KOMUTATOR OPERATOR MOMENTUM SUDUT TERHADAP POSISI DAN HAMILTONIAN PARTIKEL BEBAS

SKRIPSI

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PROGRAM STUDI PENDIDIKAN FISIKA
JURUSAN PENDIDIKAN MIPA
FAKULTAS KEGURUAN DAN ILMU PENDIDIKAN
UNIVERSITAS JEMBER
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diajukan guna melengkapi tugas akhir dan memenuhi salah satu syarat
untuk menyelesaikan Program Studi Pendidikan Fisika (S1)
dan mencapai gelar Sarjana Pendidikan

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2019

PERSEMBAHAN

Skripsi ini saya persembahkan untuk:

1. Ayahanda tercinta Muhammad Shodiq dan Ibunda tersayang Alfiyah;
2. Kakak saya Ismatul Alifah dan adik-adik saya M. Tri Wahyu Ilahi dan Adilah Nafisah;
3. Guru-guru saya sejak Sekolah Dasar sampai dengan Perguruan Tinggi;
4. Almamater Fakultas Keguruan dan Ilmu Pendidikan Universitas Jember;
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MOTO

"Tidak ada yang perlu ditakuti di dunia ini,
segalanya hanya perlu dimengerti (dipelajari)."

~Marie Curie~

PERNYATAAN

Saya yang bertandatangan di bawah ini:

Nama : Vela Rizqiyah

NIM : 150210102082

menyatakan dengan sesungguhnya bahwa karya ilmiah yang berjudul "Komutator Operator Momentum Sudut Terhadap Posisi dan Hamiltonian Partikel Bebas" adalah benar-benar hasil karya sendiri, kecuali kutipan yang sudah saya sebutkan sumbernya, belum pernah diajukan pada institusi mana pun, dan bukan karya jiplakan. Saya bertanggung jawab atas keabsahan dan kebenaran isinya sesuai dengan sikap ilmiah yang harus dijunjung tinggi. Demikian pernyataan ini saya buat dengan sebenarnya tanpa adanya tekanan dan paksaan dari pihak mana pun serta bersedia mendapat sanksi akademik jika ternyata di kemudian hari pernyataan ini tidak benar.

Jember, Januari 2019

Yang menyatakan,

Vela Rizqiyah

NIM 150210102082

SKRIPSI

KOMUTATOR OPERATOR MOMENTUM SUDUT TERHADAP POSISI DAN HAMILTONIAN PARTIKEL BEBAS

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Dosen Pembimbing Utama	: Drs. Bambang Supriadi, M.Sc.
Dosen Pembimbing Anggota	: Drs. Trapsilo Prihandono, M.Si.

PENGESAHAN

Skripsi berjudul "Komutator Operator Momentum Sudut Terhadap Posisi dan Hamiltonian Partikel Bebas" telah diuji dan disahkan pada:

Hari, tanggal : Senin, 21 Januari 2019

Tempat : Fakultas Keguruan dan Ilmu Pendidikan Universitas Jember

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RINGKASAN

Komutator Operator Momentum Sudut Terhadap Posisi dan Hamiltonian Partikel Bebas; Vela Rizqiyah; 150210102082; 2019: 35 Halaman; Program Studi Pendidikan Fisika, Jurusan Pendidikan Matematika dan Ilmu Pengetahuan Alam, Fakultas Keguruan dan Ilmu Pendidikan, Universitas Jember.

Mekanika kuantum merupakan bahasan mikroskopik yang lahir dari gagasan Werner Karl Heisenberg tentang pengembangan mekanika matriks dan Erwin Schrodinger tentang pendekatan mekanika gelombang dengan persamaan Schrodinger. Pengukuran dalam mekanika kuantum tidak dapat dilakukan secara serempak dan hasilnya tidak pasti namun hanya berupa kemungkinan dengan menggunakan beberapa persamaan matematis yaitu persamaan Schrodinger dan metode operator. Operator berperan sebagai representasi variabel dinamis. Variabel dinamis (*observabel*) yang sering dijumpai dalam mekanika kuantum antara lain operator posisi, operator momentum, dan operator energi. Terdapat hubungan komutasi dalam mekanika kuantum yang hasilnya dapat dijadikan sebagai tolak ukur tertentu. Tujuan penelitian ini diantaranya (1) menentukan penyelesaian komutator operator momentum sudut terhadap posisi dengan $n = 1$ dan $n = 2$; (2) menentukan penyelesaian komutator operator momentum sudut terhadap Hamiltonian partikel bebas.

Jenis penelitian ini adalah *basic research* pada bidang fisika teori berupa pengembangan teori mekanika kuantum. Penelitian ini dilakukan di Laboratorium Fisika Dasar, Program Studi Pendidikan Fisika, Universitas Jember. Sumber data yang digunakan pada penelitian ini berasal dari buku, jurnal, dan internet tentang operator momentum sudut, operator Hamiltonian, partikel bebas, dan komutator. Metode pengambilan data penelitian ini yaitu menghitung operator momentum sudut yang dikomutasikan dengan posisi dan dengan Hamiltonian untuk partikel bebas. Setelah didapat hasil dari komutator tersebut, kemudian datanya dianalisis untuk mengetahui komutator tersebut komut atau tidak komut. Desain penelitian yang digunakan di antaranya tahap persiapan, tahap pengembangan teori, tahap

hasil pengembangan teori, tahap validasi hasil pengembangan teori, tahap pembahasan, dan tahap kesimpulan.

Berdasarkan hasil penelitian, komutator operator momentum sudut terhadap posisi akan bernilai nol atau komut jika dalam momentum sudut yang berkomutasi tidak terdapat komponen posisi yang dijadikan pasangan komutasinya sehingga dapat diukur secara bersamaan. Sedangkan komutator operator momentum sudut terhadap Hamiltonian partikel bebas akan bernilai nol atau komut hanya dengan momentum sudut orde 1 (L_x, L_y, L_z) sehingga ketiga momentum tersebut merupakan tetapan gerak.

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Penulis

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BAB 1. PENDAHULUAN

1.1 Latar Belakang

Pada abad kesembilan belas, banyak fenomena fisika tidak bisa dijelaskan secara klasik. Hal yang mendasar dari kegagalan fisika klasik adalah tidak dapat menjelaskan spektrum radiasi termal pada benda yang bersuhu sangat tinggi. Pada tahun 1900, Max Planck mampu menjelaskan penyebaran intensitas radiasi oleh benda hitam dan menyatakan bahwa energi bersifat diskrit. Pada tahun 1905, Albert Einstein dengan teori efek fotolistriknya menjelaskan bahwa energi cahaya datang dalam bentuk kuanta disebut foton. Pada tahun 1913, Niels Bohr juga menggunakan kuantisasi untuk mengulas teori atom hidrogen. Pada tahun 1924, Louis de Broglie memunculkan konsep dualisme cahaya.

Pada tahun 1925, mekanika kuantum lahir dari gagasan Werner Karl Heisenberg tentang pengembangan mekanika matriks dan Erwin Schrodinger tentang pendekatan mekanika gelombang dengan persamaan Schrodinger. Persamaa Schrodinger tersebut dapat mendefinisikan suatu fungsi gelombang partikel (Prastowo *et al.*, 2018). Heisenberg menyatakan bahwa posisi atau letak elektron tidak dapat ditentukan secara pasti. Heisenberg menentukan sifat suatu partikel melalui suatu variabel yaitu posisi partikel (x) dan momentum (p). Hipotesisnya menjelaskan bahwa tidak mungkin mengetahui posisi atau momentum suatu partikel dengan tepat secara serempak atau bersamaan (Purwanto, 2006: 37). Dalam mekanika klasik, pengukuran dapat dilakukan secara bersamaan atau serempak dengan hasil yang pasti. Sedangkan dalam mekanika kuantum, pengukuran tidak bisa dilakukan secara bersamaan dan hasil yang didapat hanya berupa suatu kemungkinan. Mekanika kuantum dapat menjelaskan fenomena-fenomena atau besaran mikroskopik yang tidak dapat dijelaskan secara klasik dengan menggunakan alat ukur yang terbatas dan hasil ukurnya hanya berupa kemungkinan dari besaran tersebut. Meskipun memiliki alat ukur yang terbatas, besarannya dapat dihitung dengan menggunakan beberapa persamaan matematis yaitu persamaan Schrodinger dan metode operator. Karena besaran yang diukur bersifat mikroskopis, maka hasil yang diperoleh tidak pasti dan hanya berupa probabilitasnya saja. Ketidakpastian tersebut tidak hanya dipengaruhi oleh lingkungan namun juga dari dalam sistem. Heisenberg menjelaskan bahwa suatu batas ketelitian yang dapat digunakan untuk melakukan sejumlah percobaan telah ditetapkan oleh alam, sebaik apa pun rancangan alat ukur tetap memiliki batas ketelitian (Krane, 1992:145).

Terdapat beberapa postulat yang mendasari perumusan mekanika kuantum, yaitu representasi keadaan, representasi variabel dinamis, evolusi sistem, dan tetapan gerak. Postulat representasi variabel dinamis mendasari metode operator. Variabel dinamis direpresentasikan oleh operator linier yang akan bekerja pada fungsi dari sistem dan mengubahnya menjadi fungsi gelombang lain (Purwanto, 2006: 109). Posisi, momentum linier, momentum sudut, dan energi merupakan variabel dinamis yang dapat diukur secara bersamaan dan pasti pada mekanika klasik. Namun dalam mekanika kuantum, besaran-besaran tersebut dihitung menggunakan fungsi gelombang. Fungsi gelombang dalam mekanika kuantum mewakili keadaan sistem, sedangkan operator mewakili hal yang teramati.

Komutator adalah hubungan komutasi dari beberapa operator. Komutator bisa bernilai nol atau bersifat komut jika observable (variabel dinamis) yang bersangkutan dapat diukur secara bersamaan. Dalam mekanika kuantum, komutatornya tidak bersifat komut karena observablenya tidak bisa diukur secara bersamaan sehingga tidak bernilai nol. Mustamin (2017: 22) menjelaskan bahwa x mewakili operator posisi dan $-i\hbar\frac{\partial}{\partial x}$ mewakili operator momentum. Operator posisi dan operator momentum dapat menentukan operator untuk observable lain.

Sunarmi dalam Nugraha (2017) mengemukakan bahwa operator dalam mekanika kuantum mewakili setiap besaran fisis yang teramati. Dalam mekanika klasik, operator dinyatakan dengan $Q(x,p)$ sedangkan dalam mekanika kuantum dengan $\hat{Q}(\hat{x},\hat{p})$. Komutator sebaiknya dikenakan pada suatu fungsi agar dapat bekerja. Fungsi tersebut dapat berupa fungsi gelombang yang diperoleh dari persamaan Schrodinger. Persamaan Schrodinger merupakan pondasi utama dalam mekanika kuantum (Supriadi *et al*., 2017).

Persamaan Schrodinger dibagi menjadi dua, yaitu persamaan Schrodinger bergantung waktu dan persamaan Schrodinger tak bergantung waktu (Supriadi *et al.*, 2018). Persamaan Schrodinger tak bergantung waktu merupakan bentuk paling sederhananya. Secara fisis, fungsi gelombang tersebut dipengaruhi potensial yang bergantung pada posisi r saja dan tidak berubah terhadap waktu. Sedangkan persamaan Schrodinger bergantung waktu menghasilkan fungsi gelombang yang dipengaruhi potensial yang bentuknya berubah terhadap waktu t.

Krane (1992: 182) menjelaskan bahwa partikel bebas bergerak tanpa adanya gaya yang mengenainya $\Sigma F = 0$ dan tetapan potensial yang bekerja adalah konstan. Bebas memilih tetapan potensial sama dengan nol, karena potensial selalu ditentukan dengan tambahan satu tetapan integrasi sebarang. Persamaan Schrodinger partikel bebas dapat dituliskan sebagai operator Hamiltonian sama dengan energi kinetik.

Pada penelitian sebelumnya, terdapat beberapa penelitian mengenai komutator operator momentum, antara lain: Nugraha (2017) dalam penelitiannya tentang *Komutator Operator Momentum Sudut dalam Koordinat Bola dengan Fungsi Gelombang Atom Hidrogen* menyimpulkan bahwa untuk keadaan dasar komutator operator momentum sudut atom Hidrogen semuanya bernilai nol atau bersifat komut yang berarti dapat diukur secara bersamaan, sedangkan dalam keadaan tereksitasi pertama dan kedua, fungsi gelombangnya tidak *compatible* dengan komutator tersebut karena terdapat beberapa komutator yang bukan termasuk persoalan Eigen; Mei dan Yu (2012) tentang *The Definition of Universal Momentum Operator of Quantum Mechanics and the Essence of Micro-Particle's Spin* menyimpulkan bahwa ketika fungsi non Eigen dikenai operator akan menghasilkan nilai non Eigen dan nilai momentum rata-rata berupa bilangan kompleks; Liu *et al.* (2010) tentang *Raising and Lowering Operator for Orbital Angular Momentum Quantum Numbers* menyimpulkan bahwa bilangan kuantum menetukan operator penaikan dan operator penurunan momentum sudut dan seluruhnya berupa fungsi harmonik bola dalam keadaan dasar; Morales dan Amaya-Tapia (1999) tentang *Relationship of The Commutation Rules to Classical-Like Representations of Quantum Angular Momenta Addition* menyimpulkan bahwa aturan komutator lebih baik daripada model vektor dalam msenggambarkan hubungan antara momentum sudut dalam mekanika kuantum; Enk dan Nienhuis (1994) tentang *Commutation Rules and Eigenvalues of Spin and Orbital Angular Momentum of Radiation Fields* menyimpulkan bahwa dengan menentukan aturan komutasi, arti fisis operator L dan S dapat diklarisifikasi lebih lanjut, operator L dan S tidak memenuhi aturan komutasi, sedangkan momentum sudut total J memenuhi aturan komutasi yang sebenarnya.

Berdasarkan uraian tersebut, dengan menggunakan hubungan komutasi dapat ditentukan besaran-besaran dalam mekanika kuantum yang dapat diukur secara bersamaaan dan yang tidak dapat diukur secara bersamaan. Oleh karena itu, perlu dilakukan penelitian dengan mengambil judul Komutator Operator Momentum Sudut Terhadap Posisi dan Hamiltonian Partikel Bebas. Penelitian ini merupakan pengembangan dari penelitian sebelumnya yang menggunakan komutator antar operator momentum sudut, sedangkan penelitian ini menggunakan komutator antara operator momentum sudut dengan posisi dan operator Hamiltonian partikel bebas yang. Penelitian ini menggunakan koordinat Kartesian agar dapat ditransformasikan ke berbagai koordinat.

1.2 Rumusan Masalah

Berdasarkan uraian latar belakang maka dapat dirumuskan beberapa permasalahan, antara lain:

a. Bagaimana penyelesaian komutator operator momentum sudut terhadap posisi dengan $n = 1$ dan $n = 2$?
b. Bagaimana penyelesaian komutator operator momentum sudut terhadap Hamiltonian partikel bebas?

1.3 Batasan Masalah

Agar penelitian lebih terfokus dan perumusan masalah dapat terjawab, maka penulis membatasi masalah sebagai berikut:

a. Persamaan Schrodinger yang digunakan adalah persamaan Schrodinger bebas waktu atau keadaan tunak.

b. Fungsi gelombang yang digunakan tidak ternormalisasi
c. Operator momentum sudut dalam koordinat Kartesian
d. Hubungan komutasi operator momentum sudut dengan posisi dan Hamiltonian hanya menggunakan masing-masing dua buah operator.
e. Operator momentum sudut dan posisi yang diteliti dalam penelitian ini yaitu orde $n = 1$ dan $n = 2$.

1.4 Tujuan Penelitian

Adapun tujuan dari penelitian ini sebagai berikut:

a. Menentukan penyelesaian komutator operator momentum sudut terhadap posisi dengan $n = 1$ dan $n = 2$.
b. Menentukan penyelesaian komutator operator momentum sudut terhadap Hamiltonian partikel bebas.

1.5 Manfaat Penelitian

Manfaat yang dapat diperoleh dari penelitian ini adalah:

a. Bagi peneliti, dapat menambah wawasan, pengetahuan, dan pengalaman tentang fisika kuantum khususnya komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas.
b. Bagi pembaca, dapat dijadikan sumber belajar dalam mempelajari fisika kuantum pokok bahasan komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas dan dapat dijadikan referensi dalam melakukan penelitian lebih lanjut dengan tema yang sama.
c. Bagi lembaga, dapat memberikan sumbangan penelitian dan referensi dalam perkuliahan fisika kuantum pokok bahasan komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas.

BAB 2. TINJAUAN PUSTAKA

2.1 Persamaan Schrodinger

Persamaan Schrodinger merupakan persamaan diferensial orde dua yang harus dipecahkan dalam kasus fisika kuantum nonrelativistik. Solusi dari persamaan Schrodinger berupa fungsi gelombang yang memberikan informasi tentang perilaku gelombang dari partikel (Krane, 1992: 170). Pemecahan persamaan Schrodinger harus memenuhi tiga syarat sebagai berikut:

a. Tidak melanggar hukum kekekalan energi

Hukum kekekalan energi menyatakan bahwa jumlah total energi kinetik dan potensial dari suatu partikel selalu bersifat kekal. Persamaan hukum kekekalan energi dapat dituliskan sebagai berikut.

$$T + V = E \tag{2.1}$$

dengan *T*, *V*, dan *E* berturut-turut menyatakan energi kinetik, energi potensial, dan energi total. Pada kasus nonrelativistik, persamaan (2.1) dituliskan sebagai:

$$\frac{p^2}{2m} + V = E \tag{2.2}$$

b. Taat asas terhadap hipotesis de Broglie

Semua bentuk persamaan diferensial harus taat pada hipotesis de Broglie. Pemecahan secara matematis bagi sebuah partikel dengan momentum p harus berbentuk sebuah fungsi gelombang dengan panjang gelombang λ yang sama dengan h/p. Dengan menggunakan $p = \hbar k$ dimana k adalah bilangan gelombang, maka energi kinetik dari gelombang de Broglie partikel bebas menjadi:

$$T = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{2.3}$$

c. Linier dan bernilai tunggal

Dalam pengertian matematis, persamaan Schrodinger harus berperilaku baik. Pemecahan persamaan Schrodinger harus memberikan informasi tentang probabilitas ditemukannya sebuah partikel. Meskipun probabilitas dapat berubah secara kontinu dan partikelnya berubah secara tiba-tiba dari suatu titik dan muncul kembali di titik lainnya, namun fungsi gelombang tersebut harus bernilai tunggal artinya tidak boleh ada dua probabilitas untuk menemukan partikel di satu titik yang sama. Fungsi gelombangnya harus memiliki sifat superposisi gelombang sebagai indikator sifat gelombang yang linier dan berkelakuan baik (Krane, 1992: 172).

2.1.1 Persamaan Schrodinger Bergantung Waktu

Persamaan Schrodinger dapat diperoleh dari persamaan gelombang datar untuk materi yang merambat ke arah r, yakni melalui persamaan:

$$\Psi(r,t) = Ae^{\frac{i}{\hbar}(pr-Et)} \tag{2.4}$$

Dari persamaan gelombang datar (2.4) jika diturunkan terhadap waktu diperoleh persamaan:

$$\frac{\partial\Psi}{\partial t} = -\frac{iE}{\hbar}\Psi \tag{2.5}$$

Energi total *E* dapat diganti dengan jumlah energi kinetik *T* dan energi potensial *V*, sehingga dapat ditulis:

$$\frac{\partial\Psi}{\partial t} = -\frac{i(T+V)}{\hbar}\Psi \tag{2.6}$$

Atau

$$(T+V)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{2.7}$$

Hubungan antara energi kinetik dan momentum adalah $T = \frac{p^2}{2m}$, maka persamaan (2.7) dapat ditulis ulang sebagai:

$$\left(\frac{p^2}{2m} + V\right)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{2.8}$$

Suku pertama persamaan (2.8) setara dengan

$$\frac{p^2}{2m}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi \tag{2.9}$$

dengan Laplacian ∇^2 diberikan oleh

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.10}$$

Persamaan (2.9) memberikan “operator” momentum, yaitu

$$\hat{p} = -i\hbar\frac{\partial}{\partial r} \tag{2.11}$$

Selanjutnya persamaan (2.8) dapat ditulis kembali menjadi persamaan Schrodinger bergantung waktu, yaitu:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{2.12}$$

(Juwono, 2017: 43-44)

2.1.2 Persamaan Schrodinger Bebas Waktu

Persamaan Schrodinger (2.12) memiliki empat peubah, yaitu tiga peubah ruang (x, y, z) dan satu peubah waktu (t). Jika energi potensial V bukan merupakan fungsi waktu, maka dua kelompok peubah tersebut dapat saling dipisahkan, sehingga didapatkan persamaan Schrodinger bebas waktu. Misalkan fungsi gelombang dapat dituliskan sebagai

$$\Psi(x,y,z,t) = \psi(x,y,z)\varphi(t) \tag{2.13}$$

Dengan memasukkan persamaan (2.13) ke dalam persamaan (2.12) didapatkan

$$\varphi(t)\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)\right]\psi(x,y,z) = i\hbar\psi(x,y,z)\frac{\partial}{\partial t}\varphi(t) \tag{2.14}$$

Jika kedua ruas persamaan (2.14) dibagi dengan $\psi(x,y,z)\varphi(t)$, maka diperoleh

$$\frac{1}{\psi(x,y,z)}\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)\right]\psi(x,y,z) = i\hbar\frac{1}{\varphi(t)}\frac{\partial}{\partial t}\varphi(t) \tag{2.15}$$

Kedua ruas persamaan (2.15) bergantung pada peubah yang berbeda, yaitu ruas kiri bergantung pada peubah ruang dan ruas kanan bergantung pada peubah waktu. Kedua ruas persamaan tersebut sama dengan suatu tetapan, yaitu energi total sistem E. Oleh karena itu, diperoleh persamaan sebagai berikut

$$i\hbar\frac{1}{\varphi(t)}\frac{\partial}{\partial t}\varphi(t) = E \tag{2.16}$$

dan

$$\frac{1}{\psi(x,y,z)}\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{(x,y,z)}\right]\psi(x,y,z) = E \tag{2.17}$$

Persamaan (2.17) dapat ditulis kembali menjadi persamaan Schrodinger bebas waktu, yaitu

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{(x,y,z)}\right)\psi = E\psi \tag{2.18}$$

dengan $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

(Juwono, 2017: 46)

2.1.3 Aplikasi Persamaan Schrodinger

Persamaan Schrodinger dapat digunakan untuk menyelesaikan beberapa kasus tertentu. Diantaranya, analisis partikel pada tangga potensial, sumur potensial, dan partikel bebas. Purwanto (2006: 70) menjelaskan bahwa jika partikel-partikel bermassa m ditembakkan dari kiri ke kanan dan bergerak di dalam suatu tangga potensial, maka perilaku partikel dapat dibedakan menjadi dua kasus berdasarkan harga E, yaitu $E \leq V_0$ dan $E > V_0$.

Sugiyono (2016: 307) menyatakan sumur potensial sebagai grafik potensial yang membentuk kurva kotak menyerupai sebuah sumur. Terdapat dua jenis sumur potensial, yaitu sumur potensial tak terhingga dan sumur potensial berhingga. Sumur potensial tak berhingga ditandai dengan energi potensial yang tak terhingga, $V_{(x)} = \infty$ pada batas $-\infty \leq x \leq -a$ atau pada batas $a \leq x \leq \infty$. Sedangkan pada batas $-a \leq x \leq a$, energi potensial bernilai nol, $V_{(x)} = 0$. Sumur potensial berhingga ditandai dengan energi potensial sama dengan nol, $V_{(x)} = 0$ pada batas $-\infty \leq x \leq -a$ dan atau pada batas $a \leq x \leq \infty$. Sedangkan pada batas $-a \leq x \leq a$, energi potensial bernilai $-V_0$, $V_{(x)} = -V_0$.

Nilai potensial dari persamaan Schrodinger dapat dipilih secara acak karena tidak akan merubah solusi persamaan Schrodingernya. Nilai potensial dalam partikel bebas yang merupakan bentuk paling sederhana dari persamaan Schrodinger adalah nol, dimana tidak ada gaya interaksi di dalamnya (Sani dan Kadri, 2017: 122). Saat $V = 0$, persamaan Schrodinger bebas waktu menjadi:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad (2.19)$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \qquad (2.20)$$

dengan memperhatikan persamaan umum gelombang:

$$\frac{d^2\psi}{dx^2} = -k^2\psi \qquad (2.21)$$

bilangan gelombang k dapat didefinisikan dengan menggunakan hubungan:

$$k^2 = \frac{2mE}{\hbar^2} \qquad (2.22)$$

bilangan gelombang tersebut dapat mendefinisikan energi dan momentum dari sebuah partikel bebas (Mc Mahon, 2005: 66).

Salah satu solusi dari persamaan (2.21) dengan k^2 selalu positif adalah

$$\psi_{(x)} = A \sin kx + B \cos kx \tag{2.23}$$

Dari persamaan (2.22), diperoleh nilai energi yang diperkenankan adalah

$$E = \frac{\hbar^2 k^2}{2m} \tag{2.24}$$

Energi partikel diperbolehkan memiliki semua nilai artinya tidak terkuantisasi karena pemecahan persamaan (2.24) tidak memberi batasan pada k. Persamaan (2.24) merupakan energi kinetik sebuah partikel dengan momentum $p = \hbar k$ atau setara dengan $p = {h}/{\lambda}$ yang berupa persamaan Schrodinger bagi partikel bebas dan menghasilkan pemecahan yang berkaitan dengan suatu gelombang de Broglie. Dalam menentukan nilai A dan B persamaan (2.23), terdapat beberapa kesulitan karena integral normalisasi tidak dapat dihitung dari $-\infty$ hingga ∞ bagi fungsi gelombang partikel bebas (Krane, 1992: 183).

