Several classes of graphs and their *r*-dynamic chromatic numbers

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Several classes of graphs and their r-dynamic chromatic numbers

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Abstract. Let G be a simple, connected and undirected graph. Let r, k be natural numbers. By a proper k-coloring of a graph G, we mean a map $c : V(G) \to S$, where $|S| = k$, such that any two adjacent vertices receive different colors. An r-dynamic k-coloring is a proper k-coloring c of G such that $|c(N(v))| \geq min\{r, d(v)\}$ for each vertex v in $V(G)$, where $N(v)$ is the neighborhood of v and $c(S) = \{c(v) : v \in S\}$ for a vertex subset S. The r-dynamic chromatic number, written as $\chi_r(G)$, is the minimum k such that G has an r-dynamic k-coloring. By simple observation it is easy to see that $\chi_r(G) \leq \chi_{r+1}(G)$, however $\chi_{r+1}(G) - \chi_r(G)$ does not always show a small difference for any r. Thus, finding an exact value of $\chi_r(G)$ is significantly useful. In this paper, we will study some of them especially when G are prism graph, three-cyclical ladder graph, joint graph and circulant graph.

Keywords: r-dynamic chromatic number, graph coloring, special graphs.

1. Introduction

The r-dynamic chromatic number, introduced by Montgomery [8] and written as $\chi_r(G)$, is the least k such that G has an r-dynamic k-coloring. Note that the 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number of graph has been studied under the name a dynamic chromatic number, denoted by $\chi_d(G)$. In [8], he conjectured $\chi_2(G) \leq \chi(G) + 2$ when G is regular, which remains open. Akbari *et.al.* [4] proved Montgomery's conjecture for bipartite regular graphs, as well as Lai, *et.al.* [9] proved $\chi_2(G) \leq \Delta(G)+1$ for $\Delta(G) \leq 3$ when no component is the 5-cycle. Some other results can be site in [1, 2, 3, 14] .

By a greedy coloring algorithm, Jahanbekama [7] proved that $\chi_r(G) \leq r\Delta(G) + 1$, and equality holds for $\Delta(G) > 2$ if and only if G is r-regular with diameter 2 and girth 5. They improved the bound to $\chi_r(G) \leq \Delta(G) + 2r - 2$ when $\delta(G) > 2r \ln n$ and

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$\chi_r(G) \leq \Delta(G) + r$ when $\delta(G) > r^2 \ln n$. For further results of r-dynamic chromatic number can be seen in [6, 10, 11, 12, 5] .

The following observation is useful to find the exact values of r-dynamic chromatic number.

Observation 1. *Let $\delta(G)$ and $\Delta(G)$ be a minimum and maximum degree of a graph G, respectively. Then the followings hold*

- $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$,
- $\chi(G) \leq \chi_2(G) \leq \chi_3(G) \leq \cdots \leq \chi_{\Delta(G)}(G)$,
- $\chi_{r+1}(G) \geq \chi_r(G)$ *and if* $r \geq \Delta(G)$ *then* $\chi_r(G) = \chi_{\Delta(G)}(G)$.

Taherkhani in [13], proved the following theorem

Theorem 1. *[13] Let G be a d-regular graph and r be a positive integer with $2 \leq r \leq \frac{\delta}{\log(2er(\Delta^2+1))}$. Then the r-dynamic chromatic number of G is $\chi_r(G) \leq \chi(G) + (r-1)\lceil e\frac{\Delta}{\delta}\log(2er(\Delta^2+1))\rceil$, where e euler's number.*

2. The Results

We are ready to show our main theorems. There are four theorems found in this study. These deals with prism graph, three-cyclical ladder graph, joint graph and circulant graph.

Theorem 2. *Let $\mathbf{P}_{n,2}$ be a prism graph, the r-dynamic chromatic number is:*

$$\chi(\mathbf{P}_{n,2}) = \begin{cases} 2, & n \text{ even} \\ 3, & n \text{ odd} \end{cases} \qquad \chi_d(\mathbf{P}_{n,2}) = \begin{cases} 3, & n = 3k, k \in N \\ 4, & n \text{ otherwise} \end{cases}$$

For $r \geq 3$, we have

$$\chi_r(\mathbf{P}_{n,2}) = \begin{cases} 4, & n = 4k, k \in N \\ 6, & n = 3, 7, 11 \\ 5, & n \text{ otherwise} \end{cases}$$

