimal, since F contains $G \cup P_3$ for every $G \in \mathscr{R}(3K_2, P_3)$, a contradiction. For cases (iii) and (iv), by Theorem 2.1(i) $F - \{v_2, v_3, v_6\}$ and $F - \{v_1, v_4, w\}$ must contain a P_3. But the P_3 in both $F - \{v_2, v_3, v_6\}$ and $F - \{v_1, v_4, w\}$ yields F containing $C_4 \cup 2P_3$, $H_4 \cup P_3$, or $H_5 \cup P_3$, a contradiction.

Since F does not contain both A_1 and A_2, F must contain A_3. ■

The next lemma, we prove that graphs F_1 and F_2 as depicted in Fig. 4 are the only graphs with circumference 6 in $\mathscr{R}(4K_2, P_3)$.

Lemma 2.14. *Let F_1 and F_2 be graphs as depicted in Fig. 4. Then, F_1 and F_2 are the only connected graphs with circumference 6 in $\mathscr{R}(4K_2, P_3)$.*

Proof. We first show that $F_1, F_2 \in \mathscr{R}(4K_2, P_3)$. We can easily prove that F_1 and F_2 satisfy Theorem 2.1(i)–(vii). But if one edge of F_1 or F_2 is deleted, then the resulted graph satisfy Theorem 2.1(viii). So, $F_1, F_2 \in \mathscr{R}(4K_2, P_3)$.

Let $F \in \mathscr{R}(4K_2, P_3)$ be a connected graph with circumference 6 but $F \neq F_1$ and $F \neq F_2$. By Lemma 2.13, F contains A_3. By Theorem 2.1(i), there must be a P_3 in $F - \{v_1, v_2, v_4\}$. Then, up to isomorphism, F contains a vertex $x \in V(F \setminus A_3)$ adjacent to $w \in V(F)$. Next, $F - \{v_1, v_4, w\}$ must contain a P_3, by Theorem 2.1(i). Therefore, there must be a vertex $y \in V(F \setminus (A_3 \cup x))$ such that either (i) $y \sim v_2$, (ii) $y \sim v_3$, (iii) $y \sim v_5$, or (iv) $y \sim v_6$. From all cases, we obtain graphs F_1 and F_2 (up to isomorphism), a contradiction. ■

Lemma 2.15. *$\mathscr{R}(4K_2, P_3)$ contains no connected graph with circumference 7.*

Proof. Suppose to the contrary that $\mathscr{R}(4K_2, P_3)$ contains a connected graph F with circumference 7. So, $F \supseteq C_7$ where $V(C_7) = \{v_1, v_2, \ldots, v_7\}$. By Theorem 2.1(vii), F has order at least 8. Now, we assume $v \in V(F)$ and v adjacent to v_4, then $F - E(C_7)$ must contain a P_3 by Theorem 2.1(vii). Then, one of the following 2 cases must hold: (i) $v_6 \sim v$ or (ii) $v_7 \sim v$. So, we have a graph A or B is contained in F where $V(A) = V(B) = V(C_7) \cup \{v\}$, $E(A) = E(C_7) \cup \{vv_4, vv_6\}$ and $E(B) = E(C_7) \cup \{vv_4, vv_7\}$.

We first consider $F \supseteq A$. By Corollary 2.5, $F - \{v_4, v_6\}$ must contain a Ramsey minimal graph $G \in \mathscr{R}(2K_2, P_3) = \{C_4, C_5, 2P_3\}$. By the minimality of F, there is no $G \in \mathscr{R}(2K_2, P_3)$ satisfying this condition.

We next consider $F \supseteq B$. By Theorem 2.1(i), there must be a P_3 in $F - \{v_2, v_4, v_7\}$. Then, F must contain an edge connecting v_1 to v_6. Thus, we have $E(F) \supseteq E(B) \cup \{v_1v_6\}$. Furthermore, by Theorem 2.1(ii), $F - \{v_2, v_4\} - E(X)$ must contain a P_3 for $V(X) = \{v_1, v_6, v_7\}$. Since F does not contain a cycle of length greater than 7, there must be a vertex $u \in V(F \setminus B)$ such that one of the following 4 cases must hold: (a) $u \sim v_5$, (b) $u \sim v_6$, (c) $u \sim v_7$, or (d) $u \sim v$ (see Fig. 5). We now have F containing a graph B_1, B_2, B_3, or B_4.

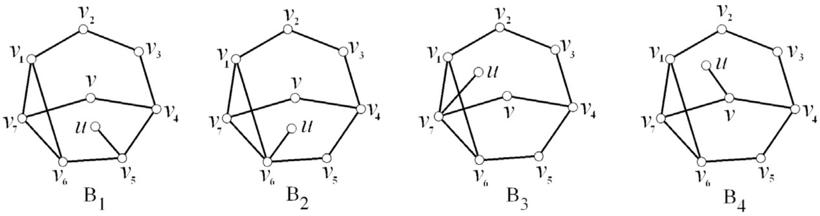

Fig. 5. Graphs B_1, B_2, B_3, and B_4.