2.2 Operator

Operator adalah lambang dari suatu perintah matematika untuk dilakukannya operasi matematik tertentu pada objek yang ada di belakangnya. Objek yang dimaksud dapat berupa fungsi, parameter, angka, dan sebagainya. Operator berperan penting dalam mekanika kuantum sebagai representasi observable atau variabel dinamis (Juwono, 2017: 131). Perbedaan utama antara mekanika klasik dan mekanika kuantum terletak pada variabel dinamis (*observable*). Observable mekanika klasik dinyatakan dengan fungsi, sedangkan observable mekanika kuantum dinyatakan dengan operator matematis. Mc Mahon (2005: 46) mendefinisikan operator sebagai instruksi matematis yang dikenakan pada suatu fungsi dan menghasilkan fungsi baru. Operator sederhana dapat diperoleh dari turunan suatu fungsi. Operator sering dituliskan dengan menggunakan huruf kapital dan kadang-kadang dengan tanda sisipan.

2.2.1 Operator Linier

Semua operator dalam mekanika kuantum merupakan operator linier. Operator tersebut bekerja pada fungsi-fungsi dari sistem dan mengubahnya menjadi fungsi gelombang lain.

$$(A\varphi \rightarrow \psi) \tag{2.25}$$

Operator $\hat{O}$ yang dioperasikan pada sebuah fungsi gelombang dapat dituliskan sebagai berikut:

$$\hat{O}\psi(\vec{r},t) = \psi'(\vec{r},t) \tag{2.26}$$

tanda aksen (‘) digunakan untuk membedakan fungsi kedua dengan fungsi asalnya.

Postulat kedua mekanika kuantum menjelaskan bahwa suatu variabel *O* yang dapat diamati atau diukur (variabel dinamis) direpresentasikan oleh operator linier $\hat{O}_{(\hat{x},\hat{p})}$. Sedangkan postulat ketiga menyebutkan bahwa nilai rata-rata dari variabel dinamis tersebut pada keadaan ψ dapat didefinisikan dengan nilai harap:

$$\bar{O} = \langle \hat{O} \rangle = \int \psi^* \hat{O} \psi \, dV \tag{2.27}$$

Operator linier $\hat{O}$ merupakan operator dengan sifat:

$$\hat{O}(a\psi + b\varphi) = a\hat{O}\psi + b\hat{O}\varphi \tag{2.28}$$

dengan ψ,φ merupakan fungsi dan a,b suatu konstanta (Mustamin, 2017: 6).

Sani dan Kadri (2017: 101) menyebutkan sifat-sifat operator linier selain persamaan (2.28) sebagai berikut:

$$\hat{O}(a\psi) = a\hat{O}\psi \tag{2.29}$$

$$\hat{O}(\psi + \varphi) = \hat{O}\psi + \hat{O}\varphi \tag{2.30}$$

$$(\hat{O}_1 + \hat{O}_2)\psi = \hat{O}_1\psi + \hat{O}_2\psi \tag{2.31}$$

2.2.2 Operator Momentum Sudut

Operator utama dalam mekanika kuantum ada dua, yaitu operator momentum $\hat{p}$ dan operator energi atau Hamiltonian $\hat{H}$. Operator momentum linier dinyatakan sebagai:

$$\hat{p} = -i\hbar\nabla \tag{2.32}$$

dan setiap komponen momentum linier dapat dituliskan:

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x};\ \hat{p}_y = -i\hbar\frac{\partial}{\partial y};\ \hat{p}_z = -i\hbar\frac{\partial}{\partial z} \tag{2.33}$$

Operator momentum linier dan operator posisi digunakan untuk mendefinisikan operator momentum sudut (Mc Mahon, 2005: 259-260). Dengan kata lain operator momentum sudut dibentuk dari operator momentum linier dan operator posisi. Fisika klasik mendefinisikan momentum sudut sebagai:

$$L = \vec{r} \times \vec{p} \tag{2.34}$$

dimana $\vec{r}$ adalah vektor perpindahan dan $\vec{p}$ adalah momentum linier. Komponen momentum sudut dalam koordinat kartesius dapat diperoleh dari perhitungan *cross product* secara eksplisit:

$$L = \vec{r} \times \vec{p}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y)\hat{\imath} + (zp_x - xp_z)\hat{\jmath} + (xp_y - yp_x)\hat{k} \tag{2.35}$$

Selanjutnya diperoleh:

$$L_x = yp_z - zp_y \tag{2.36}$$

$$L_y = zp_x - xp_z \tag{2.37}$$

$$L_z = xp_y - yp_x \tag{2.38}$$

dengan menggunakan representasi koordinat dari operator momentum, komponen operator momentum sudut dalam mekanika kuantum dapat dituliskan dengan menggunakan subtitusi biasa untuk masing-masing komponen momentum linier.

$$\hat{L}_x = y\left(-i\hbar\frac{\partial}{\partial z}\right) - z\left(-i\hbar\frac{\partial}{\partial y}\right) = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{2.39}$$

$$\hat{L}_y = z\left(-i\hbar\frac{\partial}{\partial x}\right) - x\left(-i\hbar\frac{\partial}{\partial z}\right) = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{2.40}$$

$$\hat{L}_z = x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right) = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{2.41}$$

Operator momentum sudut kuadrat dapat dituliskan:

$$\hat{L}^2 = \hat{L}_x{}^2 + \hat{L}_y{}^2 + \hat{L}_z{}^2 \tag{2.42}$$

dengan mendefinisikan $L^2 = \vec{L}.\vec{L}$

(Mustamin, 2017: 25-26)

2.2.3 Operator Hamiltonian

Operator Hamiltonian merupakan operator energi yang memiliki nilai konstan dan salah satu operator utama dalam perumusan matematis mekanika kuantum. Bentuk operator Hamiltonian dalam persamaan Schrodinger (2.18) dapat ditulis secara sederhana sebagai

$$\hat{H}\psi = E\psi \tag{2.43}$$

dimana $\hat{H}$ adalah operator Hamiltonian yang dapat ditulis sebagai berikut

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V \tag{2.44}$$

Hayward (2002: 115) menjelaskan bahwa fungsi Hamilton merupakan bentuk alternatif persamaan gerak Newton dengan menggunakan fungsi energi total. Dalam sistem klasik, fungsi Hamilton hanya berupa energi total sistem yang dinyatakan dalam koordinat partikel dan momentum konjugasi. Energi kinetik dari suatu partikel dengan massa *m* dapat ditulis sebagai:

$$T = \frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right) \tag{2.45}$$

dimana p_x, p_y, dan p_z masing-masing adalah komponen dari momentum linier dalam arah sumbu *x*, *y*, dan *z*. Energi potensial $V_{(x,y,z)}$ merupakan beberapa fungsi yang tidak ditentukan oleh koordinat. Fungsi Hamiltonian dapat didefinisikan secara sederhana sebagai jumlah dari energi kinetik dan energi potensial.

Secara keseluruhan fungsi Hamilton *H* dapat diubah menjadi operator Hamiltonian $\hat{H}$ dengan menerapkan beberapa pola sederhana sebagai berikut:

a. Menggunakan persamaan klasik *H* untuk semua partikel dan masing-masing komponen momentum linier dalam koordinat Kartesius.
b. Komponen momentum linier dapat diganti dengan operator momentum linier $\hat{p} = -i\hbar\frac{\partial}{\partial r}$
c. Koordinat posisi tidak perlu diubah.

Ketiga pola tersebut jika diterapkan, maka bentuk fungsi Hamilton akan menjadi persamaan Schrodinger yang juga berupa operator Hamiltonian (Hayward, 2002: 116). Dengan demikian operator Hamiltonian dapat ditulis sebagai berikut

$$\hat{H} = T + V = \frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V_{(x,y,z)} \tag{2.46}$$

2.3 Komutator

Misalkan $\hat{A}$ dan $\hat{B}$ adalah operator, maka komutator dari $\hat{A}$ dan $\hat{B}$ dapat dituliskan sebagai:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{2.47}$$

Jika $[\hat{A}, \hat{B}] = 0$, maka operator $\hat{A}$ dan $\hat{B}$ dikatakan komut. Secara umum dalam mekanika kuantum $\hat{A}\hat{B} \neq \hat{B}\hat{A}$ sehingga $[\hat{A}, \hat{B}] \neq 0$. Sifat-sifat dari komutator (Mc Mahon, 2005: 167), antara lain:

a. $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ (2.48)

b. $[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$ (2.49)

c. $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$ (2.50)

d. Jika $\hat{x}$ adalah operator posisi dan $\hat{p}$ adalah operator momentum, maka $[\hat{x}, \hat{p}] = i\hbar$, $[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$.

Transtum dan Jean (2005) menyebutkan sifat dari komutator selain keempat sifat di atas, sebagai berikut:

a. $[\hat{A}, \alpha\hat{B} + \hat{C}] = \alpha[\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$ (2.51)

b. $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ (2.52)

2.3.1 Komutator Operator Momentum Sudut

Mustamin (2017: 20) menyebutkan hubungan komutasi dari operator posisi dan momentum linier untuk setiap komponennya dapat dituliskan sebagai berikut:

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar \tag{2.53}$$

$$[\hat{x}, \hat{p}_y] = [\hat{x}, \hat{p}_z] = [\hat{y}, \hat{p}_x] = [\hat{y}, \hat{p}_z] = [\hat{z}, \hat{p}_x] = [\hat{z}, \hat{p}_y] = 0 \tag{2.54}$$

Sedangkan hubungan komutasi dari operator momentum sudut untuk setiap komponennya dapat dituliskan sebagai berikut:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z; \ [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x; \ [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \tag{2.55}$$

Sani dan Kadri (2017: 192) menyebutkan hubungan komutasi operator momentum sudut kuadrat dengan setiap komponen momentum sudut dapat dituliskan sebagai berikut:

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0 \tag{2.56}$$

dengan

$$\hat{L}^2 = 2\hbar^2 r.\nabla = 2\hbar^2 \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right) \tag{2.57}$$

2.3.2 Hubungan Komutasi Operator Hamiltonian dengan Tetapan Gerak

Dengan mengambil operator Hamiltonian yang berupa diferensial terhadap waktu yang serupa dengan persamaan (2.12), maka

$$\frac{\partial}{\partial t}\langle \hat{O} \rangle = \frac{\partial}{\partial t}\int \psi^* \, \hat{O}\psi \, dt \tag{2.58}$$

dimana $\hat{O}$ adalah suatu operator linier.

Dengan menggunakan persamaan:

$$(\hat{H}^*)\psi^* = -i\hbar \left(\frac{\partial}{\partial t}\right) \psi^* \tag{2.59}$$

dan memperhatikan sifat Hermitiannya operator Hamiltonian, akan terjabarkan rumus:

$$\frac{\partial \langle \hat{O} \rangle}{\partial t} = \frac{i}{\hbar}\langle [\hat{H}, \hat{O}] \rangle + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle \tag{2.60}$$

dengan menggunakan analogi kaidah Poison dalam mekanika klasik, diperoleh persamaan sebagai berikut

$$\frac{d\hat{O}}{dt} = \{\hat{O}, \hat{H}\} + \frac{\partial \hat{O}}{\partial t} \tag{2.61}$$

Apabila $\hat{O}$ tidak merupakan fungsi eksplisit dari waktu t, yaitu $\frac{d\hat{O}}{dt} = 0$, maka berlaku hubungan

$$\frac{\partial \langle \hat{O} \rangle}{\partial t} = \frac{i}{\hbar}\langle [\hat{H}, \hat{O}] \rangle \tag{2.62}$$

Jika $\hat{O}$ adalah tetapan terhadap waktu, maka kedua ruas persamaan (2.62) adalah nol. Jadi setiap besaran fisika yang konstan terhadap waktu, operatornya akan bersifat komut dengan operator Hamiltonian. Dengan kata lain setiap besaran yang komut terhadap operator Hamiltonian merupakan tetapan terhadap waktu. Untuk $\hat{O}$ adalah $\hat{x}$ atau momentum $\hat{p}$, diperoleh hubungan persamaan:

$$\frac{\partial\langle\hat{x}\rangle}{\partial t} = \frac{i}{\hbar}\langle[\hat{H},\hat{x}]\rangle = \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m} + V, \hat{x}\right]\right\rangle \tag{2.63}$$

V komut terhadap $\hat{x}$, yaitu $[V,\hat{x}] = 0$, sedangkan

$$[\hat{p}^2,\hat{x}] = \hat{p}[\hat{p},\hat{x}] + [\hat{p},\hat{x}]\hat{p} = \left(2\,{}^{\hbar}/_{i}\right)[p] \tag{2.64}$$

sehingga diperoleh persamaan

$$\frac{\partial\langle\hat{x}\rangle}{\partial t} = \left\langle\frac{p}{m}\right\rangle \tag{2.65}$$

Persamaan (2.65) sesuai dengan yang diharapkan yaitu merupakan persamaan gerak dalam mekanika.

Selanjutnya persamaan (2.60) juga dapat ditulis:

$$\frac{\partial\langle\hat{p}\rangle}{\partial t} = \frac{i}{\hbar}\langle[\hat{H},\hat{p}]\rangle = \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m} + V, \hat{p}\right]\right\rangle = -\frac{i}{\hbar}\langle[\hat{p},V]\rangle \tag{2.66}$$

dari persamaan berikut:

$$[\hat{p}]V\psi - V[\hat{p}]\psi = -i\hbar\left(\frac{dV}{dx}\right)\psi - i\hbar V\left(\frac{d\psi}{dx}\right) + i\hbar V\left(\frac{d\psi}{dx}\right) = -i\hbar\left(\frac{dV}{dx}\right)\psi \tag{2.67}$$

diperoleh persamaan:

$$[\hat{p},V] = -i\hbar\left(\frac{dV}{dx}\right)\psi \tag{2.68}$$

Sehingga

$$\frac{d\langle\hat{p}\rangle}{dt} = -\frac{\partial V}{\partial x} = \langle F_x\rangle \tag{2.69}$$

Persamaan (2.69) sesuai dengan persamaan gerak dalam mekanika (Soedojo, 2001: 101-102).

BAB 3. METODE PENELITIAN

3.1 Jenis Penelitian

Jenis penelitian ini adalah penelitian dasar (*basic research*), dengan hasil pengembangan teori yang ada dalam tinjauan pustaka. Penelitian ini dilakukan untuk mengetahui hubungan komutasi operator momentum sudut dengan posisi dan Hamiltonian partikel bebas.

3.2 Tempat dan Waktu Penelitian

Penelitian ini dilakukan di Laboratorium Fisika Dasar, Program Studi Pendidikan Fisika pada Semester Ganjil bulan Oktober sampai bulan November 2018.

3.3 Objek Penelitian

Objek penelitian ini adalah materi fisika modern dan fisika kuantum yang berkaitan dengan persamaan Schrodinger tak bergantung waktu, komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas.

3.4 Definisi Operasional

Agar tidak terjadi kesalahan dalam mengartikan istilah-istilah dalam penelitian ini, diberikan definisi operasional mengenai variabel penelitian sebagai berikut:

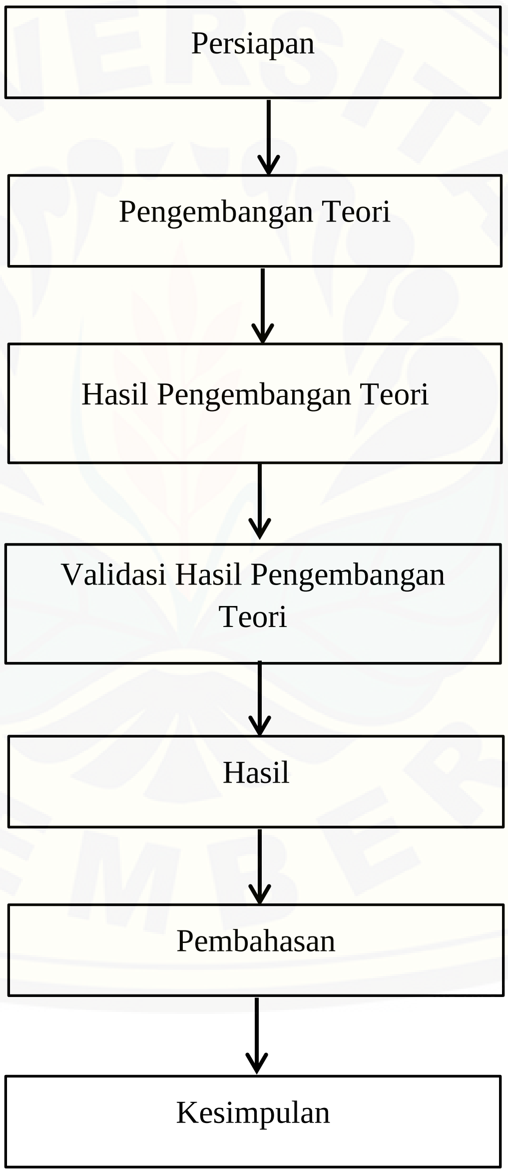
a. Komutator Operator Momentum Sudut

Komutator adalah stuktur matematika atau metode dalam mekanika kuantum untuk mengetahui hubungan komut atau tidak komut suatu variabel. Operator dalam mekanika kuantum mewakili hal-hal yang teramati. Komutator operator momentum sudut adalah metode untuk mengetahui hubungan komut atau tidak komut operator momentum sudut dengan operator atau variabel lain. Komutator akan bersifat komut jika bernilai nol, berarti operator-operator tersebut dapat diukur secara serempak. Sedangkan jika komutator tidak bernilai nol maka bersifat tidak komut, berarti operator-operator tersebut tidak dapat diukur secara serempak.

b. Hamiltonian Partikel Bebas

Fungsi Hamiltonian merupakan fungsi energi total yaitu jumlah energi kinetik dan energi potensial. Dalam partikel bebas fungsi Hamiltonian hanya terdiri dari fungsi energi kinetik partikel karena tidak terdapat gaya interaksi yang menyebabkan adanya energi potensial.

3.5 Langkah Penelitian



3.5.1 Persiapan

Tahap ini adalah tahap untuk mempersiapkan bahan-bahan yang akan dijadikan sumber referensi mengenai penelitian dengan mengumpulkan buku yang relevan, internet, dan jurnal berskala nasional ataupun internasional terkait dengan penelitian ini.

3.5.2 Pengembangan Teori

Tahap ini adalah tahap untuk mengembangkan teori yang telah ada di berbagai sumber referensi mengenai komutator operator momentum sudut. Teori yang dikembangkan adalah komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas. Langkah awal adalah menentukan operator momentum sudut dalam koordinat kartesian. Langkah selanjutnya yaitu menetukan komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas.

3.5.3 Hasil Pengembangan Teori

Dari pengembangan teori yang telah dilakukan, dapat diperoleh persamaan matematis komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas. Hasil pengembangan teori dapat digunakan untuk menentukan hubungan komutasi operator momentum sudut dengan posisi dan Hamiltonian partikel bebas.

3.5.4 Validasi Hasil Pengembangan Teori

Tahap ini adalah tahap untuk membandingkan persamaan matematis operator momentum sudut dan persamaan matematis komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas antara hasil pengembangan teori dengan hasil penelitian yang diperoleh dari buku, internet, atau jurnal.

3.5.5 Hasil

Tahap ini adalah tahap perhitungan secara manual untuk menentukan komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas.

3.5.6 Pembahasan

Hasil dari pengambilan data akan dibahas secara rinci dalam kaitannya mengenai komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas.

3.5.7 Kesimpulan

Hasil dari pembahasan kemudian disimpulkan untuk menjawab perumusan masalah dalam penelitian.

3.6 Teknik Analisis Data

3.6.1 Analisis Data

a. Operator Momentum Sudut

1) Operator $\hat{L}_x$

$$\hat{L}_x = y\left(-i\hbar\frac{\partial}{\partial z}\right) - z\left(-i\hbar\frac{\partial}{\partial y}\right) = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

2) Operator $\hat{L}_y$

$$\hat{L}_y = z\left(-i\hbar\frac{\partial}{\partial x}\right) - x\left(-i\hbar\frac{\partial}{\partial z}\right) = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

3) Operator $\hat{L}_z$

$$\hat{L}_z = x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right) = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

4) Operator $\hat{L}^2$

$$\hat{L}^2 = \hat{L}_x{}^2 + \hat{L}_y{}^2 + \hat{L}_z{}^2$$

b. Hamiltonian Partikel Bebas

$$\hat{H} = \frac{1}{2m}\left(p_x + p_y + p_z\right)$$

c. Komutator Operator Momentum Linier Dengan Posisi

$$[\hat{x},\hat{p}_x] = [\hat{y},\hat{p}_y] = [\hat{z},\hat{p}_z] = i\hbar$$

$$[\hat{x},\hat{p}_y] = [\hat{x},\hat{p}_z] = [\hat{y},\hat{p}_x] = [\hat{y},\hat{p}_z] = [\hat{z},\hat{p}_x] = [\hat{z},\hat{p}_y] = 0$$

3.6.2 Teknik Penyajian Data

Teknik penyajian menampilkan tabel data untuk menentukan komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas.

Tabel 3.1 Contoh tabel hasil komutator operator momentum sudut terhadap posisi

No.	Komutator	Hasil
1	$[\hat{L}_x, x]$	
2	$[\hat{L}_y, x]$	
3	$[\hat{L}_z, x]$	
4	$[\hat{L}_x, y]$	
5	$[\hat{L}_y, y]$	
6	$[\hat{L}_z, y]$	
7	$[\hat{L}_x, z]$	
8	$[\hat{L}_y, z]$	
9	$[\hat{L}_z, z]$	

Tabel 3.2 Contoh tabel hasil komutator operator momentum sudut kuadrat terhadap posisi

No.	Komutator	Hasil
1	$[\hat{L}_x{}^2, x]$	
2	$[\hat{L}_y{}^2, x]$	
3	$[\hat{L}_z{}^2, x]$	
4	$[\hat{L}^2, x]$	
5	$[\hat{L}_x{}^2, y]$	
6	$[\hat{L}_y{}^2, y]$	
7	$[\hat{L}_z{}^2, y]$	
8	$[\hat{L}^2, y]$	
9	$[\hat{L}_x{}^2, z]$	
10	$[\hat{L}_y{}^2, z]$	
11	$[\hat{L}_z{}^2, z]$	
12	$[\hat{L}^2, z]$	

Tabel 3.3 Contoh tabel hasil komutator operator momentum sudut terhadap posisi kuadrat

No.	Komutator	Hasil
1	$\left[\hat{L}_x, x^2\right]$	
2	$\left[\hat{L}_y, x^2\right]$	
3	$\left[\hat{L}_z, x^2\right]$	
4	$\left[\hat{L}_x, y^2\right]$	
5	$\left[\hat{L}_y, y^2\right]$	
6	$\left[\hat{L}_z, y^2\right]$	
7	$\left[\hat{L}_x, z^2\right]$	
8	$\left[\hat{L}_y, z^2\right]$	
9	$\left[\hat{L}_z, z^2\right]$	

Tabel 3.4 Contoh tabel hasil komutator operator momentum sudut kuadrat terhadap posisi kuadrat

No.	Komutator	Hasil
1	$\left[\hat{L}_x{}^2, x^2\right]$	
2	$\left[\hat{L}_y{}^2, x^2\right]$	
3	$\left[\hat{L}_z{}^2, x^2\right]$	
4	$\left[\hat{L}^2, x^2\right]$	
5	$\left[\hat{L}_x{}^2, y^2\right]$	
6	$\left[\hat{L}_y{}^2, y^2\right]$	
7	$\left[\hat{L}_z{}^2, y^2\right]$	
8	$\left[\hat{L}^2, y^2\right]$	
9	$\left[\hat{L}_x{}^2, z^2\right]$	
10	$\left[\hat{L}_y{}^2, z^2\right]$	
11	$\left[\hat{L}_z{}^2, z^2\right]$	
12	$\left[\hat{L}^2, z^2\right]$	

Tabel 3.5 Contoh tabel hasil komutator operator momentum sudut terhadap Hamiltonian partikel bebas

Komutator	Hasil
$\left[\hat{L}_x, \hat{H}\right]$	
$\left[\hat{L}_y, \hat{H}\right]$	
$\left[\hat{L}_z, \hat{H}\right]$	

$\left[\hat{L}_x{}^2, \hat{H}\right]$	
$\left[\hat{L}_y{}^2, \hat{H}\right]$	
$\left[\hat{L}_z{}^2, \hat{H}\right]$	
$\left[\hat{L}^2, \hat{H}\right]$	

3.7 Validasi Hasil Pengembangan Teori

Validasi dalam penelitian ini menggunakan data dari buku yang relevan mengenai operator momentum sudut dalam koordinat Kartesian, hasil beberapa komutator operator momentum sudut terhadap posisi, dan hasil beberapa komutator operator momentum sudut terhadap Hamiltonian Partikel bebas.

3.7.1 Validasi Operator Momentum Sudut

Berikut ini merupakan tabel perbandingan operator momentum sudut dalam koordinat Kartesian dari buku teks (Tabel 3.6) dan perhitungan matematis manual (Tabel 3.7).

Tabel 3.6 Operator momentum sudut dalam koordinat Kartesian

No.	Operator Momentum Sudut	Operator Momentum Sudut dalam Koordinat Kartesian
1	$\hat{L}_x$	$-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$
2	$\hat{L}_y$	$-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$
3	$\hat{L}_z$	$-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$
4	$\hat{L}^2$	$2\hbar^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$

(Sani dan Kadri, 2017: 192)

Tabel 3.7 Validasi operator momentum sudut dalam koordinat Kartesian

No.	Operator Momentum Sudut	Operator Momentum Sudut dalam Koordinat Kartesian
1	$\hat{L}_x$	$-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$
2	$\hat{L}_y$	$-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$

3	$\hat{L}_z$	$-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$
4	$\hat{L}^2$	$2\hbar^2\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)$

3.7.2 Validasi Beberapa Komutator Operator Momentum Sudut Terhadap Posisi

Berikut ini merupakan tabel perbandingan beberapa komutator operator momentum sudut terhadap posisi dari buku teks (Tabel 3.8) dan perhitungan matematis manual (Tabel 3.9).

