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INTERNATIONAL CONFERENCE ON RESEARCH, IMPLEMENTATION AND EDUCATION OF MATHEMATICS AND SCIENCES 2014

ICRIEMS 2014

Yogyakarta, 18-20 May 2014

Global Trends and Issues on Mathematics and Sciences and the Education

PROCEEDING

Conference Proceedings

INTERNATIONAL CONFERENCE ON RESEARCH, IMPLEMENTATION AND EDUCATION OF MATHEMATICS AND SCIENCES (ICRIEMS) 2014

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Global Trends and Issues
on Mathematics and Science
and The Education

Faculty of Mathematics and Natural Sciences
Yogyakarta State University

ICRIEMS 2014 : Global Trends and Issues on Mathematics and Science and The Education

- Mathematics & Mathematics Education
- Physics & Physics Education
- Chemistry & Chemistry Education
- Biology & Biology Education
- Science Education

Preface

Bless upon God Almighty such that this proceeding on International Conference on Research, Implementation, and Education of Mathematics and Sciences (ICRIEMS) 2014 may be compiled according to the schedule provided by the organizing committee. All of the articles in this proceeding are obtained by selection process by the reviwer team and already ben presented in the Conference on 18 – 20 May 2014 in the Faculty of Mathematics and Natural Sciences, Yogyakarta State University. This proceeding consists of 344 parallel papers, and comprises 9 fields, that is mathematics, mathematics education, physics, physics education, chemistry, chemistry education, biology, biology education, and science education.

The theme of ICRIEMS 2014 is 'Global Trends and Issues of Mathematics and Science and the Education'. The main articles in this conference are given by five keynote speakers, which are Prof. Dean Zollman (Physics Department, Kansas State University), Prof. David F. Treagust (Center of Education, Curtin University), Prof. Dr. Amy Cutter-Mackenzie (School of Education, Southern Cross University, Australia), Prof. Tran Vui (Hue University, Vietnam), and Asst. Prof. Dr. Duangjai Nacapricha (Faculty of Science, Mahidol University). The conference is also supported by the LPTK (Lembaga Pendidikan Tenaga Kependidikan) Forum from Faculty of Mathematics and Sciences that consists of 12 universities all over Indonesia. Each member of the Forum contributed one invited speakers, such that there are an additional 10 invited speakers presenting in the forum. Besides the keynote and invited speakers, there are also 344 parallel articles that presented the latest research results in the field of mathematics and sciences, and the education. These parallel session speakers come from researchers from Indonesia and abroad, including Malaysia and Australia.

Hopefully, this proceeding may contribute in disseminating research results and studies in the field of Mathematics and Sciences and the Education such that they are accessible by many people and useful for the Nation Building.

Yogyakarta, June 2014

The Editor Team

Forewords from The Head of Committee

Assalamu'alaikum wa Rahmatullahi wa Barakatuh
May God bless upon us.

Your excellency The president of UNY Prof. Dr. Rochmat Wahab, M. Pd., M.A., ladies and gentlemen, good morning and welcome to State University Yogyakarta. This seminar entitled International Conference on Research, Implementation, and Education of Mathematics and Science (ICRIEMS): global trends and issues on mathematics and science and the education is organized by the Faculty of Mathematics and Science, State University of Yogyakarta working together with 12 members of the Association of the Faculty of Math and Sciences from Teacher Education Program (LPTK). This seminar is also dedicated to the golden aniversary of UNY; 1 among 90 academic activities dedicated to the aniversary.

Ladies and gentlemen, on behalf of the committee of this conference, I would like to express highest appreciation and gratitudes to the keynote speakers, including:

1. Prof. David F. Treagust (Center of Science Education Curtin University)
2. Prof. Dean Zollman (Physics Dept, Kansas University, US)
3. Dr. Amy Cutter-Mackenzie (School of Education, Southern Cross University, Australia)
4. Asst. Prof. Dr. Duangjai Nacapricha (Faculty of Science , Mahidol University)
5. Prof. Tran Vui (College of Education, Hue University, Hue City, Vietnam)

Secondly, I would like also to give sincere thanks and gratitudes to the speakers from 10 College of Educations, including:

