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[1] D.H. Ackley, G.E. Hinton and T.J. Sejnowski, A learning algorithm for Boltzmann machine, Cognitive Science, 9 (1985), 147-169.

[2] F.L. Crane, H. Low, P. Navas, I.L. Sun, Control of cell growth by plasma membrane NADH oxidation, Pure and Applied Chemical Sciences, 1 (2013), 31-42. http://dx.doi.org/10.12988/pacs.2013.3310

[3] D.O. Hebb, The Organization of Behavior, Wiley, New York, 1949.

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On the Henstock-Kurzweil Integral of $C[a, b]$ Space-valued Functions

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Abstract

Let $C[a, b]$ be the set of all real-valued continuous functions defined on a closed interval $[a, b] \subset \mathbb{R}$. In this paper, we construct the Henstock-Kurzweil integral on a closed interval $[f, g] \subset C[a, b]$ and investigate some of its properties of the Henstock-Kurzweil integral. Futhermore, we prove a monotone convergence theorem of the Henstock-Kurzweil integral on $C[a, b]$.

Mathematics Subject Classification: 06A06, 28B15, 28C20, 40A10, 46G10, 46G12

Keywords: $C[a, b]$ space-valued function, δ-fine partition, the Henstock-Kurzweil integral, monotone convergence theorem

1 Introduction

There have been many contributions to the study of integration for mappings, taking values in ordered spaces. Among the authors, we quote Riečan [8], Duchon and Riečan [5], Riečan and Vrábelová [9]. Henstock-Kurzweil-type integral for Riesz spaces-valued functions, defined on an interval $[a, b] \subset \mathbb{R}$, was studied in detail by Boccuto, Riečan and Vrábelová [3]. In the book, they assumed that Riesz spaces are Dedekind complete, that is, every bounded above subset of Riesz spaces has a supremum.

In this paper, we will construct the Henstock-Kurzweil integral of $C[a, b]$ space-valued functions, where $C[a, b]$ means the collection of all real-valued continuous functions defined on a closed interval $[a, b]$. Before, we show that $C[a, b]$ as a Riesz space but it is not Dedekind complete.

Some properties of elements of $C[a, b]$ were studied by Bartle and Sherbert [2]. They mentioned some of its properties are bounded, it has an absolute maximum and an absolute minimum, it can be approximated uniformly by step functions, uniformly continuous, and Riemann integrable. A property of $C[a, b]$ is not a complete Dedekind Riesz space. Further discussion of $C[a, b]$ can be shown in classical Banach spaces such as Albiac and Kalton [1], Diestel [4], Lindenstrauss and Tzafriri [6], Meyer-Nieberg [7], and others.

2 Preliminaries

Before we begin the discussion, we give an introductory about $C[a, b]$ as a Riesz space and a commutative Riesz algebra. It is well-known that $C[a, b]$ is a commutative algebra with e as its unit element, where $e(x) = 1$ for every $x \in [a, b]$, over a field $\mathbb{R}$. If $f, g \in C[a, b]$ we define

$$f \le g \Leftrightarrow f(x) \le g(x), f < g \Leftrightarrow f(x) < g(x), \text{ and } f = g \Leftrightarrow f(x) = g(x)$$

for every $x \in [a, b]$. The relation "$\le$" is a partial ordering in $C[a, b]$. Therefore $(C[a, b], \le)$, briefly $C[a, b]$, is a partially ordered set. From now on, $f \le g$ can be written by $g \ge f$, in the case $f < g$ is similar. If f and g are elements of $C[a, b]$ such that $f \le g$ or $f \ge g$, we say that f and g are comparable. If neither $f \le g$ nor $f \ge g$, then f and g are incomparable. Further, the $C[a, b]$ satisfies

$$f \le g \Rightarrow f + h \le g + h \text{ for every } h \in C[a, b],$$

$$f \le g \Rightarrow \alpha f \le \alpha g \text{ for every } \alpha \in \mathbb{R}^+.$$

Therefore, $C[a, b]$ is also Riesz space. If $f, g \in C[a, b]$, we define fg by

$$(fg)(x) = f(x)g(x), \text{ for every } x \in [a, b].$$

Hence, $C[a,b]$ is called a commutative Riesz algebra with e as its unit element. More depth discussion of Riesz spaces and commutative Riesz algebras can be found in [7] and [11].