Proof. A prism graph, denoted by $\mathbf{P}_{n,2}, n \geq 3$, is a connected graph with vertex set $V(\mathbf{P}_{n,2}) = \{x_i, y_i, 1 \leq i \leq n\}$, and edge set $E(\mathbf{P}_{n,2}) = \{x_i x_{i+1}, y_i y_{i+1}; 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{y_n y_1\} \cup \{x_i y_i; 1 \leq i \leq n\}$. The order and size of $\mathbf{P}_{n,2}, n \geq 3$ are $|V(\mathbf{P}_{n,2})| = 2n$ and $|E(\mathbf{P}_{n,2})| = 3n$. A prism graph is regular graph of degree 3, thus $\mathbf{P}_{n,2}$, $\delta(\mathbf{P}_{n,2}) = \Delta(\mathbf{P}_{n,2}) = 3$. By Observation 1, $\chi_r(\mathbf{P}_{n,2}) \geq \min\{\Delta(\mathbf{P}_{n,2}), r\} + 1 = \min\{3, r\} + 1$. To find the exact value of r-dynamic chromatic number of $\mathbf{P}_{n,2}$, we define three cases, namely $\chi(\mathbf{P}_{n,2}), \chi_2(\mathbf{P}_{n,2})$ and $\chi_{r\geq 3}(\mathbf{P}_{n,2})$. For $\chi(\mathbf{P}_{n,2})$, the lower bound $\chi(\mathbf{P}_{n,2}) \geq \min\{3, 1\} + 1 = 2$. We will prove that $\chi(\mathbf{P}_{n,2}) \leq 2$ by defining a map $c_1 : V(\mathbf{P}_{n,2}) \to \{1, 2, \ldots, k\}$ for $n \geq 3$, by the following:

$$c_1(x_1, x_2, \ldots, x_n) = \begin{cases} 21 \ \ldots \ 21, & n \text{ even} \\ 12 \ \ldots \ 12\ 3, & n \text{ odd} \end{cases}$$

$$c_1(y_1, y_2, \ldots, y_n) = \begin{cases} 12 \ \ldots \ 12, & n \text{ even} \\ 3\ 12 \ldots \ 12, & n \text{ odd} \end{cases}$$

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It is easy to see that c_1 gives $\chi(\mathbf{P}_{n,2}) \leq 2$ for n even, but for n odd, we could not avoid to have $\chi(\mathbf{P}_{n,2}) \leq 3$, otherwise there are at least two adjacent vertices assigned the same colors. Thus $\chi(\mathbf{P}_{n,2}) = 2$ for n even and $\chi(\mathbf{P}_{n,2}) = 3$, for n odd.

For $\chi_2(\mathbf{P}_{n,2})$, the lower bound $\chi_2(\mathbf{P}_{n,2}) \geq \min\{3,2\}+1 = 3$. We will prove that $\chi_2(\mathbf{P}_{n,2}) \leq 3$ by defining a map $c_2 : V(\mathbf{P}_{n,2}) \to \{1,2,\dots,k\}$ where $n \geq 3$, by the following

$$c_2(x_1,x_2,\dots,x_{n-1}) = \begin{cases} 12123, \; n=5 \\ 123 \; \dots \; 123, \; n \equiv 0 (\text{mod } 3), \\ 123 \; \dots \; 123 \; 4, \; n \equiv 1 (\text{mod } 3), \\ 123 \; \dots \; 123 \; 41234, \; n \equiv 2 (\text{mod } 3). \end{cases}$$

$$c_2(y_1,y_2,\dots,y_{n-1}) = \begin{cases} 23434, \; n=5 \\ 312 \; \dots \; 312, \; n \equiv 0 (\text{mod } 3), \\ 4 \; 123 \dots \; 123, \; n \equiv 1 (\text{mod } 3), \\ 4 \; 123 \dots \; 123 \; 4123, \; n \equiv 2 (\text{mod } 3). \end{cases}$$

It is easy to see that c_2 gives $\chi_2(\mathbf{P}_{n,2}) \leq 3$, for $n = 3k, k \in N$, but apart $n = 3k$ we could not avoid to have $\chi_2(\mathbf{P}_{n,2}) \leq 4$ otherwise there are at least two adjacent vertices assigned the same colors. Thus $\chi_2(\mathbf{P}_{n,2}) = 3$ for $n = 3k$ and $\chi_2(\mathbf{P}_{n,2}) = 4$ for otherwise.

