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Elegant labeling of some graphs

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Elegant labeling of some graphs

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Abstract. An elegant labeling on graph G with α vertices and β edges is an injective (one-to-one) mapping from the set of vertices $V(G)$ to the set of non-negative integers $\{0, 1, 2, 3, ..., \beta\}$ in such a way that the set of values on the edges obtained by the sums mod $\beta + 1$ of the labels of their end vertices is $\{1, 2, 3, 4, \ldots, \beta\}$. The purpose of this research is to prove that several graphs, namely diamond ladder admit an elegant labeling. We also look for relationships between Edge Antimagic Vertex Labeling and Elegant Labeling.

Keywords: *Elegant labeling, Shackle of Graph, and Diamond Ladder*

1. Introduction

Labeling is one topic on graph theory. Labeling is one topic in graph theory that still growing. So many kind of graph labeling among others: magic labeling, antimagic labeling, edge antimagic vertex labeling, edge antimagic total labeling, super $(a, d) - H$ antimagic total labeling, harmonius labeling, and etc. The research related to super $(a, d) - H$ antimagic total labeling can be seen at [3, 2, 7]. Super $(a, d) - H-$ antimagic total labeling is developed by (a, d) edge antimagic vertex labeling. For a graph $G = (V, E)$, a bijection γ from $V(G) \cup E(G)$ into $\{1, 2, ..., |V(G)| + |E(G)|\}$ is called (a, d)-edge-antimagic total labeling of G if the edge-weights $w(xy) = \gamma(x) + \gamma(y) + \gamma(xy), xy \in E(G)$, form an arithmetic progression starting from a and having common difference d. An (a, d)-edge-antimagic total labeling is called super (a, d)-edge-antimagic total if $g(V(G)) = \{1, 2, ..., |V(G)|\}$ [6]. One of the labeling topic chosen in this study is an elegant labeling.

An elegant labeling on graph G with α vertices and β edges is an injective (one-to-one) mapping from the set of vertices $V(G)$ to the set of non-negative integers $\{0, 1, 2, 3, ..., \beta\}$ in such a way that the set of values on the edges obtained by the sums mod $\beta + 1$ of the labels of their end vertices is $\{1, 2, 3, 4, \ldots, \beta\}$ [1].

Balakrishnan *et al* in [1] have discovered that some graphs namely $K_{n,m}$; $H_{n,n}$; Total Graph $(T(P_n))$; Bistar $(B_{n,n})$; $< K_{1,n,2t} >$, $S_{n-1}(K_{1,2m})$; $< S_{m-2}(K_{1,n}; 2^k) >$ are elegant. They also have found some results such as: If G is elegant, then for any elegant labeling of $G, g(v) \neq 0$ for any $V \in V(G)$, Every simple graph is a subgraph of an elegant graph, and $B_{n,n}$ is not elegant when n is odd. Some results related to elegant labeling are can be seen in [5, 4]. Gomathi *et al* in [8] have found that certain families of graphs are shown to be elegant.

Modulo is a number operation that produces the remainder of the division of one number by another number. Given two numbers a and b, a modulo b is the remainder of division a by b. A simple example of modulo numbers is on a 7-day system. Within 1 week, starting from Monday to Sunday, there are 7 days. Modulo numbers can be used to guess the day. First step, change Monday to Sunday into numbers from 0 to 7. Monday = 0, Tuesday= 1, Wednesday=2, Thirsday=3, Friday=4, Saturday = 5, and Sunday = 6.To determine x days to come, add the days that have been converted into numbers with x days to come, and then divide the sum by seven. Check the remaining results of the division by looking at the Table 1. The remaining of the division is x days to come. Suppose today is Tuesday, the next ten days are days? to determine the next ten days is to use the following calculation: (1+ 10): 7 = 1 remaining 4. Based on Table 1, the next 10 days will be Friday. The aim on this research are to proof that some

Table 1. Table of Remainder

Days	Remainder
Monday	0
Tuesday	1
Wednesday	2
Thirsday	3
Friday	4
Saturday	5
Sunday	6

graphs namely Diamond Ladder graph Dl_t and $Shack(F_4, e, t)$ admit elegant labeling. In this research, we also analyze the relation between Edge Antimagic Vertex Labeling (EAVL) and Elegant Labeling.