By Theorem 2.1(i), $F - \{v_1, v_4, v_6\}$ (for $F \supseteq B_1$ or $F \supseteq B_2$) must contain a P_3. Since F has circumference 7, there must be a vertex $w \in V(F)$ adjacent to one of 4 vertices: v_2, v_3, v_5, v_7. By Theorem 2.1(i), $F - \{v_1, v_4, v_7\}$ (for $F \supseteq B_3$ or $F \supseteq B_4$) must contain a P_3. Since F has circumference 7, there must be a vertex $w \in V(F)$ adjacent to one of 4 vertices: v_2, v_3, v_5, v_6. On both cases, it implies that F contains F_1 or F_2, a contradiction. Hence, $\mathscr{R}(4K_3, P_3)$ contains no connected graph with circumference 7. ■

In the following lemmas, we determine all graphs F with circumference 8, 9, 10, and 11 belonging to $\mathscr{R}(4K_2, P_3)$. These graphs are constructed by applying Theorem 2.1 and we obtain graphs $F_3, F_4, \ldots, F_{37}$ in Fig. 6 and graphs $F_{38}, F_{39}, \ldots, F_{54}$ in Fig. 7 as elements of $\mathscr{R}(4K_2, P_3)$ with circumference 8 and 9, respectively. Moreover, we will show that the only cycle of order 10 and 11 as elements of $\mathscr{R}(4K_2, P_3)$ with circumference 10 and 11, respectively.

Lemma 2.16. *Let F be a connected graph with circumference* 8 *or* 9. *If $F \in \mathscr{R}(4K_2, P_3)$ then* $|E(F)| \geq 11$.

Proof. Let F be a connected graph with circumference 8 or 9. If F has circumference 8, then F contains a cycle C_8. Let $V(C_8) = \{v_1, v_2, \ldots, v_8\}$. We now assume $V(X_i) = \{v_i, v_{i+1}, v_{i+2}\}$ for $i = 1, 2, \ldots, 6$, $V(X_7) = \{v_7, v_8, v_1\}$, $V(X_8) = \{v_8, v_1, v_2\}$, and $V(Y_i) = V(C_8 \setminus X_i)$ for $i = 1, 2, \ldots, 8$.

By Theorem 2.1(i) and (iv), $F - \{v_2, v_5, v_8\}$ and $F - E(X_4 \cup Y_4)$ must contain a P_3. Then, up to isomorphism, there is a new edge in F, namely v_4v_7. Therefore, we now have $E(F) \supseteq E(C_8) \cup \{v_4v_7\}$. Next, by Theorem 2.1(i), there must be a P_3 in $F - \{v_2, v_4, v_7\}$ and $F - \{v_1, v_4, v_7\}$. Then, one of the following 3 cases must hold (1) $v_5 \sim v_8$, (2) $v_6 \sim v_8$, or (3) $v_5 \sim u$. We have $|E(F)| = 10$. Now, let us consider F when $E(F) \supseteq E(C_8) \cup \{v_4v_7, v_5v_8\}$. This graph does not satisfy Theorem 2.1(ii), since $F - \{v_5, v_7\} - E(X_1)$ does not contain a P_3. Next, let us consider F when $E(F) \supseteq E \cup \{v_4v_7, v_6v_8\}$. This graph does not satisfy Theorem 2.1(ii), since $F - \{v_2, v_4\} - E(X_6)$ does not contain a P_3. Last, let us consider F when $E(F) \supseteq E \cup \{v_4v_7, v_5u\}$. This graph does not satisfy Theorem 2.1(i), since $F - \{v_2, v_5, v_7\}$ does not contain a P_3. For all cases, we conclude that $|E(F)| \geq 11$.

If F has circumference 9, then F contains a cycle C_9. Let us consider $|E(F)| = 10$. Then, there exists at least one vertex in C_9 of degree 3, say v_1. But this graph does not satisfy Theorem 2.1(i), since $F - \{v_1, v_4, v_7\}$ does not contain a P_3. Therefore $|E(F)| \geq 11$. ■

Lemma 2.17. *Let $F_3, F_4, \ldots, F_{37}$ be graphs as depicted in Fig. 6. Then, these graphs are the only connected graphs with circumference* 8 in $\mathscr{R}(4K_2, P_3)$.

Proof. Let $F \in \{F_3, F_4, \ldots, F_{37}\}$. We can easily show that F satisfy Theorem 2.1. So, $F \in \mathscr{R}(4K_2, P_3)$.