1. Universitas Negeri Surabaya (UNESA): Prof. Dr. Muchlas Samani, and 33 speakers
2. Universitas Negeri Jakarta (UNJ): Prof. Dr. Gerardus Pola, and 7 speaker
3. Universitas Pendidikan Indonesia (UPI): Dr. Hary Firman, and
4. Universitas Negeri Malang (UM): Prof. Effendi, Ph.D
5. Universitas Negeri Padang (UNP): Prof. Tjeerd Plomp
6. Universitas Negeri Semarang (UNNES): Prof. Dr. Supriyadi Rustad

7. Universitas Pendidikan Singaraja (UNDIKSA): Prof. Dr. I Nengah Suparta, M.Si
8. Universitas Negeri Makasar (UNM): Oslan Junaidi, Ph.D
9. Universitas Negeri Gorontalo (UNG): Prof. Dr. Sarson Pomalto, M.Pd
10. Universitas Negeri Yogyakarta (UNY): Dr. Jaslin Ikhsan

Next, I also would like to thanks to our special guests and speakers from:

1. Universitas Pendidikan Sultan Indris (UPSI), Malaysia
2. University of Mahidol, Thailand
3. University of Malaysia in Trengganu

Next, I would like to thanks and welcome to 379 speakers from the entire Indonesia and all participants registered in this seminar.

Ladies and gentlemen, recently the number of research and publication on mathematics and science and the education is vulnarable. It is nescessary for us to organise, to share, and to publish the results of the research in this conference. I hope the conference will bear fruitful results and promote networking and future collaborations for all participants from diverse background of expertise, intitutions, and countries to promote science, mathematics, and the education.

Finally, I am delighted to thank the committee members who have been working very hard to ensure the succes of the conference.

Please enjoy the conference and enjoy Yogyakarta, the city of education, tourism, and culture. Thank you very much.

Assalamu'alaikum wa rahmatullahi wa barrakatuh

Dr. Slamet Suyanto, M. Ed.

Forewords from The Dean of Faculty of Mathematics and Natural Sciences, Yogyakarta State University

Assalamu'alaikum warahmatullahi wabarakatuh

May peace and God's blessings be upon us all.

On behalf of the Organizing Committee, first of all allow me to extend my warmest greeting and welcome to the International Conference on Research, Implementation, and Education of Mathematics and Sciences 2014, held in Yogyakarta State University, one of the qualified education universities in Indonesia.

To celebrate the 50th Commemoration of Yogyakarta State University, our faculty, in collaboration with Forum of MIPA LPTK, has the opportunity to conduct International Conference on Research, Implementation, and Education of Mathematics and Sciences 2014. This conference proudly presents five keynote speeches by five fabulous speakers: Prof. Dean Zollman, Prof. David F. Treagust, Prof. Dr. Amy Cutter-Mackenzie, Prof. Tran Vui, and Asst. Prof. Dr. Duangjai Nacapricha, around 380 parallel speakers with 344 orally presented articles.

Distinguished guest, ladies and gentlemen,

The independence of a country is impossible to gain if the education does not become the priority and it is not supported with the development of technology. We all know that the technology development could be achieved if it is supported by the improvement of firm fundamental knowledge. The empowerment of fundamental knowledge could not be separated from research which is related to the development of technology and the learning process in school and universities.

This conference is aimed to pull together researchers, educators, policy makers, and practitioners to share their critical thinking and research outcomes. Therefore, we are able to understand and examine the development of fundamental principle, knowledge, and technology. By perceiving the matters and condition in research and education field of mathematics and sciences, we could take a part in conducting qualified education to reach out the real independence of our nation.

Distinguished guest, ladies, and gentlemen

This conference will be far from success and we could not accomplish what we do without the support from various parties. So let me extend my deepest gratitude and highest appreciation to all committee members. I would also like to thank each of participants for

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$\boldsymbol{C[a,b]}$-VALUED MEASURE AND SOME OF ITS PROPERTIES

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Abstract

Let $C[a,b]$ be the set of all real-valued continuous functions defined on a closed interval $[a,b]$. It is a commutative Riesz algebra space with unit element e, where $e(x) = 1$ for every $x \in [a,b]$. As in the real numbers system $\mathbb{R}$, we define $\bar{C}[a,b]$ of the extended of $C[a,b]$. In this paper, we shall generalize the notions of outer measure, measure, measurable sets and measurable functions from $C[a,b]$ into $\bar{C}[a,b]$. This paper is a part of our study in Henstock-Kurzweil integral of functions define on a closed interval $[f,g] \subset C[a,b]$ which values in $\bar{C}[a,b]$.