In the paper, if $f, g \in C[a,b]$ with $f < g$, we define bounded intervals in $C[a,b]$ as follows

$$(f,g) = \{h \in C[a,b] : f < h < g\}, \text{ is called an open interval}$$

and

$$[f,g] = \{h \in C[a,b] : f \leq h \leq g\}, \text{ is called a closed interval.}$$

We say that two intervals in $C[a,b]$ are disjoint if their intersection is empty, that is, if they have no common elements. Similary, we will say that two intervals in $C[a,b]$ are non-overlapping if their intersection is either empty or contains at most one element.
For $f, g \in C[a,b]$, we define $\frac{f}{g}$, $f \vee g$, $f \wedge g$, and $|f|$ as follows

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \text{ for every } x \in [a,b] \text{ whenever } g(x) \neq 0,$$

$$(f \vee g)(x) = \sup_{x\in[a,b]}\{f(x), g(x)\},$$

$$(f \wedge g)(x) = \inf_{x\in[a,b]}\{f(x), g(x)\},$$

$$|f|(x) = |f(x)| \text{ for every } x \in [a,b].$$

Bartle and Sherbert [2] showed that if f and g are elements in $C[a,b]$, then $f+g, fg, \frac{f}{g}, f \vee g, f \wedge g$ and $|f|$ are also elements in $C[a,b]$.

A sequence $\{f_n\}$ of elements of $C[a,b]$ is said to be convergent to $f \in C[a,b]$, if for every $\epsilon > 0$ there is a positive integer K such that for every $n \geq K$, the terms f_n satisfy

$$|f_n - f| < \epsilon e.$$

A sequence $\{f_n\}$ which converges to f in $C[a,b]$ will be written

$$\lim_{n\to\infty} f_n = f \quad \text{ or } \quad f_n \to f \text{ as } n \to \infty.$$

More depth discussion of the sequence properties can be found in [10].

Let $[f,g]$ be the closed interval subset of $C[a,b]$. A division of $[f,g]$ is any finite set $\{h_0, h_1, \cdots, h_n\} \subset [f,g]$, where $h_0 = f, h_n = g$ and $h_{i-1} < h_i$ for all $i = 1, 2, \cdots, n$. A partition of $[f,g]$ is a finite collection $\{([h_{i-1}, h_i], t_i) : i = 1, 2, \cdots, n\}$ of interval and element pairs such that $t_i \in [h_{i-1}, h_i]$ for every $i = 1, 2, \cdots, n$, where $\{h_0, h_1, \cdots, h_n\}$ is a division of $[f,g]$. Let θ be the null element in $C[a,b]$, where $\theta(x) = 0$ for every $x \in [a,b]$. A function $\delta : I \to C[a,b]$ is said to be a gauge on I if $\delta(h) > \theta$ for every $h \in I$.

Definition 2.1 *Let δ be a gauge on $[f,g]$. A partition $\mathcal{D} = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \cdots, n\}$ is said to be δ-fine of $[f,g]$ if $[h_{i-1}, h_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i = 1, 2, \cdots, n$.*

Observe, if δ_1 and δ_2 are two gauges on $[f,g] \subset C[a,b]$ with $\delta_1(h) \leq \delta_2(h)$ for every $h \in [f,g]$, then for every δ_1-fine partition $\mathcal{D}$ of $[f,g]$ is a δ_2-fine partition $\mathcal{D}$ of $[f,g]$.

Lemma 2.2 *If $\{[f_n, g_n]\} \subset C[a,b]$ is a sequence of closed intervals such that*

i. $[f_{n+1}, g_{n+1}] \subseteq [f_n, g_n]$ *for every* $n \in \mathbb{N}$,

ii. $\lim_{n\to\infty} |f_n - g_n| = \theta$,

then there is a unique $h \in C[a,b]$ such that $h \in [f_n, g_n]$ for every $n \in \mathbb{N}$.

Proof. From condition (i), for every $x \in [a,b]$ we have a sequence of closed intervals $\{[f_n(x), g_n(x)]\} \subset \mathbb{R}$ such that

$$[f_{n+1}(x), g_{n+1}(x)] \subseteq [f_n(x), g_n(x)], \text{ for every } n \in \mathbb{N}.$$

Based on [2], there is a unique number $h(x)$ that lies in all of the intervals $[f_n(x), g_n(x)]$. It is clear that h is a real-valued function defined on a closed interval $[a,b]$. Next, we will prove that $h \in C[a,b]$.
Based on condition (ii), if given $\epsilon > 0$ arbitrary, there is a positive integer N such that for every $n \geq N$ we have

$$|f_n - g_n| < \frac{\epsilon e}{3}.$$

Therefore, for every positive integer $n \geq N$ and for every $x \in [a,b]$ we obtain

$$|f_n(x) - g_n(x)| < \frac{\epsilon}{3}.$$

Since $f_n(x) \leq h(x) \leq g_n(x)$ for every $x \in [a,b]$ and for every positive integer $n \geq N$, we obtain

$$|f_n(x) - h(x)| < \frac{\epsilon}{3}. \tag{1}$$

On the other hand, since $f_n \in C[a,b]$ for every n, then for every $\epsilon > 0$ there is an $\eta > 0$ such that whenever $x, y \in [a,b]$ with $|x - y| < \eta$, we have

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{3}. \tag{2}$$

By the inequalities (1) and (2), if $n \geq N$ and $x, y \in [a,b]$ with $|x-y| < \eta$, then we obtain

$$|h(x) - h(y)| \leq |h(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - h(y)| < \epsilon.$$

Therefore, $h \in C[a,b]$ and $h \in [f_n, g_n]$ for every $n \in \mathbb{N}$. □

The next theorem guarantees the existence of δ-fine partitions of $[f,g]$ for each gauge δ on $[f,g]$.