Next, we prove that the connected graphs with circumference 8 in $\mathscr{R}(4K_2, P_3)$ are $F_3, F_4, \ldots, F_{37}$. Suppose that there exists a graph $F \in \mathscr{R}(4K_2, P_3)$ with circumference 8 other than $F_3, F_4, \ldots, F_{37}$. So $F \supseteq C_8$, where $V(C_8) = \{v_1, v_2, \ldots, v_8\}$. We now assume $V(X_i) = \{v_i, v_{i+1}, v_{i+2}\}$ for $i = 1, 2, \ldots, 6$, $V(X_7) = \{v_7, v_8, v_1\}$, $V(X_8) = \{v_8, v_1, v_2\}$, and $V(Y_i) = V(C_8 \setminus X_i)$ for $i = 1, 2, \ldots, 8$. F can have order 8 or greater than 8.

For case F has order 8, by Theorem 2.1(i) and (iv), $F - \{v_2, v_5, v_8\}$ and $F - E(X_4 \cup Y_4)$ must contain a P_3. Then, up to isomorphism, there is a new edge in F, namely v_4v_7. Therefore, we now have $E(F) \supseteq E(C_8) \cup \{v_4v_7\}$. Next, by Theorem 2.1(i), there must be a P_3 in $F - \{v_2, v_4, v_7\}$ and $F - \{v_1, v_4, v_7\}$. Then, one of the following 2 cases must hold (1) $v_5 \sim v_8$ or (2) $v_6 \sim v_8$. Now, let us consider F when $E(F) \supseteq E(C_8) \cup \{v_4v_7, v_5v_8\}$. By Theorem 2.1(ii) and (v), $F - \{v_5, v_7\} - E(X_1)$ and $F - \{v_2\} - E(Y_1)$ must contain a P_3. But the P_3 in $F - \{v_5, v_7\} - E(X_1)$ and $F - \{v_2\} - E(Y_1)$ implies that F is the graph F_3 or not minimal, a contradiction. Next, let us consider F when

