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A SOLUTION OF LINEAR PROGRAMMING WITH INTERVAL VARIABLES USING THE INTERIOR POINT METHOD BASED ON INTERVAL BOUNDARY CALCULATIONS

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Abstract

Linear programming with interval variables is the development of linear programming with interval coefficients. In this paper, linear programming with interval variables is solved using the interior point method based on interval boundary calculations. The solution's initial procedure is to change the linear programming model with interval variables into a pair of classical linear programming models. This pair

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of two classical linear programming models corresponds to an interval boundary called the least upper bound and the greatest lower bound. The least upper bound is determined using the largest feasible region and the greatest lower bound is determined by using the smallest feasible region. The least upper bound and the greatest lower bound are obtained using the interior point method. Finally, the optimal solution in the form of intervals is obtained by constructing the two models. This paper provides an alternative solution to solve the linear interval programming problem by using the interior point method.

1. Introduction

An optimization problem could be categorized as a linear programming model if it satisfies assumptions of proportionality, additivity, divisibility, and certainty [8]. The certainty assumption is that all coefficients and variables in the model are known. However, in real situations, sometimes the values of coefficients and variables are not known with certainty. The uncertainty problem could be overcome using the interval approach for the coefficients and variables. This approach supported the concept and theory of interval analysis developed in [13]. Linear programming with interval coefficients is linear programming where the coefficients of the objective function and the constraint function are intervals. In case the coefficients and variables in the objective function and also in the constraint function are both in the form of intervals, we term the programming to be linear programming with interval variables.

Researches on linear programming with interval coefficients have been carried out in [2, 3, 12, 14]. The research on linear programming with interval variables was inspired by [15-19]. The researchers in [15-19] used the simplex method to solve the problem. The interior point method has been used to solve the linear programming in [4, 6-8, 10, 11]. Meanwhile, several researchers used the interior point method to solve the linear programming of fuzzy numbers [20]. The advantages of the interior point method are used in solving linear programming problems with many variables. For several

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reasons that have been stated, this paper considers the interior point method as an alternative solution to linear programming with interval variables.

This paper presents a solution to linear programming with interval variables using an interior point method. The steps to obtain the optimal solution in the form of intervals are: first, to transform the problem into linear programming with interval variables model, meaning that the coefficients and variables in the objective function and the constraints function are intervals. The second step is to change linear programming with interval variables into a pair of the classical linear programming models. This pair of two classical linear programming models corresponds to an interval boundary called the *least upper bound* and the *greatest lower bound*. The least upper bound is determined using the largest feasible region and the greatest lower bound is determined by using the smallest feasible region. Finally, the least upper bound and the greatest lower bound are obtained using the interior point method. A more detailed explanation of the last step is provided in Subsection 2.3.

2. Preliminaries

In this section, we review some of the concepts needed, such as interval arithmetic and formula of linear programming with interval variables. For more details, we refer to [1, 8, 13, 18].

2.1. Interval arithmetic

The basic concepts, definition, and properties of interval number, interval arithmetic, and comparison of two intervals can be found in [1, 13, 18]. Let R denote the set of all real numbers.

Definition 2.1. A closed real interval $\underline{x} = [x_I, x_S]$ is defined by

$$\underline{x} = [x_I, x_S] = \{x \in R \mid x_I \leq x \leq x_S;\ x_I, x_S \in R\}, \tag{2.1}$$

where x_I and x_S are called *infimum* and *supremum* of $\underline{x}$, respectively.

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Definition 2.2. An interval of real numbers $\underline{x}$ is called an *unbounded interval* if the infimum x_I or supremum x_S does not exist, i.e., is infinite.

Definition 2.3. A real interval $\underline{x} = [x_I, x_S]$ is called *degenerate*, if $x_I = x_S$.

Definition 2.4. The *width* of an interval $\underline{x}$ is the real number $w(\underline{x}) = \frac{1}{2}(x_S - x_I)$.

Definition 2.5. The *midpoint* of an interval $\underline{x}$ is the real number $m(\underline{x}) = \frac{1}{2}(x_I + x_S)$.