The Results

In this section, there are three theorems.

Theorem 1. *Diamond Ladder Dl_t is Elegant*

Proof. Diamond Ladder graph is denoted by Dl_t for $t \geq 2$. Diamond Ladder graph is developed by Ladder graph. Diamond Ladder graph consits of t diamond. Diamond Ladder graph Dl_t has vertex set as follows: $\{y_k; k = 1, 2, \ldots, t\} \cup \{z_k; k = 1, 2, \ldots, t\} \cup \{u_l; l = 1, 2, \ldots, 2t\}$. The edge set of Diamond Ladder graph Dl_t is as follows: $\{y_k y_{k+1}; k = 1, 2, \ldots, t-1\} \cup \{z_k z_{k+1}; k = 1, 2, \ldots, t-1\} \cup \{y_k z_k; k = 1, 2, \ldots, t\} \cup$

$\{y_k u_{2k-1}; k = 1, 2, \ldots, t\} \cup \{y_k u_{2k}; k = 1, 2, \ldots, t\} \cup \{z_k u_{2k-1}; k = 1, 2, \ldots, t\} \cup \{z_k u_{2k}; k = 1, 2, \ldots, t\} \cup \{u_{2k} u_{2k+1}; k = 1, \ldots, t-1\}$. Based on the vertex and edge set, we can determine the number of vertex and edge of graph Dl_t thus the number of vertex and edge of graph Dl_t are in the following: $(V(Dl_t)) = \alpha = 4t$ and $(E(Dl_t)) = \beta = 8t-3$. Now we are ready to define the vertex labeling under the function $\gamma_1 : V(Dl_t) \rightarrow \{0, 1, 2, \ldots, (8t-3)\}$. Let γ_1 be an injective function (one to one mapping) which defined in the following:

$$\begin{aligned} \gamma_1(y_k) &= 1 + 4(k-1) = 4k-3; k = 1, 2, \ldots, t \\ \gamma_1(z_k) &= 2 + 4(k-1) = 2(2k-1); k = 1, 2, \ldots, t \end{aligned}$$

$$\gamma_1(u_l) = \begin{cases} 0 + 4(\frac{l+1}{2} - 1) = 2(l-1); l = 1, 2, \ldots, 2t & \text{and } l \text{ odd} \\ 3 + 4(\frac{l}{2} - 1) = 2l - 1; l = 1, 2, \ldots, 2t & \text{and } l \text{ even} \end{cases}$$

Next, can be determined the edge labeling of graph Dl_t. The edge labeling is obtained from the sum of two vertex which adjacent:

$$\begin{aligned} \gamma_1(y_k y_{k+1}) &= [(4k-3) + (4k+1)] \; mod(8t-2) = [8k-2] \; mod(8t-2); k = [1, t-1] \\ \gamma_1(z_k z_{k+1}) &= [(4k-2) + (4k+2)] \; mod(8t-2) = [8k] \; mod(8t-2); k = [1, t-1] \\ \gamma_1(y_k z_k) &= [(4k-3) + (4k-2)] \; mod(8t-2) = [8k-5] \; mod(8t-2); k = [1, t-1] \\ \gamma_1(y_k u_{2k-1}) &= [(4k-3) + (4k-4)] \; mod(8t-2) = [8k-7] \; mod(8t-2); k = [1, t] \\ \gamma_1(y_k u_{2k}) &= [4 + 8(k-1)] \; mod(8t-2) = [8k-4] \; mod(8t-2); k = [1, t] \\ \gamma_1(z_k u_{2k-1}) &= [2 + 8(k-1)] \; mod(8t-2) = [8k-6] \; mod(8t-2); k = [1, t] \\ \gamma_1(z_k u_{2k}) &= [5 + 8(k-1)] \; mod(8t-2) = [8k-3] \; mod(8t-2); k = [1, t] \\ \gamma_1(u_{2k} u_{2k+1}) &= [7 + 8(k-1)] \; mod(8t-2) = [8k-1] \; mod(8t-2); k = [1, t-1] \end{aligned}$$

