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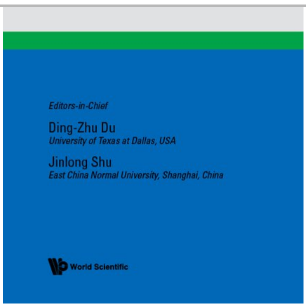

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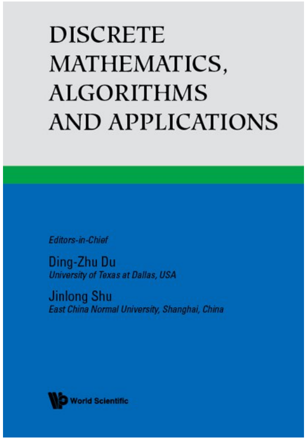

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On r-dynamic vertex coloring of some flower graph families

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Let G be a simple, connected undirected graph with m vertices and n edges. Let vertex coloring c of a graph G be a mapping $c : V(G) \to S$, where $|S| = k$ and it is k-colorable. Vertex coloring is proper if none of the any two neighboring vertices receives the similar color. An r-dynamic coloring is a proper coloring such that $|c(Nbd(v))| \geq \min\{r, \deg_G(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minutest coloring k of G which is r-dynamic k-colorable and denoted by $\chi_r(G)$. By a simple view, we exhibit that $\chi_r(G) \leq \chi_{r+1}(G)$, howbeit $\chi_{r+1}(G) - \chi_r(G)$ cannot be arbitrarily small. Thus, finding the result of $\chi_r(G)$ is useful. This study gave the result of r-dynamic chromatic number for the central graph, Line graph, Subdivision graph, Line of subdivision graph, Splitting graph and Mycielski graph of the Flower graph F_n denoted by $C(F_n)$, $L(F_n)$, $S(F_n)$, $L(S(F_n))$, $S(F_n)$ and $\mu(F_n)$, respectively.

Keywords: r-dynamic coloring; central graph; line graph; subdivision graph; line of subdivision graph; splitting graph; Mycielski graph.

Mathematics Subject Classification 2020: 05C15

1. Introduction

In this paper, the graphs [4] are considered simplistic, finite with δ(minimum)(G), Δ(maximum)(G) and χ(chromatic number)(G). Montgomery was the first person who brought out the r-dynamic coloring [10]. An r-dynamic k-coloring is a mapping c from the vertex set $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that (i) if $uv \in E(G)$,

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then $c(u) \neq c(v)$ and (ii) $|c(Nbd(v))| \geq \min\{r, \deg_G(v)\}$, for each $v \in V(G)$, where $Nbd(v)$ is the set of all vertices adjacent to v, $\deg_G(v)$ is the degree of the vertex and r is the positive integer. The premier one is the adjacency condition and the paired one is the couple-adjacency condition. The r-dynamic coloring is the minimum coloring of the graph which is denoted as $\chi_r(G)$. When $r = 1$, then 1-dynamic chromatic number is same as the chromatic number of the graph G and if r is equal to two, it is a dynamic chromatic number. The following authors have also investigated on r-dynamic coloring [1, 7, 12]. Lai *et al.* [9] explained upper bounds of $\chi_r(G)$ in the pursuing lemma.

Lemma 1. $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$.

The upper bounds and lower bounds of r-dynamic chromatic number of some graphs have been explained in many research papers. Dafik *et al.* [5] studied the lower bounds of r-dynamic coloring for some graphs and gave an open problem to find the sharp lower bound for the connected graph. Alishahi [2] established that all graph G with $\chi(G) \geq 4, \chi_2(G) \leq \chi(G) + \gamma(G)$, where $\gamma(G)$ is the domination number G and also showed, for k-regular graph, $\chi(G) \geq 4, \chi_2(G) \leq \chi(G) + \alpha(G^2)$, where α is the independence number. For d-regular graph, the bounds for dynamic chromatic number in terms of independence number $\chi_d(G) \leq \chi(G) + 2\log_2 \alpha(G) + 3$ were proved in [6]. In this work, the bounds of r-dynamic chromatic number are examined which gives an easier study on r-dynamic coloring.

Flower graph [3] is obtained from Helm graph H_n by joining the pendent edge to the hub vertex and it has $2n+1$ vertices and $4n$ edges and denoted by F_n. Figure 1 illustrate the flower graph F_4.

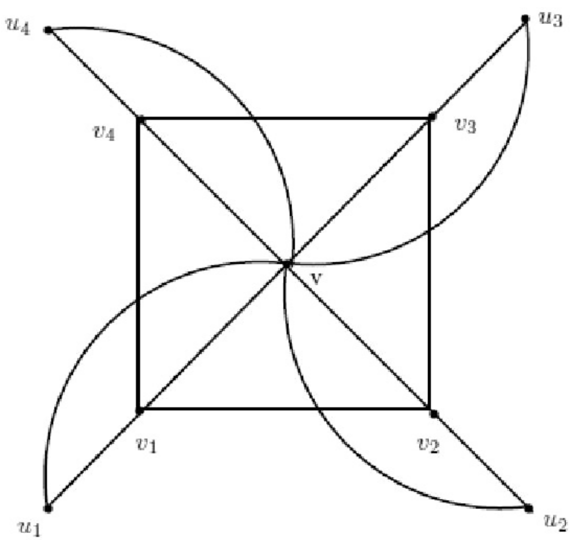

Fig. 1. Flower graph F_4.

Central graph [13] is obtained from adding a new vertex to each edge at once and connects the non-adjacency vertices of the given graph.

Line graph [7] is obtained from the original graph and the vertices are adjacent if the edges of the given graph are proximity.