Tabel 3.8 Beberapa komutator operator momentum sudut terhadap posisi dalam koordinat Kartesian

No.	Komutator Operator Momentum Sudut Terhadap Posisi	Komutator Operator Momentum Sudut Terhadap Posisi dalam Koordinat Kartesian
1	$[\hat{L}_x, x]$	0
2	$[\hat{L}_y, x]$	$-i\hbar z$
3	$[\hat{L}_z, x]$	$i\hbar y$
4	$[\hat{L}_x, y]$	$i\hbar z$
5	$[\hat{L}_y, y]$	0
6	$[\hat{L}_z, y]$	$-i\hbar x$
7	$[\hat{L}_x, z]$	$-i\hbar y$
8	$[\hat{L}_y, z]$	$i\hbar x$
9	$[\hat{L}_z, z]$	0

(Mc Mahon, 2005: 261)

Tabel 3.9 Validasi beberapa komutator operator momentum sudut terhadap posisi dalam koordinat Kartesian

No.	Komutator Operator Momentum Sudut Terhadap Posisi	Komutator Operator Momentum Sudut Terhadap Posisi dalam Koordinat Kartesian
1	$[\hat{L}_x, x]$	0
2	$[\hat{L}_y, x]$	$-i\hbar z$
3	$[\hat{L}_z, x]$	$i\hbar y$
4	$[\hat{L}_x, y]$	$i\hbar z$
5	$[\hat{L}_y, y]$	0
6	$[\hat{L}_z, y]$	$-i\hbar x$
7	$[\hat{L}_x, z]$	$-i\hbar y$
8	$[\hat{L}_y, z]$	$i\hbar x$

9	$[\hat{L}_z, z]$	0

Dari Tabel 3.8 dan tabel 3.9 dapat dilihat bahwa komutator operator momentum sudut terhadap posisi akan bernilai nol (komut) apabila terdapat komponen momentum sudut yang sama dengan posisi yang dijadikan pasangan komutasi.

3.7.3 Validasi Komutator Operator Momentum Sudut Terhadap Hamiltonian

Berikut ini merupakan tabel perbandingan beberapa komutator operator momentum sudut terhadap Hamiltonian dari buku teks (Tabel 3.10) dan perhitungan matematis manual (Tabel 3.11).

Tabel 3.10 Beberapa komutator operator momentum sudut terhadap Hamiltonian

No.	Komutator Operator Momentum Sudut Terhadap Hamiltonian	Komutator Operator Momentum Sudut Terhadap Hamiltonian dalam Koordinat Kartesian
1	$[\hat{L}_x, \hat{H}]$	0
2	$[\hat{L}_y, \hat{H}]$	0
3	$[\hat{L}_z, \hat{H}]$	0

(Gasiorowicz, 1996: 171)

Tabel 3.11 Validasi beberapa komutator operator momentum sudut terhadap Hamiltonian

No.	Komutator Operator Momentum Sudut Terhadap Hamiltonian	Komutator Operator Momentum Sudut Terhadap Hamiltonian dalam Koordinat Kartesian
1	$[\hat{L}_x, \hat{H}]$	0
2	$[\hat{L}_y, \hat{H}]$	0
3	$[\hat{L}_z, \hat{H}]$	0

Dari Tabel 3.10 dan 3.11 bisa dimaknai bahwa ketiga komponen momentum sudut merupakan tetapan gerak karena komut dengan operator Hamiltonian (komutasinya bernilai nol).

BAB 5. PENUTUP

5.1 Kesimpulan

Pada penelitian ini telah dikaji komutator operator momentum sudut dalam koordinat bola terhadap fungsi gelombang atom hidrogen. Berdasarkan data hasil penelitian dapat disimpulkan:

a. Komutator operator momentum sudut terhadap posisi dengan orde $n = 1$ dan $n = 2$ bernilai nol atau komut jika dalam komponen momentum sudut tersebut tidak terdapat posisi yang dijadikan pasangan komutasinya. Kekomutan tersebut dapat menunjukkan bahwa momentum sudut dan posisi yang berpasangan dapat ditentukan secara serempak atau simultan.
b. Komutator operator momentum sudut terhadap Hamiltonian partikel bebas bernilai nol atau komut jika dikomutasikan dengan momentum sudut $\hat{L}_x$; $\hat{L}_y$ dan $\hat{L}_z$. Kekomutan dengan Hamiltonian tersebut menunjukkan bahwa ketiga momentum sudut tersebut termasuk dalam tetapan gerak. Sedangkan saat operator momentum sudut maupun momentum sudut total dikuadratkan lalu dikomutasikan dengan Hamiltonian akan menghasilkan nilai negatif namun tidak berkaitan dengan arah vektor. Sehingga momentum sudut kuadrat dan momentum sudut total kuadrat bukan termasuk tetapan gerak.

5.2 Saran

Berdasarkan kesimpulan, penulis memberikan saran agar diadakan penelitian lanjut dengan mengubah operator posisi x dan operator momentum sudut L orde $n > 2$; operator Hamiltonian orde $n > 1$; atau dengan menambahkan fungsi gelombang pada komutator yang sudah ada dengan mentransformasikan ke koordinat bola.

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LAMPIRAN A. MATRIK PENELITIAN

JUDUL	TUJUAN PENELITIAN	VARIABEL	DATA DAN TEKNIK PENGAMBILAN DATA	METODE PENELITIAN
Komutator Operator Momentum Sudut Terhadap Posisi dan Hamiltonian Partikel Bebas	– Menentukan komutator operator momentum sudut terhadap posisi – Menentukan komutator operator momentum sudut terhadap Hamiltonian partikel bebas	– Variabel bebas: operator momentum sudut – Variabel terikat: komutator operator momentum sudut terhadap posisi dan Hamiltonian partikel bebas	– Data diperoleh dari: 1. Buku 2. Jurnal 3. Internet – Teknik pengambilan data: Secara analitik dengan perhitungan matematis manual	1. Jenis penelitian: *basic research* 2. Analisis: Penelitian ini menggunakan persamaan teori sebagai berikut: a. Persamaan Schrodinger partikel bebas b. Operator momentum sudut c. Hamiltonian Partikel bebas d. Sifat komutator

LAMPIRAN B. OPERATOR MOMENTUM SUDUT KUADRAT

1. $\hat{L}_x{}^2 = (yp_z - zp_y)(yp_z - zp_y)$

$= yp_zyp_z - zp_yyp_z - yp_zzp_y + zp_yzp_y$

$= y\left(-i\hbar\frac{d}{dz}\right)y\left(-i\hbar\frac{d}{dz}\right) - z\left(-i\hbar\frac{d}{dy}\right)y\left(-i\hbar\frac{d}{dz}\right) - y\left(-i\hbar\frac{d}{dz}\right)z\left(-i\hbar\frac{d}{dy}\right)$

$+z\left(-i\hbar\frac{d}{dy}\right)z\left(-i\hbar\frac{d}{dy}\right)$

$= 0 + i\hbar z\left(-i\hbar\frac{d}{dz}\right) + i\hbar y\left(-i\hbar\frac{d}{dy}\right) + 0 = i\hbar zp_z + i\hbar yp_y$

$= \hbar^2\left(z\frac{d}{dz} + y\frac{d}{dy}\right)$

2. $\hat{L}_y{}^2 = (zp_x - xp_z)(zp_x - xp_z)$

$= zp_xzp_x - xp_zzp_x - zp_xxp_z + xp_zxp_z$

$= z\left(-i\hbar\frac{d}{dx}\right)z\left(-i\hbar\frac{d}{dx}\right) - x\left(-i\hbar\frac{d}{dz}\right)z\left(-i\hbar\frac{d}{dx}\right) - z\left(-i\hbar\frac{d}{dx}\right)x\left(-i\hbar\frac{d}{dz}\right)$

$+x\left(-i\hbar\frac{d}{dz}\right)x\left(-i\hbar\frac{d}{dz}\right)$

$= 0 + i\hbar x\left(-i\hbar\frac{d}{dx}\right) + i\hbar z\left(-i\hbar\frac{d}{dz}\right) + 0 = i\hbar xp_x + i\hbar zp_z$

$= \hbar^2\left(x\frac{d}{dx} + z\frac{d}{dz}\right)$

3. $\hat{L}_z{}^2 = (xp_y - yp_x)(xp_y - yp_x)$

$= xp_yxp_y - yp_xxp_y - xp_yyp_x + yp_xyp_x$

$= x\left(-i\hbar\frac{d}{dy}\right)x\left(-i\hbar\frac{d}{dy}\right) - y\left(-i\hbar\frac{d}{dx}\right)x\left(-i\hbar\frac{d}{dy}\right) -$

$x\left(-i\hbar\frac{d}{dy}\right)y\left(-i\hbar\frac{d}{dx}\right) + y\left(-i\hbar\frac{d}{dx}\right)y\left(-i\hbar\frac{d}{dx}\right)$

$= 0 + i\hbar y\left(-i\hbar\frac{d}{dy}\right) + i\hbar x\left(-i\hbar\frac{d}{dx}\right) + 0 = i\hbar yp_y + i\hbar xp_x$

$= \hbar^2(y\frac{d}{dy} + x\frac{d}{dx})$

4. $\hat{L}^2 = \hat{L}_x{}^2 + \hat{L}_y{}^2 + \hat{L}_z{}^2$

$= i\hbar zp_z + i\hbar yp_y + i\hbar xp_x + i\hbar zp_z + i\hbar yp_y + i\hbar xp_x$

$= 2i\hbar xp_x + 2i\hbar yp_y + 2i\hbar zp_z$

$= 2i\hbar(xp_x + yp_y + zp_z)$

$= 2\hbar^2\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\right)$

LAMPIRAN C. KOMUTATOR OPERATOR MOMENTUM SUDUT TERHADAP POSISI

1. $[\boldsymbol{L_x}, \boldsymbol{x}]$

Cara I

$$[L_x, x] = [(yp_z - zp_y), x]$$
$$= [yp_z, x] - [zp_y, x]$$
$$= (y[p_z, x] + [y, x]p_z) - (z[p_y, x] + [z, x]p_y)$$
$$= (y(0) + (0)p_z) - (z(0) + (0)p_y)$$
$$= 0$$

Cara II

$$[L_x, x]\psi = (L_x x - xL_x)\psi$$
$$= \Big((yp_z - zp_y)x - x(yp_z - zp_y)\Big)\psi$$
$$= (yp_z - zp_y)x\psi - x(yp_z - zp_y)\psi$$
$$= yp_z x\psi - zp_y x\psi - xyp_z\psi + xzp_y\psi$$
$$= y\left(-i\hbar\frac{d}{dz}\right)x\psi - z\left(-i\hbar\frac{d}{dy}\right)x\psi - xy\left(-i\hbar\frac{d}{dz}\right)\psi + xz\left(-i\hbar\frac{d}{dy}\right)\psi$$
$$= -i\hbar y\frac{d}{dz}(x\psi) + i\hbar z\frac{d}{dy}(x\psi) + i\hbar xy\frac{d}{dz}(\psi) - i\hbar xz\frac{d}{dy}(\psi)$$
$$= -i\hbar y\left(0 + x\frac{d\psi}{dz}\right) + i\hbar z\left(0 + x\frac{d\psi}{dy}\right) + i\hbar xy\frac{d\psi}{dz} - i\hbar xz\frac{d\psi}{dy}$$
$$= -i\hbar xy\frac{d\psi}{dz} + i\hbar xz\frac{d\psi}{dy} + i\hbar xy\frac{d\psi}{dz} - i\hbar xz\frac{d\psi}{dy}$$
$$= 0$$

$$[L_x, x] = 0$$

2. $[\boldsymbol{L_y}, \boldsymbol{x}]$

Cara I

$$[L_y, x] = [(zp_x - xp_z), x]$$
$$= [zp_x, x] - [xp_z, x]$$
$$= (z[p_x, x] + [z, x]p_x) - (x[p_z, x] + [x, x]p_z)$$
$$= (z(-i\hbar) + (0)p_x) - (x(0) + (0)p_z)$$
$$= -i\hbar z$$

Cara II

$$[L_y, x]\psi = (L_y x - xL_y)\psi$$
$$= L_y x\psi - xL_y\psi$$
$$= (zp_x - xp_z)x\psi - x(zp_x - xp_z)\psi$$
$$= zp_x x\psi - xp_z x\psi - xzp_x\psi + x^2 p_z\psi$$
$$= z\left(-i\hbar\frac{d}{dx}\right)x\psi - x\left(-i\hbar\frac{d}{dz}\right)x\psi - xz\left(-i\hbar\frac{d}{dx}\right)\psi + x^2\left(-i\hbar\frac{d}{dz}\right)\psi$$
$$= -i\hbar z\frac{d}{dx}(x\psi) + i\hbar x\frac{d}{dz}(x\psi) + i\hbar xz\frac{d}{dx}(\psi) - i\hbar x^2\frac{d}{dz}(\psi)$$
$$= -i\hbar z\left(\psi + x\frac{d\psi}{dx}\right) + i\hbar x\left(0 + x\frac{d\psi}{dz}\right) + i\hbar xz\frac{d\psi}{dx} - i\hbar x^2\frac{d\psi}{dz}$$
$$= -i\hbar z\psi - i\hbar xz\frac{d\psi}{dx} + i\hbar x^2\frac{d\psi}{dz} + i\hbar xz\frac{d\psi}{dx} - i\hbar x^2\frac{d\psi}{dz}$$
$$= -i\hbar z\psi$$
$$[L_y, x] = -i\hbar z$$

3. $[L_z, x]$

Cara I

$$[L_z, x] = [(xp_y - yp_x), x]$$
$$= [xp_y, x] - [yp_x, x]$$
$$= (y[p_y, x] + [x, x]p_y) - (y[p_x, x] + [y, x]p_x)$$
$$= (y(0) + (0)p_y) - (y(-i\hbar) + (0)p_x)$$
$$= i\hbar y$$

Cara II

$$[L_z, x]\psi = (L_z x - xL_z)\psi$$
$$= L_z x\psi - xL_z\psi$$
$$= (xp_y - yp_x)x\psi - x(xp_y - yp_x)\psi$$
$$= xp_y x\psi - yp_x x\psi - x^2 p_y\psi + xyp_x\psi$$
$$= x\left(-i\hbar\frac{d}{dy}\right)x\psi - y\left(-i\hbar\frac{d}{dx}\right)x\psi - x^2\left(-i\hbar\frac{d}{dy}\right)\psi$$
$$+xy\left(-i\hbar\frac{d}{dx}\right)\psi$$
$$= -i\hbar x\frac{d}{dy}(x\psi) + i\hbar y\frac{d}{dx}(x\psi) + i\hbar x^2\frac{d}{dy}(\psi) - i\hbar xy\frac{d}{dx}(\psi)$$
$$= -i\hbar x\left(0 + x\frac{d\psi}{dy}\right) + i\hbar y\left(\psi + x\frac{d\psi}{dx}\right) + i\hbar x^2\frac{d\psi}{dy} - i\hbar xy\frac{d\psi}{dx}$$

$$= -i\hbar x^2 \frac{d\psi}{dy} + i\hbar y\psi + i\hbar xy \frac{d\psi}{dx} + i\hbar x^2 \frac{d\psi}{dy} - i\hbar xy \frac{d\psi}{dx}$$

$$= i\hbar y\psi$$

$$[L_z, x] = i\hbar y$$

4. $[\boldsymbol{L_x}, \boldsymbol{y}]$

Cara I

$$[L_x, y] = [(yp_z - zp_y), y]$$

$$= [yp_z, y] - [zp_y, y]$$

$$= (y[p_z, y] + [y, y]p_z) - (z[p_y, y] + [z, y]p_y)$$

$$= (y(0) + (0)p_z) - (z(-i\hbar) + (0)p_y)$$

$$= i\hbar z$$

Cara II

$$[L_x, y]\psi = (L_x y - yL_x)\psi$$

$$= L_x y\psi - yL_x\psi$$

$$= (yp_z - zp_y)y\psi - y(yp_z - zp_y)\psi$$

$$= yp_z y\psi - zp_y y\psi - y^2 p_z\psi + yzp_y\psi$$

$$= y\left(-i\hbar \frac{d}{dz}\right)y\psi - z\left(-i\hbar \frac{d}{dy}\right)y\psi - y^2\left(-i\hbar \frac{d}{dz}\right)\psi + yz\left(-i\hbar \frac{d}{dy}\right)\psi$$

$$= -i\hbar y \frac{d}{dz}(y\psi) + i\hbar z \frac{d}{dy}(y\psi) + i\hbar y^2 \frac{d}{dz}(\psi) - i\hbar yz \frac{d}{dy}(\psi)$$

$$= -i\hbar y\left(0 + y\frac{d\psi}{dz}\right) + i\hbar z\left(\psi + y\frac{d\psi}{dy}\right) + i\hbar y^2 \frac{d\psi}{dz} - i\hbar yz \frac{d\psi}{dy}$$

$$= -i\hbar y^2 \frac{d\psi}{dz} + i\hbar z\psi + i\hbar yz \frac{d\psi}{dy} + i\hbar y^2 \frac{d\psi}{dz} - i\hbar yz \frac{d\psi}{dy}$$

$$= i\hbar z\psi$$

$$[L_x, y] = i\hbar z$$

5. $[\boldsymbol{L_y}, \boldsymbol{y}]$

Cara I

$$[L_y, y] = [(zp_x - xp_z), y]$$

$$= [zp_x, y] - [xp_z, y]$$

$$= (z[p_x, y] + [z, y]p_x) - (x[p_z, y] + [x, y]p_z)$$

$$= (z(0) + (0)p_x) - (x(0) + (0)p_z)$$

$= 0$

Cara II

$$[L_y, y]\psi = (L_y y - yL_y)\psi$$
$$= L_y y\psi \ - yL_y\psi$$
$$= (zp_x - xp_z)y\psi \ - y(zp_x - xp_z)\psi$$
$$= zp_x\, y\psi - xp_z y\psi \ - yzp_x\psi \ + xyp_z\psi$$
$$= z\left(-i\hbar\frac{d}{dx}\right)y\psi - x\left(-i\hbar\frac{d}{dz}\right)y\psi \ - yz\left(-i\hbar\frac{d}{dx}\right)\psi \ + xy\left(-i\hbar\frac{d}{dz}\right)\psi$$
$$= -i\hbar z\frac{d}{dx}(y\psi) + i\hbar x\frac{d}{dz}(y\psi) + i\hbar yz\frac{d}{dx}(\psi) \ - i\hbar xy\frac{d}{dz}(\psi)$$
$$= -i\hbar z\left(0 + y\frac{d\psi}{dx}\right) + i\hbar x\left(0 + y\frac{d\psi}{dz}\right) + i\hbar yz\frac{d\psi}{dx} \ - i\hbar xy\frac{d\psi}{dz}$$
$$= -i\hbar yz\frac{d\psi}{dx} + i\hbar xy\frac{d\psi}{dz} + i\hbar yz\frac{d\psi}{dx} \ - i\hbar xy\frac{d\psi}{dz}$$
$$= 0$$

$[L_y, y] = 0$

6. $[\boldsymbol{L_z}, \boldsymbol{y}]$

Cara I

$$[L_z, y] = [(xp_y - yp_x), y]$$
$$= [xp_y, y] - [yp_x, y]$$
$$= (x[p_y, y] + [x, y]p_y) - (y[p_x, y] + [y, y]p_x)$$
$$= (x(-i\hbar) + (0)p_y) - (y(0) + (0)p_x)$$
$$= -i\hbar x$$

Cara II

$$[L_z, y]\,\psi = (L_z y - yL_z)\psi$$
$$= L_z y\psi - yL_z\psi$$
$$= (xp_y - yp_x)y\psi - y(xp_y - yp_x)\psi$$
$$= xp_y y\psi - yp_x y\psi - xyp_y\psi + y^2 p_x\psi$$
$$= x\left(-i\hbar\frac{d}{dy}\right)y\psi - y\left(-i\hbar\frac{d}{dx}\right)y\psi - xy\left(-i\hbar\frac{d}{dy}\right)\psi + y^2\left(-i\hbar\frac{d}{dx}\right)\psi$$

$$= -i\hbar x \frac{d}{dy}(y\psi) + i\hbar y \frac{d}{dx}(y\psi) + i\hbar xy \frac{d}{dy}(\psi) - i\hbar y^2 \frac{d}{dx}(\psi)$$

$$= -i\hbar x \left(\psi + y\frac{d\psi}{dy}\right) + i\hbar y \left(0 + y\frac{d\psi}{dx}\right) + i\hbar xy \frac{d\psi}{dy} - i\hbar y^2 \frac{d\psi}{dx}$$

$$= -i\hbar x\psi - i\hbar xy \frac{d\psi}{dy} + i\hbar y^2 \frac{d\psi}{dx} + i\hbar xy \frac{d\psi}{dy} - i\hbar y^2 \frac{d\psi}{dx}$$

$$= -i\hbar x\psi$$

$[L_z, y] = -i\hbar x$

7. $[\boldsymbol{L_x}, \boldsymbol{z}]$

Cara I

$$[L_x, z] = \left[\left(yp_z - zp_y\right), z\right]$$

$$= [yp_z, z] - \left[zp_y, z\right]$$

$$= \left(y[p_z, z] + [y, z]p_z\right) - \left(z\left[p_y, z\right] + [z, z]p_y\right)$$

$$= \left(y(-i\hbar) + (0)p_z\right) - \left(z(0) + (0)p_y\right)$$

$$= -i\hbar y$$

Cara II

$$[L_x, z]\psi = (L_x z - zL_x)\psi$$

$$= L_x z\psi - zL_x\psi$$

$$= \left(yp_z - zp_y\right)z\psi - z\left(yp_z - zp_y\right)\psi$$

$$= yp_z z\psi - zp_y z\psi - yzp_z\psi + z^2 p_y\psi$$

$$= y\left(-i\hbar\frac{d}{dz}\right)z\psi - z\left(-i\hbar\frac{d}{dy}\right)z\psi - yz\left(-i\hbar\frac{d}{dz}\right)\psi + z^2\left(-i\hbar\frac{d}{dy}\right)\psi$$

$$= -i\hbar y \frac{d}{dz}(z\psi) + i\hbar z \frac{d}{dy}(z\psi) + i\hbar yz \frac{d}{dz}(\psi) - i\hbar z^2 \frac{d}{dy}(\psi)$$

$$= -i\hbar y \left(\psi + z\frac{d\psi}{dz}\right) + i\hbar z \left(0 + z\frac{d\psi}{dy}\right) + i\hbar yz \frac{d\psi}{dz} - i\hbar z^2 \frac{d\psi}{dy}$$

$$= -i\hbar y\psi - i\hbar yz \frac{d\psi}{dz} + i\hbar z^2 \frac{d\psi}{dy} + i\hbar yz \frac{d\psi}{dz} - i\hbar z^2 \frac{d\psi}{dy}$$

$$= -i\hbar y\psi$$

$[L_x, z] = -i\hbar y$

8. $[\boldsymbol{L_y}, \boldsymbol{z}]$

Cara I

$$\left[L_y, z\right] = [(zp_x - xp_z), z]$$

$$= [zp_x, z] - [xp_z, z]$$

$= (z[p_x, z] + [z, z]p_x) - (x[p_z, z] + [x, z]p_z)$

$= (z(0) + (0)p_x) - (x(-i\hbar) + (0)p_z)$

$= i\hbar x$

Cara II

$[L_y, z]\psi = (L_y z - zL_y)\psi$

$= L_y z\psi - zL_y\psi$

$= (zp_x - xp_z)z\psi - z(zp_x - xp_z)\psi$

$= zp_x z\psi - xp_z z\psi - z^2 p_x\psi + xzp_z\psi$

$= z\left(-i\hbar\frac{d}{dx}\right)z\psi - x\left(-i\hbar\frac{d}{dz}\right)z\psi - z^2\left(-i\hbar\frac{d}{dx}\right)\psi + xz\left(-i\hbar\frac{d}{dz}\right)\psi$

$= -i\hbar z\frac{d}{dx}(z\psi) + i\hbar x\frac{d}{dz}(z\psi) + i\hbar z^2\frac{d}{dx}(\psi) - i\hbar xz\frac{d}{dz}(\psi)$

$= -i\hbar z\left(0 + z\frac{d\psi}{dx}\right) + i\hbar x\left(\psi + z\frac{d\psi}{dz}\right) + i\hbar z^2\frac{d\psi}{dx} - i\hbar xz\frac{d\psi}{dz}$

$= -i\hbar z^2\frac{d\psi}{dx} + i\hbar x\psi + i\hbar xz\frac{d\psi}{dz} + i\hbar z^2\frac{d\psi}{dx} - i\hbar xz\frac{d\psi}{dz}$

$= i\hbar x\psi$

$[L_y, z] = i\hbar x$

9. **$[L_z, z]$**

Cara I

$[L_z, z] = [(xp_y - yp_x), z]$

$= [xp_y, z] - [yp_x, z]$

$= (x[p_y, z] + [x, z]p_y) - (y[p_x, z] + [y, z]p_x)$

$= (x(0) + (0)p_y) - (y(0) + (0)p_x)$

$= 0$

Cara II

$[L_z, z]\psi = (L_z z - zL_z)\psi$

$= L_z z\psi - zL_z\psi$

$= (xp_y - yp_x)z\psi - z(xp_y - yp_x)\psi$

$= xp_y z\psi - yp_x z\psi - xzp_y\psi - yzp_x\psi$

$$= x\left(-i\hbar\frac{d}{dy}\right)z\psi \; - y\left(-i\hbar\frac{d}{dx}\right)z\psi \; - xz\left(-i\hbar\frac{d}{dx}\right)\psi -$$

$$yz\left(-i\hbar\frac{d}{dx}\right)\psi$$

$$= -i\hbar x\frac{d}{dy}(z\psi) + i\hbar y\frac{d}{dx}(z\psi) + i\hbar xz\frac{d}{dx}(\psi) - i\hbar yz\frac{d}{dx}(\psi)$$

$$= -i\hbar x\left(0 + z\frac{d\psi}{dy}\right) + i\hbar y\left(0 + z\frac{d\psi}{dx}\right) + i\hbar xz\frac{d\psi}{dx} - i\hbar yz\frac{d\psi}{dx}$$

$$= -i\hbar xz\frac{d\psi}{dy} + i\hbar yz\frac{d\psi}{dx} + i\hbar xz\frac{d\psi}{dx} - i\hbar yz\frac{d\psi}{dx}$$

$$= 0$$

$$[L_z, z] = 0$$

LAMPIRAN D. KOMUTATOR OPERATOR MOMENTUM SUDUT KUADRAT TERHADAP POSISI

1. $[\boldsymbol{L_x}^2, \boldsymbol{x}]$

Cara I

$$\begin{aligned}
[L_x{}^2, x] &= \left[\left(i\hbar(zp_z + yp_y)\right), x\right] \\
&= i\hbar[(zp_z + yp_y), x] \\
&= i\hbar([zp_z, x] + [yp_y, x]) \\
&= i\hbar\left((z[p_z, x] + [z, x]p_z) + (y[p_y, x] + [y, x]p_y)\right) \\
&= i\hbar\left((z(0) + (0)p_z) + (y(0) + (0)p_y)\right) \\
&= 0
\end{aligned}$$