For $\chi_r(\mathbf{P}_{n,2})$ and $r \geq 3$, the lower bound $\chi_3(\mathbf{P}_{n,2}) \geq \min\{3,3\}+1 = 4$. We will prove that $\chi_3(\mathbf{P}_{n,2}) \leq 4$ by defining a map $c_3 : V(\mathbf{P}_{n,2}) \to \{1,2,\dots,k\}$ for $n \geq 3$, by the following.

$$c_3(x_1,x_2,\dots,x_n) = \begin{cases} 123, \; n=3, \\ 1234 \; \dots \; 1234, \; n \equiv 0 (\text{mod } 4), \\ 1234 \; \dots \; 1234 \; 5, \; n \equiv 1 (\text{mod } 4), \\ 123456, \; n=6, \\ 1234563, \; n=7, \\ 1234 \; \dots \; 1234 \; 512345, \; n \equiv 2 (\text{mod } 4), n \geq 10, \\ 12345123456, \; n=11, \\ 1234 \; \dots \; 1234 \; 51234512345, \; n \equiv 3 (\text{mod } 4), n \geq 15. \end{cases}$$

$$c_3(y_1,y_2,\dots,y_{n-1}) = \begin{cases} 456, \; n=3, \\ 34 \; 1234 \; \dots \; 1234 \; 12, \; n \equiv 0 (\text{mod } 4), \\ 45 \; 1234 \; \dots \; 1234 \; 123, \; n \equiv 1 (\text{mod } 4), \\ 561234, \; n=6, \\ 6412345, \; n=7, \\ 4512345 \; 1234 \; \dots \; 1234 \; 123, \; n \equiv 0 (\text{mod } 4), n \geq 10, \\ 56123451234, \; n=11, \\ 45 \; 1234 \; \dots \; 1234 \; 512345123, \; n \equiv 0 (\text{mod } 4), n \geq 15. \end{cases}$$

It is easy to see that c_3 gives $\chi_3(\mathbf{P}_{n,2}) \leq 4$, for $n = 4k, k \in N$, but for $n = 3,6,7,11$ we are forced to have $\chi_3(\mathbf{P}_{n,2}) \leq 6$ as well as $\chi_3(\mathbf{P}_{n,2}) \leq 5$ for n otherwise. Thus $\chi_3(\mathbf{P}_{n,2}) = 4$, for $n = 4k$, $\chi_3(\mathbf{P}_{n,2}) = 6$ for $n = 3,6,7,11$ and $\chi_3(\mathbf{P}_{n,2}) = 5$ for n otherwise. By Observation 1, since $r \geq \Delta(\mathbf{P}_{n,2}) = 3$, it immediately gives $\chi_3(\mathbf{P}_{n,2})$=$\chi_r(\mathbf{P}_{n,2})$ for $n \geq 3$. □

Theorem 3. *Let G be three-cyclical ladder graph (TCL_n) for $n \geq 2$, r-dynamic chromatic number of TCL_n is*

$$\chi(TCL_n) = \chi_d(TCL_n) = 3, \chi_3(TCL_n) = 4, \chi_4(TCL_n) = 5, \chi_r(TCL_n) = 6, r \geq 5$$

Proof. The graph three-cyclical ladder graph, denoted by TCL_n, is connected graph with vertex set $V(TCL_n) = \{x_i, y_j, z_j; 1 \leq i \leq n; 1 \leq j \leq n+1\}$ and edge set $E(TCL_n) = \{y_j z_j; 1 \leq j \leq n+1\} \cup \{y_j y_{j+1}; 1 \leq i \leq n\} \cup \{x_i y_i; x_i z_i; x_i y_{i+1}; x_i z_{i+1}; 1 \leq i \leq n\}$. Thus, $p = |V(TCL_n)| = 3n+2, q = |E(TCL_n)| = 6n+1, \Delta(TCL_n) = 5$.

By Observation 1, $\chi_r(TCL_n) \geq \min\{\Delta(TCL_n), r\} + 1 = \min\{5, r\} + 1$. To find the exact value of r-dynamic chromatic number of TCL_n, we define three cases, namely for $\chi(TCL_n), \chi_d(TCL_n), \chi_3(TCL_n)$ and $\chi_4(TCL_n)$.