Theorem 2.3 *If δ is a gauge on $[f,g] \subset C[a,b]$, then there is a δ-fine partition of $[f,g]$*

Proof. Suppose that $[f,g]$ does not have a δ-fine partition. We devide $[f,g]$ into $[f, \frac{f+g}{2}]$ and $[\frac{f+g}{2}, g]$. We can choose an interval $[f_1, g_1]$ from the set $\{[f, \frac{f+g}{2}], [\frac{f+g}{2}, g]\}$ so that $[f_1, g_1]$ does not have a δ-fine partition. Using induction, we construct intervals $[f_1, g_1], [f_2, g_2], \cdots$ in $C[a,b]$ so that for every $n \in \mathbb{N}$ is satisfied the following properties:

i. $[f_{n+1}, g_{n+1}] \subseteq [f_n, g_n]$,

ii. there is no δ-fine partition of $[f_n, g_n]$, and

iii. $g_n - f_n = \frac{g-f}{2^n}$.

From conditions (i) and (iii), by Lemma 2.2, there is an element $h_0 \in C[a,b]$ such that $h_0 \in [f_n, g_n]$ for every $n \in \mathbb{N}$. On the other hand, $\delta(h_0) > \theta$, it follows from condition (iii), there is a positive integer $N \in \mathbb{N}$ such that $\{([f_N, g_N], h_0)\}$ is a δ-fine partition of $[f_N, g_N]$, a contradiction to (ii). This contradiction completes the proof. □

3 The Henstock-Kurzweil Integral

The aim of this section is to introduce the Henstock-Kurzweil integral for $C[a,b]$ space-valued functions defined on a closed interval $[f,g]$ subset of $C[a,b]$.

Let $\mathcal{D} = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \cdots, n\}$ be a partition of $[f,g]$. If F is a $C[a,b]$ space-valued function defined on $[f,g]$, we write

$$S(F, \mathcal{D}) = \sum_{i=1}^{n} F(t_i)(h_i - h_{i-1}).$$

Next, we define the Henstock-Kurzweil integral of a $C[a,b]$ space-valued function in the following.

Definition 3.1 *A function* $F : [f, g] \subset C[a, b] \to C[a, b]$ *is said to be Henstock-Kurzweil integrable, briefly HK-integrable, on* $[f, g]$ *if there is* $s \in C[a, b]$ *with the following property: for every* $\epsilon > 0$ *there is a gauge* δ *on* $[f, g]$ *such that if* $\mathcal{D} = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \cdots, n\}$ *is any* δ*-fine partition of* $[f, g]$*, then*

$$|S(F, \mathcal{D}) - s| < \epsilon e.$$

It is important to know that the element s in Definition 3.1 is uniquely determined.

The collection of all functions that are HK-integrable on $[f, g]$ will be denoted by $\mathcal{HK}[f, g]$. The element $s \in C[a, b]$ which is mentioned in Definition 3.1, is called Henstock-Kurzweil integral, briefly HK-integral, of F over $[f, g]$ and it is written by

$$s = (\mathcal{HK}) \int_f^g F.$$

We now give some basic properties of the Henstock-Kurzweil integral.

Theorem 3.2 *If* $F, G \in \mathcal{HK}[f, g]$ *and* $\alpha \in \mathbb{R}$*, then* $\alpha F, F + G \in \mathcal{HK}[f, g]$*. Furthermore,*

$$(\mathcal{HK}) \int_f^g (F + G) = (\mathcal{HK}) \int_f^g F + (\mathcal{HK}) \int_f^g G$$

and

$$(\mathcal{HK}) \int_f^g \alpha F = \alpha (\mathcal{HK}) \int_f^g F.$$

Proof. Let $\epsilon > 0$ be given. Since $F \in \mathcal{HK}[f, g]$, there is a gauge δ_1 on $[f, g]$ such that for every δ_1-fine partition $\mathcal{D}_1$ on $[f, g]$ we have

$$\left| S(F, \mathcal{D}_1) - (\mathcal{HK}) \int_f^g F \right| < \frac{\epsilon e}{2}.$$

Similary, there is a gauge δ_2 on $[f, g]$ such that for every δ_2-fine partition $\mathcal{D}_2$ on $[f, g]$ we have

$$\left| S(G, \mathcal{D}_2) - (\mathcal{HK}) \int_f^g G \right| < \frac{\epsilon e}{2}.$$