Cara II

$$\begin{aligned}
[L_x{}^2, x]\,\psi &= (L_x{}^2 x - xL_x{}^2)\psi \\
&= L_x{}^2 x\psi - xL_x{}^2\psi \\
&= \left(i\hbar(zp_z + yp_y)\right)x\psi - x\left(i\hbar(zp_z + yp_y)\right)\psi \\
&= i\hbar zp_z x\psi + i\hbar yp_y x\psi - i\hbar xzp_z\psi - i\hbar xyp_y\psi \\
&= i\hbar z\left(-i\hbar\frac{d}{dz}\right)x\psi + i\hbar y\left(-i\hbar\frac{d}{dy}\right)x\psi - i\hbar xz\left(-i\hbar\frac{d}{dz}\right)\psi - \\
&\quad i\hbar xy\left(-i\hbar\frac{d}{dy}\right)\psi \\
&= \hbar^2 z\frac{d}{dz}(x\psi) + \hbar^2 y\frac{d}{dy}(x\psi) - \hbar^2 xz\frac{d}{dz}(\psi) - \hbar^2 xy\frac{d}{dy}(\psi) \\
&= \hbar^2 z\left(0 + x\frac{d\psi}{dz}\right) + \hbar^2 y\left(0 + x\frac{d\psi}{dy}\right) - \hbar^2 xz\frac{d\psi}{dz} - \hbar^2 xy\frac{d\psi}{dy} \\
&= \hbar^2 xz\frac{d\psi}{dz} + \hbar^2 xy\frac{d\psi}{dy} - \hbar^2 xz\frac{d\psi}{dz} - \hbar^2 xy\frac{d\psi}{dy} \\
&= 0
\end{aligned}$$

$$[L_x{}^2, x] = 0$$

2. $[\boldsymbol{L_y}^2, \boldsymbol{x}]$

Cara I

$$\begin{aligned}
[L_y{}^2, x] &= [(i\hbar(xp_x + zp_z)), x] \\
&= i\hbar[(xp_x + zp_z), x]
\end{aligned}$$

$$= i\hbar([xp_x, x] + [zp_z, x])$$
$$= i\hbar\big((x[p_x, x] + [x, x]p_x) + (z[p_z, x] + [z, x]p_z)\big)$$
$$= i\hbar\big((x(-i\hbar) + (0)p_x) + (z(0) + (0)p_z)\big)$$
$$= \hbar^2 x$$

Cara II

$$[L_y{}^2, x]\,\psi = (L_y{}^2 x - xL_y{}^2)\psi$$
$$= L_y{}^2 x\psi - xL_y{}^2\psi$$
$$= \big(i\hbar(xp_x + zp_z)\big)x\psi - x\big(i\hbar(xp_x + zp_z)\big)\psi$$
$$= i\hbar xp_x x\psi + i\hbar zp_z x\psi - i\hbar x^2 p_x\psi - i\hbar xzp_z\psi$$
$$= i\hbar x\left(-i\hbar\frac{d}{dx}\right)x\psi + i\hbar z\left(-i\hbar\frac{d}{dz}\right)x\psi - i\hbar x^2\left(-i\hbar\frac{d}{dx}\right)\psi - i\hbar xz\left(-i\hbar\frac{d}{dz}\right)\psi$$
$$= \hbar^2 x\frac{d}{dx}(x\psi) + \hbar^2 z\frac{d}{dz}(x\psi) - \hbar^2 x^2\frac{d}{dx}(\psi) - \hbar^2 xz\frac{d}{dz}(\psi)$$
$$= \hbar^2 x\left(\psi + x\frac{d\psi}{dx}\right) + \hbar^2 z\left(0 + x\frac{d\psi}{dz}\right) - \hbar^2 x^2\frac{d\psi}{dx} - \hbar^2 xz\frac{d\psi}{dz}$$
$$= \hbar^2 x\psi + \hbar^2 x^2\frac{d\psi}{dx}(+) + \hbar^2 xz\frac{d\psi}{dz} - \hbar^2 x^2\frac{d\psi}{dx} - \hbar^2 xz\frac{d\psi}{dz}$$
$$= \hbar^2 x\psi$$
$$[L_y{}^2, x] = \hbar^2 x$$

3. $[\boldsymbol{L_z{}^2}, \boldsymbol{x}]$

Cara I

$$[L_z{}^2, x] = \left[\big(i\hbar(yp_y + xp_x)\big), x\right]$$
$$= i\hbar[(yp_y + xp_x), x]$$
$$= i\hbar([yp_y, x] + [xp_x, x])$$
$$= i\hbar\left((y[p_y, x] + [y, x]p_y) + (x[p_x, x] + [x, x]p_x)\right)$$
$$= i\hbar\left((y(0) + (0)p_y) + (x(-i\hbar) + (0)p_x)\right)$$
$$= \hbar^2 x$$

Cara II

$$[L_z{}^2, x]\psi = (L_z{}^2 x - xL_z{}^2)\psi$$

$$= L_z{}^2 x\psi - xL_z{}^2\psi$$
$$= \left(i\hbar\left(yp_y + xp_x\right)\right)x\psi - x\left(i\hbar\left(yp_y + xp_x\right)\right)\psi$$
$$= i\hbar y p_y x\psi + i\hbar x p_x x\psi - i\hbar xy p_y\psi - i\hbar x^2 p_x\psi$$
$$= i\hbar y\left(-i\hbar\frac{d}{dy}\right)x\psi + i\hbar x\left(-i\hbar\frac{d}{dx}\right)x\psi - i\hbar xy\left(-i\hbar\frac{d}{dy}\right)\psi -$$
$$i\hbar x^2\left(-i\hbar\frac{d}{dx}\right)\psi$$
$$= \hbar^2 y\frac{d}{dy}(x\psi) + \hbar^2 x\frac{d}{dx}(x\psi) - \hbar^2 xy\frac{d}{dy}(\psi) - \hbar^2 x^2\frac{d}{dx}(\psi)$$
$$= \hbar^2 y\left(0 + x\frac{d\psi}{dy}\right) + \hbar^2 x\left(\psi + x\frac{d\psi}{dx}\right) - \hbar^2 xy\frac{d\psi}{dy} - \hbar^2 x^2\frac{d\psi}{dx}$$
$$= \hbar^2 xy\frac{d\psi}{dy} + \hbar^2 x\psi + \hbar^2 x^2\frac{d\psi}{dx} - \hbar^2 xy\frac{d\psi}{dy} - \hbar^2 x^2\frac{d\psi}{dx}$$
$$= \hbar^2 x\psi$$

$$\left[L_z{}^2, x\right] = \hbar^2 x$$

4. $[\boldsymbol{L^2}, \boldsymbol{x}]$

Cara I

$$[L^2, x] = \left[\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right), x\right]$$
$$= 2i\hbar\left[\left(xp_x + yp_y + zp_z\right), x\right]$$
$$= 2i\hbar\left([xp_x, x] + \left[yp_y, x\right] + [zp_z, x]\right)$$
$$= 2i\hbar\left((x[p_x, x] + [x, x]p_x) + (0) + (0)\right)$$
$$= 2i\hbar(x(-i\hbar) + (0)p_x)$$
$$= 2\hbar^2 x$$

Cara II

$$[L^2, x]\psi = (L^2 x - xL^2)\psi$$
$$= L^2 x\psi - xL^2\psi$$
$$= \left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)x\psi - x\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)\psi$$
$$= 2i\hbar x p_x x\psi + 2i\hbar y p_y x\psi + 2i\hbar z p_z x\psi - 2i\hbar x^2 p_x\psi -$$
$$2i\hbar xy p_y\psi - 2i\hbar xz p_z\psi$$
$$= 2i\hbar x\left(-i\hbar\frac{d}{dx}\right)x\psi + 2i\hbar y\left(-i\hbar\frac{d}{dy}\right)x\psi + 2i\hbar z\left(-i\hbar\frac{d}{dz}\right)x\psi -$$
$$2i\hbar x^2\left(-i\hbar\frac{d}{dx}\right)\psi - 2i\hbar xy\left(-i\hbar\frac{d}{dy}\right)\psi - 2i\hbar xz\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= 2\hbar^2 x \frac{d}{dx}(x\psi) + 2\hbar^2 y \frac{d}{dy}(x\psi) + 2\hbar^2 z \frac{d}{dz}(x\psi) - 2\hbar^2 x^2 \frac{d}{dx}(\psi) -$$
$$2\hbar^2 xy \frac{d}{dy}(\psi) - 2\hbar^2 xz \frac{d}{dz}(\psi)$$
$$= 2\hbar^2 x \left(\psi + x\frac{d\psi}{dx}\right) + 2\hbar^2 y \left(0 + x\frac{d\psi}{dy}\right) + 2\hbar^2 z \left(0 + x\frac{d\psi}{dz}\right) -$$
$$2\hbar^2 x^2 \frac{d\psi}{dx} - 2\hbar^2 xy \frac{d\psi}{dy} - 2\hbar^2 xz \frac{d\psi}{dz}$$
$$= 2\hbar^2 x\psi + 2\hbar^2 x^2 \frac{d\psi}{dx} + 2\hbar^2 xy \frac{d\psi}{dy} + 2\hbar^2 xz \frac{d\psi}{dz} - 2\hbar^2 x^2 \frac{d\psi}{dx} -$$
$$2\hbar^2 xy \frac{d\psi}{dy} - 2\hbar^2 xz \frac{d\psi}{dz}$$
$$= 2\hbar^2 x\psi$$
$$[L^2, x] = 2\hbar^2 x$$

5. $[L_x{}^2, y]$

Cara I

$$[L_x{}^2, y] = \left[\left(i\hbar\left(zp_z + yp_y\right)\right), y\right]$$
$$= i\hbar\left[\left(zp_z + yp_y\right), y\right]$$
$$= i\hbar\left([zp_z, y] + [yp_y, y]\right)$$
$$= i\hbar\left((0) + \left(y[p_y, y] + [y, y]p_y\right)\right)$$
$$= i\hbar\left(y(-i\hbar) + (0)p_y\right)$$
$$= \hbar^2 y$$

Cara II

$$[L_x{}^2, y]\psi = \left(L_x{}^2 y - yL_x{}^2\right)\psi$$
$$= L_x{}^2 y\psi - yL_x{}^2\psi$$
$$= \left(i\hbar\left(zp_z + yp_y\right)\right) y\psi - y\left(i\hbar\left(zp_z + yp_y\right)\right)\psi$$
$$= i\hbar zp_z y\psi + i\hbar yp_y y\psi - i\hbar yzp_z\psi - i\hbar y^2 p_y\psi$$
$$= i\hbar z\left(-i\hbar\frac{d}{dz}\right) y\psi + i\hbar y\left(-i\hbar\frac{d}{dy}\right) y\psi - i\hbar yz\left(-i\hbar\frac{d}{dz}\right)\psi -$$
$$i\hbar y^2\left(-i\hbar\frac{d}{dy}\right)\psi$$
$$= \hbar^2 z\frac{d}{dz}(y\psi) + \hbar^2 y\frac{d}{dy}(y\psi) - \hbar^2 yz\frac{d}{dz}(\psi) - \hbar^2 y^2\frac{d}{dy}(\psi)$$
$$= \hbar^2 z\left(0 + y\frac{d\psi}{dz}\right) + \hbar^2 y\left(\psi + y\frac{d\psi}{dy}\right) - \hbar^2 yz\frac{d\psi}{dz} - \hbar^2 y^2\frac{d\psi}{dy}$$

$$= \hbar^2 yz\frac{d\psi}{dz} + \hbar^2 y\psi + \hbar^2 y^2\frac{d\psi}{dy} - \hbar^2 yz\frac{d\psi}{dz} - \hbar^2 y^2\frac{d\psi}{dy}$$

$$= \hbar^2 y\psi$$

$$[L_x{}^2, y] = \hbar^2 y$$

6. $[\boldsymbol{L_y{}^2}, \boldsymbol{y}]$

Cara I

$$[L_y{}^2, y] = [(i\hbar(xp_x + zp_z)), y]$$

$$= i\hbar[(xp_x + zp_z), y]$$

$$= i\hbar([xp_x, y] + [zp_z, y])$$

$$= i\hbar(0 + 0)$$

$$= 0$$

Cara II

$$[L_y{}^2, y]\psi = (L_y{}^2 y - yL_y{}^2)\psi$$

$$= L_y{}^2 y\psi - yL_y{}^2\psi$$

$$= (i\hbar(xp_x + zp_z))y\psi - y(i\hbar(xp_x + zp_z))\psi$$

$$= i\hbar xp_x y\psi + i\hbar zp_z y\psi - i\hbar xyp_x\psi - i\hbar yzp_z\psi$$

$$= i\hbar x\left(-i\hbar\frac{d}{dx}\right)y\psi + i\hbar z\left(-i\hbar\frac{d}{dz}\right)y\psi - i\hbar xy\left(-i\hbar\frac{d}{dx}\right)\psi - i\hbar yz\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= \hbar^2 x\frac{d}{dx}(y\psi) + \hbar^2 z\frac{d}{dz}(y\psi) - \hbar^2 xy\frac{d}{dx}(\psi) - \hbar^2 yz\frac{d}{dz}(\psi)$$

$$= \hbar^2 x\left(0 + y\frac{d}{dx}\psi\right) + \hbar^2 z\left(0 + y\frac{d}{dz}\psi\right) - \hbar^2 xy\frac{d\psi}{dx} - \hbar^2 yz\frac{d\psi}{dz}$$

$$= \hbar^2 xy\frac{d\psi}{dx} + \hbar^2 yz\frac{d\psi}{dz} - \hbar^2 xy\frac{d\psi}{dx} - \hbar^2 yz\frac{d\psi}{dz}$$

$$= 0$$

$$[L_z{}^2, y] = 0$$

7. $[\boldsymbol{L_z{}^2}, \boldsymbol{y}]$

Cara I

$$[L_z{}^2, y] = \left[\left(i\hbar(yp_y + xp_x)\right), y\right]$$

$$= i\hbar[(yp_y + xp_x), y]$$

$$= i\hbar([yp_y, y] + [xp_x, y])$$

$$= i\hbar\left(\left(y[p_y, y] + [y, y]p_y\right) + (0)\right)$$

$$= i\hbar\left(y(-i\hbar) + (0)p_y\right)$$

$$= \hbar^2 y$$

Cara II

$$[L_z{}^2, y]\psi = (L_z{}^2 y - yL_z{}^2)\psi$$

$$= L_z{}^2 y\psi - yL_z{}^2\psi$$

$$= \left(i\hbar\left(yp_y + xp_x\right)\right) y\psi - y\left(i\hbar\left(yp_y + xp_x\right)\right)\psi$$

$$= i\hbar yp_y y\psi + i\hbar xp_x y\psi - i\hbar y^2 p_y\psi + i\hbar xyp_x\psi$$

$$= i\hbar y\left(-i\hbar\frac{d}{dy}\right) y\psi + i\hbar x\left(-i\hbar\frac{d}{dx}\right) y\psi - i\hbar y^2\left(-i\hbar\frac{d}{dy}\right)\psi + i\hbar xy\left(-i\hbar\frac{d}{dx}\right)\psi$$

$$= \hbar^2 y\frac{d}{dy}(y\psi) + \hbar^2 x\frac{d}{dx}(y\psi) - \hbar^2 y^2\frac{d}{dy}(\psi) + \hbar^2 xy\frac{d}{dx}(\psi)$$

$$= \hbar^2 y\left(\psi + y\frac{d\psi}{dy}\right) + \hbar^2 x\left(0 + y\frac{d\psi}{dx}\right) - \hbar^2 y^2\frac{d\psi}{dy} + \hbar^2 xy\frac{d\psi}{dx}$$

$$= \hbar^2 y\psi + \hbar^2 y^2\frac{d\psi}{dy} + \hbar^2 xy\frac{d\psi}{dx} - \hbar^2 y^2\frac{d\psi}{dy} + \hbar^2 xy\frac{d\psi}{dx}$$

$$= \hbar^2 y\psi$$

$$[L_z{}^2, y] = \hbar^2 y$$

8. $[\boldsymbol{L}^2, \boldsymbol{y}]$

Cara I

$$[L^2, y] = \left[\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right), y\right]$$

$$= 2i\hbar\left[\left(xp_x + yp_y + zp_z\right), y\right]$$

$$= 2i\hbar\left([xp_x, y] + [yp_y, y] + [zp_z, y]\right)$$

$$= 2i\hbar\left((0) + \left(y[p_y, y] + [y, y]p_y\right) + (0)\right)$$

$$= 2i\hbar\left(y(-i\hbar) + (0)p_y\right)$$

$$= 2\hbar^2 y$$

Cara II

$$[L^2, y]\psi = (L^2 y - yL^2)\psi$$

$$= L^2 y\psi - yL^2\psi$$

$= \left(2i\hbar(xp_x + yp_y + zp_z)\right) y\psi - y\left(2i\hbar(xp_x + yp_y + zp_z)\right)\psi$

$= 2i\hbar xp_x y\psi + 2i\hbar yp_y y\psi + 2i\hbar zp_z y\psi - 2i\hbar xyp_x\psi -$
$2i\hbar y^2 p_y\psi - 2i\hbar yzp_z\psi$

$= 2i\hbar x\left(-i\hbar\frac{d}{dx}\right)y\psi + 2i\hbar y\left(-i\hbar\frac{d}{dy}\right)y\psi + 2i\hbar z\left(-i\hbar\frac{d}{dz}\right)y\psi -$
$2i\hbar xy\left(-i\hbar\frac{d}{dx}\right)\psi - 2i\hbar y^2\left(-i\hbar\frac{d}{dy}\right)\psi - 2i\hbar yz\left(-i\hbar\frac{d}{dz}\right)\psi$

$= 2\hbar^2 x\frac{d}{dx}(y\psi) + 2\hbar^2 y\frac{d}{dy}(y\psi) + 2\hbar^2 z\frac{d}{dz}(y\psi) - 2\hbar^2 xy\frac{d}{dx}(\psi) -$
$2\hbar^2 y^2\frac{d}{dy}(\psi) - 2\hbar^2 yz\frac{d}{dz}(\psi)$

$= 2\hbar^2 x\left(0 + y\frac{d\psi}{dx}\right) + 2\hbar^2 y\left(\psi + y\frac{d\psi}{dy}\right) + 2\hbar^2 z\left(0 + y\frac{d\psi}{dz}\right) -$
$2\hbar^2 xy\frac{d\psi}{dx} - 2\hbar^2 y^2\frac{d\psi}{dy} - 2\hbar^2 yz\frac{d\psi}{dz}$

$= 2\hbar^2 xy\frac{d\psi}{dx} + 2\hbar^2 y\psi + 2\hbar^2 y^2\frac{d\psi}{dy} + 2\hbar^2 yz\frac{d\psi}{dz} - 2\hbar^2 xy\frac{d\psi}{dx} -$
$2\hbar^2 y^2\frac{d\psi}{dy} - 2\hbar^2 yz\frac{d\psi}{dz}$

$= 2\hbar^2 y\psi$

$[L^2, y] = 2\hbar^2 y$

9. $\boldsymbol{[L_x{}^2, z]}$

Cara I

$[L_x{}^2, z] = \left[\left(i\hbar(zp_z + yp_y)\right), z\right]$

$= i\hbar\left[(zp_z + yp_y), z\right]$

$= i\hbar\left([zp_z, z] + [yp_y, z]\right)$

$= i\hbar\left((z[p_z, z] + [z, z]p_z) + (0)\right)$

$= i\hbar(z(-i\hbar) + (0)p_z)$

$= \hbar^2 z$

Cara II

$[L_x{}^2, z]\psi = (L_x{}^2 z - zL_x{}^2)\psi$

$= L_x{}^2 z\psi - zL_x{}^2\psi$

$= \left(i\hbar(zp_z + yp_y)\right)z\psi - z\left(i\hbar(zp_z + yp_y)\right)\psi$

$$= i\hbar z p_z z\psi + i\hbar y p_y z\psi - i\hbar z^2 p_z \psi - i\hbar y z p_y \psi$$

$$= i\hbar z\left(-i\hbar\frac{d}{dz}\right)z\psi + i\hbar y\left(-i\hbar\frac{d}{dy}\right)z\psi - i\hbar z^2\left(-i\hbar\frac{d}{dz}\right)\psi - i\hbar yz\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= \hbar^2 z\frac{d}{dz}(z\psi) + \hbar^2 y\frac{d}{dy}(z\psi) - \hbar^2 z^2\frac{d}{dz}(\psi) - \hbar^2 yz\frac{d}{dz}(\psi)$$

$$= \hbar^2 z\left(\psi + z\frac{d\psi}{dz}\right) + \hbar^2 y\left(0 + z\frac{d\psi}{dy}\right) - \hbar^2 z^2\frac{d\psi}{dz} - \hbar^2 yz\frac{d\psi}{dz}$$

$$= \hbar^2 z\psi + \hbar^2 z^2\frac{d\psi}{dz} + \hbar^2 yz\frac{d\psi}{dy} - \hbar^2 z^2\frac{d\psi}{dz} - \hbar^2 yz\frac{d\psi}{dz}$$

$$= \hbar^2 z\psi$$

$$[L_x{}^2, z] = \hbar^2 z$$

10. $[L_y{}^2, z]$

Cara I

$$[L_y{}^2, z] = [(i\hbar(xp_x + zp_z)), z]$$

$$= i\hbar[(xp_x + zp_z), z]$$

$$= i\hbar([xp_x, z] + [zp_z, z])$$

$$= i\hbar((0) + (z[p_z, z] + [z, z]p_z))$$

$$= i\hbar(z(-i\hbar) + (0)p_z)$$

$$= \hbar^2 z$$

Cara II

$$[L_y{}^2, z]\psi = (L_y{}^2 z - zL_y{}^2)\psi$$

$$= L_y{}^2 z\psi - zL_y{}^2\psi$$

$$= (i\hbar(xp_x + zp_z))z\psi - z(i\hbar(xp_x + zp_z))\psi$$

$$= i\hbar x p_x z\psi + i\hbar z p_z z\psi - i\hbar x z p_x \psi - i\hbar z^2 p_z \psi$$

$$= i\hbar x\left(-i\hbar\frac{d}{dx}\right)z\psi + i\hbar z\left(-i\hbar\frac{d}{dz}\right)z\psi - i\hbar xz\left(-i\hbar\frac{d}{dx}\right)\psi - i\hbar z^2\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= \hbar^2 x\frac{d}{dx}(z\psi) + \hbar^2 z\frac{d}{dz}(z\psi) - \hbar^2 xz\frac{d}{dx}(\psi) - \hbar^2 z^2\frac{d}{dz}(\psi)$$

$$= \hbar^2 x\left(0 + z\frac{d\psi}{dx}\right) + \hbar^2 z\left(\psi + z\frac{d\psi}{dz}\right) - \hbar^2 xz\frac{d\psi}{dx} - \hbar^2 z^2\frac{d\psi}{dz}$$

$$= \hbar^2 xz\frac{d\psi}{dx} + \hbar^2 z\psi + \hbar^2 z^2\frac{d\psi}{dz} - \hbar^2 xz\frac{d\psi}{dx} - \hbar^2 z^2\frac{d\psi}{dz}$$

$$= \hbar^2 z\psi$$

$$[L_z{}^2, z] = \hbar^2 z$$

11. $[\boldsymbol{L_z{}^2}, \boldsymbol{z}]$

Cara I

$$[L_z{}^2, z] = \left[\left(i\hbar\left(yp_y + xp_x\right)\right), z\right]$$

$$= i\hbar\left[\left(yp_y + xp_x\right), z\right]$$

$$= i\hbar\left(\left[yp_y, z\right] + [xp_x, z]\right)$$

$$= i\hbar(0 + 0)$$

$$= 0$$

Cara II

$$[L_z{}^2, z]\,\psi = (L_z{}^2 z - zL_z{}^2)\psi$$

$$= L_z{}^2 z\psi - zL_z{}^2\psi$$

$$= \left(i\hbar\left(yp_y + xp_x\right)\right) z\psi - z\left(i\hbar\left(yp_y + xp_x\right)\right)\psi$$

$$= i\hbar yp_y z\psi + i\hbar xp_x z\psi - i\hbar yzp_y\psi - i\hbar xzp_x\psi$$

$$= i\hbar y\left(-i\hbar\frac{d}{dy}\right) z\psi + i\hbar x\left(-i\hbar\frac{d}{dx}\right) z\psi - i\hbar yz\left(-i\hbar\frac{d}{dy}\right)\psi - i\hbar xz\left(-i\hbar\frac{d}{dx}\right)\psi$$

$$= \hbar^2 y\frac{d}{dy}(z\psi) + \hbar^2 x\frac{d}{dx}(z\psi) - \hbar^2 yz\frac{d}{dy}(\psi) - \hbar^2 xz\frac{d}{dx}(\psi)$$

$$= \hbar^2 y\left(0 + z\frac{d\psi}{dy}\right) + \hbar^2 x\left(0 + z\frac{d\psi}{dx}\right) - \hbar^2 yz\frac{d\psi}{dy} - \hbar^2 xz\frac{d\psi}{dx}$$

$$= \hbar^2 yz\frac{d\psi}{dy} + \hbar^2 xz\frac{d\psi}{dx} - \hbar^2 yz\frac{d\psi}{dy} - \hbar^2 xz\frac{d\psi}{dx}$$

$$= 0$$

$$[L_z{}^2, z] = 0$$

12. $[\boldsymbol{L^2}, \boldsymbol{z}]$

Cara I

$$[L^2, z] = \left[\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right), z\right]$$

$$= 2i\hbar\left[\left(xp_x + yp_y + zp_z\right), z\right]$$

$$= 2i\hbar\left([xp_x, z] + \left[yp_y, z\right] + [zp_z, z]\right)$$

$$= 2i\hbar\big((0) + (0) + (z[p_z, z] + [z, z]p_z)\big)$$
$$= 2i\hbar(z(-i\hbar) + (0)p_z)$$
$$= 2\hbar^2 z$$