Key words: outer measure, measure, measurable set, measurable function

INTRODUCTION

Some properties of real-valued continuous function defined on a closed interval were studied by several authors. Bartle and Sherbert [2] mention some of its properties are bounded, it has an absolute maximum and an absolut minimum, it can be approximated arbitrarily closely by step functions, uniformly continuous, and Riemann integrable.

In this paper, $C[a,b]$ denotes the set of all real-valued continuous functions defined on a closed interval $[a,b]$. Further discussion of $C[a,b]$ can be shown in classical Banach spaces such as Albiac and Kalton [1], Diestel [4], Lindenstrauss and Tzafriri [5], Meyer-Nieberg [6], and others.

In development of mathematical analysis, sometimes we need to extend of definition, such as measure. For example, Boccuto, Minotti and Sambucini [3] define Riemann sum of a function $\boldsymbol{f}: \boldsymbol{T} \to \boldsymbol{R}$, where $\boldsymbol{R}$ is a Riesz space, is

$$\boldsymbol{S(f,D) = \sum_{i=1}^{n} f(t_i)\mu(E_i)}$$

where $\boldsymbol{D = \{(E_i, t_i), i = 1, 2, \dots, n\}}$ is $\boldsymbol{\delta}$-fine partition of $\boldsymbol{T}$ and $\boldsymbol{\mu}$ is a Riesz-valued measure $\boldsymbol{R}$, that is $\boldsymbol{\mu}: \boldsymbol{\Sigma} \to \boldsymbol{R}$ where $\boldsymbol{\Sigma}$ is the $\boldsymbol{\sigma}$-algebra of all Borel subsets of $\boldsymbol{T}$. In their definition, they assumed that $\boldsymbol{R}$ is Dedekind complete Riesz space. Now, if we take $\boldsymbol{R = C[a,b]}$ that is Riesz space but not Dedekind complete, interesting for us to discuss a $\boldsymbol{C[a,b]}$-valued measure.

The aim of this paper is to construct a $\boldsymbol{C[a,b]}$-valued measure and to discuss some of its properties, including measurable sets and measurable functions. The construction of the $\boldsymbol{C[a,b]}$-valued measure could be applied to construct integral of $\boldsymbol{C[a,b]}$-valued functions.

PRELIMINARIES

Before we begin the discussion, we give an introductory about $C[a,b]$ as a Riesz space and a commutative Riesz algebra. Let $C[a,b]$ be the set of all real-valued continuous functions defined on a closed interval $[a,b]$. It is well known that $C[a,b]$ is a commutative algebra with unit element e, where $e(x) = 1$ for every $x \in [a,b]$, over a field $\mathbb{R}$. If $f,g \in C[a,b]$, we define

$$f \leq g \Leftrightarrow f(x) \leq g(x)\,, \qquad f < g \Leftrightarrow f(x) < g(x) \quad \text{and} \quad f = g \Leftrightarrow f(x) = g(x)$$

for every $x \in [a,b]$. The relation " $\leq$ " is a partial ordering in $C[a,b]$ because it satisfies

(i) $f \leq f$ for every $f \in C[a,b]$,
(ii) $f \leq g$ and $g \leq h \Rightarrow f \leq h$ for every $f,g,h \in C[a,b]$,
(iii) $f \leq g$ and $g \leq f \Rightarrow f = g$.

Therefore $(C[a,b], \leq)$, briefly $C[a,b]$, is a partially ordered set. Further, the $C[a,b]$ satisfies

(i) $f \leq g \Rightarrow f + h \leq g + h$ for every $h \in C[a,b]$,
(ii) $f \leq g \Rightarrow \alpha f \leq \alpha g$ for every $\alpha \in \mathbb{R}^+$.