Definition 2.6. Let $I(R)$ be the set of all intervals of R. Then a *real interval vector* $\underline{\boldsymbol{x}} \in I(R^n)$, is a vector in the form $\underline{\boldsymbol{x}} = (\underline{x}_i)_{n\times 1}$, where $\underline{x_i} = [x_{iI}, x_{iS}] \in I(R)$, $i = 1, 2, \ldots, n$.

Definition 2.7. Let $\underline{x}, \underline{y} \in I(R)$, where $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$. Then

(a) $\underline{x} + \underline{y} = [x_I + y_I, x_S + y_S]$ (addition),

(b) $\underline{x} - \underline{y} = [x_I - y_S, x_S - y_I]$ (subtraction),

(c)

$$\underline{x} \cdot \underline{y} = [\min\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}, \max\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}]$$

(multiplication),

(d) $\underline{x}/\underline{y} = [x_I, x_S][1/y_S, 1/y_I]$, $0 \notin \underline{y}$ (division).

Definition 2.8. Let $\underline{x}, \underline{y} \in I(R)$, $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$. We say $\underline{x} \leq \underline{y}$ if and only if $ux_I + vx_S \leq uy_I + vy_S$, where $u, v \in (0, 1]$ and $u \leq v$.

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2.2. Linear programming with interval variables

The general form of linear programming with interval variables is defined as follows:

Maximize (objective function)

$$\underline{Z} = \sum_{j=1}^{n} [c_{jI}, c_{jS}][x_{jI}, x_{jS}] \tag{2.2}$$

subject to

$$\sum_{j=1}^{n} [a_{ijI}, a_{ijS}][x_{jI}, x_{jS}] \leq [b_I, b_S], \tag{2.3}$$

$$[x_{jI}, x_{jS}] \geq 0, \tag{2.4}$$

where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. $[x_{jI}, x_{jS}]$ is a decision variable, $[c_{jI}, c_{jS}]$ is a coefficient of the objective function, $[a_{ijI}, a_{ijS}]$ is a coefficient of the constraint function, $[b_I, b_S]$ is a coefficient of resource constraint, $\underline{Z}$ is the objective function, and

$$[x_{jI}, x_{jS}], [c_{jI}, c_{jS}], [a_{ijI}, a_{ijS}], [b_I, b_S] \in I(R).$$

Theorem 2.9 and Theorem 2.10 stated below have been presented in [18]. Theorem 2.9 determines the largest feasible region and the smallest feasible region of linear programming with interval variables. Theorem 2.10 described some property of the objective function of linear programming with interval variables model.

Theorem 2.9. *Suppose the interval inequality of linear programming with interval variables*:

$$\sum_{j=1}^{n} [a_{jI}, a_{jS}][x_{jI}, x_{jS}] \leq [b_I, b_S], \tag{2.5}$$

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where $[x_{jI}, x_{jS}] \geq 0$, *for every* $j = 1, 2, \ldots, n$. *Then*

(a) *the largest feasible region is a region that satisfies the following inequality*:

$$\sum_{j=1}^{n} \min\{a_{jI}x_{jI}, a_{jI}x_{jS}\} \leq b_S, \tag{2.6}$$

(b) *the smallest feasible region is a region that satisfies the following inequality*:

$$\sum_{j=1}^{n} \max\{a_{jS}x_{jI}, a_{jS}x_{jS}\} \leq b_I. \tag{2.7}$$

Theorem 2.10. *Suppose an objective function of linear programming with interval variables*:

$$\underline{Z} = \sum_{j=1}^{n} [c_{jI}, c_{jS}][x_{jI}, x_{jS}], \tag{2.8}$$

where $[x_{jI}, x_{jS}] \geq 0$ *for every* $j = 1, 2, \ldots, n$. *Then for every* $\underline{\boldsymbol{x}} = (\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n})^T$ *in the feasible region*

$$\sum_{j=1}^{n} \min\{c_{JI}x_{JI}, c_{JI}x_{JS}\} \leq \sum_{j=1}^{n} \max\{c_{JS}x_{JI}, c_{JI}x_{JS}\}. \tag{2.9}$$