Subdivision graph [14] is a graph obtained by inserting a new vertex with the degree 2 for each edge of the given graph.

Line graph of Subdivision graph [14] is a graph with vertices that are acquired from the edges of the subdivision graph and the vertices are adjacent if the corresponding edges are in proximity. It is also known as **Paraline graph.**

Splitting graph [8] is a graph, for each vertex $V = (v_1, v_2, \ldots, v_n)$ of G, take a new vertex $V' = (v'_1, v'_2, \ldots, v'_n)$. Join each vertex of v'_i to v_j if and only if the vertex v_j is adjacent to v_i.

Mycielski graph [11] is a graph which has two conditions; (i) add a vertex as given in splitting graph (ii) add another common vertex z and join the common vertex z to all points of V'.

2. Some Results on Flower Graph Families

Lemma 2. *Let $C(F_n)$ be the central graph of a flower graph F_n. The lower bound for r-dynamic chromatic number of the central graph of flower graph is*

$$\chi_r[C(F_n)] \geq \begin{cases} n & 1 \leq r \leq \delta - 1, \\ \Delta + 1 & \delta \leq r \leq \Delta - 1, \\ \Delta + 3 & r \geq \Delta. \end{cases}$$

Proof. $V(F_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$. The order of the graph is $|V(C(F_n))| = 6n+1$. The edge sets $\{v_iu_i, vv_i, vu_i, v_iv_{i+1}(1 \leq i \leq n-1), v_nv_1\}$ can be subdivided by $\{e_i, w'_i, e'_i, w_i, w_n\}$ and the size of the graph is $|E(C(F_n))| = 13n + 1$. For $1 \leq r \leq \delta - 1$, the vertices $V = v_i, e_i$ persuade a clique with order n in $C(F_n)$. Hence, $\chi_r[C(F_n)] \geq n$. For $\delta \leq r \leq \Delta - 1$ based on Lemma 1, $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$. Thus, $\chi_r[C(F_n)] \geq \min\{r, \Delta[C(F_n)]\} + 1 = \Delta + 1$. For $r \geq \Delta$ based on Lemma 1, $\chi_r[C(F_n)] \geq \min\{r, \Delta[C(F_n)]\} + 1 = \Delta + 3$. Thus, it completes the proof. □

Theorem 1. *For $n \geq 4$, let $C(F_n)$ be the central graph of a Flower graph F_n. Then,*

$$\chi_r[C(F_n)] = \begin{cases} n & r = 1, \\ 2n+1 & \text{for } \delta = 2 \leq r \leq 2n-1, \\ 2n+3 & \text{for } r = \Delta = 2n, \quad n \text{ is even}, \\ 2n+4 & \text{for } r = \Delta = 2n, \quad n \text{ is odd}. \end{cases}$$

Proof. The r-dynamic coloring of $C(F_n)$ is explained in the following cases:

Case 1. When $r = 1$.

Based on the Lemma 2 the lower bound of $\chi_r[C(F_n)] \geq n$. The upper bound can be found out from the following coloring:

- Color the hub vertex v with color 1.
- Color the vertices v_i and u_i with i colors for $(1 \leq i \leq n)$.
- Color the vertices w_i and e_i for $(2 \leq i \leq n)$ with $i-1$ colors and w_1, e_1 with color n. Thus, $2n-2$ vertices are colored.
- Color the vertices e'_i, w'_i for $(1 \leq i \leq n-1)$ with color n and e'_n, w'_n with color 2. Hence, couple-adjacency condition is fulfilled and therefore, $\chi_r[C(F_n)] \leq n$. Hence, $\chi_r[C(F_n)] = n$.

Case 2. When $\delta = 2 \leq r \leq n-1$.

According to Lemma 2, the lower bound of $\chi_r[C(F_n)] \geq \Delta + 1$. The r-dynamic $(2n+1)$ coloring is justify to find the upper bound of $C(F_n)$:

- Assign the color i for the vertices v_i and e'_i for $(1 \leq i \leq n)$.
- Color the vertices u_i and w'_i with $n+1, n+2, \ldots, 2n$ colors for $(1 \leq i \leq n)$.
- Assign the color $2n+1$ to the hub vertex v. Figure 2 illustrate the central graph of flower graph F_4.

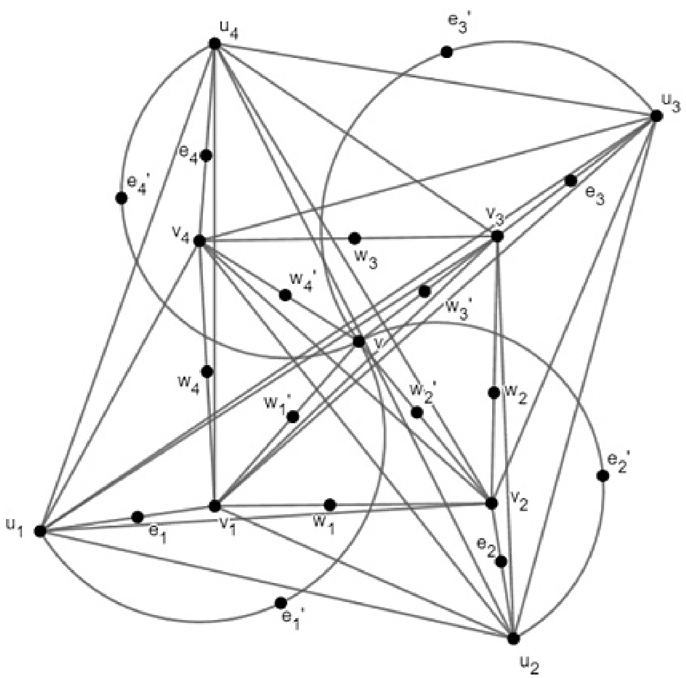

Fig. 2. Central graph of Flower graph F_4.