Therefore, $C[a,b]$ is also Riesz space. If $f,g \in C[a,b]$, we define fg with

$$(fg)(x) = f(x)g(x) \text{ for every } x \in [a,b].$$

Hence, $C[a,b]$ will be called a commutative Riesz algebra with unit element e. The Riesz spaces and commutative Riesz algebras more in-depth discussion can be found in [6] and [9].

So far, if $f,g \in C[a,b]$ with $f < g$, we define

$(f,g) = \{h \in C[a,b]: f < h < g\}$,
$[f,g] = \{h \in C[a,b]: f \leq h \leq g\}$,
$(f,\infty) = \{h \in C[a,b]: f < \infty\}$,
$[f,\infty) = \{h \in C[a,b]: f \leq h\}$,
$(-\infty,g) = \{h \in C[a,b]: h < g\}$,
$(-\infty,g] = \{h \in C[a,b]: h \leq g\}$.

If $f,g \in C[a,b]$, we define $f \vee g, f \wedge g, |f|, f^+, f^-$ with

(i) $(f \vee g)(x) = \sup\limits_{x \in [a,b]} \{f(x), g(x)\}$,
(ii) $(f \wedge g)(x) = \inf\limits_{x \in [a,b]} \{f(x), g(x)\}$,
(iii) $|f|(x) = |f(x)|$ for every $x \in [a,b]$,
(iv) $f^+(x) = \begin{cases} f(x), \text{if } f(x) \geq 0 \\ 0, \quad \text{if } f(x) < 0 \end{cases}$,
(v) $f^-(x) = \begin{cases} 0, \quad \text{if } f(x) \geq 0 \\ -f(x), \text{if } f(x) < 0 \end{cases}$.

Bartle and Sherbert [2] showed that if $f,g \in C[a,b]$, then $f \vee g$, $f \wedge g, |f|, f^+$ and f^- are members of $C[a,b]$. Explanation of infimum/supremum of set and limit of sequence on $C[a,b]$ can be shown in [8].

DISCUSSION

We shall construct a $\bar{C}[a,b]$-valued outer measure. We need an extention of the system of $C[a,b]$ as follows:

$$\bar{C}[a,b] = C[a,b] \cup \{-\infty, \infty\}$$

and we call it the **extended system of $\boldsymbol{C[a,b]}$**. If $f,g \in \bar{C}[a,b]$, we have $f+g \in \bar{C}[a,b]$ and $fg \in \bar{C}[a,b]$. The enlargement of the operations between $\pm\infty \in \bar{C}[a,b]$ and $f \in C[a,b]$ are defined as follows:

(i) $-\infty < f < \infty$ for every $f \in C[a,b]$,
(ii) $f + \infty = \infty$ and $f - \infty = -\infty$ for every $f \in C[a,b]$,
(iii) $f \cdot \infty = \infty$ and $f \cdot -\infty = -\infty$ for every $f \in C[a,b]$ and $f > \theta$,
(iv) $f \cdot \infty = -\infty$ and $f \cdot -\infty = \infty$ for every $f \in C[a,b]$ and $f < \theta$,
(v) $\infty + \infty = \infty$ and $-\infty + (-\infty) = -\infty$, and
(vi) $\theta \cdot \infty = \theta \cdot -\infty = \theta$

where θ is null element of $C[a,b]$ with $\theta(x) = 0$ for every $x \in [a,b]$.

Definition 1. *A function $\mu^*: 2^{C[a,b]} \to \bar{C}[a,b]$ is called a **$\boldsymbol{C[a,b]}$-valued outer measure**, briefly **outer measure**, if it satisfies the following properties:*

(i) $\mu^*(A) \geq \theta$ for every $A \in 2^{C[a,b]}$,
$\mu(\emptyset) = \theta$,
(ii) $A, B \in 2^{C[a,b]}$ where $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$, and
(iii) $\{A_n\} \subset 2^{C[a,b]} \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

If $f,g \in C[a,b]$ with $f < g$ and $I = (f,g)$ is a open interval, we defined a $C[a,b]$-valued interval function ℓ by

$$\ell(I) = g - f.$$

Next theorem is a example there is a outer measure on $C[a,b]$.