2.3. Interior point algorithm

The interior point algorithm starts with an initial interior point $\tilde{\mathbf{X}}^0$ which satisfies the interior point conditions, such that $\mathbf{A}\tilde{\mathbf{X}}^0 = \mathbf{b}$, $\tilde{\mathbf{X}}_j^0 > 0$ for $j = 1, 2, \ldots, n + m$. The condition $\tilde{\mathbf{X}}_j^0 > 0$ makes the variable values $(x_1, x_2, \ldots, x_{n+m})$ always remain within the boundary of the feasible area. The interior point algorithm produces a feasible interior point sequence,

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namely $\widetilde{\mathbf{X}}^1$ (obtained from the 1st iteration), $\widetilde{\mathbf{X}}^2$ (obtained from the 2nd iteration), …, $\widetilde{\mathbf{X}}^k$ (obtained from the kth iteration), $\widetilde{\mathbf{X}}^{k+1}$ (obtained from the $(k+1)$th iteration), and so on, with iterations following $i+1$ $(i = 0, 1, 2, \ldots)$. The steps for the interior point algorithm are as follows [8].

Initial step. Determine the initial interior point $\widetilde{X}^0 = (x_1, x_2, \ldots, x_{n+m})$ which satisfies the boundary constraints of the problem and calculate the value of Z_0 with

$$Z_0 = C^T \widetilde{X}^0. \tag{2.10}$$

Step 1. Determine the diagonal matrix $\mathbf{D}_{i+1}$, i.e.,

$$\mathbf{D}_{i+1} = \operatorname{diag}(\widetilde{\mathbf{X}}^i) = \begin{bmatrix} x_1 & 0 & 0 & & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n+m} \end{bmatrix}. \tag{2.11}$$

Step 2. Determine the matrices $\mathbf{A}_{i+1}$ and $\mathbf{C}_{i+1}$, i.e.,

$$\left.\begin{aligned} \mathbf{A}_{i+1} &= \mathbf{A}\mathbf{D}_{i+1} \\ \mathbf{C}_{i+1} &= \mathbf{D}_{i+1}\mathbf{C} \end{aligned}\right\}. \tag{2.12}$$

Step 3. Determine the projection matrix $\mathbf{P}_{i+1}$ and projected gradient $\mathbf{C}_{P_{i+1}}$.

(1) The projection matrix, i.e.,

$$\mathbf{P}_{i+1} = \mathbf{I} - \mathbf{A}_{i+1}^T(\mathbf{A}_{i+1}\mathbf{A}_{i+1}^T)^{-1}\mathbf{A}_{i+1}, \tag{2.13}$$

where $\mathbf{I}$ is an identity matrix.

(2) The projected gradient, i.e.,

$$\mathbf{C}_{P_{i+1}} = \mathbf{P}_{i+1}\mathbf{C}_{i+1}. \tag{2.14}$$

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Step 4. Determine the scalar $\mathbf{V}_{i+1}$ and matrix $\mathbf{M}_{i+1}$:

$$\left.\begin{aligned} \mathbf{V}_{i+1} &= |\min(\mathbf{C}_{P_{i+1}})| \\ \mathbf{M}_{i+1} &= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\alpha}{\mathbf{V}_{i+1}} \mathbf{C}_{P_{i+1}} \end{aligned}\right\}, \tag{2.15}$$

where $\mathbf{M}_{i+1}$ is an $(n+m) \times 1$ matrix and $\alpha \in (0, 1)$.

Step 5. Determine the candidate interior point for the next iteration, i.e.,

$$\widetilde{\mathbf{X}}^{i+1} = \mathbf{D}_{i+1}\mathbf{M}_{i+1}. \tag{2.16}$$

Step 6. Determine the optimum value, i.e.,

$$\mathbf{Z}_{i+1} = \mathbf{C}^T \widetilde{\mathbf{X}}^{i+1}. \tag{2.17}$$

(1) If $Z_{i+1} > Z_i$, then proceed to the next iteration following the same Steps 1-5 in the previous iteration. $\widetilde{X}^{i+1}$ is chosen to be the next iteration's interior point.

(2) If $Z_{i+1} \le Z_i$, then the iteration stops, and the optimum solution is obtained, namely the optimum point and the optimum value $(Z$ and $\widetilde{\mathbf{X}}^i)$.