For $\chi(TCL_n), \chi_d(TCL_n)$, the lower bound $\chi_1(TCL_n) \geq \min\{5, 2\} + 1 = 3$. We will show that $\chi_1(TCL_n) \leq 3$, by defining a map $c_4 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 2$ by the following

$$c_4(x_i) = 3, 1 \leq i \leq n$$

$$c_4(y_j) = \begin{cases} 1, & j \equiv 1(mod\, 2),\ 1 \leq j \leq n+1, \\ 2, & j \equiv 0(mod\, 2),\ 1 \leq j \leq n+1. \end{cases}$$

$$c_4(z_j) = \begin{cases} 1, & j \equiv 0(mod\, 2),\ 1 \leq j \leq n+1, \\ 2, & j \equiv 1(mod\, 2),\ 1 \leq j \leq n+1. \end{cases}$$

It easy to see that c_4 gives $\chi(TCL_n) \leq 3$ and $\chi_d(TCL_n) \leq 3$. Thus $\chi(TCL_n) = 3$ and $\chi_d(TCL_n) = 3$.

For $r = 3$, the lower bound $\chi_3(TCL_n) \geq \min\{5, 3\} + 1 = 4$. We will show that $\chi_3(TCL_n) \leq 4$, by defining a map $c_5 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 2$ by the following

$$c_5(x_i) = \begin{cases} 1, & i \equiv 2(mod\, 3),\ 1 \leq i \leq n, \\ 2, & i \equiv 0(mod\, 3),\ 1 \leq i \leq n, \\ 3, & i \equiv 1(mod\, 3),\ 1 \leq i \leq n. \end{cases}$$

$$c_5(y_j) = \begin{cases} 1, & j \equiv 1(mod\, 3),\ 1 \leq j \leq n+1, \\ 2, & j \equiv 2(mod\, 3),\ 1 \leq j \leq n+1, \\ 3, & j \equiv 0(mod\, 3),\ 1 \leq j \leq n+1. \end{cases}$$

$$c_5(z_j) = 4, \text{for } 1 \leq i \leq n+1$$

It is easy to understand that c_5 gives $\chi_3(TCL_n) \leq 4$. Thus $\chi_3(TCL_n) = 4$.

For $r = 4$, the lower bound $\chi_4(TCL_n) \geq \min\{5, 4\} + 1 = 5$. We will show that $\chi_4(TCL_n) \leq 5$, by defining a map $c_6 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 2$ by the following

$$c_6(x_i) = \begin{cases} 3, & i \equiv 1(mod\, 3),\ 1 \leq i \leq n, \\ 4, & i \equiv 2(mod\, 3),\ 1 \leq i \leq n, \\ 5, & i \equiv 0(mod\, 3),\ 1 \leq i \leq n. \end{cases}$$

$$c_6(y_j) = \begin{cases} 1, & j \equiv 1(mod\, 2),\ 1 \leq j \leq n+1, \\ 2, & j \equiv 0(mod\, 2),\ 1 \leq j \leq n+1. \end{cases}$$

$$c_6(z_j) = \begin{cases} 3, & j \equiv 0 (mod\, 3),\ 1 \le j \le n+1, \\ 4, & j \equiv 1 (mod\, 3),\ 1 \le j \le n+1, \\ 5, & j \equiv 2 (mod\, 3),\ 1 \le j \le n+1. \end{cases}$$

It is easy to see that c_6 gives $\chi_4(TCL_n) \le 5$. Thus $\chi_4(TCL_n) = 5$.

For $r = 5$, the lower bound $\chi_5(TCL_n) \ge \min\{5, 5\} + 1 = 6$. We will show that $\chi_5(TCL_n) \le 6$, by defining a map $c_7 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \ge 2$ by the following

$$c_7(x_i) = \begin{cases} 4, & i \equiv 1 (mod\, 3),\ 1 \le i \le n, \\ 5, & i \equiv 2 (mod\, 3),\ 1 \le i \le n, \\ 6, & i \equiv 0 (mod\, 3),\ 1 \le i \le n. \end{cases}$$

$$c_7(y_j) = \begin{cases} 1, & j \equiv 1 (mod\, 3),\ 1 \le j \le n+1, \\ 2, & j \equiv 2 (mod\, 3),\ 1 \le j \le n+1, \\ 3, & j \equiv 0 (mod\, 3),\ 1 \le j \le n+1. \end{cases}$$

$$c_7(z_j) = \begin{cases} 4, & j \equiv 0 (mod\, 3),\ 1 \le j \le n+1, \\ 5, & j \equiv 1 (mod\, 3),\ 1 \le j \le n+1, \\ 6, & j \equiv 2 (mod\, 3),\ 1 \le j \le n+1. \end{cases}$$