Theorem 2. *A function $\mu^*: 2^{C[a,b]} \to \bar{C}[a,b]$ that defined*

$$\mu^*(A) = \bigwedge\left\{\sum_{k=1}^{\infty} \ell(I_k) : I_k \text{ is open interval for every } k \in \mathbb{N} \text{ and } A \subset \bigcup_{k=1}^{\infty} I_k\right\}$$

is a outer measure on $C[a,b]$.

Proof. It is clear that $\mu^*(A) \geq \theta$ for every $A \in 2^{C[a,b]}$. Since $\emptyset \subset A$ for every $A \in 2^{C[a,b]}$, then $\emptyset \subset (-\frac{e}{n}, \frac{e}{n})$ for every $n \in \mathbb{N}$. Therefore, we have

$$\mu^*(\emptyset) = \inf_{n} \ell\left((-\frac{e}{n}, \frac{e}{n})\right) = \inf_{n}\{\frac{2e}{n}\} = \theta.$$

Given two sets $A, B \subset C[a,b]$ arbitrary where $A \subset B$. If $\mu^*(B) = \infty$, it is true that $\mu^*(A) \leq \mu^*(B)$. If $\mu^*(B) < \infty$, then for every real number $\varepsilon > 0$ there is a sequence $\{(f_n, g_n)\} \subset C[a,b]$ such that $B \subset \bigcup_{n=1}^{\infty}(f_n, g_n)$ and $\sum_{n=1}^{\infty}(g_n - f_n) < \mu^*(B) + \varepsilon e$. Since $A \subset B$, then we have $A \subset \bigcup_{n=1}^{\infty}(f_n, g_n)$, hence $\mu^*(A) \leq \sum_{n=1}^{\infty}(g_n - f_n)$. Thus, we have

$$\mu^*(A) \leq \mu^*(B).$$

Given a sequence $\{A_n\} \subset C[a,b]$ arbitrary. If there is $m \in \mathbb{N}$ such that $\mu^*(A_m) = \infty$, it is true that $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. If $\mu^*(A_n) < \infty$ for every $n \in \mathbb{N}$, then for every real number $\varepsilon > 0$ there is a sequence $\{(f_{n,k}, g_{n,k})\} \subset C[a,b]$ such that $A_n \subset \bigcup_{k=1}^{\infty}(f_{n,k}, g_{n,k})$ for every $n \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty}(g_{n,k}-f_{n,k})<\mu^*(A_n)+\frac{\varepsilon e}{2^n}.$$

For every $n\in\mathbb{N}$, we obtain $\bigcup_{n=1}^{\infty}A_n\subset\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}(f_{n,k},g_{n,k})$ and

$$\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right)\leq\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(g_{n,k}-f_{n,k})<\sum_{n=1}^{\infty}\left(\mu^*(A_n)+\frac{\varepsilon e}{2^n}\right)=\sum_{n=1}^{\infty}\mu^*(A_n)+\varepsilon e,$$

that is, $\mu^*(\bigcup_{n=1}^{\infty}A_n)\leq\sum_{n=1}^{\infty}\mu^*(A_n)$. ■

Next, we know a $C[a,b]$-valued measure (the explanation of σ-algebra of set $\mathcal{A}$, measurable space $(X,\mathcal{A})$ and their properties can be shown in [7]).

Definition 3. *Let $X\subseteq C[a,b]$ be a nonempty set and $(X,\mathcal{A})$ be a measurable space. A function $\mu:\mathcal{A}\to\bar{C}[a,b]$ is called a* ***$C[a,b]$-valued measure on*** *$(X,\mathcal{A})$, briefly* ***measure on*** *X, if*

(i) *$\mu(A)\geq\theta$ for every $A\in\mathcal{A}$*
$\mu(\emptyset)=\theta$,

(ii) *$\{A_n\}\subset\mathcal{A}$ where $A_m\cap A_n=\emptyset$ for $m\neq n$, then $\mu(\bigcup_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\mu(A_n)$*