Define a gauge δ on $[f, g]$ by setting $\delta(h) = \delta_1(h) \wedge \delta_2(h)$ for every $h \in [f, g]$. Then, for every δ-fine partition $\mathcal{D}$ of $[f, g]$ we obtain

$$\begin{aligned} \left| S(F + G, \mathcal{D}) - \left((\mathcal{HK}) \int_f^g F + (\mathcal{HK}) \int_f^g G \right) \right| &\leq \left| S(F, \mathcal{D}) - (\mathcal{HK}) \int_f^g F \right| \\ &\quad + \left| S(G, \mathcal{D}) - (\mathcal{HK}) \int_f^g G \right| \\ &< \frac{\epsilon e}{2} + \frac{\epsilon e}{2} = \epsilon e. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $F + G \in \mathcal{HK}[f,g]$ and

$$(\mathcal{HK})\int_f^g (F+G) = (\mathcal{HK})\int_f^g F + (\mathcal{HK})\int_f^g G.$$

Let α be a real number. Since $F \in \mathcal{HK}[f,g]$, there is a gauge δ_0 on $[f,g]$ such that for every δ_0-fine partition $\mathcal{D}$ of $[f,g]$ we have

$$\left| S(F,\mathcal{D}) - (\mathcal{HK})\int_f^g F \right| < \frac{\epsilon e}{|\alpha|+1}.$$

If $\mathcal{B}$ is a δ_0-fine partition of $[f,g]$, then

$$\begin{aligned}\left| S(\alpha F,\mathcal{B}) - \alpha(\mathcal{HK})\int_f^g F \right| &= \left| \alpha S(F,\mathcal{B}) - \alpha(\mathcal{HK})\int_f^g F \right| \\ &= |\alpha|\left| S(F,\mathcal{B}) - (\mathcal{HK})\int_f^g F \right| \\ &< |\alpha|\frac{\epsilon e}{|\alpha|+1} < \epsilon e.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\alpha F \in \mathcal{HK}[f,g]$ and

$$(\mathcal{HK})\int_f^g \alpha F = \alpha(\mathcal{HK})\int_f^g F.$$

The proof is complete. □

Theorem 3.3 *Let $f < r < g$. If $F \in \mathcal{HK}[f,r]$ and $F \in \mathcal{HK}[r,g]$, then $F \in \mathcal{HK}[f,g]$ and*

$$(\mathcal{HK})\int_f^g F = (\mathcal{HK})\int_f^r F + (\mathcal{HK})\int_r^g F.$$

Proof. Let $\epsilon > 0$ be given. Since $F \in \mathcal{HK}[f,r]$, there is a gauge δ_1 on $[f,r]$ such that for every δ_1-fine partition $\mathcal{D}_1$ of $[f,r]$ we have

$$\left| S(F,\mathcal{D}_1) - (\mathcal{HK})\int_f^r F \right| < \frac{\epsilon e}{2}.$$

Similarly, since $F \in \mathcal{HK}[r,g]$, there is a gauge δ_2 on $[r,g]$ such that for every δ_2-fine partition $\mathcal{D}_2$ of $[r,g]$ we have

$$\left| S(F,\mathcal{D}_2) - (\mathcal{HK})\int_r^g F \right| < \frac{\epsilon e}{2}.$$

Define a gauge on $[f,g]$ by setting

$$\delta(h) = \begin{cases} \delta_1(h) \wedge (r-h) & \text{, if } f \le h < r \\ \delta_1(h \wedge r) \wedge \delta_2(h \vee r) & \text{, if } h = r \text{ or } h \text{ is incomparable to } r \\ \delta_2(h) \wedge (h-r) & \text{, if } r < h \le g \end{cases}$$

Take an arbitrary δ-fine partition $\mathcal{D} = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \cdots, p\}$ of $[f,g]$. Our choice of δ implies that $r = h_i$ for some $i \in \{1, 2, \cdots, p\}$, we conclude that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ for some δ-fine partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ of $[f,r]$ and $[r,g]$, respectively. Consequently

$$\begin{aligned} \left| S(F,\mathcal{D}) - \left((\mathcal{HK}) \int_f^r F + (\mathcal{HK}) \int_r^g F \right) \right| &\le \left| S(F,\mathcal{D}_1) - (\mathcal{HK}) \int_f^r F \right| \\ &\quad + \left| S(F,\mathcal{D}_2) - (\mathcal{HK}) \int_r^g F \right| \\ &< \frac{\epsilon e}{2} + \frac{\epsilon e}{2} = \epsilon e. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $F \in \mathcal{HK}[f,g]$ and

$$(\mathcal{HK}) \int_f^g F = (\mathcal{HK}) \int_f^r F + (\mathcal{HK}) \int_r^g F.$$

□

The following theorem gives a necessary and sufficient condition for a function F to be HK-integrable on $[f,g]$.