It clearly shows that c_7 gives $\chi_5(TCL_n) \le 6$. Thus $\chi_5(TCL_n) = 6$. Since for $r \ge 5$, we have $r \ge \Delta(TCL_n)$. By Observation 1, $\chi_r(TCL_n) = \chi_5(TCL_n) = 6$. It concludes the proof. □

Theorem 4. *Let $P_n + C_m$ be a joint graph of P_n and C_m, the r-dynamic chromatic number is*

$$\chi_{1 \le r \le 4}(P_n + C_m) = \begin{cases} 5, & m = 3k, k \in N, \\ 6, & m \text{ otherwise.} \end{cases}$$

$$\chi_5(P_n + C_m) = \begin{cases} 6, & m = 3, \\ 8, & m = 5, \\ 7, & m \text{ otherwise.} \end{cases}$$

For $r \ge 6$, we have

$$\chi_r(P_n + C_m) = \begin{cases} r + m - 2, & 3 \le m \le r - 2, m \ge r - 1, n \ge m - 1, \\ 2r - 3, & m \text{ lainnya}, n \ge r - 1. \end{cases}$$

Proof. The graph $P_n + C_m$ is a connected graph with vertex set $V(P_n + C_m) = \{x_i; 1 \le i \le n\} \cup \{y_j; 1 \le j \le m\}$ and edge set $E(P_n + C_m) = \{x_i x_{i+1}; 1 \le i \le n-1\} \cup \{y_j y_{j+1}; 1 \le j \le m\} \cup \{y_m y_1\} \cup \{x_i y_j; 1 \le i \le n; 1 \le j \le m\}$. The order and size of this graph are $p = |V(P_n + C_m)| = m + n, q = |E(P_n + C_m)| = mn + m - 1$. Since all vertices in P_n joint with all vertices in C_m, it gives $\Delta(P_n + C_m) = m + 2$

By Observation 1, $\chi_r(P_n + C_m) \ge \min\{\Delta(P_n + C_m), r\} + 1 = \min\{m + 2, r\} + 1$. To find the exact value of r-dynamic chromatic number of $P_n + C_m$, we define three cases, namely for $\chi_{1 \le r \le 4}(P_n + C_m), \chi_5(P_n + C_m)$ and $\chi_{r \ge 6}(P_n + C_m)$.

For $\chi_{1\leq r\leq 4}(P_n+C_m)$, define a map $c_8 : V(P_n+C_m) \to \{1,2,\dots,k\}$ where $n \geq 3$, by the following:

$$c_8(x_0,x_1,x_2,\dots,x_{n-1}) = \begin{cases} 123 \ \dots \ 123, \ n\equiv 0(\text{mod } 3), \ m \equiv 2(\text{mod } 3), \\ 123 \ \dots \ 123\ 1, \ n\equiv 1(\text{mod } 3), \ m \equiv 2(\text{mod } 3), \\ 123 \ \dots \ 123\ 12, \ n\equiv 2(\text{mod } 3), \ m \equiv 2(\text{mod } 3), \\ 12 \ \dots \ 12, \ n \text{ even}, \ m \text{ otherwise}, \\ 12 \ \dots \ 12\ 1, \ n \text{ odd}, \ m \text{ otherwise}. \end{cases}$$

$$c_8(y_0,y_1,y_2,\dots,y_{n-1}) = \begin{cases} 345 \ \dots \ 345, \ m\equiv 0(\text{mod } 3), \\ 345 \ \dots \ 345\ 6, \ m\equiv 1(\text{mod } 3), \\ 45 \ \dots \ 45\ 6, \ m\equiv 2(\text{mod } 3), m \text{ odd}, \\ 45 \ \dots \ 45\ 46, \ m\equiv 2(\text{mod } 3), m \text{ even}. \end{cases}$$

It is easy to see that c_8 gives $\chi_{1\leq r\leq 4}(P_n+C_m) = 5$, for $m = 3k, k \in N$ and $\chi_{1\leq r\leq 4}(P_n+C_m) = 6$ for m otherwise.