- Color the vertex e_1 with color n and color the remaining vertices of e_i for $(2 \le i \le n)$ with $1, 2, \ldots, n-1$ colors. Thus, $2n$ colors are used. To satisfy the couple-adjacency coloring, we need one color. So, the coloring for the remaining n vertices is as follows:

  (1) When n is odd, color the vertices w_i, for $(i = 1, 3, \ldots, n-2)$ with $2n+1$ color. To attain r-coloring, color the left over vertices w_{n-1} and w_n with color 1 and color 2.
  (2) As n is even, color the vertices w_i for $(1 \le i \le n)$ with $2n+1$ color.

Thus, the condition of couple-adjacency coloring is satisfied and we obtain $\chi_r[C(F_n)] \le \Delta + 1$. Hence, $\chi_r[C(F_n)] = \Delta + 1 = 2n + 1$.

Case 3. When $r = \Delta = 2n$.

Based on Lemma 2, the lower bound of $\chi_r[C(F_n)] \ge \Delta + 3$. The r-dynamic $(2n+1)$ coloring is given in the following to find the upper bound of $C(F_n)$:

- Color the vertices v, u_i, w'_i, v_i and e'_i for $(1 \le i \le n)$ with the colors as given in case-2 and color the vertices e_i with $2n+1$ color for $(1 \le i \le n)$.

  (1) As n is even, color the vertices w_i with $2n+2$ color, for $(i = 1, 3, \ldots, n-1)$ and the vertices w_i with $2n+3$ color for $(i = 2, 4, \ldots, n)$. Hence, $\chi_r[C(F_n)] \le \Delta + 3$. Therefore, $\chi_r[C(F_n)] = \Delta + 3 = 2n + 3$.
  (2) As n is odd, to satisfy the couple-adjacency condition one must add one color. So, color the vertices w_i with color $2n+2$ for $(i = 1, 3, \ldots, n-2)$ and for $(i = 2, 4, 6, \ldots, n-1)$ with color $2n+3$. Therefore, to obtain couple-adjacency condition, color the vertex w_n with $2n+4$ color. Thus, $\chi_r[C(F_n)] \le \Delta + 4$. Hence, $\chi_r[C(F_n)] = 2n + 4$. □

Lemma 3. *Let $L(F_n)$ be the line graph of flower graph. The lower bound for r-dynamic chromatic number of the line graph of flower graph is*

$$\chi_r[L(F_n)] \ge \begin{cases} 2n & 1 \le r \le \Delta - 3, \\ \Delta + 1 & r = \Delta - 2, \\ \Delta + 3 & \Delta - 1 \le r \le \Delta. \end{cases}$$

Proof. Let the line graph of flower graph be $L(F_n)$. The vertices of $L(F_n)$ are nothing but the edges of (F_n); they are $\{w_i, w'_i, e_i, e'_i\}$, for $(1 \le i \le n)$. The order of the graph $L(F_n)$ is $|V(L(F_n))| = 4n$ and the size of the graph $L(F_n)$ is $|E(L(F_n))| = 2n+3$. For $1 \le r \le \Delta - 3$, the vertices e'_i, w'_i persuade a clique of order $2n$ in $L(F_n)$. Thus, $\chi_r[L(F_n)] \ge 2n$. For $r = \Delta - 2$ based on Lemma 1, $\chi_r(G) \ge \min\{r, \Delta(G)\} + 1$ which implies $\chi_r[L(F_n)] \ge \min\{r, \Delta[L(F_n)]\} + 1 = \Delta + 1$. For $\Delta - 1 \le r \le \Delta$ based on Lemma 1, $\chi_r[L(F_n)] \ge \min\{r, \Delta[L(F_n)]\} + 1 = \Delta + 3$. Hence, it completes the proof. □

Theorem 2. *For $n \geq 5$, let $L(F_n)$ be the line graph of a Flower graph F_n. Then,*

$$\chi_r[L(F_n)] = \begin{cases} 2n, & \text{for } 1 \leq r \leq 2n-1, \\ 2n+2, & \text{for } r = 2n \quad \text{and} \quad n \text{ is even}, \\ 2n+3, & \text{for } r = 2n \quad \text{and} \quad n \text{ is odd}, \\ 2n+5, & \text{for } 2n+1 \leq r \leq 2n+2 = \triangle, \\ & \text{and} \quad n \equiv 0 \pmod 6 \quad \text{and} \quad n = 5, \\ 2n+6, & \text{for } 2n+1 \leq r \leq 2n+2 = \triangle, \quad \text{and} \quad n \not\equiv 0 \pmod 6). \end{cases}$$

Proof. The r-dynamic coloring of $L(F_n)$ is explained in the following.

Case 1. When $1 \leq r \leq 2n-1$.

According to the Lemma 3, the lower bound of $\chi_r[L(F_n)] \geq 2n$. To find the upper bound color the vertices w_i', for $(1 \leq i \leq n)$ orderly with colors i and e_i', for $(1 \leq i \leq n)$ with the colors $n+m$ for $(1 \leq m \leq n)$. Next, the vertex w_2 is with color n, w_1 with $n-1$ color and the left over vertices of w_i for $(3 \leq i \leq n)$ are with $1, 2, \ldots, n-2$ color. At last, the n vertices of e_i are uncolored. Hence, color the vertices e_i, for $(3 \leq i \leq n)$ with $n+m$ colors, where $1 \leq m \leq n-2$ and color the vertices e_1 and e_2 with the colors $2n-1$ and $2n$. Thus, $\chi_r[L(F_n)] \leq 2n$. Hence, $\chi_r[L(F_n)] = 2n$. Figure 3 illustrate the line graph of flower graph F_4.