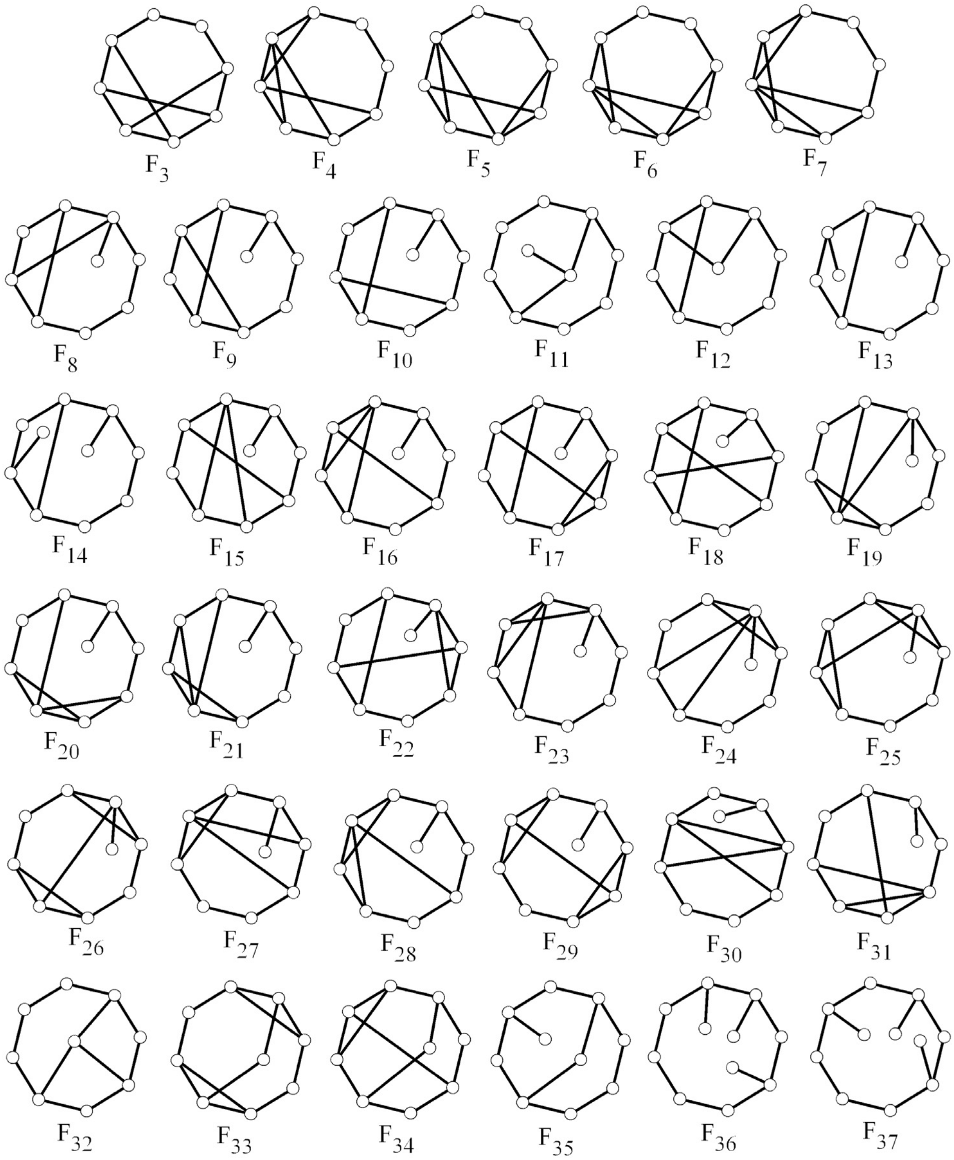

Fig. 6. All connected graphs with circumference 8 in $\mathscr{R}(4K_2, P_3)$.

$E(F) \supseteq E \cup \{v_4v_7, v_6v_8\}$. By Theorem 2.1(ii) and (vi), $F - \{v_2, v_4\} - E(X_6)$ and $F - E(X_6 \cup Y_6)$ must contain a P_3. Then one of the following 2 cases must hold: (1) $v_5 \sim v_7$ or (2) $v_5 \sim v_8$. For both cases, there must be a P_3 in $F - \{v_2\} - E(Y_1)$ and $F - E(X_1 \cup Y_1)$ by Theorem 2.1(v) and (vi). But the P_3 in $F - \{v_2\} - E(Y_1)$ and $F - E(X_1 \cup Y_1)$ implies that F is one of the graphs F_4, F_5, F_6, F_7 (up to isomorphism) or not minimal, a contradiction.

For case F has order greater than 8, there exist at least one vertex $u \in F$ but $u \notin C_8$ adjacent to a vertex in C_8. We assume $u \sim v_2$. By Theorem 2.1(v), $F - \{v_2\} - E(Y_1)$ must contain a P_3. Then one of the following 8 cases must hold: (1) $v_1 \sim v_6$, (2) $v_1 \sim v_3$, (3) $v_1 \sim v_7$, (4) $v_1 \sim v_5$, (5) $v_1 \sim v_4$, (6) $v_1 \sim w$, (7) $v_4 \sim u$, or (8) $v_4 \sim w$. Otherwise F is the graph F_{11} or not minimal. So, F contains one of graphs $D_1, D_2, \ldots, D_8$ where $E(D_1) = E(C_8) \cup \{uv_2, v_1v_6\}$, $E(D_2) = E(C_8) \cup \{uv_2, v_1v_3\}$, $E(D_3) = E(C_8) \cup \{uv_2, v_1v_7\}$, $E(D_4) = E(C_8) \cup \{uv_2, v_1v_5\}$, $E(D_5) = E(C_8) \cup \{uv_2, v_1v_4\}$, $E(D_6) = E(C_8) \cup \{uv_2, wv_1\}$, $E(D_7) = E(C_8) \cup \{uv_2, uv_4\}$, or $E(D_8) = E(C_8) \cup \{uv_2, wv_4\}$.

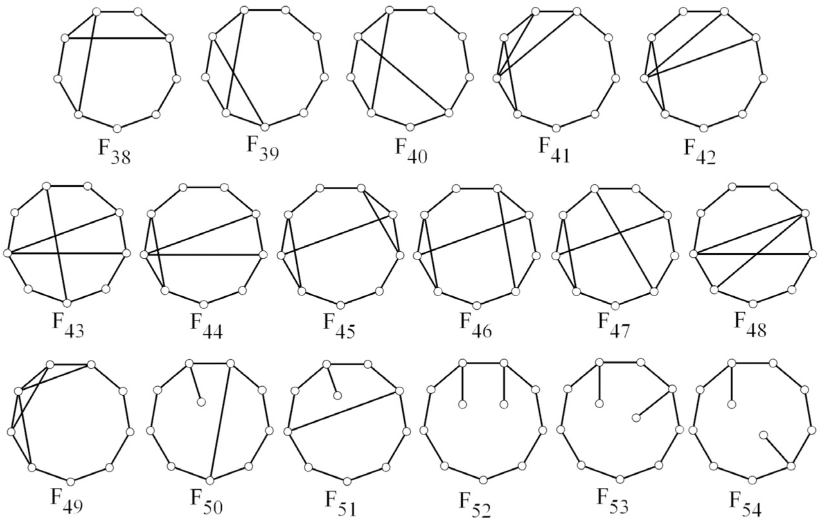

Fig. 7. All connected graphs with circumference 9 in $\mathscr{R}(4K_2, P_3)$.