For $\chi_5(P_n+C_m)$, define a map $c_9 : V(P_n+C_m) \to \{1,2,\dots,k\}$ where $n \geq 3$, by the following:

$$c_9(x_0,x_1,x_2,\dots,x_{n-1}) = \begin{cases} 123 \ \dots \ 123, \ n\equiv 0(\text{mod } 3), \\ 123 \ \dots \ 123\ 1, \ n\equiv 1(\text{mod } 3), \\ 123 \ \dots \ 123\ 12, \ n\equiv 2(\text{mod } 3). \end{cases}$$

$$c_9(y_0,y_1,y_2,\dots,y_{n-1}) = \begin{cases} 456, \ m=3, \\ 45678, \ m=5, \\ 456 \ \dots \ 456\ 457, \ m\equiv 0(\text{mod } 3), m\geq 6, \\ 456 \ \dots \ 456\ 4567, \ m\equiv 1(\text{mod } 3), \\ 456 \ \dots \ 45\ 74567, \ m\equiv 2(\text{mod } 3), m\geq 8. \end{cases}$$

It is easy to see that c_9 gives $\chi_5(P_n+C_m) = 6$, for $m = 3$, $\chi_5(P_n+C_m) = 8$, for $m = 5$, and $\chi_5(P_n+C_m) = 7$ for m otherwise.

The last for $\chi_6(P_n+C_m)$, define a map $c_{10} : V(P_n+C_m) \to \{1,2,\dots,k\}$ where $m \geq 3, n \geq r-2$, by the following

$$c_{10}(x_i) = \begin{cases} 1, \ i\equiv 1(\text{mod } r-2), \\ 2, \ i\equiv 2(\text{mod } r-2), \\ 3, \ i\equiv 3(\text{mod } r-2), \\ \vdots \\ r-3, \ i=n-1, \\ r-2, \ i=n. \end{cases}$$

$$c_{10}(y_j) = \begin{cases} r-1, \ i \equiv 1(\text{mod } 3), 1 \leq i \leq n-r+4, \\ r, \ i \equiv 2(\text{mod } 3), 1 \leq i \leq n-r+3, \\ r+1, \ i \equiv 3(\text{mod } 3), 1 \leq i \leq n-r+2, \\ r+2, \ i = n-r+1, \\ r+3, \ i = n-r, \\ r+4, \ i = n-r-1, \\ \vdots \\ 2n-2, \ i = n-1, \\ 2n-3, \ i = n. \end{cases}$$

It easy to see that c_{10} gives $\chi_6(P_n+C_m) = r+m-2$ for $3 \leq m \leq r-2, m \geq r-1, n \geq m-1$ and $\chi_6(P_n+C_m) = 2r-3$ for $n \geq r-1$, m otherwise. By Observation 1, since $r \geq \Delta(P_n+C_m) = m+2$, it immediately gives $\chi_6(P_n+C_m)=\chi_r(P_n+C_m)$ for $n \geq 4$. □

Theorem 5. *Let $C_n(1, \frac{n}{2})$ be a circulant graph of order 3, the r-dynamic chromatic number is*

$$\chi(C_n(1, \frac{n}{2})) = \begin{cases} 4, \ n = 4, \\ 2, \ n = 4k+2, k \in N, \\ 3, \ n = 4k+4, k \in N. \end{cases} \qquad \chi_d(C_n(1, \frac{n}{2})) = 4$$

For $r \geq 3$, we have

$$\chi_r(C_n(1, \frac{n}{2})) = \begin{cases} n, \ n = 4, 6, 8, \\ 4, \ n = 8k+4, k \in N, \\ 5, \ n = 8k+6, k \in N, \\ 6, \ n \text{ otherwise.} \end{cases}$$

Proof. The graph $C_n(1, \frac{n}{2})$ is a connected graph with vertex set $V(C_n(1, \frac{n}{2})) = \{x_i, 0 \leq i \leq n-1\}$ and edge set $E(C_n(1, \frac{n}{2})) = \{x_i x_{i+1 (\text{mod } n)}, 0 \leq i \leq n-1\} \cup \{x_i x_{i+\frac{n}{2} (\text{mod } n)}, 0 \leq i \leq \frac{n}{2}\}$. The order and size of the graph $C_n(1, \frac{n}{2})$ are $p = |V(C_n(1, \frac{n}{2}))| = n, q = |E(C_n(1, \frac{n}{2}))| = \frac{3n}{2}$. Since $C_n(1, \frac{n}{2})$ is a regular graph of degree 3, thus $\delta(C_n(1, \frac{n}{2})) = \Delta(C_n(1, \frac{n}{2})) = 3$.