Theorem 3.4 (Cauchy Criterion). *$F \in \mathcal{HK}[f,g]$ if and only if for every $\epsilon > 0$ there is a gauge δ on $[f,g]$ such that for every two δ-fine partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[f,g]$, we have*

$$|S(F,\mathcal{P}) - S(F,\mathcal{Q})| < \epsilon e.$$

Proof. ($\Rightarrow$) Let $\epsilon > 0$ be given. Since $F \in \mathcal{HK}[f,g]$, there is a gauge δ on $[f,g]$ such that for every δ-fine partition $\mathcal{D}$ of $[f,g]$ we have

$$\left| S(F,\mathcal{D}) - (\mathcal{HK}) \int_f^g F \right| < \frac{\epsilon e}{2}.$$

If $\mathcal{P}$ and $\mathcal{Q}$ are two δ-fine patitions of $[f,g]$, we obtain

$$\begin{aligned} \left| S(F,\mathcal{P}) - S(F,\mathcal{Q}) \right| &\le \left| S(F,\mathcal{P}) - (\mathcal{HK}) \int_f^g F \right| + \left| S(F,\mathcal{Q}) - (\mathcal{HK}) \int_f^g F \right| \\ &< \frac{\epsilon e}{2} + \frac{\epsilon e}{2} = \epsilon e. \end{aligned}$$

($\Leftarrow$) For each $n \in \mathbb{N}$, there is a gauge δ_n on $[f,g]$ such that for each pair of δ_n-fine partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[f,g]$ we have

$$|S(F,\mathcal{P}) - S(F,\mathcal{Q})| < \frac{e}{n}.$$

For each $n \in \mathbb{N}$, define a gauge δ_n^* on $[f,g]$ by setting

$$\delta_1^*(h) = \delta_1(h), \text{ for every } h \in [f,g],$$

and

$$\delta_n^*(h) = \delta_{n-1}^*(h) \wedge \delta_n(h), \text{ for every } h \in [f,g],\ n = 2,3,\cdots$$

Consequently, for every $m,n \in \mathbb{N}$ with $m \geq n$ we obtain

$$\delta_m^*(h) \leq \delta_n^*(h), \qquad \text{for every } h \in [f,g].$$

For every $n \in \mathbb{N}$, let $\mathcal{D}_n$ be a δ_n^*-fine partition of $[f,g]$ and we define

$$r_n = S(F,\mathcal{D}_n).$$

Cleary, if $m > n$ then both $\mathcal{D}_m$ and $\mathcal{D}_n$ are δ_n^*-fine partitions of $[f,g]$. Hence,

$$|r_n - r_m| < \frac{e}{n}, \qquad \text{for } m > n.$$

Therefore, $\{r_n\}$ is a Cauchy sequence. According to [6], that is a Cauchy sequence if and only if a convergent sequence, then the sequence $\{r_n\}$ converges to some $r \in C[a,b]$. Passing to the limit as $m \to \infty$ in the above inequality, we have

$$|r_n - r| < \frac{e}{n}, \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, given $\epsilon > 0$, let $K \in \mathbb{N}$ with $K > 2e/\epsilon$. For any δ_K^*-fine partition $\mathcal{D}$ of $[f,g]$, we obtain

$$|S(F,\mathcal{D}) - r| \leq |r - r_K| + |S(F,\mathcal{D}) - S(F,\mathcal{D}_K)| < \frac{e}{K} + \frac{e}{K} < \epsilon e.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $F \in \mathcal{HK}[f,g]$ and

$$r = (\mathcal{HK})\int_f^g F.$$

The proof is complete. □

The following theorem is a consequence of Theorem 3.4.

Theorem 3.5 *If $F \in \mathcal{HK}[f,g]$ and $[r,s] \subseteq [f,g]$, then $F \in \mathcal{HK}[r,s]$*

Proof. Let $\epsilon > 0$ be given. Since $F \in \mathcal{HK}[f,g]$, by Theorem 3.4, there is a gauge δ on $[f,g]$ such that for every $\mathcal{P}$ and $\mathcal{Q}$ are two δ-fine partitions of $[f,g]$ we have

$$|S(F,\mathcal{P}) - S(F,\mathcal{Q})| < \epsilon e.$$

Since $[r,s]$ is a subinterval of $[f,g]$, there is a finite collection $\{[u_1,v_1],[u_2,v_2],\cdots,[u_K,v_K]\}$ of pairwise non-overlapping subintervals of $[f,g]$ such that $[r,s] \notin \{[u_1,v_1],[u_2,v_2],\cdots,[u_K,v_K]\}$ and

$$[f,g] = [r,s] \cup \bigcup_{i=1}^{K} [u_i,v_i].$$

For every $i \in \{1,2,\cdots,K\}$ we fix a δ-fine partition $\mathcal{D}_i$ of $[u_i,v_i]$. If $\mathcal{P}'$ and $\mathcal{Q}'$ are δ-fine partitions of $[r,s]$ then $\mathcal{P}' \cup \bigcup_{i=1}^{K} \mathcal{D}_i$ and $\mathcal{Q}' \cup \bigcup_{i=1}^{K} \mathcal{D}_i$ are δ-fine partitions of $[f,g]$. Thus

$$\begin{aligned} |S(F,\mathcal{P}') - S(F,\mathcal{Q}')| &= \left| S(F,\mathcal{P}') + \sum_{i=1}^{K} S(F,\mathcal{D}_i) - S(F,\mathcal{Q}') - \sum_{i=1}^{K} S(F,\mathcal{D}_i) \right| \\ &= \left| S(F,\mathcal{P}' \cup \bigcup_{i=1}^{K} \mathcal{D}_i) - S(F,\mathcal{Q}' \cup \bigcup_{i=1}^{K} \mathcal{D}_i) \right| < \epsilon e. \end{aligned}$$