The sequence of the edge labeling of Diamond Ladder graph with $4t$ vertices can be written as follows: $S = \{y_k u_{2k-1}, z_k u_{2k-1}, y_k z_k, y_k u_{2k}, z_k u_{2k}; k = 1, 2, \ldots, t\} \cup \{y_k y_{k+1}; u_{2k} u_{2k+1}, zkzk+1; k = 1, 2, \ldots, t-1\}$. Based on the edge labeling and the sequence of the edge labeling, obviously γ_1 is an injective (one to one mapping) function. Hence, γ_1 is an elegant labeling and it is complete the proof. □

We illustrate the Diamond Ladder graph with twelve vertices as shown in Figure 1 with $V(Dl_3) = \{y_1, \ldots, y_3, z_1, \ldots, z_3, u_1, u_2, \ldots, u_6\}$. The number of edge of graph Dl_3 is twenty one. The sequence of edge labeling is exhibited below:

$\gamma_1(y_1 u_1) = 0 + 1 = 1$	$\gamma_1(z_1 u_1) = 0 + 2 = 2$	$\gamma_1(y_1 z_1) = 1 + 2 = 3$
$\gamma_1(y_1 u_2) = 1 + 3 = 4$	$\gamma_1(z_1 u_2) = 2 + 3 = 5$	$\gamma_1(y_1 z_1) = 1 + 2 = 3$
$\gamma_1(y_1 y_2) = 1 + 5 = 6$	$\gamma_1(u_2 u_3) = 3 + 4 = 7$	$\gamma_1(z_1 z_2) = 2 + 6 = 8$
$\gamma_1(y_2 u_2) = 4 + 5 = 9$	$\gamma_1(z_2 u_2) = 4 + 6 = 10$	$\gamma_1(y_2 z_2) = 5 + 6 = 11$
$\gamma_1(y_2 u_3) = 5 + 7 = 12$	$\gamma_1(z_2 u_3) = 6 + 7 = 13$	$\gamma_1(y_2 z_2) = 5 + 9 = 14$
$\gamma_1(u_4 u_5) = 7 + 8 = 15$	$\gamma_1(z_2 z_3) = 6 + 10 = 16$	$\gamma_1(y_3 u_5) = 8 + 9 = 17$
$\gamma_1(u_5 z_3) = 8 + 10 = 18$	$\gamma_1(y_3 z_3) = 9 + 10 = 19$	$\gamma_1(y_3 u_6) = 9 + 11 = 20$
$\gamma_1(z_3 u_6) = 10 + 11 = 21$		

Theorem 2. *Shackle of Fan graph with four vertices $Shack(F_4, e, t)$ with $t \geq 2$ is Elegant*