Let $X\subseteq C[a,b]$ be a nonempty set and μ a measure on measurable space $(X,\mathcal{A})$. A measure μ is called a **finite measure** if $\mu(X)<\infty$ and a measure μ is called a **σ-finite measure** if there is a sequence of measurable sets $\{X_n\}\subset\mathcal{A}$ such that $X=\bigcup_{n=1}^{\infty}X_n$ and $\mu(X_n)<\infty$ for every $n\in\mathbb{N}$. If μ is a measure on $(X,\mathcal{A})$, a pair $(X,\mathcal{A},\mu)$ is called a **measure space**. A measure space $(X,\mathcal{A},\mu)$ is called **complete** if $B\in\mathcal{A}$ with $\mu(B)=\theta$ and $A\subset B$ implies $A\in\mathcal{A}$. Some properties of measure on $(C[a,b],\mathcal{A})$ are given in Theorem 4, Theorem 5 and Theorem 6.

Theorem 4. *Let $(C[a,b],\mathcal{A},\mu)$ be a measure space. If $A,B\in\mathcal{A}$ and $A\subseteq B$ then $\mu(A)\leq\mu(B)$.*

Proof. If $A\subseteq B$ then $B=A\cup(B-A)$ where $A\cap(B-A)=\emptyset$. So we have

$$\mu(B)=\mu(A)+\mu(B-A)\geq\mu(A).$$ ■

Theorem 5. *Let $(C[a,b],\mathcal{A},\mu)$ be a measure space. If $\{A_n\}\subset\mathcal{A}$ where $A_{i+1}\subseteq A_i$ for every $i\in\mathbb{N}$ and $\mu(A_1)<\infty$ then*

$$\mu\left(\bigcap_{n=1}^{\infty}A_n\right)=\lim_{n\to\infty}\mu(A_n)$$

if it has limit.

Proof. Set $A=\bigcap_{i=1}^{\infty}A_i$, we have $A_1=A\cup(\bigcup_{i=1}^{\infty}(A_i-A_{i+1}))$ with $A\cap(A_i-A_{i+1})=\emptyset$ for every $i\in\mathbb{N}$. Then $\mu(A_1)=\mu(A)+\sum_{i=1}^{\infty}\mu(A_i-A_{i+1})$ and $A_{i+1}\cap(A_i-A_{i+1})=\emptyset$, we have

$$\mu(A_i-A_{i+1})=\mu(A_i)-\mu(A_{i+1}).$$

If $\lim\limits_{n\to\infty}\mu(A_n)$ exist, we have

$$\begin{aligned}\mu(A_1)&=\mu(A)+\sum_{i=1}^{\infty}(\mu(A_i)-\mu(A_{i+1}))=\mu(A)+\lim_{n\to\infty}\sum_{i=1}^{n-1}(\mu(A_i)-\mu(A_{i+1}))\\&=\mu(A)+\mu(A_1)-\lim_{n\to\infty}\mu(A_n),\end{aligned}$$

that is, $\mu(\bigcap_{n=1}^{\infty}A_n)=\mu(A)=\lim\limits_{n\to\infty}\mu(A_n)$. ■

Theorem 6. *Let $(C[a,b],\mathcal{A},\mu)$ be a measure space. If $\{A_n\} \subset \mathcal{A}$ then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Proof. Set $B_1 = A_1$ and $B_n = A_n - (\bigcup_{i=1}^{n-1} A_i)$ for every $n \geq 2$. Then $B_n \subseteq A_n$ for every n and $B_i \cap B_j = \emptyset$ for every $i \neq j$. Thus $\mu(B_n) \leq \mu(A_n)$ for every n and

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n). \qquad \blacksquare$$

Measurable Sets

We introduce a definition of a μ^*-measurable set.

Definition 7. *A set $E \subset C[a,b]$ is said* ***μ^*-measurable*** *if every $A \subseteq C[a,b]$ we have*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C).$$

Some μ^*-measurable sets on $C[a,b]$ is given in Theorem 8 as follows.