Based on Theorem 3.4, we conclude that $F \in \mathcal{HK}[r,s]$. □

Theorem 3.6 *If $F \in \mathcal{HK}[f,g]$ where $F(h) \geq \theta$ for every $h \in [f,g]$ then*

$$(\mathcal{HK}) \int_f^g F \geq \theta.$$

Proof. Let $\epsilon > 0$ be given. Since $F \in \mathcal{HK}[f,g]$, there is a gauge δ on $[f,g]$ such that for every δ-fine partition $\mathcal{D} = \{([h_{i-1},h_i],t_i) : i = 1,2,\cdots,n\}$ of $[f,g]$, we have

$$|S(F,\mathcal{D}) - (\mathcal{HK}) \int_f^g F| < \epsilon e.$$

Since $F(h) \geq \theta$ for every $h \in [f,g]$ then

$$S(F,\mathcal{D}) = \sum_{i=1}^{n} F(t_i)(h_i - h_{i-1}) \geq \theta.$$

Therefore

$$\theta \leq S(F,\mathcal{D}) < (\mathcal{HK}) \int_f^g F + \epsilon e.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$(\mathcal{HK})\int_f^g F \geq \theta.$$

□

Theorem 3.7 *If $F, G \in \mathcal{HK}[f,g]$ and $F(h) \leq G(h)$ for every $h \in [f,g]$, then*

$$(\mathcal{HK})\int_f^g F \leq (\mathcal{HK})\int_f^g G.$$

Proof. Define a function H on $[f,g]$ by setting $H(h) = G(h) - F(h)$ for every $h \in [f,g]$. It is clear that $H(h) \geq \theta$ for every $h \in [f,g]$. Since $F, G \in \mathcal{HK}[f,g]$, according to Theorem 3.2 then $H \in \mathcal{HK}[f,g]$. Further, based on Theorem 3.6 we obtain

$$(\mathcal{HK})\int_f^g H = (\mathcal{HK})\int_f^g G - (\mathcal{HK})\int_f^g F \geq \theta.$$

Therefore

$$(\mathcal{HK})\int_f^g F \leq (\mathcal{HK})\int_f^g G.$$

□

4 A Monotone Convergence Theorem

The aim of this section is to prove monotone convergence theorem in $C[a,b]$ space for the Henstock-Kurzweil. Before, we introduce the notion of Henstock-Kurzweil primitive and we prove the Henstock's lemma.

If $F \in \mathcal{HK}[f,g]$, based on Theorem 3.3 and Theorem 3.5, then F is HK-integrable on $[f,h]$ for every $h \in [f,g]$. We define a function $\mathcal{F}$ on $[f,g]$ by

$$\mathcal{F}(h) = (\mathcal{HK})\int_f^h F$$

for every $h \in [f,g]$ is called Henstock-Kurzweil primitive, briefly HK-primitive, on $[f,g]$. For simplicity, if $I = [s,t]$ we write $\mathcal{F}(I) = \mathcal{F}(s,t) = \mathcal{F}(t) - \mathcal{F}(s)$.

A partial partition of $[f,g]$ is a finite collection $\{([u_i,v_i],t_i) : i = 1,2,\cdots,p\}$ of interval and element pairs such that $t_i \in [u_i,v_i]$ for $i = 1,2,\cdots,p$ and $\{[u_1,v_1],\cdots,[u_p,v_p]\}$ is a collection of non-overlapping subintervals of $[f,g]$. The partial partition $\{([u_i,v_i],t_i) : i = 1,2,\cdots,p\}$ is said to be δ-fine if $[u_i,v_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for $i = 1,2,\cdots,p$.

Lemma 4.1 (Henstock's Lemma). *Let $F \in \mathcal{HK}[f,g]$ with HK-primitive $\mathcal{F}$. For $\epsilon > 0$, let δ be a gauge on $[f,g]$ such that if $\mathcal{D} = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \cdots, n\}$ is a δ-fine partition of $[f,g]$, then*

$$|S(F, \mathcal{D}) - \mathcal{F}(f,g)| < \epsilon e.$$

Suppose $\mathcal{D}' = \{([h_{i_j-1}, h_{i_j}], t_{i_j}) : j : 1, 2, \cdots, k, 1 \le k \le n\}$ is a δ-fine partial partition of $[f,g]$ where $\mathcal{D}' \subseteq \mathcal{D}$. Then

$$\Big|\sum_{j=1}^{k}\Big(F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j})\Big)\Big| \le \epsilon e.$$

and

$$\sum_{j=1}^{k}\Big|F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j})\Big| \le 2\epsilon e.$$