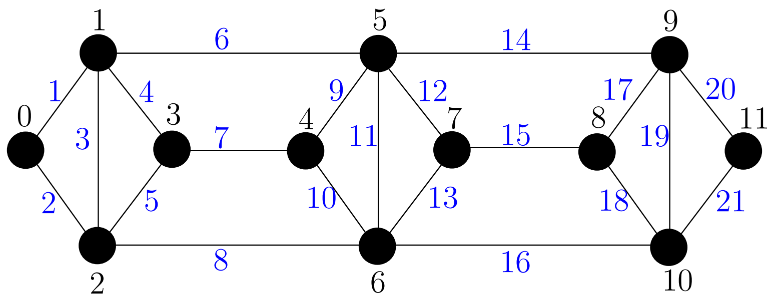

Figure 1. The elegant labeling of graph Dl_3

Proof. Fan graph with four vertices is constructed by joining one vertex to every vertex of Path with four vertices. Fan graph with four vertices can also be written: $F_4 = K_1 + P_4$. For every $1 \leq k \leq t-1$, $G_k = F_4^k$ and $G_{k+1} = F_4^{k+1}$ share exactly one common edge, called a linkage edge, where the $t-1$ linkage edges are all distinct. The edge that is not incident with the center vertex u_l is the lingkage edge. $Shack(F_4, e, s)$ has vertex set as follows: $\{y_k; k = 1, 2, \ldots, t+1\} \cup \{z_k; k = 1, 2, \ldots, t+1\} \cup \{u_k; k = 1, 2, \ldots, t\}$. The edge set of $Shack(F_4, e, s)$ is as follows: $\{y_k z_k; k = 1, 2, \ldots, t+1\} \cup \{z_k z_{k+1}; k = 1, \ldots, t\} \cup \{y_k u_k; k = 1, 2, \ldots, t\} \cup \{y_{k+1} u_k; k = 1, 2, \ldots, t\} \cup \{z_k u_k; k = 1, 2, \ldots, t\} \cup \{z_{k+1} u_k; k = 1, 2, \ldots, t\}$. Based on the vertex and edge set, we can determine the number of vertex and edge of graph $Shack(F_4, e, s)$ thus the number of vertex and edge of graph $Shack(F_4, e, s)$ are in the following: $V(Shack(F_4, e, s)) = \alpha = 3t + 2$ and $E(Shack(F_4, e, s)) = \beta = 6t + 1$. Now we are ready to define the vertex labeling under the function $\gamma_1 : V(Shack(F_4, e, s)) \rightarrow \{0, 1, 2, \ldots, (6t+1)\}$. Let γ_2 be an injective function (one to one mapping) which defined in the following:

$$\gamma_2(y_k) = \begin{cases} 0 + 6(\frac{k+1}{2} - 1) = 3(k-1); k = 1, 2, \ldots, t+1 \text{ and } k \text{ odd} \\ 4 + 6(\frac{k}{2} - 1) = 3k - 2; k = 1, 2, \ldots, t+1 \text{ and } k \text{ even} \end{cases}$$

$$\gamma_2(z_k) = \begin{cases} 1 + 6(\frac{k+1}{2} - 1) = 3k - 2; k = 1, 2, \ldots, t+1 \text{ and } k \text{ odd} \\ 3 + 6(\frac{k}{2} - 1) = 3(k-1); k = 1, 2, \ldots, t+1 \text{ and } k \text{ even} \end{cases}$$

$$\gamma_2(u_k) = 2 + 3(k-1) = 3k - 1$$

Next, can be determined the edge labeling of graph $Shack(F_4, e, s)$. The edge labeling is obtained from the sum of two vertex which adjacent:

$$\begin{aligned}
\gamma_2(y_kz_k) &= [(3k-3)+(3k-2)]\ mod(6t+2) = [6k-5]\ mod(6t+2); k=[1,t-1]\\
\gamma_2(z_kz_{k+1}) &= [(3k-2)+(3k)]\ mod(6t+2) = [6k-2]\ mod(6t+2); k=[1,t]\\
\gamma_2(y_ku_k) &= [(3k-3)+(3k-1)]\ mod(6t+2) = [6k-4]\ mod(6t+2); k=[1,t], k\ odd\\
\gamma_2(y_ku_k) &= [(3k-2)+(3k-1)]\ mod(6t+2) = [6k-3]\ mod(6t+2); k=[1,t], k\ even\\
\gamma_2(y_{k+1}u_k) &= [(3k+1)+(3k-1)]\ mod(6t+2) = [6k]\ mod(6t+2); k=[1,t], k\ odd\\
\gamma_2(y_{k+1}u_k) &= [(3k)+(3k-1)]\ mod(6t+2) = [6k-1]\ mod(6t+2); k=[1,t], k\ even\\
\gamma_2(z_ku_k) &= [(3k-2)+(3k-1)]\ mod(6t+2) = [6k-3]\ mod(6t+2); k=[1,t], k\ odd\\
\gamma_2(z_ku_k) &= [(3k-3)+(3k-1)]\ mod(6t+2) = [6k-4]\ mod(6t+2); k=[1,t], k\ even\\
\gamma_2(z_{k+1}z_k) &= [(3k)+(3k-1)]\ mod(6t+2) = [6k-1]\ mod(6t+2); k=[1,t], k\ odd\\
\gamma_2(z_{k+1}z_k) &= [(3k+1)+(3k-1)]\ mod(6t+2) = [6k]\ mod(6t+2); k=[1,t], k\ even
\end{aligned}$$