Cara II

$$[L^2, z]\psi = (L^2 z - zL^2)\psi$$
$$= L^2 z\psi - zL^2\psi$$
$$= \left(2i\hbar\big(xp_x + yp_y + zp_z\big)\right) z\psi - z\left(2i\hbar\big(xp_x + yp_y + zp_z\big)\right)\psi$$
$$= 2i\hbar xp_x z\psi + 2i\hbar yp_y z\psi + 2i\hbar zp_z z\psi - 2i\hbar xzp_x\psi - 2i\hbar yzp_y\psi - 2i\hbar z^2 p_z\psi$$
$$= 2i\hbar x\left(-i\hbar\frac{d}{dx}\right)z\psi + 2i\hbar y\left(-i\hbar\frac{d}{dy}\right)z\psi + 2i\hbar z\left(-i\hbar\frac{d}{dz}\right)z\psi - 2i\hbar xz\left(-i\hbar\frac{d}{dx}\right)\psi - 2i\hbar yz\left(-i\hbar\frac{d}{dy}\right)\psi - 2i\hbar z^2\left(-i\hbar\frac{d}{dz}\right)\psi$$
$$= 2\hbar^2 x\frac{d}{dx}(z\psi) + 2\hbar^2 y\frac{d}{dy}(z\psi) + 2\hbar^2 z\frac{d}{dz}(z\psi) - 2\hbar^2 xz\frac{d}{dx}(\psi) - 2\hbar^2 yz\frac{d}{dy}(\psi) - 2\hbar^2 z^2\frac{d}{dz}(\psi)$$
$$= 2\hbar^2 x\left(0 + z\frac{d\psi}{dx}\right) + 2\hbar^2 y\left(0 + z\frac{d\psi}{dy}\right) + 2\hbar^2 z\left(\psi + z\frac{d\psi}{dz}\right) - 2\hbar^2 xz\frac{d\psi}{dx} - 2\hbar^2 yz\frac{d\psi}{dy} - 2\hbar^2 z^2\frac{d\psi}{dz}$$
$$= 2\hbar^2 xz\frac{d\psi}{dx} + 2\hbar^2 yz\frac{d\psi}{dy} + 2\hbar^2 z\psi + 2\hbar^2 z^2\frac{d\psi}{dz} - 2\hbar^2 xz\frac{d\psi}{dx} - 2\hbar^2 yz\frac{d\psi}{dy} - 2\hbar^2 z^2\frac{d\psi}{dz}$$
$$= 2\hbar^2 z\psi$$

$$[L^2, z] = 2\hbar^2 z$$

LAMPIRAN E. KOMUTATOR OPERATOR MOMENTUM SUDUT TERHADAP POSISI KUADRAT

1. $[\boldsymbol{L_x}, \boldsymbol{x^2}]$

Cara I

$$\begin{aligned}[L_x, x^2] &= \left[\left(yp_z - zp_y\right), x^2\right] \\ &= [yp_z, x^2] - \left[zp_y, x^2\right] \\ &= \left(y[p_z, x^2] + [y, x^2]p_z\right) - \left(z\left[p_y, x^2\right] + [z, x^2]p_y\right) \\ &= \left(y(0) + (0)p_z\right) - \left(z(0) + (0)p_y\right) \\ &= 0\end{aligned}$$

Cara II

$$\begin{aligned}[L_x, x^2]\,\psi &= (L_x x^2 - x^2 L_x)\psi \\ &= L_x x^2\psi - x^2 L_x\psi \\ &= \left(yp_z - zp_y\right)x^2\psi - x^2\left(yp_z - zp_y\right)\psi \\ &= yp_z x^2\psi - zp_y x^2\psi - x^2 yp_z\psi + x^2 zp_y\psi \\ &= y\left(-i\hbar\frac{d}{dz}\right)x^2\psi - z\left(-i\hbar\frac{d}{dy}\right)x^2\psi - x^2 y\left(-i\hbar\frac{d}{dz}\right)\psi + \\ &\quad x^2 z\left(-i\hbar\frac{d}{dy}\right)\psi \\ &= -i\hbar y\frac{d}{dz}(x^2\psi) + i\hbar z\frac{d}{dy}(x^2\psi) + i\hbar x^2 y\frac{d}{dz}(\psi) - i\hbar x^2 z\frac{d}{dy}(\psi) \\ &= -i\hbar y\left(0 + x^2\frac{d\psi}{dz}\right) + i\hbar z\left(0 + x^2\frac{d\psi}{dy}\right) + i\hbar x^2 y\frac{d\psi}{dz} - i\hbar x^2 z\frac{d\psi}{dy} \\ &= -i\hbar x^2 y\frac{d\psi}{dz} + i\hbar x^2 z\frac{d\psi}{dy} + i\hbar x^2 y\frac{d\psi}{dz} - i\hbar x^2 z\frac{d\psi}{dy} \\ &= 0\end{aligned}$$

$[L_x, x^2] = 0$

2. $\left[\boldsymbol{L_y}, \boldsymbol{x^2}\right]$

Cara I

$$\begin{aligned}\left[L_y, x^2\right] &= [(zp_x - xp_z), x^2] \\ &= [zp_x, x^2] - [xp_z, x^2] \\ &= (z[p_x, x^2] + [z, x^2]p_x) - (x[p_z, x^2] + [x, x^2]p_z)\end{aligned}$$

$= (z(-2i\hbar x) + (0)p_x) - (x(0) + (0)p_z)$

$= -2i\hbar xz$

Cara II

$[L_y, x^2]\,\psi = (L_y x^2 - x^2 L_y)\psi$

$= L_y x^2\psi - x^2 L_y\psi$

$= (zp_x - xp_z)x^2\psi - x^2(zp_x - xp_z)\psi$

$= zp_x x^2\psi - xp_z x^2\psi - x^2 zp_x\psi + x^3 p_z\psi$

$= z\left(-i\hbar\frac{d}{dx}\right)x^2\psi - x\left(-i\hbar\frac{d}{dz}\right)x^2\psi - x^2 z\left(-i\hbar\frac{d}{dx}\right)\psi +$

$x^3\left(-i\hbar\frac{d}{dz}\right)\psi$

$= -i\hbar z\frac{d}{dx}(x^2\psi) + i\hbar x\frac{d}{dz}(x^2\psi) + i\hbar x^2 z\frac{d}{dx}(\psi) - i\hbar x^3\frac{d}{dz}(\psi)$

$= -i\hbar z\left(2x\psi + x^2\frac{d\psi}{dx}\right) + i\hbar x\left(0 + x^2\frac{d\psi}{dz}\right) + i\hbar x^2 z\frac{d\psi}{dx} - i\hbar x^3\frac{d\psi}{dz}$

$= -2i\hbar xz\psi - i\hbar x^2 z\frac{d\psi}{dx} + i\hbar x^3\frac{d\psi}{dz} + i\hbar x^2 z\frac{d\psi}{dx} - i\hbar x^3\frac{d\psi}{dz}$

$= -2i\hbar xz\psi$

$[L_y, x^2] = -2i\hbar xz$

3. $[L_z, x^2]$

Cara I

$[L_z, x^2] = [(xp_y - yp_x), x^2]$

$= [xp_y, x^2] - [yp_x, x^2]$

$= (x[p_y, x^2] - [x, x^2]p_y) - (y[p_x, x^2] + [y, x^2]p_x)$

$= (x(0) + (0)p_y) - (y(-2i\hbar x) + (0)p_x)$

$= 2i\hbar xy$

Cara II

$[L_z, x^2]\psi = (L_z x^2 - x^2 L_z)\psi$

$= L_z x^2\psi - x^2 L_z\psi$

$= (xp_y - yp_x)x^2\psi - x^2(xp_y - yp_x)\psi$

$= xp_y x^2\psi - yp_x x^2\psi - x^3 p_y\psi + x^2 yp_x\psi$

$= x\left(-i\hbar\frac{d}{dy}\right)x^2\psi - y\left(-i\hbar\frac{d}{dx}\right)x^2\psi - x^3\left(-i\hbar\frac{d}{dy}\right)\psi +$

$x^2y\left(-i\hbar\frac{d}{dx}\right)\psi$

$= -i\hbar x\frac{d}{dy}(x^2\psi) + i\hbar y\frac{d}{dx}(x^2\psi) + i\hbar x^3\frac{d}{dy}(\psi) - i\hbar x^2y\frac{d}{dx}(\psi)$

$= -i\hbar x\left(0 + x^2\frac{d\psi}{dy}\right) + i\hbar y\left(2x\psi + x^2\frac{d\psi}{dx}\right) + i\hbar x^3\frac{d\psi}{dy} - i\hbar x^2y\frac{d\psi}{dx}$

$= -i\hbar x^3\frac{d\psi}{dy} + i\hbar y\left(2x\psi + x^2\frac{d\psi}{dx}\right) + i\hbar x^3\frac{d\psi}{dy} - i\hbar x^2y\frac{d\psi}{dx}$

$= -i\hbar x^3\frac{d\psi}{dy} + 2i\hbar xy\psi + i\hbar x^2y\frac{d\psi}{dx} + i\hbar x^3\frac{d\psi}{dy} - i\hbar x^2y\frac{d\psi}{dx}$

$= 2i\hbar xy\psi$

$[L_z, x^2] = 2i\hbar xy$

4. $[\boldsymbol{L_x}, \boldsymbol{y^2}]$

Cara I

$[L_x, y^2] = [(yp_z - zp_y), y^2]$

$= [yp_z, y^2] - [zp_y, y^2]$

$= (y[p_z, y^2] + [y, y^2]p_z) - (z[p_y, y^2] + [z, y^2]p_y)$

$= (y(0) + (0)p_z) - (z(-2i\hbar y) + (0)p_y)$

$= 2i\hbar yz$

Cara II

$[L_x, y^2]\psi = (L_xy^2 - y^2L_x)\psi$

$= L_xy^2\psi - y^2L_x\psi$

$= (yp_z - zp_y)y^2\psi - y^2(yp_z - zp_y)\psi$

$= yp_zy^2\psi - zp_yy^2\psi - y^3p_z\psi + y^2zp_y\psi$

$= y\left(-i\hbar\frac{d}{dz}\right)y^2\psi - z\left(-i\hbar\frac{d}{dy}\right)y^2\psi - y^3\left(-i\hbar\frac{d}{dz}\right)\psi +$

$y^2z\left(-i\hbar\frac{d}{dy}\right)\psi$

$= -i\hbar y\frac{d}{dz}(y^2\psi) + i\hbar z\frac{d}{dy}(y^2\psi) + i\hbar y^3\frac{d}{dz}(\psi) - i\hbar y^2z\frac{d}{dy}(\psi)$

$= -i\hbar y\left(0 + y^2\frac{d\psi}{dz}\right) + i\hbar z\left(2y\psi + y^2\frac{d\psi}{dy}\right) + i\hbar y^3\frac{d\psi}{dz} - i\hbar y^2z\frac{d\psi}{dy}$

$= -i\hbar y^3\frac{d\psi}{dz} + 2i\hbar yz\psi + i\hbar y^2z\frac{d\psi}{dy} + i\hbar y^3\frac{d\psi}{dz} - i\hbar y^2z\frac{d\psi}{dy}$

$$= 2i\hbar yz\psi$$

$$[L_x, y^2] = 2i\hbar yz$$

5. $\boldsymbol{[L_y, y^2]}$

Cara I

$$[L_y, y^2] = [(zp_x - xp_z), y^2]$$
$$= [zp_x, y^2] - [xp_z, y^2]$$
$$= (z[p_x, y^2] + [z, y^2]p_x) - (x[p_z, y^2] + [x, y^2]p_z)$$
$$= (z(0) + (0)p_x) - (x(0) + (0)p_z)$$
$$= 0$$

Cara II

$$[L_y, y^2]\psi = (L_y y^2 - y^2 L_y)\psi$$
$$= L_y y^2\psi - y^2 L_y\psi$$
$$= (zp_x - xp_z)y^2\psi - y^2(zp_x - xp_z)\psi$$
$$= zp_x y^2\psi - xp_z y^2\psi - y^2 zp_x\psi + xy^2 p_z\psi$$
$$= z\left(-i\hbar\frac{d}{dx}\right)y^2\psi - x\left(-i\hbar\frac{d}{dz}\right)y^2\psi - y^2 z\left(-i\hbar\frac{d}{dx}\right)\psi + xy^2\left(-i\hbar\frac{d}{dz}\right)\psi$$
$$= -i\hbar z\frac{d}{dx}(y^2\psi) + i\hbar x\frac{d}{dz}(y^2\psi) + i\hbar y^2 z\frac{d}{dx}(\psi) - i\hbar xy^2\frac{d}{dz}(\psi)$$
$$= -i\hbar z\left(0 + y^2\frac{d\psi}{dx}\right) + i\hbar x\left(0 + y^2\frac{d\psi}{dz}\right) + i\hbar y^2 z\frac{d\psi}{dx} - i\hbar xy^2\frac{d\psi}{dz}$$
$$= -i\hbar y^2 z\frac{d\psi}{dx} + i\hbar xy^2\frac{d\psi}{dz} + i\hbar y^2 z\frac{d\psi}{dx} - i\hbar xy^2\frac{d\psi}{dz}$$
$$= 0$$

$$[L_y, y^2] = 0$$

6. $\boldsymbol{[L_z, y^2]}$

Cara I

$$[L_z, y^2] = [(xp_y - yp_x), y^2]$$
$$= [xp_y, y^2] - [yp_x, y^2]$$
$$= (x[p_y, y^2] + [x, y^2]p_y) - (y[p_x, y^2] + [y, y^2]p_x)$$
$$= (x(-2i\hbar y) + (0)p_y) - (y(0) + (0)p_x)$$
$$= -2i\hbar xy$$

Cara II

$$[L_z, y^2]\psi = (L_z y^2 - y^2 L_z)\psi$$
$$= L_z y^2\psi - y^2 L_z\psi$$
$$= (xp_y - yp_x)y^2\psi - y^2(xp_y - yp_x)\psi$$
$$= xp_y y^2\psi - yp_x y^2\psi - xy^2 p_y\psi + y^3 p_x\psi$$
$$= x\left(-i\hbar\frac{d}{dy}\right)y^2\psi - y\left(-i\hbar\frac{d}{dx}\right)y^2\psi - xy^2\left(-i\hbar\frac{d}{dy}\right)\psi + y^3\left(-i\hbar\frac{d}{dx}\right)\psi$$
$$= -i\hbar x\frac{d}{dy}(y^2\psi) + i\hbar y\frac{d}{dx}(y^2\psi) - i\hbar xy^2\frac{d}{dy}(\psi) - i\hbar y^3\frac{d}{dx}(\psi)$$
$$= -i\hbar x\left(2y\psi + y^2\frac{d\psi}{dy}\right) + i\hbar y\left(0 + y^2\frac{d\psi}{dx}\right) - i\hbar xy^2\frac{d\psi}{dy} - i\hbar y^3\frac{d\psi}{dx}$$
$$= -2i\hbar xy\psi - i\hbar xy^2\frac{d\psi}{dy} + i\hbar y^3\frac{d\psi}{dx} - i\hbar xy^2\frac{d\psi}{dy} - i\hbar y^3\frac{d\psi}{dx}$$
$$= -2i\hbar xy\psi$$

$$[L_z, y^2] = -2i\hbar xy$$

7. $[\boldsymbol{L_x}, \boldsymbol{z^2}]$

Cara I

$$[L_x, z^2] = [(yp_z - zp_y), z^2]$$
$$= [yp_z, z^2] - [zp_y, z^2]$$
$$= (y[p_z, z^2] + [y, z^2]p_z) - (z[p_y, z^2] + [z, z^2]p_y)$$
$$= (y(-2i\hbar z) + (0)p_z) - (z(0) + (0)p_y)$$
$$= -2i\hbar yz$$

Cara II

$$[L_x, z^2]\psi = (L_x z^2 - z^2 L_x)\psi$$
$$= L_x z^2\psi - z^2 L_x\psi$$
$$= (yp_z - zp_y)z^2\psi - z^2(yp_z - zp_y)\psi$$
$$= yp_z z^2\psi - zp_y z^2\psi - yz^2 p_z\psi + z^3 p_y\psi$$
$$= y\left(-i\hbar\frac{d}{dz}\right)z^2\psi - z\left(-i\hbar\frac{d}{dy}\right)z^2\psi - yz^2\left(-i\hbar\frac{d}{dz}\right)\psi + z^3\left(-i\hbar\frac{d}{dy}\right)\psi$$

$$= -i\hbar y\frac{d}{dz}(z^2\psi) + i\hbar z\frac{d}{dy}(z^2\psi) + i\hbar yz^2\frac{d}{dz}(\psi) - i\hbar z^3\frac{d}{dy}(\psi)$$

$$= -i\hbar y\left(2z\psi + z^2\frac{d\psi}{dz}\right) + i\hbar z\left(0 + z^2\frac{d\psi}{dy}\right) + i\hbar yz^2\frac{d\psi}{dz} - i\hbar z^3\frac{d\psi}{dy}$$

$$= -2i\hbar yz\psi - i\hbar yz^2\frac{d\psi}{dz} + i\hbar z^3\frac{d\psi}{dy} + i\hbar yz^2\frac{d\psi}{dz} - i\hbar z^3\frac{d\psi}{dy}$$

$$= -2i\hbar yz\psi$$

$$[L_x, z^2] = -2i\hbar yz$$

8. $[\boldsymbol{L_y}, \boldsymbol{z^2}]$

Cara I

$$[L_y, z^2] = [(zp_x - xp_z), z^2]$$

$$= [zp_x, z^2] - [xp_z, z^2]$$

$$= (z[p_x, z^2] + [z, z^2]p_x) - (x[p_z, z^2] + [x, z^2]p_z)$$

$$= (z(0) + (0)p_x) - (x(-2i\hbar z) + (0)p_z)$$

$$= 2i\hbar xz$$

Cara II

$$[L_y, z^2]\psi = (L_y z^2 - z^2 L_y)\psi$$

$$= L_y z^2\psi - z^2 L_y\psi$$

$$= (zp_x - xp_z)z^2\psi - z^2(zp_x - xp_z)\psi$$

$$= zp_x z^2\psi - xp_z z^2\psi - z^3 p_x\psi + xz^2 p_z\psi$$

$$= z\left(-i\hbar\frac{d}{dx}\right)z^2\psi - x\left(-i\hbar\frac{d}{dz}\right)z^2\psi - z^3\left(-i\hbar\frac{d}{dx}\right)\psi + xz^2\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= -i\hbar z\frac{d}{dx}(z^2\psi) + i\hbar x\frac{d}{dz}(z^2\psi) + i\hbar z^3\frac{d}{dx}(\psi) - i\hbar xz^2\frac{d}{dz}(\psi)$$

$$= -i\hbar z\left(0 + z^2\frac{d\psi}{dx}\right) + i\hbar x\left(2z\psi + z^2\frac{d\psi}{dz}\right) + i\hbar z^3\frac{d\psi}{dx} - i\hbar xz^2\frac{d\psi}{dz}$$

$$= -i\hbar z^3\frac{d\psi}{dx} + 2i\hbar xz\psi + i\hbar xz^2\frac{d\psi}{dz} + i\hbar z^3\frac{d\psi}{dx} - i\hbar xz^2\frac{d\psi}{dz}$$

$$= 2i\hbar xz\psi$$

$$[L_y, z^2] = 2i\hbar xz$$

9. $[\boldsymbol{L_z}, \boldsymbol{z^2}]$

Cara I

$$[L_z, z^2] = [(xp_y - yp_x), z^2]$$

$$= [xp_y, z^2] - [yp_x, z^2]$$

$$= \left(x[p_y, z^2] + [x, z^2]p_y\right) - \left(y[p_x, z^2] + [y, z^2]p_x\right)$$

$$= \left(x(0) + (0)p_y\right) - \left(y(0) + (0)p_x\right) = 0$$

Cara II

$$[L_z, z^2]\psi = (L_z z^2 - z^2 L_z)\psi$$

$$= L_z z^2\psi - z^2 L_z\psi$$

$$= (xp_y - yp_x)z^2\psi - z^2(xp_y - yp_x)\psi$$

$$= xp_y z^2\psi - yp_x z^2\psi - xz^2 p_y\psi + yz^2 p_x\psi$$

$$= x\left(-i\hbar\frac{d}{dy}\right)z^2\psi - y\left(-i\hbar\frac{d}{dx}\right)z^2\psi - xz^2\left(-i\hbar\frac{d}{dy}\right)\psi + yz^2\left(-i\hbar\frac{d}{dx}\right)\psi$$

$$= -i\hbar x\frac{d}{dy}(z^2\psi) + i\hbar y\frac{d}{dx}(z^2\psi) + i\hbar xz^2\frac{d}{dy}(\psi) - i\hbar yz^2\frac{d}{dx}(\psi)$$

$$= -i\hbar x\left(0 + z^2\frac{d}{dy}\psi\right) + i\hbar y\left(0 + z^2\frac{d}{dx}\psi\right) + i\hbar xz^2\frac{d\psi}{dy} - i\hbar yz^2\frac{d\psi}{dx}$$

$$= -i\hbar xz^2\frac{d}{dy}\psi + i\hbar yz^2\frac{d}{dx}\psi + i\hbar xz^2\frac{d\psi}{dy} - i\hbar yz^2\frac{d\psi}{dx}$$

$$= 0$$

$$[L_z, z^2] = 0$$

LAMPIRAN F. KOMUTATOR OPERATOR MOMENTUM SUDUT KUADRAT TERHADAP POSISI KUADRAT

1. $[L_x{}^2, x^2]$

Cara I

$$[L_x{}^2, x^2] = \left[\left(i\hbar\left(zp_z + yp_y\right)\right), x^2\right]$$
$$= i\hbar\left[\left(zp_z + yp_y\right), x^2\right]$$
$$= \ i\hbar\left([zp_z, x^2] + [yp_y, x^2]\right)$$
$$= i\hbar\left(\left(z[p_z, x^2] + [z, x^2]p_z\right) + \left(y[p_y, x^2] + [y, x^2]p_y\right)\right)$$
$$= i\hbar\left(\left(z(0) + (0)p_z\right) + \left(y(0) + (0)p_y\right)\right)$$
$$= 0$$