Proof. There are $n-k$ intervals $[h_{m-1}, h_m] \subset [f,g]$ with $([h_{m-1}, h_m], t_m) \in \mathcal{D} \backslash \mathcal{D}'$. Denote $M = \{m : ([h_{m-1}, h_m], t_m) \in \mathcal{D} \backslash \mathcal{D}'\}$. Let $\eta > 0$ be given. By Theorem 3.5, there is a gauge δ_m on $[h_{m-1}, h_m]$ with $\delta_m \le \delta$, such that for every δ_m-fine partition $\mathcal{D}_m$ of $[h_{m-1}, h_m]$ we have

$$|S(F, \mathcal{D}_m) - \mathcal{F}(h_{m-1}, h_m)| < \frac{\eta e}{n-k}, \qquad \text{for every } m \in M.$$

Put $\mathcal{P} = \bigcup_{m \in M} \mathcal{D}_m \cup \mathcal{D}'$. Then $\mathcal{P}$ is a δ-fine partition of $[f,g]$ and $S(F, \mathcal{P}) = S(F, \mathcal{D}') + \sum_{m \in M} S(F, \mathcal{D}_m)$. By the additivity of HK-integral we obtain

$$\begin{aligned}
&\Big|\sum_{j=1}^{k}\Big(F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j})\Big)\Big| \\
&\le \Big|S(F, \mathcal{P}) - \mathcal{F}(f,g)\Big| + \sum_{m \in M}\Big|S(F, \mathcal{D}_m) - \mathcal{F}(h_{i_j-1}, h_{i_j})\Big| \\
&< \epsilon e + (n-k)\frac{\eta e}{n-k} = \epsilon e + \eta e.
\end{aligned}$$

Since $\eta > 0$ is arbitrary, the proof of the first inequality is complete.

To prove the second inequality of the Henstock's lemma, split $\mathcal{D}$ into $\mathcal{Q}_1$ and $\mathcal{Q}_2$, where

$$\mathcal{Q}_1 = \Big\{([h_{i_j-1}, h_{i_j}], t_{i_j}) \in \mathcal{D} : F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j}) \ge \theta\Big\},$$

$$\mathcal{Q}_2 = \Big\{([h_{i_j-1}, h_{i_j}], t_{i_j}) \in \mathcal{D} : F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j}) < \theta\Big\}$$

and denote $M_1 = \{i_j : ([h_{i_j-1}, h_{i_j}], t_{i_j}) \in \mathcal{Q}_1\}$, $M_2 = \{i_j : ([h_{i_j-1}, h_{i_j}], t_{i_j}) \in \mathcal{Q}_2\}$. Since $\mathcal{Q}_1 \subseteq \mathcal{D}$ and $\mathcal{Q}_2 \subseteq \mathcal{D}$, by the first inequality

$$\begin{aligned}
&\sum_{j=1}^{k} \Big| F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j}) \Big| \\
&= \sum_{i_j \in M_1} \Big| F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j}) \Big| \\
&\quad + \sum_{i_j \in M_2} \Big| F(t_{i_j})(h_{i_j} - h_{i_j-1}) - \mathcal{F}(h_{i_j-1}, h_{i_j}) \Big| \\
&\le \epsilon e + \epsilon e = 2\epsilon e.
\end{aligned}$$

The lemma is proved. □

A sequence $\{F_n\}$ of functions on $[f,g] \subset C[a,b]$ is said to be increasing on $[f,g]$ if every $h \in [f,g]$ then $F_1(h) \le F_2(h) \le \cdots$. Similary, a sequence $\{G_n\}$ of functions on $[f,g] \subset C[a,b]$ is said to be decreasing on $[f,g]$ if every $h \in [f,g]$ then $G_1(h) \ge G_2(h) \ge \cdots$. If a sequence of functions is either increasing or decreasing on $[f,g]$, we say that it is monotone on $[f,g]$.

Theorem 4.2 (Monotone Convergence Theorem). *If the following conditions are satisfied:*

i. $\lim_{n\to\infty} F_n = F$ *on* $[f,g]$ *and* $F_n \in \mathcal{HK}[f,g]$ *for every* $n \in \mathbb{N}$;

ii. $\{F_n\}$ *is monotone on* $[f,g]$;

iii. $\{\mathcal{F}_n(f,g)\}$ *converges to* s *whenever* $n \to \infty$, *where* $\mathcal{F}_n$ *is HK-primitive of* F_n *on* $[f,g]$ *for every* $n \in \mathbb{N}$,

then $F \in \mathcal{HK}[f,g]$ *and*

$$(\mathcal{HK}) \int_f^g F = s.$$

Proof. Assume $\{F_n\}$ is increasing on $[f,g]$. It follows that $\{\mathcal{F}_n(f,g)\}$ is increasing and converges to s. Let $\epsilon > 0$ be given. Choose a positive integer K such that

$$\theta \le s - \mathcal{F}_K(f,g) \le \epsilon e. \tag{3}$$

Since $F_n \in \mathcal{HK}[f,g]$ with its HK-primitive $\mathcal{F}_n$ for every $n \in \mathbb{N}$, there is a gauge δ_n on $[f,g]$ such that for every δ_n-fine partition $\mathcal{D}$ of $[f,g]$ we have

$$\Big| S(F_n, \mathcal{D}) - \mathcal{F}_n(f,g) \Big| < \frac{\epsilon e}{2^n}.$$