The sequence of the edge labeling of $Shack(F_4,e,s)$ graph with $3t+2$ vertices can be written as follows: $S = \{y_kz_k, y_ku_k, z_ku_k, z_kz_{k+1}, z_{k+1}u_k, u_ky_{k+1}; k = 1,2,\dots,t\} \cup y_{t+1}z_{t+1}\}$. Based on the edge labeling and the sequence of the edge labeling, obviously γ_2 is an injective (one to one mapping) function. Hence, γ_2 is an elegant labeling and it is complete the proof. □

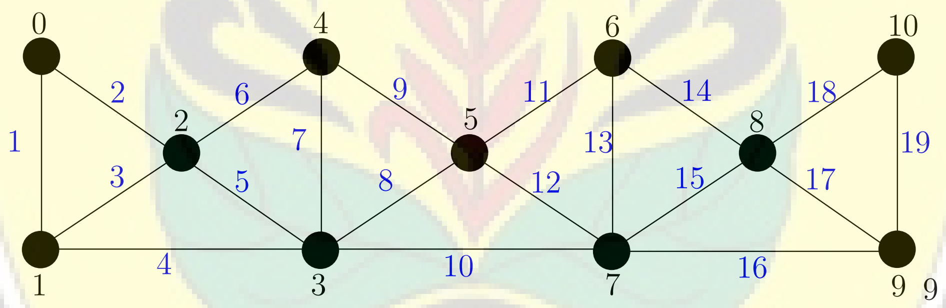

Figure 2. The elegant labeling of graph $Shack(F_4,e,s)$

We illustrate the Diamond Ladder graph with twelve vertices as shown in Figure 1 with $V(Shack(F_4,e,s)) = \{y_1,\dots,y_4,z_1,\dots,z_4,u_1,\dots,u_3\}$. The number of edge of graph $Shack(F_4,e,s)$ is nineteen. The sequence of edge labeling is exhibited below:

$\gamma_2(y_1z_1)=0+1=1$	$\gamma_2(y_1u_1)=0+2=2$	$\gamma_2(z_1u_1)=1+2=3$
$\gamma_2(z_1z_2)=1+3=4$	$\gamma_2(z_2u_1)=2+3=5$	$\gamma_2(y_2u_1)=2+4=6$
$\gamma_2(y_2z_2)=3+4=7$	$\gamma_2(z_2u_2)=3+5=8$	$\gamma_2(y_2u_2)=4+5=9$
$\gamma_2(z_2z_3)=3+7=10$	$\gamma_2(y_3u_2)=5+6=11$	$\gamma_2(z_3u_2)=5+7=12$
$\gamma_2(y_3z_3)=6+7=13$	$\gamma_2(y_2u_3)=6+8=14$	$\gamma_2(z_3u_3)=7+8=15$
$\gamma_2(z_3z_4)=7+9=16$	$\gamma_2(z_4u_3)=8+9=17$	$\gamma_2(y_4u_3)=8+10=18$
$\gamma_2(y_4z_4)=9+10=19$		

Theorem 3. *If the graph G admits $(a, d)-$ Edge Antimagic Vertex Labeling (EAVL) with $a = 3$ and $d = 1$, then graph G admits elegant labeling.*