Cara II

$$[L_x{}^2, x^2]\psi = (L_x{}^2x^2 - x^2L_x{}^2)\psi$$
$$= L_x{}^2x^2\psi - x^2L_x{}^2\psi$$
$$= \left(i\hbar\left(zp_z + yp_y\right)\right)x^2\psi - x^2\left(i\hbar\left(zp_z + yp_y\right)\right)\psi$$
$$= i\hbar zp_zx^2\psi + i\hbar yp_yx^2\psi - i\hbar x^2zp_z\psi - i\hbar x^2yp_y\psi$$
$$= i\hbar z\left(-i\hbar\frac{d}{dz}\right)x^2\psi + i\hbar y\left(-i\hbar\frac{d}{dy}\right)x^2\psi - i\hbar x^2z\left(-i\hbar\frac{d}{dz}\right)\psi - i\hbar x^2y\left(-i\hbar\frac{d}{dy}\right)\psi$$
$$= \hbar^2z\frac{d}{dz}(x^2\psi) + \hbar^2y\frac{d}{dy}(x^2\psi) - \hbar^2x^2z\frac{d}{dz}(\psi) - \hbar^2x^2y\frac{d}{dy}(\psi)$$
$$= \hbar^2z\left(0 + x^2\frac{d\psi}{dz}\right) + \hbar^2y\left(0 + x^2\frac{d\psi}{dy}\right) - \hbar^2x^2z\frac{d\psi}{dz} - \hbar^2x^2y\frac{d\psi}{dy}$$
$$= \hbar^2x^2z\frac{d\psi}{dz} + \hbar^2x^2y\frac{d\psi}{dy} - \hbar^2x^2z\frac{d\psi}{dz} - \hbar^2x^2y\frac{d\psi}{dy}$$
$$= 0$$

$$[L_x{}^2, x^2] = 0$$

2. $[L_y{}^2, x^2]$

Cara I

$$[L_y{}^2, x^2] = [(i\hbar(xp_x + zp_z)), x^2]$$
$$= i\hbar[(xp_x + zp_z), x^2] = i\hbar([xp_x, x^2] + [zp_z, x^2])$$

$$= i\hbar\big((x[p_x, x^2] + [x, x^2]p_x) + (z[p_z, x^2] + [z, x^2]p_z)\big)$$

$$= i\hbar\big((x(-2i\hbar x) + (0)p_x) + (z(0) + (0)p_z)\big)$$

$$= 2\hbar^2 x^2$$

Cara II

$$[L_y{}^2, x^2]\psi = (L_y{}^2 x^2 - x^2 L_y{}^2)\psi$$

$$= L_y{}^2 x^2 \psi - x^2 L_y{}^2 \psi$$

$$= \big(i\hbar(xp_x + zp_z)\big)x^2\psi - x^2\big(i\hbar(xp_x + zp_z)\big)\psi$$

$$= i\hbar x p_x x^2 \psi + i\hbar z p_z x^2 \psi - i\hbar x^3 p_x \psi - i\hbar x^2 z p_z \psi$$

$$= i\hbar x \left(-i\hbar \frac{d}{dx}\right) x^2\psi + i\hbar z \left(-i\hbar \frac{d}{dz}\right) x^2\psi - i\hbar x^3 \left(-i\hbar \frac{d}{dx}\right)\psi -$$

$$i\hbar x^2 z \left(-i\hbar \frac{d}{dz}\right)\psi$$

$$= \hbar^2 x \frac{d}{dx}(x^2\psi) + \hbar^2 z \frac{d}{dz}(x^2\psi) - \hbar^2 x^3 \frac{d}{dx}(\psi) - \hbar^2 x^2 z \frac{d}{dz}(\psi)$$

$$= \hbar^2 x \left(2x\psi + x^2 \frac{d\psi}{dx}\right) + \hbar^2 z \left(0 + x^2 \frac{d\psi}{dz}\right) - \hbar^2 x^3 \frac{d\psi}{dx} - \hbar^2 x^2 z \frac{d\psi}{dz}$$

$$= 2\hbar^2 x^2 \psi + \hbar^2 x^3 \frac{d\psi}{dx} + \hbar^2 x^2 z \frac{d\psi}{dz} - \hbar^2 x^3 \frac{d\psi}{dx} - \hbar^2 x^2 z \frac{d\psi}{dz}$$

$$= 2\hbar^2 x^2 \psi$$

$$[L_y{}^2, x^2] = 2\hbar^2 x^2$$

3. $[\boldsymbol{L_z{}^2, x^2}]$

Cara I

$$[L_z{}^2, x^2] = \left[\Big(i\hbar\big(yp_y + xp_x\big)\Big), x^2\right]$$

$$= i\hbar[(yp_y + xp_x), x^2]$$

$$= i\hbar([yp_y, x^2] + [xp_x, x^2])$$

$$= i\hbar\Big(\big(y[p_y, x^2] + [y, x^2]p_y\big) + (x[p_x, x^2] + [x, x^2]p_x)\Big)$$

$$= i\hbar\Big(\big(y(0) + (0)p_y\big) + (x(-2i\hbar x) + (0)p_x)\Big)$$

$$= 2\hbar^2 x^2$$

Cara II

$$[L_z{}^2, x^2]\psi = (L_z{}^2 x^2 - x^2 L_z{}^2)\psi$$

$$= L_z{}^2 x^2 \psi - x^2 L_z{}^2 \psi$$

$$= \left(i\hbar\left(yp_y + xp_x\right)\right)x^2\psi - x^2\left(i\hbar\left(yp_y + xp_x\right)\right)\psi$$

$$= i\hbar yp_y x^2\psi + i\hbar xp_x x^2\psi - i\hbar x^2 yp_y\psi - i\hbar x^3 p_x\psi$$

$$= i\hbar y\left(-i\hbar\frac{d}{dy}\right)x^2\psi + i\hbar x\left(-i\hbar\frac{d}{dx}\right)x^2\psi - i\hbar x^2 y\left(-i\hbar\frac{d}{dy}\right)\psi - i\hbar x^3\left(-i\hbar\frac{d}{dy}\right)\psi$$

$$= \hbar^2 y\frac{d}{dy}(x^2\psi) + \hbar^2 x\frac{d}{dx}(x^2\psi) - \hbar^2 x^2 y\frac{d}{dy}(\psi) - \hbar^2 x^3\frac{d}{dy}(\psi)$$

$$= \hbar^2 y\left(0 + x^2\frac{d\psi}{dy}\right) + \hbar^2 x\left(2x\psi + x^2\frac{d\psi}{dx}\right) - \hbar^2 x^2 y\frac{d}{dy}(\psi) - \hbar^2 x^3\frac{d}{dy}(\psi)$$

$$= \hbar^2 x^2 y\frac{d\psi}{dy} + 2\hbar^2 x^2\psi + \hbar^2 x^3\frac{d\psi}{dx} - \hbar^2 x^2 y\frac{d}{dy}(\psi) - \hbar^2 x^3\frac{d}{dy}(\psi)$$

$$= 2\hbar^2 x^2\psi$$

$$\left[{L_z}^2, x^2\right] = 2\hbar^2 x^2$$

4. $[\boldsymbol{L^2}, \boldsymbol{x^2}]$

Cara I

$$[L^2, x^2] = \left[\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right), x^2\right]$$

$$= 2i\hbar\left[\left(xp_x + yp_y + zp_z\right), x^2\right]$$

$$= 2i\hbar\left([xp_x, x^2] + \left[yp_y, x^2\right] + [zp_z, x^2]\right)$$

$$= 2i\hbar\left((x[p_x, x^2] + [x, x^2]p_x) + (0) + (0)\right)$$

$$= 2i\hbar(x(-2i\hbar x) + (0)p_x)$$

$$= 4\hbar^2 x^2$$

Cara II

$$[L^2, x^2]\psi = (L^2x^2 - x^2L^2)\psi$$

$$= L^2x^2\psi - x^2L^2\psi$$

$$= \left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)x^2\psi - x^2\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)\psi$$

$$= 2i\hbar xp_x x^2\psi + 2i\hbar yp_y x^2\psi + 2i\hbar zp_z x^2\psi - 2i\hbar x^3 p_x\psi - 2i\hbar x^2 yp_y\psi - 2i\hbar x^2 zp_z\psi$$

$$= 2i\hbar x\left(-i\hbar\frac{d}{dx}\right)x^2\psi + 2i\hbar y\left(-i\hbar\frac{d}{dy}\right)x^2\psi + 2i\hbar z\left(-i\hbar\frac{d}{dz}\right)x^2\psi$$

$$-2i\hbar x^3\left(-i\hbar\frac{d}{dx}\right)\psi - 2i\hbar x^2 y\left(-i\hbar\frac{d}{dy}\right)\psi - 2i\hbar x^2 z\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= 2\hbar^2 x\frac{d}{dx}(x^2\psi) + 2\hbar^2 y\frac{d}{dy}(x^2\psi) + 2\hbar^2 z\frac{d}{dz}(x^2\psi) -$$

$$2\hbar^2 x^3\frac{d}{dx}(\psi) - 2\hbar^2 x^2 y\frac{d}{dy}(\psi) - 2\hbar^2 x^2 z\frac{d}{dz}(\psi)$$

$$= 2\hbar^2 x\left(2x\psi + x^2\frac{d\psi}{dx}\right) + 2\hbar^2 y\left(0 + x^2\frac{d\psi}{dy}\right) + 2\hbar^2 z\left(0 +\right.$$

$$\left.x^2\frac{d\psi}{dz}\right) - 2\hbar^2 x^3\frac{d\psi}{dx} - 2\hbar^2 x^2 y\frac{d\psi}{dy} - 2\hbar^2 x^2 z\frac{d\psi}{dz}$$

$$= 4\hbar^2 x^2\psi + 2\hbar^2 x^3\frac{d\psi}{dx} + 2\hbar^2 x^2 y\frac{d\psi}{dy} + 2\hbar^2 x^2 z\frac{d\psi}{dz} - 2\hbar^2 x^3\frac{d\psi}{dx} -$$

$$2\hbar^2 x^2 y\frac{d\psi}{dy} - 2\hbar^2 x^2 z\frac{d\psi}{dz}$$

$$= 4\hbar^2 x^2\psi$$

$$[L^2, x^2] = 4\hbar^2 x^2$$

5. $\boldsymbol{[L_x{}^2, y^2]}$

Cara I

$$[L_x{}^2, y^2] = \left[\left(i\hbar\left(zp_z + yp_y\right)\right), y^2\right]$$

$$= i\hbar\left[\left(zp_z + yp_y\right), y^2\right]$$

$$= i\hbar\left([zp_z, y^2] + [yp_y, y^2]\right)$$

$$= i\hbar\left((0) + \left(y[p_y, y^2] + [y, y^2]p_y\right)\right)$$

$$= i\hbar\left(y(-2i\hbar y) + (0)p_y\right)$$

$$= 2\hbar^2 y^2$$

Cara II

$$[L_x{}^2, y^2]\,\psi = \left(L_x{}^2 y^2 - y^2 L_x{}^2\right)\psi$$

$$= L_x{}^2 y^2\psi - y^2 L_x{}^2\psi$$

$$= \left(i\hbar\left(zp_z + yp_y\right)\right)y^2\psi - y^2\left(i\hbar\left(zp_z + yp_y\right)\right)\psi$$

$$= i\hbar zp_z y^2\psi + i\hbar yp_y y^2\psi - i\hbar y^2 zp_z\psi - i\hbar y^3 p_y\psi$$

$$= i\hbar z\left(-i\hbar\frac{d}{dz}\right)y^2\psi + i\hbar y\left(-i\hbar\frac{d}{dy}\right)y^2\psi - i\hbar y^2 z\left(-i\hbar\frac{d}{dz}\right)\psi -$$

$$i\hbar y^3\left(-i\hbar\frac{d}{dy}\right)\psi$$

$$= \hbar^2 z\frac{d}{dz}(y^2\psi) + \hbar^2 y\frac{d}{dy}(y^2\psi) - \hbar^2 y^2 z\frac{d}{dz}(\psi) - \hbar^2 y^3\frac{d}{dy}(\psi)$$

$$= \hbar^2 z\left(0 + y^2\frac{d\psi}{dz}\right) + \hbar^2 y\left(2y\psi + y^2\frac{d\psi}{dy}\right) - \hbar^2 y^2 z\frac{d\psi}{dz} - \hbar^2 y^3\frac{d\psi}{dy}$$

$$= \hbar^2 y^2 z\frac{d\psi}{dz} + 2\hbar^2 y^2\psi + \hbar^2 y^3\frac{d\psi}{dy} - \hbar^2 y^2 z\frac{d\psi}{dz} - \hbar^2 y^3\frac{d\psi}{dy}$$

$$= 2\hbar^2 y^2\psi$$

$[{L_x}^2, y^2] = 2\hbar^2 y^2$

6. $[\boldsymbol{L_y}^2, \boldsymbol{y}^2]$

Cara I

$$[{L_y}^2, y^2] = [(i\hbar(xp_x + zp_z)), y^2]$$

$$= i\hbar[(xp_x + zp_z), y^2]$$

$$= i\hbar([xp_x, y^2] + [zp_z, y^2])$$

$$= i\hbar(0 + 0)$$

$$= 0$$

Cara II

$$[{L_y}^2, y^2]\,\psi = ({L_y}^2 y^2 - y^2 {L_y}^2)\psi$$

$$= {L_y}^2 y^2\psi - y^2 {L_y}^2\psi$$

$$= (i\hbar(xp_x + zp_z))y^2\psi - y^2(i\hbar(xp_x + zp_z))\psi$$

$$= i\hbar x p_x y^2\psi + i\hbar z p_z y^2\psi - i\hbar x y^2 p_x\psi - i\hbar y^2 z p_z\psi$$

$$= i\hbar x\left(-i\hbar\frac{d}{dx}\right)y^2\psi + i\hbar z\left(-i\hbar\frac{d}{dz}\right)y^2\psi - i\hbar x y^2\left(-i\hbar\frac{d}{dx}\right)\psi - i\hbar y^2 z\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= \hbar^2 x\frac{d}{dx}(y^2\psi) + \hbar^2 z\frac{d}{dz}(y^2\psi) - \hbar^2 x y^2\frac{d}{dx}(\psi) - \hbar^2 y^2 z\frac{d}{dz}(\psi)$$

$$= \hbar^2 x\left(0 + y^2\frac{d\psi}{dx}\right) + \hbar^2 z\left(0 + y^2\frac{d\psi}{dz}\right) - \hbar^2 x y^2\frac{d\psi}{dx} - \hbar^2 y^2 z\frac{d\psi}{dz}$$

$$= \hbar^2 x y^2\frac{d\psi}{dx} + \hbar^2 y^2 z\frac{d\psi}{dz} - \hbar^2 x y^2\frac{d\psi}{dx} - \hbar^2 y^2 z\frac{d\psi}{dz}$$

$$= 0$$

$[{L_y}^2, y^2] = 0$

7. $[\boldsymbol{L_z}^2, \boldsymbol{y}^2]$

Cara I

$$[{L_z}^2, y^2] = \left[\left(i\hbar(yp_y + xp_x)\right), y^2\right]$$

$= i\hbar[(yp_y + xp_x), y^2]$

$= i\hbar([yp_y, y^2] + [xp_x, y^2])$

$= i\hbar\left(\left(y[p_y, y^2] + [y, y^2]p_y\right) + (0)\right)$

$= i\hbar\left(y(-2i\hbar y) + (0)p_y\right)$

$= 2\hbar^2 y^2$

Cara II

$[L_z{}^2, y^2]\psi = (L_z{}^2 y^2 - y^2 L_z{}^2)\psi$

$= L_z{}^2 y^2\psi - y^2 L_z{}^2\psi$

$= \left(i\hbar(yp_y + xp_x)\right) y^2\psi - y^2\left(i\hbar(yp_y + xp_x)\right)\psi$

$= i\hbar y p_y y^2\psi + i\hbar x p_x y^2\psi - i\hbar y^3 p_y\psi - i\hbar x y^2 p_x\psi$

$= i\hbar y\left(-i\hbar\frac{d}{dy}\right) y^2\psi + i\hbar x\left(-i\hbar\frac{d}{dx}\right) y^2\psi - i\hbar y^3\left(-i\hbar\frac{d}{dy}\right)\psi -$
$i\hbar x y^2\left(-i\hbar\frac{d}{dx}\right)\psi$

$= \hbar^2 y\frac{d}{dy}(y^2\psi) + \hbar^2 x\frac{d}{dx}(y^2\psi) - \hbar^2 y^3\frac{d}{dy}(\psi) - \hbar^2 x y^2\frac{d}{dx}(\psi)$

$= \hbar^2 y\left(2y\psi + y^2\frac{d\psi}{dy}\right) + \hbar^2 x\left(0 + y^2\frac{d}{dx}\psi\right) - \hbar^2 y^3\frac{d\psi}{dy} -$
$\hbar^2 x y^2\frac{d\psi}{dx}$

$= 2\hbar^2 y^2\psi + \hbar^2 y^3\frac{d\psi}{dy} + \hbar^2 x y^2\frac{d}{dx}\psi - \hbar^2 y^3\frac{d\psi}{dy} - \hbar^2 x y^2\frac{d\psi}{dx}$

$= 2\hbar^2 y^2\psi$

$[L_z{}^2, y^2] = 2\hbar^2 y^2$

8. $[\boldsymbol{L^2}, \boldsymbol{y^2}]$

Cara I

$[L^2, y^2] = \left[\left(2i\hbar(xp_x + yp_y + zp_z)\right), y^2\right]$

$= 2i\hbar[(xp_x + yp_y + zp_z), y^2]$

$= 2i\hbar([xp_x, y^2] + [yp_y, y^2] + [zp_z, y^2])$

$= 2i\hbar((0) + (-2i\hbar y^2) + (0))$

$= 4\hbar^2 y^2$

Cara II

$$
\begin{aligned}
[L^2, y^2]\psi &= (L^2y^2 - y^2L^2)\psi \\
&= L^2y^2\psi - y^2L^2\psi \\
&= \left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)y^2\psi - y^2\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)\psi \\
&= 2i\hbar xp_x y^2\psi + 2i\hbar yp_y y^2\psi + 2i\hbar zp_z y^2\psi - 2i\hbar xy^2p_x\psi - \\
&\quad 2i\hbar y^3p_y\psi - 2i\hbar y^2zp_z\psi \\
&= 2i\hbar x\left(-i\hbar\tfrac{d}{dx}\right)y^2\psi + 2i\hbar y\left(-i\hbar\tfrac{d}{dy}\right)y^2\psi + 2i\hbar z\left(-i\hbar\tfrac{d}{dz}\right)y^2\psi \\
&\quad -2i\hbar xy^2\left(-i\hbar\tfrac{d}{dx}\right)\psi - 2i\hbar y^3\left(-i\hbar\tfrac{d}{dy}\right)\psi - 2i\hbar y^2z\left(-i\hbar\tfrac{d}{dz}\right)\psi \\
&= 2\hbar^2x\tfrac{d}{dx}(y^2\psi) + 2\hbar^2y\tfrac{d}{dy}(y^2\psi) + 2\hbar^2z\tfrac{d}{dz}(y^2\psi) - \\
&\quad 2\hbar^2xy^2\tfrac{d}{dx}(\psi) - 2\hbar^2y^3\tfrac{d}{dy}(\psi) - 2\hbar^2y^2z\tfrac{d}{dz}(\psi) \\
&= 2\hbar^2x\left(0 + y^2\tfrac{d\psi}{dx}\right) + 2\hbar^2y\left(2y\psi + y^2\tfrac{d\psi}{dy}\right) + 2\hbar^2z\left(0 + \right. \\
&\quad \left. y^2\tfrac{d\psi}{dz}\right) - 2\hbar^2xy^2\tfrac{d\psi}{dx} - 2\hbar^2y^3\tfrac{d\psi}{dy} - 2\hbar^2y^2z\tfrac{d\psi}{dz} \\
&= 2\hbar^2xy^2\tfrac{d\psi}{dx} + 4\hbar^2y^2\psi + 2\hbar^2y^3\tfrac{d\psi}{dy} + 2\hbar^2y^2z\tfrac{d\psi}{dz} - 2\hbar^2xy^2\tfrac{d\psi}{dx} - \\
&\quad 2\hbar^2y^3\tfrac{d\psi}{dy} - 2\hbar^2y^2z\tfrac{d\psi}{dz} \\
&= 4\hbar^2y^2\psi
\end{aligned}
$$

$$[L^2, y^2] = 4\hbar^2y^2$$

9. $[L_x{}^2, z^2]$

Cara I

$$
\begin{aligned}
[L_x{}^2, z^2] &= \left[\left(i\hbar\left(zp_z + yp_y\right)\right), z^2\right] \\
&= i\hbar\left[\left(zp_z + yp_y\right), z^2\right] \\
&= i\hbar\left([zp_z, z^2] + [yp_y, z^2]\right) \\
&= i\hbar(-2i\hbar z^2 + 0) \\
&= 2\hbar^2z^2
\end{aligned}
$$

Cara II

$$
\begin{aligned}
[L_x{}^2, z^2]\,\psi &= (L_x{}^2z^2 - z^2L_x{}^2)\psi \\
&= L_x{}^2z^2\psi - z^2L_x{}^2\psi
\end{aligned}
$$

$$= \left(i\hbar\left(zp_z + yp_y\right)\right)z^2\psi - z^2\left(i\hbar\left(zp_z + yp_y\right)\right)\psi$$

$$= i\hbar zp_z z^2\psi + i\hbar yp_y z^2\psi - i\hbar z^3 p_z\psi - i\hbar yz^2 p_y\psi$$

$$= i\hbar z\left(-i\hbar\frac{d}{dz}\right)z^2\psi + i\hbar y\left(-i\hbar\frac{d}{dy}\right)z^2\psi - i\hbar z^3\left(-i\hbar\frac{d}{dz}\right)\psi -$$

$$i\hbar yz^2\left(-i\hbar\frac{d}{dy}\right)\psi$$

$$= \hbar^2 z\frac{d}{dz}(z^2\psi) + \hbar^2 y\frac{d}{dy}(z^2\psi) - \hbar^2 z^3\frac{d}{dz}(\psi) - \hbar^2 yz^2\frac{d}{dy}(\psi)$$

$$= \hbar^2 z\left(2z\psi + z^2\frac{d\psi}{dz}\right) + \hbar^2 y\left(0 + z^2\frac{d\psi}{dy}\right) - \hbar^2 z^3\frac{d\psi}{dz} - \hbar^2 yz^2\frac{d\psi}{dy}$$

$$= 2\hbar^2 z^2\psi + \hbar^2 z^3\frac{d\psi}{dz} + \hbar^2 yz^2\frac{d\psi}{dy} - \hbar^2 z^3\frac{d\psi}{dz} - \hbar^2 yz^2\frac{d\psi}{dy}$$

$$= 2\hbar^2 z^2\psi$$

$$\left[{L_x}^2, z^2\right] = 2\hbar^2 z^2$$

10. $\left[\boldsymbol{L_y}^{\mathbf{2}}, \boldsymbol{z}^{\mathbf{2}}\right]$

Cara I

$$\left[{L_y}^2, z^2\right] = \left[\left(i\hbar(xp_x + zp_z)\right), z^2\right]$$

$$= i\hbar[(xp_x + zp_z), z^2]$$

$$= i\hbar([xp_x, z^2] + [zp_z, z^2])$$

$$= i\hbar(0 - 2i\hbar z^2)$$

$$= 2\hbar^2 z^2$$

Cara I

$$\left[{L_y}^2, z^2\right]\psi = \left({L_y}^2 z^2 - z^2 {L_y}^2\right)\psi$$

$$= {L_y}^2 z^2\psi - z^2 {L_y}^2\psi$$

$$= \left(i\hbar(xp_x + zp_z)\right)z^2\psi - z^2\left(i\hbar(xp_x + zp_z)\right)\psi$$

$$= i\hbar x\left(-i\hbar\frac{d}{dx}\right)z^2\psi + i\hbar z\left(-i\hbar\frac{d}{dz}\right)z^2\psi - i\hbar xz^2\left(-i\hbar\frac{d}{dx}\right)\psi -$$

$$i\hbar z^3\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= \hbar^2 x\frac{d}{dx}(z^2\psi) + \hbar^2 z\frac{d}{dz}(z^2\psi) - \hbar^2 xz^2\frac{d}{dx}(\psi) - \hbar^2 z^3\frac{d}{dz}(\psi)$$

$$= \hbar^2 x\left(0 + z^2\frac{d\psi}{dx}\right) + \hbar^2 z\left(2z\psi + z^2\frac{d\psi}{dz}\right) - \hbar^2 xz^2\frac{d\psi}{dx} - \hbar^2 z^3\frac{d\psi}{dz}$$

$$= \hbar^2 xz^2\frac{d\psi}{dx} + 2\hbar^2 z^2\psi + \hbar^2 z^3\frac{d\psi}{dz} - \hbar^2 xz^2\frac{d\psi}{dx} - \hbar^2 z^3\frac{d\psi}{dz}$$

$$= 2\hbar^2 z^2\psi$$

$\left[{L_y}^2, z^2\right] = 2\hbar^2 z^2$

11. $\left[\boldsymbol{{L_z}^2}, \boldsymbol{z^2}\right]$

Cara I

$$\left[{L_z}^2, z^2\right] = \left[\left(i\hbar\left(yp_y + xp_x\right)\right), z^2\right]$$
$$= i\hbar\left[\left(yp_y + xp_x\right), z^2\right]$$
$$= i\hbar\left(\left[yp_y, z^2\right] + \left[xp_x, z^2\right]\right)$$
$$= i\hbar(0 + 0)$$
$$= 0$$

Cara II

$$\left[{L_z}^2, z^2\right]\psi = \left({L_z}^2 z^2 - z^2 {L_z}^2\right)\psi$$
$$= {L_z}^2 z^2 \psi - z^2 {L_z}^2 \psi$$
$$= \left(i\hbar\left(yp_y + xp_x\right)\right) z^2\psi - z^2\left(i\hbar\left(yp_y + xp_x\right)\right)\psi$$
$$= i\hbar yp_y z^2\psi + i\hbar xp_x z^2\psi - i\hbar yz^2 p_y\psi - i\hbar xz^2 p_x\psi$$
$$= i\hbar y\left(-i\hbar\frac{d}{dy}\right)z^2\psi + i\hbar x\left(-i\hbar\frac{d}{dx}\right)z^2\psi - i\hbar yz^2\left(-i\hbar\frac{d}{dy}\right)\psi - i\hbar xz^2\left(-i\hbar\frac{d}{dx}\right)\psi$$
$$= \hbar^2 y\frac{d}{dy}(z^2\psi) + \hbar^2 x\frac{d}{dx}(z^2\psi) - \hbar^2 yz^2\frac{d}{dy}(\psi) - \hbar^2 xz^2\frac{d}{dx}(\psi)$$
$$= \hbar^2 y\left(0 + z^2\frac{d}{dy}\psi\right) + \hbar^2 x\left(0 + z^2\frac{d}{dx}\psi\right) - \hbar^2 yz^2\frac{d\psi}{dy} - \hbar^2 xz^2\frac{d\psi}{dx}$$
$$= \hbar^2 yz^2\frac{d}{dy}\psi + \hbar^2 xz^2\frac{d}{dx}\psi - \hbar^2 yz^2\frac{d\psi}{dy} - \hbar^2 xz^2\frac{d\psi}{dx}$$
$$= 0$$