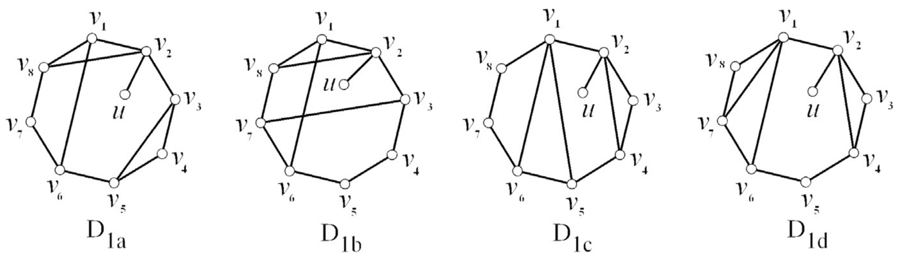

Fig. 8. Graphs D_{1a}, D_{1b}, D_{1c}, and D_{1d}.

For F contains D_1. By Theorem 2.1(i), (ii), (v) and (vi), $F - \{v_1, v_3, v_6\}$, $F - \{v_1, v_3\} - E(X_5)$, $F - \{v_6\} - E(Y_5)$, and $F - E(X_5 \cup Y_5)$ must contain a P_3. Then one of the following 4 cases must hold: (1) $v_2 \sim v_8$, $v_3 \sim v_5$, (2) $v_2 \sim v_8$, $v_3 \sim v_7$, (3) $v_4 \sim v_8$, $v_1 \sim v_5$, or (4) $v_4 \sim v_8$, $v_1 \sim v_7$, (graph D_{1a}, D_{1b}, D_{1c}, or D_{1d}, respectively in Fig. 8). Otherwise F is one of the graphs $F_8, F_9, F_{10}, F_{12}, F_{13}, \dots, F_{23}$ (up to isomorphism) or not minimal. Next, when F contains D_{1a}, D_{1b}, D_{1c}, or D_{1d}, by Theorem 2.1(ii), $F - \{v_3, v_6\} - E(X_8)$ and $F - \{v_1, v_6\} - E(X_2)$ must contain a P_3. But this leads to F which is not minimal, a contradiction.

Now, we observe when F contains D_2. By Theorem 2.1(i), (v) and (vi), $F - \{v_1, v_3, v_6\}$, $F - \{v_6\} - E(Y_5)$, $F - \{v_5\} - E(Y_4)$, $F - \{v_7\} - E(Y_6)$, and $F - E(X_1 \cup Y_1)$ must contain a P_3. Then one of the following 3 cases must hold: (1) both $v_4 \sim v_8$ and $v_2 \sim v_7$, (2) both $v_4 \sim v_8$ and $v_1 \sim v_5$, or (3) both $v_4 \sim v_8$ and $v_1 \sim v_7$ (graph D_{2a}, D_{2b}, or D_{2c}, respectively in Fig. 9). Otherwise F is one of the graphs $F_{24}, F_{25}, F_{26}, F_{33}$ (up to isomorphism) or not minimal. Next, when F contains D_{2a}, D_{2b}, or D_{2c}, there must be a P_3 in both $F - \{v_4, v_7\} - E(X_1)$ and $F - \{v_4, v_8\} - E(X_1)$ by Theorem 2.1(ii). But it causes F which is not minimal, a contradiction.

We consider F contains D_3. By Theorem 2.1(i), (ii) and (v), all graphs $F - \{v_1, v_3, v_6\}$, $F - \{v_2, v_4, v_7\}$, $F - \{v_2, v_5, v_7\}$, $F - \{v_1, v_6\} - E(X_2)$, $F - \{v_5\} - E(Y_4)$, and $F - \{v_7\} - E(Y_6)$ must contain a P_3. Then, one of the following 5 cases must hold: (1) $v_1 \sim v_3$, $v_4 \sim v_8$, (2) $v_1 \sim v_6$, $v_5 \sim v_7$, (3) $v_3 \sim v_6$, $v_5 \sim v_7$, (4) $v_4 \sim v_6$, $v_5 \sim v_8$, or (5) $v_4 \sim v_8$, $v_5 \sim v_8$. (graph D_{3a}, D_{3b}, D_{3c}, D_{3d}, or D_{3e}, respectively, in Fig. 10). Otherwise F is one of the graphs $F_{16}, F_{17}, F_{23}, F_{27}, F_{28}, F_{29}, F_{34}$ (up to isomorphism) or not minimal. Next, when F contains one of graphs in Fig. 10, by Theorem 2.1(ii) and (iii), there must be a P_3 in all graphs $F - \{v_4, v_7\} - E(X_1)$, $F - \{v_1, v_3\} - E(X_5)$, $F - \{v_2, v_7\} - E(X_4)$, and $F - \{v_2\} - E(X_4 \cup X_7)$. But the P_3 in these graphs yields F which is not minimal, a contradiction.