Proof. Edge Antimagic Vertex labeling (EAVL) is give labels to the vertices of graph $G(\alpha, \beta)$. Graph $G(\alpha, \beta)$ is graph with the number of vertex α and the number of edge β. Edge Antimagic Vertex Labeling (EAVL) is a bijective function from the set of vertex of graph G into the integer set, under the function γ, let $\gamma : V \rightarrow \{1, 2, 3, 4, \ldots, \alpha\}$ be an $(a, d)-$ Edge Antimagic Vertex Labeling for graph $G(\alpha, \beta)$ and let $w_\gamma = \{wt_\gamma(xy) = \gamma(x)+\gamma(y)|xy \in E\}$ be the set of edge weights of graph $G(\alpha, \beta)$. Now, we are ready to proof that the graph G admits $(a, d)-$ Edge Antimagic Vertex Labeling (EAVL) with $a = 3$ and $d = 1$, then graph G admits elegant labeling. As we know that the first term is three and the difference is one. By the set of weights, we get the following set $w_\gamma = \{a, a+d, a+2d, \ldots, a+(\beta-1)d\} = \{3, 3+1, 3+2, \ldots, 3+(\beta-1)\times 1\}$. It can be seen that the possibility of the smallest edge weights is three. It is mean that the probability of the label vertex is one and two. While in the elegant labeling, the function of the vertex labeling is $\gamma^* : V \rightarrow \{0, 1, 2, \ldots, \alpha\}$ thus the label vertex on the graph G must be reduced by one. It is mean that the edge weight of graph G must be reduced by two such that the set of edge weight of graph G can be written as follows: $w_\gamma = \{3 - 2, 3 + 1 - 2, 3 + 2 - 2, \ldots, 3 + (\beta - 1) - 2\} = \{1, 2, 3, \ldots, \beta\}$. Based on the results of the edge weight, it is conclude the proof. □

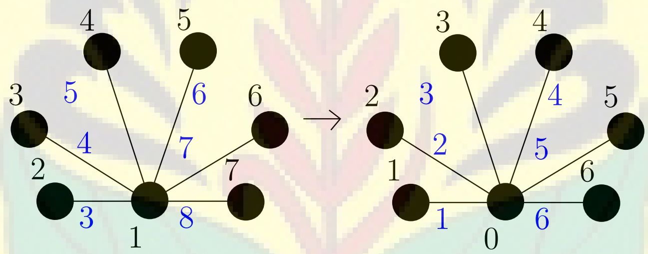

Figure 3. Graph Star S_6 admits an (a, d)- EAVL and it admits an elegant labeling

The Figure 1 is an illustration of graph star S_6 with six pendant vertices and one center vertex. Graph star S_t admits an $(a, d)-$ Edge Antimagic Vertex Labeling with $a = 3$ and $d = 1$. The vertex set of graph S_6 is exhibited below: $V(S_6) = \{y\} \cup \{z_k; 1 \leq k \leq 6\}$. The label of vertex in graph S_6, under the function λ, is as follow: $\lambda(y) = 1$ and $\lambda(z_k) = k + 1; 1 \leq k \leq 6$. Furthermore, we can determine the edge weight of graph S_6. The edge weights is as follow: $\lambda(yz_k) = \lambda(y) + \lambda(z_k) = k + 2; 1 \leq k \leq 6$. By simple method, reducing every vertex label with one, we can determine the vertex label of graph S_6 thus the labeling admits an elegant labeling.

The ladder graph L_t is a simple and undirected graph with $2t$ vertices and $t+2(t-1)$ edges. Ladder graph L_t is defined as a cartesian product from P_2 and P_t. Ladder graph looks like a ladder with t stairs.

Conclusion
Some graphs namely diamond ladder graph and $Shack(F_4, e, t)$ admit an elegant labeling. In this paper, we have also investigated the relation between Edge Antimagic Vertex Labeling and Elegant.

Open Problem
Open Problem 1. *Do all families of ladder graphs admit an elegant labeling?*

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