$\left[{L_z}^2, z^2\right] = 0$

12. $[\boldsymbol{L^2}, \boldsymbol{z^2}]$

Cara I

$$[L^2, z^2] = \left[\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right), z^2\right]$$
$$= 2i\hbar\left[\left(xp_x + yp_y + zp_z\right), z^2\right]$$
$$= 2i\hbar\left([xp_x, z^2] + \left[yp_y, z^2\right] + [zp_z, z^2]\right)$$
$$= 2i\hbar(0 + 0 - 2i\hbar z^2)$$

$$= 4\hbar^2 z^2$$

Cara II

$$[L^2, z^2]\,\psi = (L^2 z^2 - z^2 L^2)\psi$$

$$= L^2 z^2 \psi - z^2 L^2 \psi$$

$$= \left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right) z^2\psi - z^2\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)\psi$$

$$= 2i\hbar x p_x z^2\psi + 2i\hbar y p_y z^2\psi + 2i\hbar z p_z z^2\psi - 2i\hbar x z^2 p_x\psi - 2i\hbar y z^2 p_y \psi - 2i\hbar z^3 p_z\psi$$

$$= 2i\hbar x\left(-i\hbar\frac{d}{dx}\right) z^2\psi + 2i\hbar y\left(-i\hbar\frac{d}{dy}\right) z^2\psi + 2i\hbar z\left(-i\hbar\frac{d}{dz}\right) z^2\psi$$
$$-2i\hbar x z^2\left(-i\hbar\frac{d}{dx}\right)\psi - 2i\hbar y z^2\left(-i\hbar\frac{d}{dy}\right)\psi - 2i\hbar z^3\left(-i\hbar\frac{d}{dz}\right)\psi$$

$$= 2\hbar^2 x\frac{d}{dx}(z^2\psi) + 2\hbar^2 y\frac{d}{dy}(z^2\psi) + 2\hbar^2 z\frac{d}{dz}(z^2\psi) - 2i\hbar^2 z^2\frac{d}{dx}(\psi) - 2\hbar^2 y z^2\frac{d}{dy}(\psi) - 2\hbar^2 z^3\frac{d}{dz}(\psi)$$

$$= 2\hbar^2 x\left(0 + z^2\frac{d\psi}{dx}\right) + 2\hbar^2 y\left(0 + z^2\frac{d\psi}{dy}\right) + 2\hbar^2 z\left(2z\psi + z^2\frac{d\psi}{dz}\right) - 2i\hbar^2 z^2\frac{d\psi}{dx} - 2\hbar^2 y z^2\frac{d\psi}{dy} - 2\hbar^2 z^3\frac{d\psi}{dz}$$

$$= 2\hbar^2 x z^2\frac{d\psi}{dx} + 2\hbar^2 y z^2\frac{d\psi}{dy} + 4\hbar^2 z^2\psi + \hbar^2 z^3\frac{d\psi}{dz} - 2i\hbar^2 z^2\frac{d\psi}{dx} - 2\hbar^2 y z^2\frac{d\psi}{dy} - 2\hbar^2 z^3\frac{d\psi}{dz}$$

$$= 4\hbar^2 z^2\psi$$

$$[L^2, z^2] = 4\hbar^2 z^2$$

LAMPIRAN G. KOMUTATOR OPERATOR MOMENTUM SUDUT TERHADAP HAMILTONIAN PARTIKEL BEBAS

1. $[\boldsymbol{L_x}, \boldsymbol{H}]$

Cara I

$$[L_x, H] = \left[(yp_z - zp_y), \left(\frac{1}{2m}(p_x{}^2 + p_y{}^2 + p_z{}^2)\right)\right]$$

$$= \frac{1}{2m}\left[(yp_z - zp_y), (p_y{}^2 + p_z{}^2)\right]$$

$$= \frac{1}{2m}\left(\left[(yp_z - zp_y), p_y{}^2\right] + \left[(yp_z - zp_y), p_z{}^2\right]\right)$$

$$= \frac{1}{2m}\left(\left([yp_z, p_y{}^2] - [zp_y, p_y{}^2]\right) + \left([yp_z, p_z{}^2] - [zp_y, p_z{}^2]\right)\right)$$

$$= \frac{1}{2m}\left(\left(\left(y[p_z, p_y{}^2] + [y, p_y{}^2]p_z\right) - (0)\right) + \left((0) - \left(z[p_y, p_z{}^2] + [z, p_z{}^2]p_y\right)\right)\right)$$

$$= \frac{1}{2m}\left(\left(y(0) + (2i\hbar p_y)p_z\right) + \left(-z(0) - (2i\hbar p_z)p_y\right)\right)$$

$$= \frac{1}{2m}\left(2i\hbar p_y p_z - 2i\hbar p_y p_z\right)$$

$$= \frac{1}{2m}(0)$$

$$= 0$$

Cara II

$$[L_x, H]\psi = (L_x H - H L_x)\psi$$

$$= L_x H\psi - H L_x\,\psi$$

$$= L_x H\psi - H L_x\,\psi$$

$$= (yp_z - zp_y)\left(\frac{1}{2m}(p_x{}^2 + p_y{}^2 + p_z{}^2)\right)\psi - \left(\frac{1}{2m}(p_x{}^2 + p_y{}^2 + p_z{}^2)\right)(yp_z - zp_y)\psi$$

$$= \left(\frac{1}{2m}yp_zp_x{}^2\psi + \frac{1}{2m}yp_zp_y{}^2\psi + \frac{1}{2m}yp_z{}^3\psi - \frac{1}{2m}zp_yp_x{}^2\psi - \frac{1}{2m}zp_y{}^3\psi - \frac{1}{2m}zp_yp_z{}^2\psi\right) - \left(\frac{1}{2m}p_x{}^2yp_z\psi + \frac{1}{2m}p_y{}^2yp_z\psi + \frac{1}{2m}p_z{}^2yp_z\psi - \frac{1}{2m}p_x{}^2zp_y\psi - \frac{1}{2m}p_y{}^2zp_y\psi - \frac{1}{2m}p_z{}^2zp_y\psi\right)$$

$$= \left(\frac{1}{2m}y\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi + \frac{1}{2m}y\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi + \right.$$

$$\frac{1}{2m}y\left(-i\hbar\frac{d}{dz}\right)^3\psi-\frac{1}{2m}z\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi-\frac{1}{2m}z\left(-i\hbar\frac{d}{dy}\right)^3\psi-\frac{1}{2m}z\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi\Big)-\Bigg(\frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2y\left(-i\hbar\frac{d}{dz}\right)\psi+\frac{1}{2m}\left(-i\hbar\frac{d}{dy}\right)^2y\left(-i\hbar\frac{d}{dz}\right)\psi+\frac{1}{2m}\left(-i\hbar\frac{d}{dz}\right)^2y\left(-i\hbar\frac{d}{dz}\right)\psi-\frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2z\left(-i\hbar\frac{d}{dy}\right)\psi-\frac{1}{2m}\left(-i\hbar\frac{d}{dy}\right)^2z\left(-i\hbar\frac{d}{dy}\right)\psi-\frac{1}{2m}\left(-i\hbar\frac{d}{dz}\right)^2z\left(-i\hbar\frac{d}{dy}\right)\psi\Bigg)$$

$$=\Big(\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dx^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dy^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dz^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^2dy}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dy^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dydz^2}\Big)-\Big(\frac{1}{2m}i\hbar^3\frac{d^2}{dx^2}\left(y\frac{d\psi}{dz}\right)+\frac{1}{2m}i\hbar^3\frac{d^2}{dy^2}\left(y\frac{d\psi}{dz}\right)+\frac{1}{2m}i\hbar^3\frac{d^2}{dz^2}\left(y\frac{d\psi}{dz}\right)-\frac{1}{2m}i\hbar^3\frac{d^2}{dx^2}\left(z\frac{d\psi}{dy}\right)-\frac{1}{2m}i\hbar^3\frac{d^2}{dy^2}\left(z\frac{d\psi}{dy}\right)-\frac{1}{2m}i\hbar^3\frac{d^2}{dz^2}\left(z\frac{d\psi}{dy}\right)\Big)$$

$$=\Big(\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dx^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dy^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dz^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^2dy}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dy^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dydz^2}\Big)-\Big(\frac{1}{2m}i\hbar^3\frac{d}{dx}\left(y\frac{d^2\psi}{dxdz}\right)+\frac{1}{2m}i\hbar^3\frac{d}{dy}\left(\frac{d\psi}{dz}+y\frac{d^2\psi}{dydz}\right)+\frac{1}{2m}i\hbar^3\frac{d}{dz}\left(y\frac{d^2\psi}{dz^2}\right)-\frac{1}{2m}i\hbar^3\frac{d}{dx}\left(z\frac{d^2\psi}{dxdy}\right)-\frac{1}{2m}i\hbar^3\frac{d}{dy}\left(z\frac{d^2\psi}{dy^2}\right)-\frac{1}{2m}i\hbar^3\frac{d}{dz}\left(\frac{d\psi}{dy}+z\frac{d^2\psi}{dydz}\right)\Big)$$

$$=\Big(\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dx^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dy^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dz^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^2dy}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dy^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dydz^2}\Big)-\Bigg(\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dx^2dz}+\frac{1}{2m}i\hbar^3\left(\frac{d^2\psi}{dydz}+\left(\frac{d^2\psi}{dydz}+y\frac{d^3\psi}{dy^2dz}\right)\right)+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dz^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^2dy}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dy^3}-\frac{1}{2m}i\hbar^3\left(\frac{d^2\psi}{dydz}+\left(\frac{d^2\psi}{dydz}+z\frac{d^3\psi}{dydz^2}\right)\right)\Bigg)$$

$$=\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dx^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dy^2dz}+\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dz^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^2dy}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dy^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dydz^2}-\frac{1}{2m}i\hbar^3y\frac{d^3\psi}{dx^2dz}-\frac{1}{m}i\hbar^3\frac{d^2\psi}{dydz}-$$

$\frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dy^2dz} - \frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dz^3} + \frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dx^2dy} + \frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dy^3} + \frac{1}{m}i\hbar^3\frac{d^2\psi}{dydz} +$
$\frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dydz^2}$

$= \frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dx^2dz} - \frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dx^2dz} + \frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dy^2dz} - \frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dy^2dz} +$
$\frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dz^3} - \frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dz^3} - \frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dx^2dy} + \frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dx^2dy} - \frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dy^3} +$
$\frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dy^3} - \frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dydz^2} + \frac{1}{2m}i\hbar^3 z\frac{d^3\psi}{dydz^2} - \frac{1}{m}i\hbar^3\frac{d^2\psi}{dydz} + \frac{1}{m}i\hbar^3\frac{d^2\psi}{dydz}$

$= 0$

$[L_x, H] = 0$

2. $[\boldsymbol{L_y}, \boldsymbol{H}]$

Cara I

$[L_y, H] = \left[(zp_x - xp_z), \left(\frac{1}{2m}\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right)\right)\right]$

$= \frac{1}{2m}[(zp_x - xp_z), (p_x{}^2 + p_z{}^2)]$

$= \frac{1}{2m}([(zp_x - xp_z), p_x{}^2] + [(zp_x - xp_z), p_z{}^2])$

$= \frac{1}{2m}\left(([zp_x, p_x{}^2] - [xp_z, p_x{}^2]) + ([zp_x, p_z{}^2] - [xp_z, p_z{}^2])\right)$

$= \frac{1}{2m}\left(\left((0) - (x[p_z, p_x{}^2] + [x, p_x{}^2]p_z)\right) + \left((z[p_x, p_z{}^2] +\right.\right.$
$\left.\left.[z, p_z{}^2]p_x) - (0)\right)\right)$

$= \frac{1}{2m}\left((-x(0) - (2i\hbar p_x)p_z) + (z(0) + (2i\hbar p_z)p_x)\right)$

$= \frac{1}{2m}(-2i\hbar p_x p_z + 2i\hbar p_x p_z)$

$= \frac{1}{2m}(0)$

$= 0$

Cara II

$[L_y, H]\psi = (L_y H - HL_y)\psi$

$= L_y H\psi - HL_y\psi$

$= (zp_x - xp_z)\left(\frac{1}{2m}\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right)\right)\psi - \left(\frac{1}{2m}\left(p_x{}^2 + p_y{}^2 +\right.\right.$
$\left.\left.p_z{}^2\right)\right)(zp_x - xp_z)\psi$

$$= \left(\frac{1}{2m}\, zp_x{}^3\psi + \frac{1}{2m}\, zp_xp_y{}^2\psi + \frac{1}{2m}\, zp_xp_z{}^2\psi - \frac{1}{2m} xp_zp_x{}^2\psi - \frac{1}{2m} xp_zp_y{}^2\psi - \frac{1}{2m} xp_z{}^3\psi\right) - \left(\frac{1}{2m} p_x{}^2zp_x\psi + \frac{1}{2m} p_y{}^2zp_x\psi + \frac{1}{2m} p_z{}^2zp_x\psi - \frac{1}{2m} p_x{}^2xp_z\psi - \frac{1}{2m} p_y{}^2xp_z\psi - \frac{1}{2m} p_z{}^2xp_z\psi\right)$$

$$= \left(\frac{1}{2m}\, z\left(-i\hbar\frac{d}{dx}\right)^3\psi + \frac{1}{2m}\, z\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi + \frac{1}{2m}\, z\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi - \frac{1}{2m} x\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi - \frac{1}{2m} x\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi - \frac{1}{2m} x\left(-i\hbar\frac{d}{dz}\right)^3\psi\right) - \left(\frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 z\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{1}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 z\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{1}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 z\left(-i\hbar\frac{d}{dx}\right)\psi - \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 x\left(-i\hbar\frac{d}{dz}\right)\psi - \frac{1}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 x\left(-i\hbar\frac{d}{dz}\right)\psi - \frac{1}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 x\left(-i\hbar\frac{d}{dz}\right)\psi\right)$$

$$= \left(\frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dx^3} + \frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dxdy^2} + \frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dxdz^2} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dx^2dz} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dy^2dz} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dz^3}\right) - \left(\frac{1}{2m} i\hbar^3 \frac{d^2}{dx^2}\left(z\frac{d\psi}{dx}\right) + \frac{1}{2m} i\hbar^3 \frac{d^2}{dy^2}\left(z\frac{d\psi}{dx}\right) + \frac{1}{2m} i\hbar^3 \frac{d^2}{dz^2}\left(z\frac{d\psi}{dx}\right) - \frac{1}{2m} i\hbar^3 \frac{d^2}{dx^2}\left(x\frac{d\psi}{dz}\right) - \frac{1}{2m} i\hbar^3 \frac{d^2}{dy^2}\left(x\frac{d\psi}{dz}\right) - \frac{1}{2m} i\hbar^3 \frac{d^2}{dz^2}\left(x\frac{d\psi}{dz}\right)\right)$$

$$= \left(\frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dx^3} + \frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dxdy^2} + \frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dxdz^2} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dx^2dz} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dy^2dz} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dz^3}\right) - \left(\frac{1}{2m} i\hbar^3 \frac{d}{dx}\left(z\frac{d^2\psi}{dx^2}\right) + \frac{1}{2m} i\hbar^3 \frac{d}{dy}\left(z\frac{d^2\psi}{dxdy}\right) + \frac{1}{2m} i\hbar^3 \frac{d}{dz}\left(\frac{d\psi}{dx} + z\frac{d^2\psi}{dxdz}\right) - \frac{1}{2m} i\hbar^3 \frac{d}{dx}\left(\frac{d\psi}{dz} + x\frac{d^2\psi}{dxdz}\right) - \frac{1}{2m} i\hbar^3 \frac{d}{dy}\left(x\frac{d^2\psi}{dydz}\right) - \frac{1}{2m} i\hbar^3 \frac{d}{dz}\left(x\frac{d^2\psi}{dz^2}\right)\right)$$

$$= \left(\frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dx^3} + \frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dxdy^2} + \frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dxdz^2} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dx^2dz} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dy^2dz} - \frac{1}{2m} i\hbar^3 x\frac{d^3\psi}{dz^3}\right) - \left(\frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dx^3} + \frac{1}{2m} i\hbar^3 z\frac{d^3\psi}{dxdy^2} +\right.$$

$$\frac{1}{2m}i\hbar^3\left(\frac{d^2\psi}{dxdz}+\frac{d^2\psi}{dxdz}+z\frac{d^3\psi}{dxdz^2}\right)-\frac{1}{2m}i\hbar^3\left(\frac{d^2\psi}{dxdz}+\frac{d^2\psi}{dxdz}+x\frac{d^3\psi}{dx^2dz}\right)-$$

$$\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dy^2dz}-\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dz^3}\Big)$$

$$=\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^3}+\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdy^2}+\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdz^2}-\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dx^2dz}-$$

$$\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dy^2dz}-\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dz^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdy^2}-$$

$$\frac{1}{2m}i\hbar^3\left(2\frac{d^2\psi}{dxdz}+z\frac{d^3\psi}{dxdz^2}\right)+\frac{1}{2m}i\hbar^3\left(2\frac{d^2\psi}{dxdz}+x\frac{d^3\psi}{dx^2dz}\right)+$$

$$\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dy^2dz}+\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dz^3}$$

$$=\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^3}+\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdy^2}+\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdz^2}-\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dx^2dz}-$$

$$\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dy^2dz}-\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dz^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdy^2}-$$

$$\frac{1}{m}i\hbar^3\frac{d^2\psi}{dxdz}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdz^2}+\frac{1}{m}i\hbar^3\frac{d^2\psi}{dxdz}+\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dx^2dz}+$$

$$\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dy^2dz}+\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dz^3}$$

$$=\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^3}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dx^3}+\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdy^2}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdy^2}+$$

$$\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdz^2}-\frac{1}{2m}i\hbar^3z\frac{d^3\psi}{dxdz^2}-\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dx^2dz}+\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dx^2dz}-$$

$$\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dy^2dz}+\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dy^2dz}-\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dz^3}+\frac{1}{2m}i\hbar^3x\frac{d^3\psi}{dz^3}-$$

$$\frac{1}{m}i\hbar^3\frac{d^2\psi}{dxdz}+\frac{1}{m}i\hbar^3\frac{d^2\psi}{dxdz}$$

$$=0$$

$$[L_y,H]=0$$

3. $[\boldsymbol{L_z},\boldsymbol{H}]$

Cara I

$$[L_z,H]=\left[(xp_y-yp_x),\left(\frac{1}{2m}({p_x}^2+{p_y}^2+{p_z}^2)\right)\right]$$

$$=\frac{1}{2m}[(xp_y-yp_x),({p_x}^2+{p_y}^2)]$$

$$=\frac{1}{2m}\left([(xp_y-yp_x),{p_x}^2]+[(xp_y-yp_x),{p_y}^2]\right)$$

$$=\frac{1}{2m}\left(([xp_y,{p_x}^2]-[yp_x,{p_x}^2])+([xp_y,{p_y}^2]-[yp_x,{p_y}^2])\right)$$

$$= \frac{1}{2m}\left(\left(\left(x[p_y, p_x{}^2] + [x, p_x{}^2]p_y\right) - (0)\right) + \left((0) - \left(y[p_x, p_y{}^2] + [y, p_y{}^2]p_x\right)\right)\right)$$

$$= \frac{1}{2m}\left(\left(x(0) + (2i\hbar p_x)p_y\right) + \left(-y(0) - (2i\hbar p_y)p_x\right)\right)$$

$$= \frac{1}{2m}\left(2i\hbar p_x p_y - 2i\hbar p_x p_y\right)$$

$$= \frac{1}{2m}(0)$$

$$= 0$$

Cara II

$$[L_z, H]\psi = (L_z H - H L_z)\psi$$

$$= L_z H\psi - H L_z \psi$$

$$= (xp_y - yp_x)\left(\frac{1}{2m}\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right)\right)\psi - \left(\frac{1}{2m}\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right)\right)(xp_y - yp_x)\psi$$

$$= \left(\frac{1}{2m}xp_y p_x{}^2\psi + \frac{1}{2m}xp_y{}^3\psi + \frac{1}{2m}xp_y p_z{}^2\psi - \frac{1}{2m}yp_x{}^3\psi - \frac{1}{2m}yp_x p_y{}^2\psi - \frac{1}{2m}yp_x p_z{}^2\psi\right) - \left(\frac{1}{2m}p_x{}^2 xp_y\psi + \frac{1}{2m}p_y{}^2 xp_y\psi + \frac{1}{2m}p_z{}^2 xp_y\psi - \frac{1}{2m}p_x{}^2 yp_x\psi - \frac{1}{2m}p_y{}^2 yp_x\psi - \frac{1}{2m}p_z{}^2 yp_x\psi\right)$$

$$= \left(\frac{1}{2m}x\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi + \frac{1}{2m}x\left(-i\hbar\frac{d}{dy}\right)^3\psi + \frac{1}{2m}x\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi - \frac{1}{2m}y\left(-i\hbar\frac{d}{dx}\right)^3\psi - \frac{1}{2m}y\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi - \frac{1}{2m}y\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi\right) - \left(\frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 x\left(-i\hbar\frac{d}{dy}\right)\psi + \frac{1}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 x\left(-i\hbar\frac{d}{dy}\right)\psi + \frac{1}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 x\left(-i\hbar\frac{d}{dy}\right)\psi - \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 y\left(-i\hbar\frac{d}{dx}\right)\psi - \frac{1}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 y\left(-i\hbar\frac{d}{dx}\right)\psi - \frac{1}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 y\left(-i\hbar\frac{d}{dx}\right)\psi\right)$$

$$= \left(\frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dx^2dy} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdy^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdz^2}\right) - \left(\frac{1}{2m} i\hbar^3 \frac{d^2}{dx^2}\left(x\frac{d\psi}{dy}\right) + \frac{1}{2m} i\hbar^3 \frac{d^2}{dy^2}\left(x\frac{d\psi}{dy}\right) + \frac{1}{2m} i\hbar^3 \frac{d^2}{dz^2}\left(x\frac{d\psi}{dy}\right) - \frac{1}{2m} i\hbar^3 \frac{d^2}{dx^2}\left(y\frac{d\psi}{dx}\right) - \frac{1}{2m} i\hbar^3 \frac{d^2}{dy^2}\left(y\frac{d\psi}{dx}\right) - \frac{1}{2m} i\hbar^3 \frac{d^2}{dz^2}\left(y\frac{d\psi}{dx}\right)\right)$$

$$= \left(\frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dx^2dy} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdy^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdz^2}\right) - \left(\frac{1}{2m} i\hbar^3 \frac{d}{dx}\left(\frac{d\psi}{dy} + x\frac{d^2\psi}{dxdy}\right) + \frac{1}{2m} i\hbar^3 \frac{d}{dy}\left(x\frac{d^2\psi}{dy^2}\right) + \frac{1}{2m} i\hbar^3 \frac{d}{dz}\left(x\frac{d^2\psi}{dydz}\right) - \frac{1}{2m} i\hbar^3 \frac{d}{dx}\left(y\frac{d^2\psi}{dx^2}\right) - \frac{1}{2m} i\hbar^3 \frac{d}{dy}\left(\frac{d\psi}{dx} + y\frac{d^2\psi}{dxdy}\right) - \frac{1}{2m} i\hbar^3 \frac{d}{dz}\left(y\frac{d^2\psi}{dxdz}\right)\right)$$

$$= \left(\frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dx^2dy} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdy^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdz^2}\right) - \left(\frac{1}{2m} i\hbar^3 \left(\frac{d^2\psi}{dxdy} + \frac{d^2\psi}{dxdy} + x\frac{d^3\psi}{dx^2dy}\right) + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} - \frac{1}{2m} i\hbar^3 \left(\frac{d^2\psi}{dxdy} + \frac{d^2\psi}{dxdy} + y\frac{d^3\psi}{dxdy^3}\right) - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdz^2}\right)$$

$$= \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dx^2dy} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdy^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdz^2} - \frac{1}{2m} i\hbar^3 \left(2\frac{d^2\psi}{dxdy} + x\frac{d^3\psi}{dx^2dy}\right) - \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} - \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} + \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} + \frac{1}{2m} i\hbar^3 \left(2\frac{d^2\psi}{dxdy} + y\frac{d^3\psi}{dxdy^3}\right) + \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dxdz^2}$$

$$= \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dx^2dy} - \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dx^2dy} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} - \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dy^3} + \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} - \frac{1}{2m} i\hbar^3 x \frac{d^3\psi}{dydz^2} - \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} + \frac{1}{2m} i\hbar^3 y \frac{d^3\psi}{dx^3} -$$

$$\frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dxdy^2}+\frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dxdy^3}-\frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dxdz^2}+\frac{1}{2m}i\hbar^3 y\frac{d^3\psi}{dxdz^2}-$$
$$\frac{1}{m}i\hbar^3\frac{d^2\psi}{dxdy}+\frac{1}{m}i\hbar^3\frac{d^2\psi}{dxdy}$$
$$=0$$

$$[L_z,H]=0$$

4. $\boldsymbol{[L_x{}^2, H]}$

Cara I

$$[L_x{}^2,H]=\left[\left(i\hbar(zp_z+yp_y)\right),\left(\tfrac{1}{2m}(p_x{}^2+p_y{}^2+p_z{}^2)\right)\right]$$
$$=\tfrac{i\hbar}{2m}\left[(zp_z+yp_y),(p_y{}^2+p_z{}^2)\right]$$
$$=\tfrac{i\hbar}{2m}\left(\left[(zp_z+yp_y),p_y{}^2\right]+\left[(zp_z+yp_y),p_z{}^2\right]\right)$$
$$=\tfrac{i\hbar}{2m}\left(\left([zp_z,p_y{}^2]+[yp_y,p_y{}^2]\right)+\left([zp_z,p_z{}^2]+[yp_y,p_z{}^2]\right)\right)$$
$$=\tfrac{i\hbar}{2m}\left(\left((0)+\left(y[p_y,p_y{}^2]+[y,p_y{}^2]p_y\right)\right)+\left((z[p_z,p_z{}^2]+\right.\right.$$
$$\left.\left.[z,p_z{}^2]p_z)+(0)\right)\right)$$
$$=\tfrac{i\hbar}{2m}\left(\left(y(0)+(2i\hbar p_y)p_y\right)+\left(z(0)+(2i\hbar p_z)p_z\right)\right)$$
$$=\tfrac{i\hbar}{2m}\left(2i\hbar p_y{}^2+2i\hbar p_z{}^2\right)$$
$$=-2\hbar^2\left(\tfrac{1}{2m}(p_y{}^2+p_z{}^2)\right)$$
$$=-2\hbar^2 H_{yz}$$