By condition (i), for every $h \in [f,g]$ there is a positive integer $m(\epsilon,h)$ such that $m(\epsilon,h) \geq K$ and

$$|F_{m(\epsilon,h)}(h) - F(h)| < \epsilon e. \tag{4}$$

Set $\delta(h) = \delta_{m(\epsilon,h)}(h)$ for every $h \in [f,g]$. By taking any δ-fine partition $\mathcal{D} = \{([h_{i-1},h_i],t_i) : i = 1,2,\cdots,p\}$ of $[f,g]$ we obtain

$$|S(F,\mathcal{D}) - s| \quad \leq \quad I + II + III$$

where

$$I = \left| \textstyle\sum_{i=1}^p F(t_i)(h_i - h_{i-1}) - F_{m(\epsilon,t_i)}(t_i)(h_i - h_{i-1}) \right|$$

$$II = \left| \textstyle\sum_{i=1}^p F_{m(\epsilon,h)}(t_i)(h_i - h_{i-1}) - \mathcal{F}_{m(\epsilon,t_i)}(h_{i-1},h_i) \right|$$

$$III = \left| \textstyle\sum_{i=1}^p \mathcal{F}_{m(\epsilon,t_i)}(h_{i-1},h_i) - s \right|$$

By (4) we obtain

$$I \leq \sum_{i=1}^p \left| F(t_i)(h_i - h_{i-1}) - F_{m(\epsilon,t_i)}(t_i)(h_i - h_{i-1}) \right| \leq \epsilon(g-f).$$

To estimate II, set $S = \max\{k(t_1), k(t_2), \cdots, k(t_p)\} \geq K$. Then,

$$\begin{aligned} II &\leq \sum_{i=1}^p \left| F_{m(\epsilon,h)}(t_i)(h_i - h_{i-1}) - \mathcal{F}_{m(\epsilon,t_i)}(h_{i-1},h_i) \right| \\ &= \sum_{k=K}^S \sum_{k(t_i)=k} \left| F_{m(\epsilon,h)}(t_i)(h_i - h_{i-1}) - \mathcal{F}_{m(\epsilon,t_i)}(h_{i-1},h_i) \right| \end{aligned}$$

in which we have grouped together all terms corresponding to F_k for a fix k. Note that the set $\{(I_i,t_i) : k(t_i) = k\}$ is a δ-fine partition partial of $[f,g]$, so that Henstock's lemma implies

$$\sum_{k(t_i)=k} \left| F_{m(\epsilon,h)}(t_i)(h_i - h_{i-1}) - \mathcal{F}_{m(\epsilon,t_i)}(h_{i-1},h_i) \right| \leq \frac{2\epsilon e}{2^k}.$$

Summing over k,

$$II \leq \sum_{k=K}^S \frac{2\epsilon e}{2^k} < 2\epsilon e.$$

Based on Theorem 3.7 and condition (iii), the sequence $\{\mathcal{F}_n(f,g)\}$ is increasing and convergent to s. Since, the number of associated elements in $\mathcal{D}$

is finite, and so is the number of those different $m(\epsilon, h)$ in the above sum over $\mathcal{D}$.

Let q be a positive integer by

$$q = \min\{m(\epsilon, h) : ([u, v], h) \in \mathcal{D}\}.$$

Then we have

$$\mathcal{F}_q(f, g) = \sum_{i=1}^{p} \mathcal{F}_q(h_{i-1}, h_i) \leq \sum_{i=1}^{p} \mathcal{F}_{m(\epsilon,h)}(h_{i-1}, h_i) \leq \sum_{i=1}^{p} \mathcal{F}(h_{i-1}, h_i) = s.$$

Obviously, we can find m_0 such that

$$|\mathcal{F}_m(f, g) - s| < \epsilon e, \qquad \text{whenever } m \geq m_0.$$

Therefore in defining $m(\epsilon, h)$, we should choose $m(\epsilon, h) \geq m_0$. Hence

$$\Big| \sum_{i=1}^{p} \mathcal{F}_{m(\epsilon,t_i)}(h_{i-1}, h_i) - s \Big| \leq s - \mathcal{F}_q(f, g) < \epsilon e.$$

In case $\{F_n\}$ is decreasing on $[f, g]$, we define $G_n(h) = -F_n(h)$ for every $h \in [f, g]$. Therefore, $\{G_n\}$ is increasing on $[f, g]$ and the proof follows in above. □

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