Theorem 8. *The following statements are true:*

(i) *$\emptyset$ and $C[a,b]$ are μ^*-measurable,*

(ii) *If $E \subset C[a,b]$ is μ^*-measurable, then E^C is μ^*-measurable,*

(iii) *If $E_1, E_2 \subset C[a,b]$ is μ^*-measurable, then $E_1 \cup E_2$ and $E_1 \cap E_2$ are μ^*-measurable.*

Proof. We only prove (iiii). Consider $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^C)$ that implies

$$\mu^*(A \cap (E_1 \cup E_2)) \leq \mu^*(A \cap E_1) + \mu^*(A \cap E_2 \cap E_1^C) \qquad \text{(a)}$$

Since E_2 is μ^*-measurable set, then we have

$$\mu^*(A \cap E_1^C) = \mu^*\left((A \cap E_1^C) \cap E_2\right) + \mu^*\left((A \cap E_1^C) \cap E_2^C\right) \text{ or}$$

$$\mu^*\left((A \cap E_1^C) \cap E_2\right) = \mu^*(A \cap E_1^C) - \mu^*(A \cap (E_1 \cup E_2)^C). \qquad \text{(b)}$$

Subtitution (b) into (a), we have

$$\mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^C) \leq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^C) = \mu^*(A).$$

Thus, $E_1 \cup E_2$ is μ^*-measurable set. Based on this result and a statement (ii), we have $E_1 \cap E_2 = (E_1^C \cup E_2^C)^C$ μ^*-measurable. $\blacksquare$

Theorem 9. *If $E_1, E_2, \ldots, E_n \subset C[a,b]$ are disjoint and μ^*-measurable sets, then for every $A \subset C[a,b]$ we have*

$$\mu^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k).$$

Proof. We shall prove by mathematical induction.

That is clear true for $n = 1$. Next, we assume the theorem is true for $E_1, E_2, \dots, E_{n-1}$ sets, that is

$$\mu^*\left(A \cap \bigcup_{k=1}^{n-1} E_k\right) = \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$$

is true for every $A \subset C[a,b]$.

Since E_k disjoint sets, we have

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n = A \cap E_n$$

and

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^C = A \cap \bigcup_{k=1}^{n-1} E_k .$$

Since E_n is μ^*-measurable set for every n, we have

$$\mu^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \mu^*(A \cap E_n) + \mu^*\left(A \cap \bigcup_{k=1}^{n-1} E_k\right)$$

$$= \mu^*(A \cap E_n) + \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$$

$$= \sum_{k=1}^{n} \mu^*(A \cap E_k) .$$ ■

Theorem 10. *If $\{E_n\}$ is a μ^*-measurable sets sequence, then*

$$\bigcup_{n=1}^{\infty} E_n$$

is a μ^*-measurable set.

Theorem 11. *If $\mathcal{A}$ is a collection of all μ^*-measurable sets on $C[a,b]$, then $\mathcal{A}$ is a σ-algebra on $C[a,b]$.*

Proof. Since $\emptyset \in \mathcal{A}$ then $\mathcal{A} \neq \emptyset$. Based on definition, if $E \in \mathcal{A}$ we have $E^C \in \mathcal{A}$, and based on Theorem 10, if $E_n \in \mathcal{A}$ then $\bigcup_{n=1}^{\infty} E_n$ is μ^*-measurable set. ■

Let $X \subset C[a,b]$ be a nonempty set. If $\mathcal{A}$ is a collection of μ^*-measurable subsets of $C[a,b]$, we have a measurable space $(X,\mathcal{A})$ that is generated by a measure μ^* as defined in Theorem 2.

Theorem 12. *Let $X \subset C[a,b]$ be a nonempty set and $(X,\mathcal{A})$ be a measurable space. A function $\mu: \mathcal{A} \to \bar{C}[a,b]$, formulated by*

$$\mu(E) = \mu^*(E),$$

is a measure.

Proof. $\mu(E) = \mu^*(E) \geq \theta$ for every $E \in \mathcal{A}$ and $\mu(\emptyset) = \mu^*(\emptyset) = \theta$. If $\{E_k\} \subset \mathcal{A}$ and E_k are disjoint sets, with Theorem 9 and replace $A = C[a,b]$ we obtain