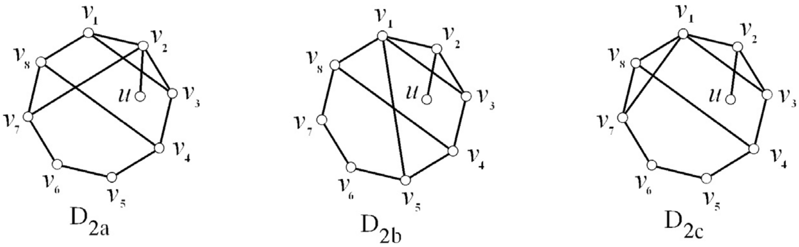

Fig. 9. Graphs D_{2a}, D_{2b}, and D_{2c}.

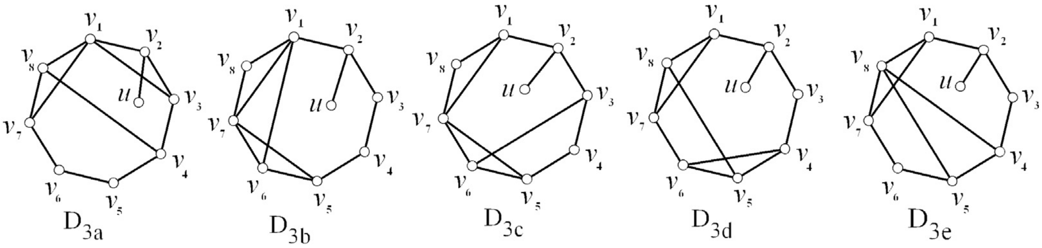

Fig. 10. Graphs D_{3a}, D_{3b}, D_{3c}, D_{3d}, and D_{3e}.

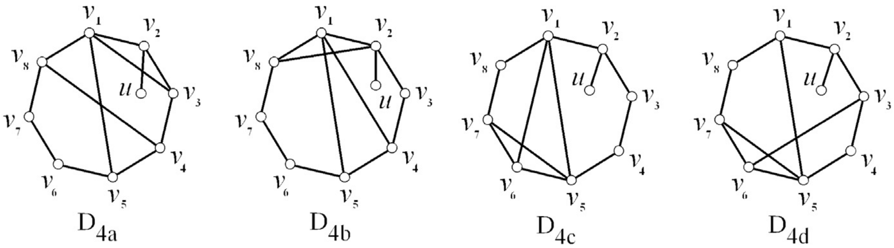

Fig. 11. Graphs D_{4a}, D_{4b}, D_{4c}, and D_{4d}.

We observe when F contains D_4. By Theorem 2.1(i), (ii), (v) and (vi), all graphs $F - \{v_2, v_5, v_7\}$, $F - \{v_2, v_5, v_8\}$, $F - \{v_1, v_3, v_6\}$, $F - \{v_1, v_6\} - E(X_2)$, $F - \{v_2, v_5\} - E(X_6)$, $F - \{v_5, v_7\} - E(X_1)$, $F - \{v_2, v_8\} - E(X_4)$, $F - \{v_7\} - E(Y_6)$, $F - \{v_5\} - E(Y_4)$, and $F - E(X_6 \cup Y_6)$ must contain a P_3. Then one of the following 4 cases must hold: (1) $v_1 \sim v_3$, $v_4 \sim v_8$, (2) $v_1 \sim v_4$, $v_2 \sim v_8$, (3) $v_1 \sim v_6$, $v_5 \sim v_7$, or (4) $v_3 \sim v_6$, $v_5 \sim v_7$ (graph D_{4a}, D_{4b}, D_{4c}, or D_{4d}, respectively, in Fig. 11). Otherwise F is one of the graphs F_{15}, F_{18}, F_{22}, F_{30}, F_{31} (up to isomorphism) or not minimal. Furthermore, when F contains D_{4a}, D_{4b}, D_{4c}, or D_{4d}, by Theorem 2.1(ii), all graphs $F - \{v_2, v_7\} - E(X_2)$, $F - \{v_1, v_3\} - E(X_5)$, and $F - \{v_5, v_8\} - E(X_1)$ must contain a P_3. But the P_3 in these graphs lead to F which is not minimal, a contradiction.