3. A Solution of Linear Programming with Interval Variables Based on Interval Boundary Calculations

The solution of linear programming with interval variables using an interior point method based on interval boundary calculations is constructed in this section, given linear programming with interval variables in equations (2.2)-(2.4). The steps of the solution are as follows:

Step 1. Determine the largest feasibility region and the smallest feasibility region from a given linear programming with interval variables. Equation (2.6) shows the largest feasibility region, and equation (2.7) shows the smallest feasibility region.

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Step 2. Determine the least upper bound and the greatest lower bound of the objective function.

(a) The least upper bound satisfies the following inequality:

$$z_S = \sum_{j=1}^{n} \max\{c_{jS}x_{jI}, c_{jS}x_{jS}\}. \quad (3.1)$$

(b) The greatest lower bound satisfies the following inequality:

$$z_I = \sum_{j=1}^{n} \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\}. \quad (3.2)$$

Step 3. Form two linear programming sub-models based on the objective function in Step 2 and the feasible solution region in Step 1. These two sub-models are called the *best optimum problem* and the *worst optimum problem*.

(a) The best optimum problem:

Maximize (objective function)

$$z_S = \sum_{j=1}^{n} \max\{c_{jS}x_{jI}, c_{jS}x_{jS}\} \quad (3.3)$$

subject to

$$\sum_{j=1}^{n} \min\{a_{ijI}x_{jI}, a_{ijI}x_{jS}\} \le b_{iS}, \; i = 1, 2, ..., m, \quad (3.4)$$

$$x_{jI}, x_{jS} \ge 0, \quad j = 1, 2, ..., n. \quad (3.5)$$

(b) The worst optimum problem:

Maximize (objective function)

$$z_I = \sum_{j=1}^{n} \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\} \quad (3.6)$$

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subject to

$$\sum_{j=1}^{n} \max\{a_{ijS} x_{jI}, a_{ijS} x_{jS}\} \leq b_{iI}, \quad i = 1, 2, ..., m, \tag{3.7}$$

$$x_{jI}, x_{jS} \geq 0, \quad j = 1, 2, ..., n. \tag{3.8}$$

Step 4. Using an interior point algorithm (Subsection 2.3), solve the best optimum problem in Step 3 to obtain the best optimum value z_S.

Step 5. Using an interior point algorithm (Subsection 2.3), solve the worst optimum problem in Step 3 to obtain the worst optimum value z_I.

Step 6. Check completion in Steps 4-5.

(a) If the best optimum solution is feasible with a z_S value and the worst optimum solution is feasible with a z_I value, then go directly to the next Step 7.

(b) If the linear programming with interval variables does not have the largest feasible region and the smallest feasible region, then the linear programming with interval variables does not have any solution.

(c) If the best optimum solution for the best problem or the worst optimum solution for the worst problem is unbounded, then select the optimum problem that is unbounded. Replace the constraint function with a new constraint function obtained from the combination of the best optimum problem constraint function and the worst optimum problem constraint function so that a new model is formed, then return to Step 4 or Step 5.

Step 7. Form the completion interval:

(a) Solving the best optimum problem, the supremum value that corresponds to the least upper bound is taken.

(b) Solving the worst optimum problem, the infimum value that corresponds to the greatest lower bound is taken.

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(c) If the solution formed (from Step 7(a) and Step 7(b)) is an interval, go to Step 8.

(d) If the solution formed (from Step 7(a) and Step 7(b)) is not in the form of an interval, then the solution to the worst optimum problem is considered.

(1) If the worst-case optimal solution for each variable has the form of an interval, then proceed to Step 8.

(2) If for the worst optimum problem, there is a part of the solution that is not in the form of an interval, the portion is given the infimum value and thus the degenerate intervals are formed.

Step 8. From Steps 4-7, the optimum solution is obtained, namely the optimum point and the optimum value in the form of intervals

$$\underline{x_i} = [x_{iI}, x_{iS}], \quad i = 1, 2, ..., m \quad \text{and} \quad \underline{Z} = [z_I, z_S]. \tag{3.9}$$

4. Numerical Example

In this section, we solve one example of interval linear programming in [5, 9, 19, 21], and compare the results.