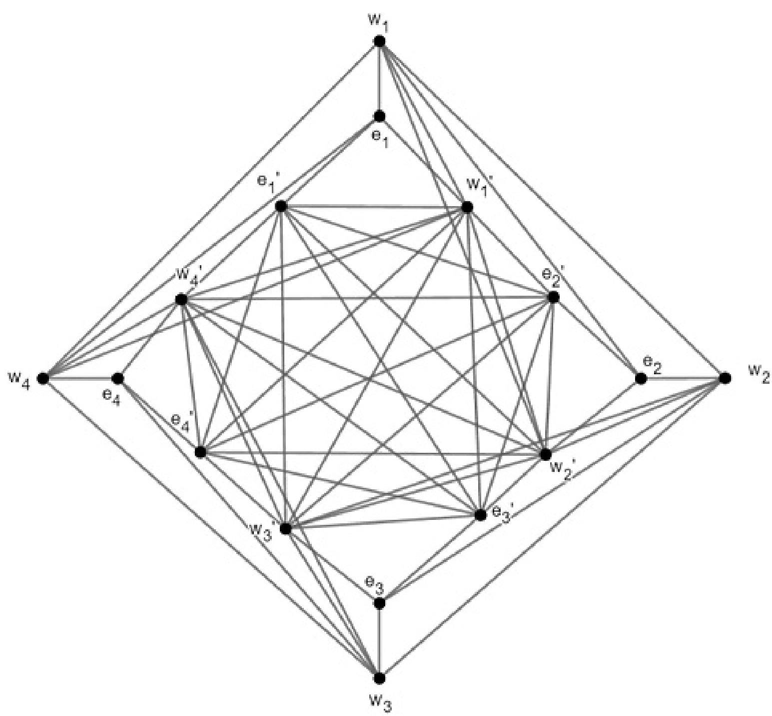

Fig. 3. Line graph of Flower graph F_4.

Case 2. When $r = 2n$.

From Lemma 3, the lower bound of $\chi_r[L(F_n)] \geq \Delta + 1$. The upper bound of $L(F_n)$ can be considered from the following coloring;

- Color the vertices w'_i and e'_i as given in Case 1. Next, color the vertices w_i for $(3 \leq i \leq n)$ with $n + m$ colors where, $1 \leq m \leq n - 2$ and then color the other vertices w_2 and w_1 with $2n$ color and $2n - 1$ color.
  (1) As n is even, the vertices e_i are with the color $2n+1$ and $2n+2$ alternatively. Hence, r-coloring has been obtained and $\chi_r[L(F_n)] \leq 2n + 2$. Therefore, $\chi_r[L(F_n)] = 2n + 2$.
  (2) As n is odd, color the vertices e_i for $(1 \leq i \leq n - 1)$with $2n + 1$ and $2n + 2$ colors alternatively. In order to get the result, color the remaining vertex e_n with $2n + 3$ color. Hence, the couple-adjacency condition is checked out and $\chi_r[L(F_n)] \leq 2n + 3$. Therefore, $\chi_r[L(F_n)] = 2n + 3$.

Case 3. When $2n + 1 \leq r \leq 2n + 2 = \triangle$.

According to Lemma 3, the lower bound of $\chi_r[L(F_n)] \geq \Delta + 3$. The upper bounds of $L(F_n)$ are demonstrated from the following way:

- The vertices w'_i and e'_i are colored in order as given in Case 1. The remaining $2n$ vertices of w_i and e_i remain uncolored, so color these vertices in the following way to attain the couple-adjacency condition:
  (1) If $n = 5$, color the vertices w_i for $(1 \leq i \leq n)$ with $2n + m$ color where $(m = 1, 2, \ldots, n)$. Next, color the vertices e_n and e_{n-1} with $2n+1$ and $2n+2$ color and color the left over vertices of e_i for $(1 \leq i \leq n - 2)$ with $2n + m$ color where, $(m = 3, 4, \ldots, n)$. Therefore, $2n + 5$ colors are obtained and $\chi_r[L(F_n)] \leq \Delta + 3$. Thus, $\chi_r[L(F_n)] = 2n + 5$.
  (2) If $n \equiv 0 \pmod 6$, then color the first $3t$ vertices of w_i with $2n+1, 2n+2$ and $2n + 3$ color in sequence, where t is the largest positive integer and $3t \leq n$. At last, remaining n vertices are uncolored. So to get $(2n + 5)$ colors, color the vertices of e_i, with $2n + 4$ and $2n + 5$ color alternatively for $(1 \leq i \leq n)$. Hence, the couple-adjacency condition is satisfied and $\chi_r[L(F_n)] \leq \Delta + 3$. Thus, $\chi_r[L(F_n)] = 2n + 5$.
  (3) In this case $n \not\equiv 0 \pmod 6$, assign the colors $2n + 1, 2n + 3$ and $2n + 2$ in sequence to the vertices w_i for $(1 \leq i \leq n - 2)$. But, two vertices of w_i are uncolored. So, to obtain the result, color the vertex w_{n-1} with $2n + 4$ color if n is odd else, with $2n + 5$ color and color the vertex w_n with $2n + 6$ color. Next, color the vertex e_1 with $2n + 2$ color and the remaining vertices of e_i for $(2 \leq i \leq n - 2)$ with $2n + 4$ and $2n + 5$ colors alternatively. Finally, two vertices of e_i are to be colored.
     (a) the vertex e_{n-1} are colored with $2n + 2$ color for $n = 7, 10, \ldots, m + 1$, where m is divisible by 3 and $m \geq 6$.