We consider F contains D_5. By Theorem 2.1(i), (ii), (v) and (vi), all graphs $F - \{v_2, v_4, v_7\}$, $F - \{v_1, v_3, v_6\}$, $F - \{v_1, v_3\} - E(X_5)$, $F - \{v_2, v_4\} - E(X_6)$, $F - \{v_4, v_7\} - E(X_1)$, $F - \{v_6\} - E(Y_5)$, $F - \{v_7\} - E(Y_6)$, and $F - E(X_6 \cup Y_6)$ must contain a P_3. But the P_3 in these graphs implies F which is one of the graphs F_{27}, F_{30} (up to isomorphism) or not minimal, a contradiction.

Now, we observe F containing D_6. All graphs $F - \{v_1, v_3, v_6\}$, $F - \{v_2, v_5, v_8\}$, $F - \{v_1, v_3\} - E(X_5)$, $F - \{v_1\} - E(X_2 \cup X_5)$ and $F - \{v_6\} - E(Y_5)$ must contain a P_3 by Theorem 2.1(i)–(iii), and (v). But the new P_3 in these graphs causes F which is the graph F_{36} or not minimal, a contradiction.

Lastly, we consider F contains D_7 or D_8. By Theorem 2.1(i), (ii), and (v), all graphs $F - \{v_2, v_4, v_7\}$, $F - \{v_2, v_4\} - E(X_6)$, and $F - \{v_7\} - E(Y_6)$ must contain a P_3. But the new P_3 in these graphs implies that F is one of the graphs F_{12}, F_{32}, F_{35}, F_{36}, F_{37} (up to isomorphism), F is not minimal or F has circumference 9, a contradiction.

For all cases, we conclude that the connected graphs with circumference 8 in $\mathcal{R}(4K_2, P_3)$ are $F_3, F_4, \ldots, F_{37}$. ■

Lemma 2.18. *Let $F_{38}, F_{39}, \ldots, F_{54}$ be graphs as depicted in Fig. 7. Then, these graphs are the only connected graphs with circumference* 9 in $\mathscr{R}(4K_2, P_3)$.

Proof. Let $F \in \{F_{38}, F_{39}, \ldots, F_{54}\}$. We can easily show that F satisfy Theorem 2.1. So, $F \in \mathscr{R}(4K_2, P_3)$.

Next, we prove that the connected graphs with circumference 9 in $\mathscr{R}(4K_2, P_3)$ are $F_{38}, F_{39}, \ldots, F_{54}$. Suppose that F having circumference 9 in $\mathscr{R}(4K_2, P_3)$ but $F \notin \{F_{38}, F_{39}, \ldots, F_{54}\}$. Since F has circumference 9 then $F \supseteq C_9$. Let $V(C_9) = \{v_1, v_2, \ldots, v_9\}$. We may assume $V(X_i) = \{v_i, v_{i+1}, v_{i+2}\}$ for $i = 1, 2, \ldots, 7$, $V(X_8) = \{v_8, v_9, v_1\}$, $V(X_9) = \{v_9, v_1, v_2\}$. F can have order 9 or greater than 9.

First, we consider F having order 9. By Theorem 2.1(i), $F - \{v_2, v_5, v_8\}$ must contain a P_3. So, one of the following 3 cases must hold: (1) $v_1 \sim v_7$, (2) $v_1 \sim v_6$, or (3) $v_7 \sim v_9$.

For case (1), we have $E(F) \supseteq E(C_9) \cup \{v_1v_7\}$. By Theorem 2.1(iii), there must be a P_3 in $F - \{v_1\} - E(X_3 \cup X_6)$. Since $F - \{v_1\} - E(X_3 \cup X_6) = 3K_2$, namely three independent edges v_2v_3, v_5v_6, and v_8v_9, then the P_3 in $F - \{v_1\} - E(X_3 \cup X_6)$ is formed by connecting two of the three edges. The eligible edge is only v_2v_9. Otherwise F is one of the graphs F_{38}, F_{39}, F_{40}, or not minimal. So, we now have $E(F) \supseteq E(C_9) \cup \{v_1v_7, v_2v_9\}$. By Theorem 2.1(iii) and (iv), both graphs $F - \{v_7\} - E(X_3 \cup X_9)$ and $F - E(X_3 \cup X_6 \cup X_9)$ must contain a P_3. The P_3 in both graphs is formed by connecting v_6 to v_8 and v_1 to v_4. Otherwise F is one of the graphs F_{41} or F_{42}, or not minimal. Thus, we obtain $E(F) \supseteq E(C_9) \cup \{v_1v_7, v_2v_9, v_6v_8, v_1v_4\}$. Next, there must be a P_3 in $F - \{v_4\} - E(X_6 \cup X_9)$ by Theorem 2.1(iii). But it implies F which is not minimal, a contradiction.