Maximize (objective function)

$$Z = [26, 30][x_{1I}, x_{1S}] - [5.5, 6][x_{2I}, x_{2S}] \tag{4.1}$$

subject to

$$\left.\begin{array}{l} [8, 10][x_{1I}, x_{1S}] - [12, 14][x_{2I}, x_{2S}] \leq [3.8, 4.2] \\ [1, 1.1][x_{1I}, x_{1S}] + [0.19, 0.2][x_{2I}, x_{2S}] \leq [6.5, 7] \\ [x_{1I}, x_{1S}], [x_{2I}, x_{2S}] \geq 0 \end{array}\right\}. \tag{4.2}$$

Step 1. Determine the largest feasibility region and the smallest feasibility region as given in Table 1.

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Table 1. The largest and the smallest feasible regions

(1) The largest feasible region	(2) The smallest feasible region
$\left.\begin{array}{l} 8x_{1I} - 14x_{2S} \le 4.2 \\ x_{1I} + 0.19x_{2I} \le 7 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \ge 0 \end{array}\right\}$	$\left.\begin{array}{l} 10x_{1S} - 12x_{2I} \le 3.8 \\ 1.1x_{1S} + 0.2x_{2S} \le 6.5 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \ge 0 \end{array}\right\}$

Step 2. Determine the least upper bound and the greatest lower bound of the objective function as given in Table 2.

Table 2. The least upper bound and the greatest lower bound

(1) The least upper bound	(2) The greatest lower bound
Maximize $z_S = 30x_{1S} - 5.5x_{2I}$	Maximize $z_I = 26x_{1I} - 6x_{2S}$

Step 3. Formation of two linear programming sub-models: the best optimum problem and the worst optimum problem as given in Table 3.

Table 3. The best optimum problem and the worst optimum problem

(1) The best optimum problem	(2) The worst optimum problem
Maximize $z_S = 30x_{1S} - 5.5x_{2I}$	Maximize $z_I = 26x_{1I} - 6x_{2S}$
subject to	subject to
$\left.\begin{array}{l} 8x_{1I} - 14x_{2S} \le 4.2 \\ x_{1I} + 0.19x_{2I} \le 7 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \ge 0 \end{array}\right\}$	$\left.\begin{array}{l} 10x_{1S} - 12x_{2I} \le 3.8 \\ 1.1x_{1S} + 0.2x_{2S} \le 6.5 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \ge 0 \end{array}\right\}$

Step 4. Solve the best optimum problem in Step 3 to obtain the best optimum Z_S value. From Step 3, an unbounded solution is obtained.

Step 5. Solve the worst optimum problem in Step 3 to get the worst optimum Z_I value. From Step 3, an unbounded solution is obtained.

Step 6. Check completion in Step 4 and Step 5.

Because the best optimum solution and the worst optimum solution are unbounded, each constraint function is added to the other model constraint functions. The best optimum problem-new and the worst optimum problem-new are formed. The next two new models were completed in Step 4 and Step 5. The best optimum problem-new and the worst optimum problem-new can be seen in Table 4.

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Table 4. The best optimum problem-new and the worst optimum problem-new

(1) The best optimum problem-new	(2) The worst optimum problem-new
Maximize $z_S = 30x_{1S} - 5.5x_{2I}$	Maximize $z_I = 26x_{1I} - 6x_{2S}$
subject to	subject to
$\left.\begin{array}{l} 8x_{1I} - 14x_{2S} \le 4.2 \\ x_{1I} + 0.19x_{2I} \le 7 \\ 10x_{1S} - 12x_{2I} \le 3.8 \\ 1.1x_{1S} + 0.2x_{2S} \le 6.5 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \ge 0 \end{array}\right\}$	$\left.\begin{array}{l} 10x_{1S} - 12x_{2I} \le 3.8 \\ 1.1x_{1S} + 0.2x_{2S} \le 6.5 \\ 8x_{1I} - 14x_{2S} \le 4.2 \\ x_{1I} + 0.19x_{2I} \le 7 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \ge 0 \end{array}\right\}$