(b) Color the vertex e_{n-1} with $2n+1$ color for $n = 8, 11, \ldots, m-1$, where m is divisible by 3 and $m \geq 9$.
(c) Color the vertex e_{n-1} with $2n+3$ color for $n = 9, 15, \ldots, m+3$, where $m \equiv 0 \pmod 6$.

(4) If n is even, color the vertex e_n with $2n+4$ color and if n is odd then use $2n+5$ color.

Therefore, the results are obtained for the line graph of flower graph by using r-dynamic coloring and hence, $\chi_r[L(F_n)] \leq 2n+6$. Therefore, $\chi_r[L(F_n)] = 2n+6$. □

Theorem 3. *For $n \geq 5$, let $S(F_n)$ be the subdivision graph of a Flower graph F_n. Then,*

$$\chi_r[S(F_n)] = \begin{cases} 2 & \text{for } r = 1, \\ n+1 & \text{for } \delta = 2 \leq r \leq n, \\ r+1 & \text{for } n+1 \leq r \leq \Delta = 2n. \end{cases}$$

Proof. The vertices of $S(F_n)$ are $v, v_i, u_i, w_i, w'_i, e_i$ and e'_i, for $(1 \leq i \leq n)$ where the vertices e_i, e'_i, w'_i are corresponding to the edge u_iv_i, vu_i, vv_i, the vertex w_i are corresponding to the edge v_iv_{i+1} for $(1 \leq i \leq n-1)$ and the vertex w_n is corresponding to the edge v_nv_1. The order of graph $S(F_n)$ is $|V(S(F_n))| = 6n+1$ and the size of the graph $S(F_n)$ is $|E(S(F_n))| = 7n$. The r-dynamic coloring of $S(F_n)$ is as follows:

Case 1. when $r = 1$.

Assign color 1 to the vertices v_i, u_i and v for $(1 \leq i \leq n)$ and the leftover vertices e_i, e'_i, w'_i and w_i for $(1 \leq i \leq n)$ with color 2. Hence, the couple-adjacency condition n be obtained and $\chi_r[S(F_n)] = 2$. If $\chi_r[S(F_n)] < 2$, then the couple-adjacency condition is not satisfied.

Case 2. When $\delta = 2 \leq r \leq n$.

- Assign the color i for the vertices v_i and e'_i where $(1 \leq i \leq n)$.
- Color the vertices u_i and w'_i for $(1 \leq i \leq n-1)$ with $2, 3, \ldots, n$ colors and the vertices u_n and w'_n with the color 1.
- The vertex e_1 is colored with color n and the vertices e_i for $(2 \leq i \leq n)$ with $1, 2, \ldots, n-1$ colors.
- Thus, $n+1$ vertices are uncolored. So, for $(1 \leq i \leq n-3)$ color the vertices w_i with the color $4, 5, \ldots, n$ and the remaining vertices w_{n-2} with color 1, w_{n-1} with color 2 and w_n with color 3. To satisfy the couple-adjacency condition, the hub vertex v is colored with $n+1$ color. Hence, $\chi_r[S(F_n)] = n+1$. If $\chi_r[S(F_n)] < n+1$, then the couple-adjacency condition is not fulfilled.

Case 3. When $n+1 \leq r \leq \Delta = 2n$.

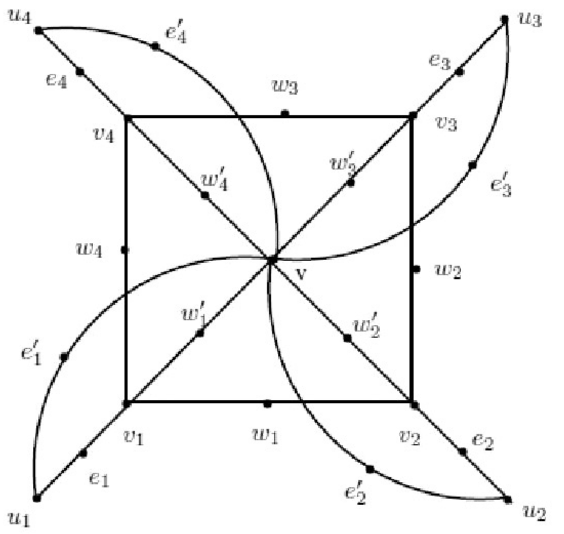

Fig. 4. Subdivision graph of Flower graph F_4.

Figure 4 illustrate the subdivision graph of flower graph F_4. Color the vertices v, v_i, u_i, w_i, w'_i and e_i for $(1 \leq i \leq n)$ as given in Case 2. In order, to maintain the couple-adjacency condition, $r+1$ colors are needed. When $r = n+1$, assign the color $n+2$ to the vertices e'_1 and e'_i for $(2 \leq i \leq n)$ with colors $2, 3, \ldots, n$. By continuing the coloring, for $r = 2n$ color the vertices e'_i with $n+2, n = 3, \ldots, 2n+1$ colors for $(1 \leq i \leq n)$. Therefore, for $r \geq n$, $\chi_r[S(F_n)] = r+1$. If $\chi_r[S(F_n)] < r+1$, then the couple-adjacency condition is not fulfilled. □

Lemma 4. *Let $L[S(F_n)]$ be the paraline graph of flower graph. The lower bound for r-dynamic chromatic number of the paraline graph of flower graph is*

$$\chi_r L[S(F_n)] \geq \begin{cases} 2n & 1 \leq r \leq \Delta - 1, \\ \Delta + 1 & r = \Delta. \end{cases}$$