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k) = \sum_{k=1}^{n} \mu(E_k)$$

for every $n \in \mathbb{N}$. ■

Measurable Functions

Before we discuss a measurable function on a μ^*-measurable set, we introduce a characteristic function and a simple function on $C[a,b]$.
Let $(C[a,b], \mathcal{A}, \mu)$ be a measure space and $E \subset C[a,b]$. A function

$$\chi_E : C[a,b] \to \bar{C}[a,b]$$

is called **characteristic function** on E if

$$\chi_E(h) = \begin{cases} e & , h \in E \\ \theta & , h \notin E \end{cases}$$

If $E_1, E_2, \dots, E_n$ are μ^*-measurable sets and $c_1, c_2, \dots, c_n \in \mathbb{R}$, a function

$$\varphi = \sum_{k=1}^{n} c_k \chi_{E_k}$$

is called a **simple function** on $E = \bigcup_{k=1}^{n} E_k$ μ^*-measurable set. A simple function φ is said in the form **canonical representation** if $E_1, E_2, \dots, E_n$ are disjoint sets. Every simple function always can be represented in canonical representation. In what follows, we shall always assume that every simple function is in the form of canonical representation if there is nothing further information. If φ and ψ are simple functions on set E, then $\alpha\varphi$ and $\varphi + \psi$ are simple functions on set E for every $\alpha \in \mathbb{R}$.

Before defining a measurable function on a μ^*-measurable set, we need a terminology what is called "almost everywhere". Let $P(h)$ denote a statement concerning the points h in a set E. We say that the statement $P(h)$ holds true **almost everywhere** on E or $P(h)$ holds true **for almost** every $h \in E$ if there is $A \subset E$ with $\mu^*(A) = \theta$ such that $P(h)$ holds true for every $h \in E - A$.

Definition 13. *A function $F : E \subseteq C[a,b] \to \bar{C}[a,b]$ is said to be* ***measurable*** *on a μ^*-measurable set E if for every number $\varepsilon > 0$ there is a simple function φ on E such that*

$$|F(h) - \varphi(h)| < \varepsilon e$$

almost everywhere on E.

By the definition, it is clear that every simple function on μ^*-measurable set E is measurable on E.

Theorem 14. *If $F, G : E \subseteq C[a,b] \to \bar{C}[a,b]$ are two functions such that F and G are measurable on μ^*-measurable set E, then for every number $\alpha \in \mathbb{R}$ functions αF and $F + G$ are measurable on μ^*-measurable set E.*

Corollary 15. *If $F_i : E \subseteq C[a,b] \to \bar{C}[a,b]$ ($i = 1,2,\dots,n$) are functions such that F_i is measurable on μ^*-measurable set E for every $i = 1,2,\dots,n$, then for every $\alpha_i \in \mathbb{R}$ ($i = 1,2,\dots,n$), a function*

$$\sum_{i=1}^{n} \alpha_i F_i$$

is measurable on μ^*-measurable set E.

Teorema 16. *If $F: [f,g] \to \bar{C}[a,b]$ is continuous function, then F is measurable on $[f,g]$.*

Proof. Interval $[f,g]$ is μ^*-measurable set. If F is continuous function on closed interval $[f,g]$ then F is uniformly continuous on $[f,g]$, that is, for every number $\varepsilon > 0$ there is number $\eta > 0$ such that for every $s,t \in [f,g]$ with $|s-t| < \eta e$ we have $|F(s) - F(t)| < \varepsilon e$. Let $\{[f_0, f_1], [f_1, f_2], \dots, [f_{n-1}, f_n] : f_0 = f_1$ and $f_n = g\}$ be a partition on $[f,g]$ such that $f_k - f_{k-1} < \varepsilon e$ for every $k = 1,2,\dots,n$. If we take $A_k = [f_{k-1}, f_k]$ and $a_k = \sup_{x\in[a,b]} \{F(f_k)(x)\} \in \mathbb{R}$ for every k, then we obtain a simple function

$$\varphi = \sum_{k=1}^{n} c_k \chi_{A_k}$$

on $[f,g]$ such that

$$|F(h) - \varphi(h)| < \varepsilon e$$

for every $h \in [f,g]$, that is, F is measurable on $[f,g]$. ∎

CONCLUSION AND SUGGESTION

From the discussion results above, we conclude:

1. There is a $C[a,b]$-valued measure that is generated by $C[a,b]$-valued outer measure.
2. A continuous function that defined on a closed interval subset of $C[a,b]$ is measurable on it.

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