Cara II

$$[L_x{}^2,H]\psi=(L_x{}^2H-HL_x{}^2)\psi$$
$$=L_x{}^2H\psi-HL_x{}^2\psi$$
$$=\left(i\hbar(zp_z+yp_y)\right)\left(\tfrac{1}{2m}(p_x{}^2+p_y{}^2+p_z{}^2)\right)\psi-\left(\tfrac{1}{2m}(p_x{}^2+\right.$$
$$\left.p_y{}^2+p_z{}^2)\right)\left(i\hbar(zp_z+yp_y)\right)\psi$$

$$= \left(\frac{i\hbar}{2m} zp_z {p_x}^2\psi + \frac{i\hbar}{2m} zp_z {p_y}^2\psi + \frac{i\hbar}{2m} z{p_z}^3\psi + \frac{i\hbar}{2m} yp_y {p_x}^2\psi + \right.$$
$$\left. \frac{i\hbar}{2m} y{p_y}^3\psi + \frac{i\hbar}{2m} yp_y {p_z}^2\psi\right) - \left(\frac{i\hbar}{2m} {p_x}^2 zp_z\psi + \frac{i\hbar}{2m} {p_y}^2 zp_z\psi + \right.$$
$$\left. \frac{i\hbar}{2m} {p_z}^2 zp_z\psi + \frac{i\hbar}{2m} {p_x}^2 yp_y\psi + \frac{i\hbar}{2m} {p_y}^2 yp_y\psi + \frac{i\hbar}{2m} {p_z}^2 yp_y\psi\right)$$
$$= \left(\frac{i\hbar}{2m} z\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi + \frac{i\hbar}{2m} z\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi + \right.$$
$$\frac{i\hbar}{2m} z\left(-i\hbar\frac{d}{dz}\right)^3\psi + \frac{i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi +$$
$$\left. \frac{i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)^3\psi + \frac{i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi\right) -$$
$$\left(\frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 z\left(-i\hbar\frac{d}{dz}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 z\left(-i\hbar\frac{d}{dz}\right)\psi + \right.$$
$$\frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 z\left(-i\hbar\frac{d}{dz}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi +$$
$$\left. \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi\right)$$
$$= \left(-\frac{\hbar^4}{2m} z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m} z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m} z\frac{d^3\psi}{dz^3} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dy^3} - \right.$$
$$\left. \frac{\hbar^4}{2m} y\frac{d^3\psi}{dydz^2}\right) - \left(-\frac{\hbar^4}{2m}\frac{d^2}{dx^2}\left(z\frac{d\psi}{dz}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dy^2}\left(z\frac{d\psi}{dz}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dz^2}\left(z\frac{d\psi}{dz}\right) - \right.$$
$$\left. \frac{\hbar^4}{2m}\frac{d^2}{dx^2}\left(y\frac{d\psi}{dy}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dy^2}\left(y\frac{d\psi}{dy}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dz^2}\left(y\frac{d\psi}{dy}\right)\right)$$
$$= \left(-\frac{\hbar^4}{2m} z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m} z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m} z\frac{d^3\psi}{dz^3} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dy^3} - \right.$$
$$\left. \frac{\hbar^4}{2m} y\frac{d^3\psi}{dydz^2}\right) - \left(-\frac{\hbar^4}{2m} z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m} z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m}\left(2\frac{d^2\psi}{dz^2} + z\frac{d^3\psi}{dz^3}\right) - \right.$$
$$\left. \frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m}\left(2\frac{d^2\psi}{dy^2} + y\frac{d^3\psi}{dy^3}\right) - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dydz^2}\right)$$
$$= -\frac{\hbar^4}{2m} z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m} z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m} z\frac{d^3\psi}{dz^3} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dy^3} -$$
$$\frac{\hbar^4}{2m} y\frac{d^3\psi}{dydz^2} + \frac{\hbar^4}{2m} z\frac{d^3\psi}{dx^2dz} + \frac{\hbar^4}{2m} z\frac{d^3\psi}{dy^2dz} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dz^2} + \frac{\hbar^4}{2m} z\frac{d^3\psi}{dz^3} +$$
$$\frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dy^2} + \frac{\hbar^4}{2m} y\frac{d^3\psi}{dy^3} + \frac{\hbar^4}{2m} y\frac{d^3\psi}{dydz^2}$$

$$= -\frac{ħ^4}{2m} z \frac{d^3\psi}{dx^2 dz} + \frac{ħ^4}{2m} z \frac{d^3\psi}{dx^2 dz} - \frac{ħ^4}{2m} z \frac{d^3\psi}{dy^2 dz} + \frac{ħ^4}{2m} z \frac{d^3\psi}{dy^2 dz} - \frac{ħ^4}{2m} z \frac{d^3\psi}{dz^3} +$$
$$\frac{ħ^4}{2m} z \frac{d^3\psi}{dz^3} - \frac{ħ^4}{2m} y \frac{d^3\psi}{dx^2 dy} + \frac{ħ^4}{2m} y \frac{d^3\psi}{dx^2 dy} - \frac{ħ^4}{2m} y \frac{d^3\psi}{dy^3} + \frac{ħ^4}{2m} y \frac{d^3\psi}{dy^3} -$$
$$\frac{ħ^4}{2m} y \frac{d^3\psi}{dy dz^2} + \frac{ħ^4}{2m} y \frac{d^3\psi}{dy dz^2} + 2\frac{ħ^4}{2m} \frac{d^2\psi}{dz^2} + 2\frac{ħ^4}{2m} \frac{d^2\psi}{dy^2}$$
$$= 2\frac{ħ^4}{2m} \frac{d^2\psi}{dy^2} + 2\frac{ħ^4}{2m} \frac{d^2\psi}{dz^2}$$
$$= -2ħ^2 \left(\frac{1}{2m}\right)\left(-iħ\frac{d}{dy}\right)^2 \psi - 2ħ^2 \left(\frac{1}{2m}\right)\left(-iħ\frac{d}{dz}\right)^2 \psi$$
$$= -2ħ^2 \left(\frac{1}{2m}\right)\left({p_y}^2 + {p_z}^2\right)\psi$$
$$= -2ħ^2 H_{yz}\psi$$
$$\left[{L_x}^2, H\right] = -2ħ^2 H_{yz}$$

5. $\left[\boldsymbol{L_y}^{\boldsymbol{2}}, \boldsymbol{H}\right]$

Cara I

$$\left[{L_y}^2, H\right] = \left[\left(iħ(xp_x + zp_z)\right), \left(\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right)\right]$$
$$= \frac{iħ}{2m}\left[(xp_x + zp_z), ({p_x}^2 + {p_z}^2)\right]$$
$$= \frac{iħ}{2m}\left([(xp_x + zp_z), {p_x}^2] + [(xp_x + zp_z), {p_z}^2]\right)$$
$$= \frac{iħ}{2m}\left(([xp_x, {p_x}^2] + [zp_z, {p_x}^2]) + ([xp_x, {p_z}^2] + [zp_z, {p_z}^2])\right)$$
$$= \frac{iħ}{2m}\left(\left((x[p_x, {p_x}^2] + [x, {p_x}^2]p_x) + (0)\right) + \left((0) + (z[p_z, {p_z}^2] +\right.\right.$$
$$\left.\left.[z, {p_z}^2]p_z)\right)\right)$$
$$= \frac{iħ}{2m}\left((x(0) + (2iħp_x)p_x) + (z(0) + (2iħp_z)p_z)\right)$$
$$= \frac{iħ}{2m}\left(2iħ{p_x}^2 + 2iħ{p_z}^2\right)$$
$$= -2ħ^2\left(\frac{1}{2m}\left({p_y}^2 + {p_z}^2\right)\right)$$
$$= -2ħ^2 H_{xz}$$

Cara II

$$\left[{L_y}^2, H\right]\psi = \left({L_y}^2 H - H{L_y}^2\right)\psi$$
$$= {L_y}^2 H\psi - H{L_y}^2\psi$$

$$= \left(i\hbar(xp_x + zp_z)\right)\left(\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right)\psi - \left(\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right)\left(i\hbar(xp_x + zp_z)\right)\psi$$

$$= \left(\frac{i\hbar}{2m}x{p_x}^3\psi + \frac{i\hbar}{2m}xp_x{p_y}^2\psi + \frac{i\hbar}{2m}xp_x{p_z}^2\psi + \frac{i\hbar}{2m}zp_z{p_x}^2\psi + \frac{i\hbar}{2m}zp_z{p_y}^2\psi + \frac{i\hbar}{2m}z{p_z}^3\psi\right) - \left(\frac{i\hbar}{2m}{p_x}^2xp_x\psi + \frac{i\hbar}{2m}{p_y}^2xp_x\psi + \frac{i\hbar}{2m}{p_z}^2xp_x\psi + \frac{i\hbar}{2m}{p_x}^2zp_z\psi + \frac{i\hbar}{2m}{p_y}^2zp_z\psi + \frac{i\hbar}{2m}{p_z}^2zp_z\psi\right)$$

$$= \left(\frac{i\hbar}{2m}x\left(-i\hbar\frac{d}{dx}\right)^3\psi + \frac{i\hbar}{2m}x\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi + \frac{i\hbar}{2m}x\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi + \frac{i\hbar}{2m}z\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi + \frac{i\hbar}{2m}z\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi + \frac{i\hbar}{2m}z\left(-i\hbar\frac{d}{dz}\right)^3\psi\right) - \left(\frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2x\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2x\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2x\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2z\left(-i\hbar\frac{d}{dz}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2z\left(-i\hbar\frac{d}{dz}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2z\left(-i\hbar\frac{d}{dz}\right)\psi\right)$$

$$= \left(-\frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dz^3}\right) - \left(-\frac{\hbar^4}{2m}\frac{d^2}{dx^2}\left(x\frac{d\psi}{dx}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dy^2}\left(x\frac{d\psi}{dx}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dz^2}\left(x\frac{d\psi}{dx}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dx^2}\left(z\frac{d\psi}{dz}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dy^2}\left(z\frac{d\psi}{dz}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dz^2}\left(z\frac{d\psi}{dz}\right)\right)$$

$$= \left(-\frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dz^3}\right) - \left(-\frac{\hbar^4}{2m}\left(2\frac{d^2\psi}{dx^2} + x\frac{d^3\psi}{dx^3}\right) - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m}\left(2\frac{d^2\psi}{dz^2} + z\frac{d^3\psi}{dz^3}\right)\right)$$

$$= -\frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} -$$
$$\frac{\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dz^3} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dx^2} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} +$$
$$\frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} + \frac{\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} + \frac{\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dz^2} + \frac{\hbar^4}{2m}z\frac{d^3\psi}{dz^3}$$
$$= -\frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} +$$
$$\frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} + \frac{\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} +$$
$$\frac{\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{\hbar^4}{2m}z\frac{d^3\psi}{dz^3} + \frac{\hbar^4}{2m}z\frac{d^3\psi}{dz^3} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dx^2} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dz^2}$$
$$= 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dx^2} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dz^2}$$
$$= -2\hbar^2\left(\frac{1}{2m}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi - 2\hbar^2\left(\frac{1}{2m}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi$$
$$= -2\hbar^2\left(\frac{1}{2m}\right)({p_x}^2 + {p_z}^2)\psi$$
$$= -2\hbar^2 H_{xz}\psi$$
$$[{L_y}^2, H] = -2\hbar^2 H_{xz}$$

6. $\boldsymbol{[{L_z}^2, H]}$

Cara I

$$[{L_z}^2, H] = \left[\left(i\hbar(yp_y + xp_x)\right), \left(\frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2)\right)\right]$$
$$= \frac{i\hbar}{2m}[(yp_y + xp_x), ({p_x}^2 + {p_y}^2)]$$
$$= \frac{i\hbar}{2m}([(yp_y + xp_x), {p_x}^2] + [(yp_y + xp_x), {p_y}^2])$$
$$= \frac{i\hbar}{2m}\left(([yp_y, {p_x}^2] + [xp_x, {p_x}^2]) + ([yp_y, {p_y}^2] + [xp_x, {p_y}^2])\right)$$
$$= \frac{i\hbar}{2m}\left(((0) + (x[p_x, {p_x}^2] + [x, {p_x}^2]p_x)) + \left((y[p_y, {p_y}^2] + [y, {p_y}^2]p_y) + (0)\right)\right)$$
$$= \frac{i\hbar}{2m}\left((x(0) + (2i\hbar p_x)p_x) + \left(y(0) + (2i\hbar p_y)p_y\right)\right)$$
$$= \frac{i\hbar}{2m}(2i\hbar {p_x}^2 + 2i\hbar {p_y}^2)$$
$$= -2\hbar^2\left(\frac{1}{2m}({p_x}^2 + {p_y}^2)\right)$$

$$= -2\hbar^2 H_{xy}$$

Cara II

$$[L_z{}^2, H]\psi = (L_z{}^2 H - H L_z{}^2)\psi$$

$$= L_z{}^2 H\psi - H L_z{}^2\psi$$

$$= \left(i\hbar(yp_y + xp_x)\right)\left(\frac{1}{2m}(p_x{}^2 + p_y{}^2 + p_z{}^2)\right)\psi - \left(\frac{1}{2m}(p_x{}^2 + p_y{}^2 + p_z{}^2)\right)\left(i\hbar(yp_y + xp_x)\right)\psi$$

$$= \left(\frac{i\hbar}{2m} yp_y p_x{}^2\psi + \frac{i\hbar}{2m} yp_y{}^3\psi + \frac{i\hbar}{2m} yp_y p_z{}^2\psi + \frac{i\hbar}{2m} xp_x{}^3\psi + \frac{i\hbar}{2m} xp_x p_y{}^2\psi + \frac{i\hbar}{2m} xp_x p_z{}^2\psi\right) - \left(\frac{i\hbar}{2m} p_x{}^2 yp_y\psi + \frac{i\hbar}{2m} p_y{}^2 yp_y\psi + \frac{i\hbar}{2m} p_z{}^2 yp_y\psi + \frac{i\hbar}{2m} p_x{}^2 xp_x\psi + \frac{i\hbar}{2m} p_y{}^2 xp_x\psi + \frac{i\hbar}{2m} p_z{}^2 xp_x\psi\right)$$

$$= \left(\frac{i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi + \frac{i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)^3\psi + \frac{i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi + \frac{i\hbar}{2m} x\left(-i\hbar\frac{d}{dx}\right)^3\psi + \frac{i\hbar}{2m} x\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi + \frac{i\hbar}{2m} x\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi\right) - \left(\frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 x\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 x\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 x\left(-i\hbar\frac{d}{dx}\right)\psi\right)$$

$$= \left(-\frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dy^3} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dydz^2} + \frac{i\hbar}{2m} x\frac{d^3\psi}{dx^3} + -\frac{\hbar^4}{2m} x\frac{d^3\psi}{dxdy^2} - \frac{\hbar^4}{2m} x\frac{d^3\psi}{dxdz^2}\right) - \left(-\frac{\hbar^4}{2m}\frac{d^2}{dx^2}\left(y\frac{d\psi}{dy}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dy^2}\left(y\frac{d\psi}{dy}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dz^2}\left(y\frac{d\psi}{dy}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dx^2}\left(x\frac{d\psi}{dx}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dy^2}\left(x\frac{d\psi}{dx}\right) - \frac{\hbar^4}{2m}\frac{d^2}{dz^2}\left(x\frac{d\psi}{dx}\right)\right)$$

$$= \left(-\frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dy^3} - \frac{\hbar^4}{2m} y\frac{d^3\psi}{dydz^2} - \frac{\hbar^4}{2m} x\frac{d^3\psi}{dx^3} - \frac{\hbar^4}{2m} x\frac{d^3\psi}{dxdy^2} - \frac{\hbar^4}{2m} x\frac{d^3\psi}{dxdz^2}\right) - \left(-\frac{\hbar^4}{2m} y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m}\left(2\frac{d^2\psi}{dy^2} + \right.\right.$$

$$y\frac{d^3\psi}{dy^3}\Big) - \frac{\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} - \frac{\hbar^4}{2m}\left(2\frac{d^2\psi}{dx^2} + y\frac{d^3\psi}{dx^3}\right) - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} -$$

$$\frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2}\Big)$$

$$= -\frac{\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m}y\frac{d^3\psi}{dy^3} - \frac{\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} -$$

$$\frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} + \frac{\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dy^2} + \frac{\hbar^4}{2m}y\frac{d^3\psi}{dy^3} + \frac{\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} +$$

$$2\frac{\hbar^4}{2m}\frac{d^2\psi}{dx^2} + \frac{\hbar^4}{2m}y\frac{d^3\psi}{dx^3} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2}$$

$$= -\frac{\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} + \frac{\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} - \frac{\hbar^4}{2m}y\frac{d^3\psi}{dy^3} + \frac{\hbar^4}{2m}y\frac{d^3\psi}{dy^3} - \frac{\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} +$$

$$\frac{\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dx^3} + \frac{\hbar^4}{2m}y\frac{d^3\psi}{dx^3} - \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} -$$

$$\frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} + \frac{\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dy^2} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dx^2}$$

$$= 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dx^2} + 2\frac{\hbar^4}{2m}\frac{d^2\psi}{dy^2}$$

$$= -2\hbar^2\left(\frac{1}{2m}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi - 2\hbar^2\left(\frac{1}{2m}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi$$

$$= -2\hbar^2\left(\frac{1}{2m}\right)\left({p_x}^2 + {p_y}^2\right)\psi$$

$$= -2\hbar^2 H_{xy}\psi$$

$$\left[{L_z}^2, H\right] = -2\hbar^2 H_{xy}$$

7. **$[\boldsymbol{L^2}, \boldsymbol{H}]$**

Cara I

$$[L^2, H] = \left[\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right), \left(\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right)\right]$$

$$= \frac{2i\hbar}{2m}\left[\left(xp_x + yp_y + zp_z\right), \left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right]$$

$$= \frac{2i\hbar}{2m}\left(\left[\left(xp_x + yp_y + zp_z\right), {p_x}^2\right] + \left[\left(xp_x + yp_y + zp_z\right), {p_y}^2\right] + \right.$$

$$\left.\left[\left(xp_x + yp_y + zp_z\right), {p_z}^2\right]\right)$$

$$= \frac{2i\hbar}{2m}\left(\left(\left[xp_x, {p_x}^2\right] + \left[yp_y, {p_x}^2\right] + \left[zp_z, {p_x}^2\right]\right) + \left(\left[xp_x, {p_y}^2\right] + \right.\right.$$

$$\left.\left.\left[yp_y, {p_y}^2\right] + \left[zp_z, {p_y}^2\right]\right) + \left(\left[xp_x, {p_z}^2\right] + \left[yp_y, {p_z}^2\right] + \left[zp_z, {p_z}^2\right]\right)\right)$$

$$= \frac{2i\hbar}{2m}\left(\left(2i\hbar{p_x}^2 + 0 + 0\right) + \left(0 + 2i\hbar{p_y}^2 + 0\right) + \left(0 + 0 + 2i\hbar{p_z}^2\right)\right)$$

$$= \frac{2i\hbar}{2m}\left(2i\hbar {p_x}^2 + 2i\hbar {p_y}^2 + 2i\hbar {p_z}^2\right)$$

$$= -4\hbar^2 \left(\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right)$$

$$= -4\hbar^2 H$$

Cara II

$$[L^2, H]\psi = (L^2 H - HL^2)\psi$$

$$= L^2 H\psi - HL^2\psi$$

$$= \left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)\left(\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right)\psi -$$

$$\left(\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\right)\left(2i\hbar\left(xp_x + yp_y + zp_z\right)\right)\psi$$

$$= \left(\frac{2i\hbar}{2m} x{p_x}^3 \psi + \frac{2i\hbar}{2m} xp_x{p_y}^2 \psi + \frac{2i\hbar}{2m} xp_x{p_z}^2 \psi + \frac{2i\hbar}{2m} yp_y{p_x}^2 \psi + \right.$$

$$\frac{2i\hbar}{2m} y{p_y}^3 + \frac{2i\hbar}{2m} yp_y{p_z}^2 \psi + \frac{2i\hbar}{2m} zp_z{p_x}^2 \psi + \frac{2i\hbar}{2m} zp_z{p_y}^2 \psi +$$

$$\left.\frac{2i\hbar}{2m} z{p_z}^3 \psi\right) - \left(\frac{2i\hbar}{2m}{p_x}^2 xp_x\psi + \frac{2i\hbar}{2m}{p_x}^2 yp_y\psi + \frac{2i\hbar}{2m}{p_x}^2 zp_z\psi + \right.$$

$$\frac{2i\hbar}{2m}{p_y}^2 xp_x\psi + \frac{2i\hbar}{2m}{p_y}^2 yp_y\psi + \frac{2i\hbar}{2m}{p_y}^2 zp_z\psi + \frac{2i\hbar}{2m}{p_z}^2 xp_x\psi +$$

$$\left.\frac{2i\hbar}{2m}{p_z}^2 yp_y\psi + \frac{2i\hbar}{2m}{p_z}^2 zp_z\psi\right)$$

$$=$$

$$\left(\frac{2i\hbar}{2m} x\left(-i\hbar\frac{d}{dx}\right)^3 \psi + \frac{2i\hbar}{2m} x\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dy}\right)^2 \psi + \right.$$

$$\frac{2i\hbar}{2m} x\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dz}\right)^2 \psi + \frac{2i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dx}\right)^2 \psi +$$

$$\frac{2i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)^3 \psi + \frac{2i\hbar}{2m} y\left(-i\hbar\frac{d}{dy}\right)\left(-i\hbar\frac{d}{dz}\right)^2 \psi +$$

$$\frac{2i\hbar}{2m} z\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dx}\right)^2 \psi + \frac{2i\hbar}{2m} z\left(-i\hbar\frac{d}{dz}\right)\left(-i\hbar\frac{d}{dy}\right)^2 \psi +$$

$$\left.\frac{2i\hbar}{2m} z\left(-i\hbar\frac{d}{dz}\right)^3 \psi\right) -$$

$$\left(\frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 x\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi + \right.$$

$$\frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 z\left(-i\hbar\frac{d}{dz}\right)\psi + \frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 x\left(-i\hbar\frac{d}{dx}\right)\psi +$$

$$\frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi + \frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dy}\right)^2 z\left(-i\hbar\frac{d}{dz}\right)\psi +$$

$$\frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 x\left(-i\hbar\frac{d}{dx}\right)\psi + \frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 y\left(-i\hbar\frac{d}{dy}\right)\psi +$$

$$\left.\frac{2i\hbar}{2m}\left(-i\hbar\frac{d}{dz}\right)^2 z\left(-i\hbar\frac{d}{dz}\right)\psi\right)$$

$$= \left(-\frac{2\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dy^3} -\right.$$

$$\left.\frac{2\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dz^3}\right) - \left(-\frac{2\hbar^4}{2m}\frac{d^2}{dx^2}\left(x\frac{d\psi}{dx}\right) -\right.$$

$$\frac{2\hbar^4}{2m}\frac{d^2}{dx^2}\left(y\frac{d\psi}{dy}\right) - \frac{2\hbar^4}{2m}\frac{d^2}{dx^2}z\left(\frac{d\psi}{dz}\right) - \frac{2\hbar^4}{2m}\frac{d^2}{dy^2}\left(x\frac{d\psi}{dx}\right) - \frac{2\hbar^4}{2m}\frac{d^2}{dy^2}\left(y\frac{d\psi}{dy}\right) -$$

$$\left.\frac{2\hbar^4}{2m}\frac{d^2}{dy^2}\left(z\frac{d}{dz}\right)\psi - \frac{2\hbar^4}{2m}\frac{d^2}{dz^2}\left(x\frac{d\psi}{dx}\right) - \frac{2\hbar^4}{2m}\frac{d^2}{dz^2}\left(y\frac{d\psi}{dy}\right) - \frac{2\hbar^4}{2m}\frac{d^2}{dz^2}\left(z\frac{d\psi}{dz}\right)\right)$$

$$= \left(-\frac{2\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dy^3} -\right.$$

$$\left.\frac{2\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dz^3}\right) - \left(-\frac{2\hbar^4}{2m}\left(2\frac{d^2\psi}{dx^2} + x\frac{d^3\psi}{dx^3}\right) -\right.$$

$$\frac{2\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{2\hbar^4}{2m}\left(2\frac{d^2\psi}{dy^2} + y\frac{d^3\psi}{dy^3}\right) - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} -$$

$$\left.\frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} - \frac{2\hbar^4}{2m}\left(2\frac{d^2\psi}{dz^2} + z\frac{d^3\psi}{dz^3}\right)\right)$$

$$= -\frac{2\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dy^3} -$$

$$\frac{2\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dz^3} + \frac{4\hbar^4}{2m}\frac{d^2\psi}{dx^2} + \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dx^3} +$$

$$\frac{2\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} + \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} + \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} + \frac{4\hbar^4}{2m}\frac{d^2\psi}{dy^2} + \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dy^3} + \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} +$$

$$\frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} + \frac{2\hbar^4}{2m}\frac{d^3\psi}{dydz^2} + \frac{4\hbar^4}{2m}\frac{d^2\psi}{dz^2} + \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dz^3}$$

$$= -\frac{2\hbar^4}{2m}x\frac{d^3\psi}{dx^3} + \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dx^3} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} + \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdy^2} - \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} + \frac{2\hbar^4}{2m}x\frac{d^3\psi}{dxdz^2} -$$

$$\frac{2\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} + \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dx^2dy} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dy^3} + \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dy^3} - \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} + \frac{2\hbar^4}{2m}y\frac{d^3\psi}{dydz^2} -$$

$$\frac{2\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} + \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dx^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} + \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dy^2dz} - \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dz^3} + \frac{2\hbar^4}{2m}z\frac{d^3\psi}{dz^3} +$$

$$\frac{4\hbar^4}{2m}\frac{d^2\psi}{dx^2} + \frac{4\hbar^4}{2m}\frac{d^2\psi}{dy^2} + \frac{4\hbar^4}{2m}\frac{d^2\psi}{dz^2}$$

$$= \frac{4\hbar^4}{2m}\frac{d^2\psi}{dx^2} + \frac{4\hbar^4}{2m}\frac{d^2\psi}{dy^2} + \frac{4\hbar^4}{2m}\frac{d^2\psi}{dz^2}$$

$$= -4\hbar^2\left(\frac{1}{2m}\right)\left(-i\hbar\frac{d}{dx}\right)^2\psi - 4\hbar^2\left(\frac{1}{2m}\right)\left(-i\hbar\frac{d}{dy}\right)^2\psi -$$

$$4\hbar^2\left(\frac{1}{2m}\right)\left(-i\hbar\frac{d}{dz}\right)^2\psi$$

$$= -4\hbar^2\left(\frac{1}{2m}\right)\left({p_x}^2 + {p_y}^2 + {p_z}^2\right)\psi = -4\hbar^2 H\psi$$

$$[L^2, H] = -4\hbar^2 H$$