Proof. The vertices of $\chi_r(L[S(F_n)])$ are a_i, b_i, c_i and d_i, which are the edges of subdivision graph $S(F_n)$. The order of the graph is $|V(L[S(F_n)])| = 8n$ and the size of the graph is $|E(L[S(F_n)])| = 2n(n+5)$. For $1 \leq r \leq \Delta - 1$ the vertices b_i, d_i for $i = 2, 4, \ldots, 2n$ persuade a clique of order $2n$ in $(L[S(F_n)])$. Thus, $\chi_r(L[S(F_n)]) \geq 2n$. For $r = \Delta$ based on Lemma 1, $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$ such that $\chi_r(L[S(F_n)]) \geq \min\{r, \Delta(L[S(F_n)])\} + 1 = \Delta + 1$. Thus, the proof is complete. □

Theorem 4. *For $n \geq 3$, let $L[S(F_n)]$ be the paraline graph of a Flower graph F_n. Then,*

$$\chi_r(L[S(F_n)]) = \begin{cases} 2n & \textit{for } 1 \leq r \leq 2n-1, \\ 2n+1 & \textit{for } r = 2n = \Delta. \end{cases}$$

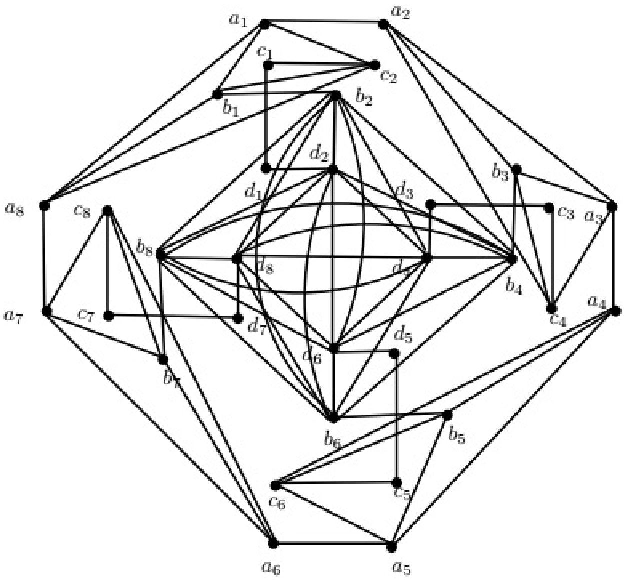

Fig. 5. Paraline graph of Flower graph F_4.

Figure 5 illustrate the paraline graph of flower graph F_4.

Proof. The r-dynamic coloring for paraline graph of flower graph is as follows:

Case 1. When $1 \leq r \leq 2n - 1$.

From Lemma 4, the lower bound of $\chi_r(L[S(F_n)]) \geq 2n$. However, one cannot find the sharpest lower bound. But, $\chi_r(L[S(F_n)]) = 2n$ has been considered. The upper bound can be found from coloring of $(L[S(F_n)])$ which is as follows:

- Color b_i for $i = 1, 3, \ldots, 2n-1$ with color 4. Color the vertices b_i for $i = 2, 4, \ldots, 2n$ with different colors to satisfy the couple-adjacency condition, so color the vertex b_2 with color 2, b_4 with color 1 and b_6 with color 3 and the remaining vertices of b_i for $i = 8, 10, \ldots, 2n$ with $5, 6, \ldots, n+1$ colors.
- Since, the vertices c_i are with least degree δ, color the vertices c_i for $i = 1, 3, \ldots, 2n-1$ with color $2, 1$ and 3 in sequence and the other vertices c_2, c_4 are with color 5 and c_6 are with color 6. Then, the leftover vertices c_i for $i = 8, 10, \ldots, 2n$ with color 5.
- Color the vertices d_i for $i = 1, 3, \ldots, 2n-1$ with color $3, 2$ and 1 alternatively. Since the vertices d_i for $i = 2, \ldots, 2n$ form a cycle (C_{n-1}), color the vertices d_2 with color 4 and the other vertices d_i for $i = 4, 6, \ldots, 2n$ with colors $n+2, n+3, \ldots, 2n$.
- At last, $2n$ vertices of a_i are uncolored. Hence, the coloring of the vertices a_i is as follows:
  (1) When $n \equiv 0 \pmod 3$, color the vertices a_i for $(1 \leq i \leq 2n)$ with colors $1, 2$ and 3 alternatively.

(2) When $n = 4, 7, 10, \ldots, m+1$, where $m \equiv 0 \pmod 3$, color the vertices a_i for $(1 \leq i \leq 2n-2)$ with colors $1, 2$ and 3 in sequence and color the vertex a_{2n-1} with color 6 and a_{2n} with color 7.

(3) When $n = 5, 8, 11, \ldots, m-1$, where $m \equiv 0 \pmod 3$ and $m \geq 6$, color the vertices a_i for $(1 \leq i \leq 2n-1)$ with colors $1, 2$ and 3 in sequence and color the vertex a_{2n} with color 6. Hence, the couple-adjacency condition is fulfilled and $\chi_r(L[S(F_n)]) \leq 2n$. Therefore, $\chi_r(L[S(F_n)]) = 2n$.

Case 2. When $r = 2n$.