Step 7. By using the interior point algorithm (Subsection 2.3) and $\alpha = 0.95$ [8], to solve the best optimum problem-new in Step 6, the best optimum value is obtained as follows:

$$x_{1I} \cong 0.2485568309, \quad x_{1S} \cong 5.9090910433,$$

$$x_{2I} \cong 4.6075757564, \quad x_{2S} \cong 4.2070175164\ 10^{-8} \cong 0.$$

Step 8. By using the interior point algorithm (Subsection 2.3) and $\alpha = 0.95$ [8], to solve the worst optimum problem-new in Step 6, the worst optimum value is obtained as follows:

$$x_{1I} \cong 6.9999999424, \quad x_{1S} \cong 0.1626671934,$$

$$x_{2I} \cong 4.8425050227\ 10^{-7} \cong 0, \quad x_{2S} \cong 3.7000001840.$$

Step 9. Form a completion interval.

The solution to the optimum point formed from Step 7 and Step 8 is not in the form of an interval, so a solution to the worst optimum problem is taken, namely the solution in Step 8 only. From Step 8, there is an optimum point that is not in the form of an interval, so at that optimum point, only the infimum value is given, so that the optimum point completion is obtained, namely

$$\underline{x_1} \cong [6.9999999424, 6.9999999424], \quad \underline{x_2} \cong [0, 3.7000001840].$$

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Step 10. Obtain the optimum point solution and the optimum value in the form of intervals, namely

$$\underline{x_1} \cong [6.9999999424, 6.9999999424], \quad \underline{x_2} \cong [0, 3.7000001840],$$

$$\underline{Z} \cong [159.7999973984, 209.9999982720].$$

The optimum point and optimum value from previous researchers' results, and this paper are presented in Table 5.

Table 5. The results of previous researchers and of this paper

No	Researcher	$\underline{x_j}$	$\underline{Z}$
1	Huang et al. [9]	$\underline{x_1} = [5.21, 6.34]$, $\underline{x_2} = [3.32, 4.03]$	$\underline{Z} = [111.4, 171.8]$
2	Zhou et al. [21]	$\underline{x_1} = [4.574, 6.336]$, $\underline{x_2} = [3.320, 3.495]$	$\underline{Z} = [97.954, 171.82]$
3	Suprajitno and bin Mohd [19]	$\underline{x_1} = [5.1816, 5.1816]$, $\underline{x_2} = [4.0013, 4.0013]$	$\underline{Z} = [110.71, 133.440]$
4	Fan and Huang [5]	$\underline{x_1} = [5.21, 6.23]$, $\underline{x_2} = [3.26, 4.03]$	$\underline{Z} = [111.38, 169.1]$
5	This paper	$\underline{x_1} \cong [6.99, 6.99]$, $\underline{x_2} \cong [0, 3.70]$,	$\underline{Z} \cong [159.799, 209.99]$

According to [21], two criteria can be used to determine the best solution for a linear programming with interval variables. These criteria include the value of the width of the interval $(w(\underline{Z}), \text{the smallest})$ and the value of the degree of uncertainty $(w(\underline{Z})/m(\underline{Z}), \text{the smallest})$. Table 6 presents a width of an interval and the degree of uncertainty from previous researchers' results, and this paper.

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Table 6. The results of a width of an interval and a degree of uncertainty

No	Researcher	$w(\underline{Z})$	$m(\underline{Z})$	Degree of uncertainty
1	Huang et al. [9]	30.2	141.6	21.3276%
2	Zhou et al. [21]	36.933	134.887	27.3807%
3	Suprajitno and bin Mohd [19]	11.36352	122.077	9.3085%
4	Fan and Huang [5]	28.86	140.24	20.579%
5	This paper	25.1	184.899	13.5749

It can be seen from Tables 4-6 that the solution of linear programming with interval variables using interior point method based on interval boundary calculations has a smaller value for a width of interval and a degree of uncertainty compared to [5, 9, 21]. So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with interval variables based on interval boundary calculations.

5. Conclusions

This paper presents the interior point method, which can solve linear programming with interval variables. Linear programming with interval variables can be solved by converting the problem into two classical linear programming models. Furthermore, both the models were solved using the interior point method. Using the criteria of interval's width and degree of uncertainty, solutions using the linear programming with interval variables get better results than previous researchers and satisfy the criteria for constraint values. So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with interval variables based on interval boundary calculations.

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