For case (2), we have $E(F) \supseteq E(C_9) \cup \{v_1v_6\}$. By Theorem 2.1(i) and (iii), both graphs $F - \{v_3, v_6, v_9\}$ and $F - \{v_6\} - E(X_2) - E(X_8)$ must contain a P_3. Then, one of the following 7 cases must hold: (a) $v_1 \sim v_4$, (b) $v_1 \sim v_5$, (c) $v_1 \sim v_7$, (d) $v_2 \sim v_7$, (e) $v_4 \sim v_7$, (f) $v_4 \sim v_8$, or (g) $v_5 \sim v_7$. Otherwise F is the graph F_{40} or not minimal. For all cases, there must be a P_3 in all graphs $F - \{v_1, v_4, v_7\}$, $F - \{v_1, v_7\} - E(X_3)$, $F - \{v_1, v_4\} - E(X_6)$ and $F - \{v_1\} - E(X_3 \cup X_6)$ by Theorem 2.1(i)–(iii). Then, the P_3 in these graphs is formed by connecting two of the three independent edges v_2v_3, v_5v_6 and v_8v_9. It causes F which is one of the graphs $F_{42}, F_{43}, \ldots, F_{49}$ (up to isomorphism) or not minimal, a contradiction.

For case (3), we have $E(F) \supseteq E(C_9) \cup \{v_7v_9\}$. There must be a P_3 in $F - \{v_7\} - E(X_3 \cup X_9)$ by Theorem 2.1(iii). Considering F does not contain graph in both cases (1) and (2), then there is only one case hold, that is $v_6v_8 \in E(F)$. So, we have $E(F) \supseteq E(C_9) \cup \{v_7v_9, v_6v_8\}$. Next, there must be a P_3 in $F - \{v_6, v_9\} - E(X_2)$ by Theorem 2.1(iii). Since $F - \{v_6, v_9\} - E(X_2) = 3K_2$ namely 3 independent edges v_1v_2, v_4v_5 and v_7v_8, then the P_3 in $F - \{v_6, v_9\} - E(X_2)$ is formed by connecting two of the three edges. It implies that F is one of the graphs F_{41}, F_{42}, F_{49}, or not minimal, a contradiction.

Secondly, we observe when F has order greater than 9. Then, there exists at least one vertex $u \in V(F)$ but $u \notin V(C_9)$ adjacent to a vertex in C_9. We assume $u \sim v_1$. There must be a P_3 in both graphs $F - \{v_1, v_4, v_7\}$ and $F - \{v_1\} - E(X_3) - E(X_6)$, by Theorem 2.1(i) and (iii). We know that $F - \{v_1, v_4, v_7\}$ and $F - \{v_1\} - E(X_3) - E(X_6)$ are three independent edges v_2v_3, v_5v_6, v_8v_9. Then, only one case must hold, that is $v_2 \sim v_9$. Otherwise F is one of the graphs $F_{50}, F_{52}, F_{53}, F_{54}$ or not minimal. Hence, we have $E(F) \supseteq E(C_9) \cup \{uv_1, v_2v_9\}$. Next, by Theorem 2.1(ii), $F - \{v_4, v_7\} - E(X_9)$ must contain a P_3. Then, only one case must hold, that is $v_6 \sim v_8$. Otherwise F is the graph F_{51} or not minimal. Therefore, we obtain $E(F) \supseteq E(C_9) \cup \{uv_1, v_2v_9, v_6v_8\}$. There must be a P_3 in $F - \{v_7\} - E(X_3) - E(X_9)$, by Theorem 2.1(iii). But, this yields that F is not minimal, a contradiction.

For all cases, we obtain the connected graphs with circumference 9 in $\mathscr{R}(4K_2, P_3)$ are $F_{38}, F_{39}, \ldots, F_{54}$. ■

Lemma 2.19. *The only graphs with circumference* 10 *or* 11 *in* $\mathscr{R}(4K_2, P_3)$ *are* C_{10} *or* C_{11}, *respectively.*

Proof. We can check easily that $C_{10}, C_{11} \in \mathscr{R}(4K_2, P_3)$. Furthermore, since every graph having circumference 10 or 11 contains C_{10} or C_{11}, respectively, the only C_{10} and C_{11} are in $\mathscr{R}(4K_2, P_3)$. ■

By Corollary 2.7, Lemma 2.8, 2.10–2.19, we obtain all graphs in $\mathscr{R}(4K_2, P_3)$. Therefore, we have the following theorem.

Theorem 2.20. $\mathscr{R}(4K_2, P_3) = \{4P_3, C_4 \cup 2P_3, C_5 \cup 2P_3, 2C_4, 2C_5, C_4 \cup C_5, C_7 \cup P_3, C_8 \cup P_3, H_1 \cup P_3, H_2 \cup P_3, H_3 \cup P_3, H_4 \cup P_3, H_5 \cup P_3\} \cup \{F_i \mid i \in [1, 54]\} \cup \{C_{10}, C_{11}\}$. ■

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