Based on the Lemma 4, the lower bound of $\chi_r(L[S(F_n)]) \geq \Delta + 1$. However, one cannot find the sharpest lower bound. But, $\chi_r(L[S(F_n)]) = \Delta + 1$ has been considered. To find the upper bound, color the vertices a_i, b_i and c_i for $(1 \leq i \leq n)$ which is given in case 1. Color d_i for $i = 1, 3, \ldots, 2n-1$ with color 4 and the leftover vertices d_i for $i = 2, 4, \ldots, 2n$ with $n+2, n+3, \ldots, 2n+1$ colors in order. Therefore, the couple-adjacency condition is satisfied and $\chi_r(L[S(F_n)]) \leq \Delta + 1$ and $\chi_r(L[S(F_n)]) = 2n + 1$. □

Theorem 5. *For $n \geq 4$, let $S[F_n]$ be the splitting graph of Flower graph F_n. Then,*

$$\chi_r(S[F_n]) = \begin{cases} 3 & \text{for } 1 \leq r \leq 2 = \delta, \quad n \text{ is even}, \\ 4 & \text{for } 1 \leq r \leq 2 = \delta, \quad n \text{ is odd}, \\ 5 & \text{for } (r \equiv 0 \bmod 3) \quad \text{and} \quad r \leq 5, \\ 7 & \text{for } 4 \leq r \leq 5, \\ r+2 & 6 \leq r \leq 4n = \Delta. \end{cases}$$

Proof. The vertices of $S[F_n]$ are $v, v_i, u_i, v', v'_i, u'_i$ for $(1 \leq i \leq n)$. The order of the graph is $|V(S[F_n])| = 4n + 2$ and the size of the graph $E(S[F_n])| = 12n$. The coloring of splitting graph of flower graph is as follows:

Case 1. When $1 \leq r \leq 2$.

- Color the hub vertex v and v' with color 3.

(1) As n is even, assign the color 1 and 2 to the vertices v_i and v'_i for $(1 \leq i \leq n)$ consecutively. Next, color the vertices u_i and u'_i with color 2 and 1 cyclically for $(1 \leq i \leq n)$. Hence, $\chi_r(S[F_n]) \leq 3$. If $\chi_r(S[F_n]) < 3$, then the couple-adjacency condition is not fulfilled. Therefore, $\chi_r(S[F_n]) = 3$.

(2) When n is odd, the vertices v_i and v'_i for $(1 \leq i \leq n-1)$ are with color 1 and color 2 consecutively and color the vertex v_n and v'_n with color 4. Color the vertices u_i and u'_i for $(1 \leq i \leq n-1)$ with color 2 and color 1 consecutively and color the remaining vertex u_n and u'_n with color 1. Hence, $\chi_r(S[F_n]) \leq 4$. If $\chi_r(S[F_n]) < 4$, then the couple-adjacency condition cannot be verified. Thus, $\chi_r(S[F_n]) = 4$.

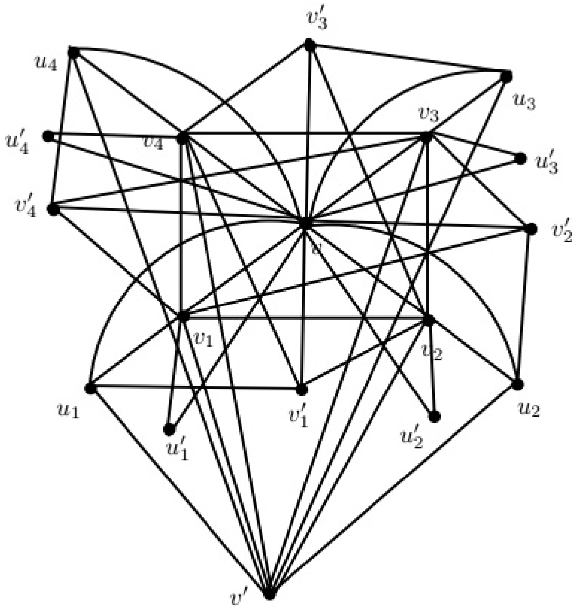

Fig. 6. Splitting graph of Flower graph F_4.

Case 2. When ($r \equiv\ 0 \bmod 3$) and $r \leq 5$. Figure 6 illustrate the spliting graph of flower graph F_4.

- Color the hub vertex v and the vertex v' with color 3 and color 4.
  (1) When n is odd, the vertices v_i and v_i' for $(1 \leq i \leq n-1)$ are with color 1 and color 2 cyclically and color the vertex v_n and v_n' with color 5. Color the vertices u_i and u_i' with color 5 for $(1 \leq i \leq n-1)$. Next, the vertex u_n and u_n' are colored with color 1.
  (2) When n is even, the vertices v_i and v_i' for $(1 \leq i \leq n)$ are with color 1 and color 2 cyclically. Color the vertices u_i and u_i' for $(1 \leq i \leq n)$ with color 5.

Hence, $\chi_r(S[F_n]) \leq 5$. If $\chi_r(S[F_n]) < 5$, then the couple-adjacency condition is not fulfilled. Then, $\chi_r(S[F_n]) = 5$.

Case 3. When $4 \leq r \leq 5$.

To satisfy the couple-adjacency condition for $r = 4$ one must need 7 colors. Hence, $\chi_{r=4}(S[F_n]) \leq 7$ for any n. If $\chi_{r=4}(S[F_n]) < 7$, the couple-adjacency condition is not satisfied. Therefore, $\chi_{r=4}(S[F_n]) = 7$. Similarly, it also satisfies the 5-dynamic coloring. Hence, $\chi_{r=4,5}(S[F_n]) = 7$ for any n.

Case 4. When $6 \leq r \leq 4n = \Delta$.

For any n, $\chi_{r=6}(S[F_n]) = 8$. If $\chi_{r=6}(S[F_n]) \leq 8$ the couple-adjacency condition is not fulfilled. Similarly, $\chi_{r=7}(S[F_n]) = 9$. Thus, one must need $r+2$ colors. Therefore, for any n $\chi_r(S[F_n]) = r + 2$. $\square$

Theorem 6. *For $n \geq 4$, let $\mu[F_n]$ be the mycielski graph of Flower graph F_n. Then,*

$$\chi_r(\mu[F_n]) = \begin{cases} 4 & \textit{for } 1 \leq r \leq 2 = \delta, \quad n \textit{ is even}, \\ 5 & \textit{for } 1 \leq r \leq 2 = \delta, \quad n \textit{ is odd}, \\ r+2 & \textit{for } r \equiv 0 \pmod 3 \quad \textit{and} \quad n \textit{ is even}, \\ r+3 & \textit{otherwise}. \end{cases}$$

Proof. The vertices of $\mu[F_n]$ are $z, v, v_i, u_i, v', v'_i, u'_i$ for $(1 \leq i \leq n)$. The order of the graph is $|V(\mu[F_n])| = 4n+3$ and size of the graph is $|E(\mu[F_n])| = 14n+1$. The coloring of mycielski graph of flower graph is as follows:

Case 1. When $1 \leq r \leq 2$.

Color the vertices u_i and u'_i for $(1 \leq i \leq n)$ with color 2 and color 1 orderly.

- When n is even, the vertices v_i and v'_i for $(1 \leq i \leq n)$ are colored with colors 1 and 2 alternatively and color the vertices u_i and u'_i for $(1 \leq i \leq n)$ with colors 2 and 1 in order. Color the hub vertex v and the vertex v' with color 3 and the common vertex z with color 4. Therefore, $\chi_r(\mu[F_n]) \leq 4$. If $\chi_r(\mu[F_n]) < 4$, the couple-adjacency condition is not fulfilled. Hence, $\chi_r(\mu[F_n]) = 4$.
- When n is odd, the vertices v_i and v' for $(1 \leq i \leq n-1)$ are colored with colors 1 and 2 orderly and color the vertex v_n and v'_n with color 3. Figure 7 illustrate the Mycielski graph of flower graph F_4. Color the hub vertex v and the vertex v' with color 4 and the common vertex z with color 5. Hence, $\chi_r(\mu[F_n]) = 5$. If $\chi_r(\mu[F_n]) < 5$, the couple-adjacency condition is not fulfilled.

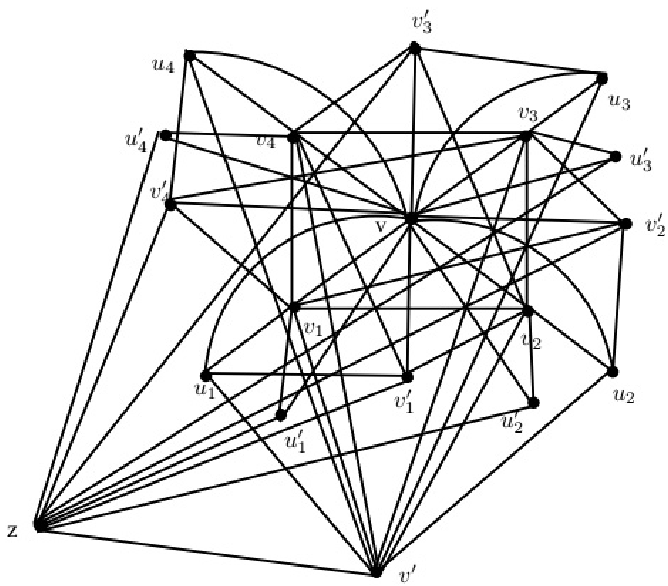

Fig. 7. Mycielski graph of Flower graph F_4.

Case 2. When $r \equiv 0 \pmod 3$ and n is even.

- Color the vertices v_i and v'_i with colors 1 and 2 alternatively for $(1 \leq i \leq n)$.
- Color the vertices u_i for $(1 \leq i \leq n)$ with color 5 and the vertices u'_i for $(1 \leq i \leq n)$ are colored with colors 2 and 1 alternatively.
- Color the hub vertex v with color 3 and the vertex v' with color 4 and the common vertex z with color 5. Thus, $\chi_r(\mu[F_n]) \leq r+2$. If $\chi_r(\mu[F_n]) < r+2$, the couple-adjacency condition is not fulfilled. Hence, $\chi_r(\mu[F_n]) = r+2$.

Case 3. otherwise.

- When $r \equiv 0 \pmod 3$ and n is odd, color the vertices u_i and u'_i for $(1 \leq i \leq n)$ which is given in case 1. Color the vertices v_i for $(1 \leq i \leq n-1)$ with colors 1 and 2 alternatively and color the vertex v_n with color 3. Color the vertices v'_i with color 6 for $(1 \leq i \leq n)$. At last, color hub vertex v and assign the color 4 for the vertex v' and the common vertex z with color 5. Thus, $\chi_r(\mu[F_n]) = r+3$. If $\chi_r(\mu[F_n]) < r+3$, the couple-adjacency condition is not verified. Hence, $\chi_r(\mu[F_n]) = r+3$.
- When $4 \leq r \leq 4n = \Delta$, the r-dynamic chromatic number for $\chi_r(\mu[F_n]) \leq r+3$ for any n. If $\chi_r(\mu[F_n]) < r+3$, the couple-adjacency condition is not fulfilled. So, one must need $r+3$ colors. Hence, for any n, $\chi_r(\mu[F_n]) = r+3$. □

3. Conclusion

Thus, the lower bound of the r-dynamic chromatic number of some flower graph families has been found and also some exact results have been determined. Further studies of distance graphs may give an additional insight to the r-dynamic coloring problem.

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