By Observation 1, $\chi_r(C_n(1, \frac{n}{2})) \geq \min\{\Delta(C_n(1, \frac{n}{2})), r\} + 1 = \min\{3, r\} + 1$. In the same way, to find the exact value of r-dynamic chromatic number of $C_n(1, \frac{n}{2})$, we define three cases, namely for $\chi(C_n(1, \frac{n}{2})), \chi_2(C_n(1, \frac{n}{2}))$ and $\chi_{r \geq 3}(C_n(1, \frac{n}{2}))$.

For $\chi(C_n(1, \frac{n}{2}))$, define a map $c_{11} : V(C_n(1, \frac{n}{2})) \to \{1, 2, \ldots, k\}$ where $n \geq 3$, by the following:

$$c_{11}(x_0, x_1, \ldots, x_{n-1}) = \begin{cases} 1234, \ n = 4, \\ 12 \ \ldots \ 12, \ n = 4k+2, k \in N. \end{cases}$$

$$\begin{aligned} c_{11}(x_0, x_1, \ldots, x_{\frac{n}{2}}) &= 12 \ \ldots \ 12 \ 13, \ n = 4k+4, k \in N. \\ c_{11}(x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_{n-1}) &= 21 \ \ldots \ 21, \ 32, \ n = 4k+4, k \in N. \end{aligned}$$

It easy to see that c_{11} gives $\chi(C_n(1,\frac{n}{2}))=4$, for $n=4$, $\chi(C_n(1,\frac{n}{2}))=2$, for $n=4k+2, k\in N$, and $\chi(C_n(1,\frac{n}{2}))=3$, for $n=4k+4, k\in N$.

For $\chi_2(C_n(1,\frac{n}{2}))$, define a map $c_{12}: V(C_n(1,\frac{n}{2}))\rightarrow\{1,2,\ldots,k\}$ where $n\geq 3$, by the following:

$$\begin{aligned} c_{12}(x_0,x_1,\ldots,x_{n-1}) &= 1234,\ \text{for } n=4\\ c_{12}(x_0,x_1,\ldots,x_{\frac{n}{2}}) &= 12\ldots\ 12,\ \text{for } n=4k+2, k\in N\\ c_{12}(x_{\frac{n}{2}+1},x_{\frac{n}{2}+2},\ldots,x_{n-1}) &= 34\ \ldots\ 34,\ \text{for } n=4k+2, k\in N \end{aligned}$$

It easy to see that c_{12} gives $\chi_2(C_n(1,\frac{n}{2}))=4$ for any n.

For $\chi_r(C_n(1,\frac{n}{2}))$, and $r\geq 3$, define a map $c_{13}: V(C_n(1,\frac{n}{2}))\rightarrow\{1,2,\ldots,k\}$ where $n\geq 3$, by the followings

- For $n=4$, $c_{13}(x_i)= i+1,\ 0\leq i\leq n-1$
- For $n=10$

$$c_{13}(x_i) = \begin{cases} 1, & i=0,7,\\ 2, & i=5,8,\\ 3, & i=1,4,\\ 4, & i=3,6,\\ 5, & i=2,9.\end{cases}$$

- For $n=8k+4, k\in N$

$$c_{13}(x_i) = \begin{cases} 1, & i\equiv 0 (\text{mod } 4), 0\leq i\leq n-4,\\ 2, & i\equiv 1 (\text{mod } 4), 1\leq i\leq n-3,\\ 3, & i\equiv 2 (\text{mod } 4), 2\leq i\leq n-2,\\ 4, & i\equiv 3 (\text{mod } 4), 3\leq i\leq n-1.\end{cases}$$

- For $n=8k+6, k\in N$

$$c_{13}(x_i) = \begin{cases} 1, & i\equiv 0 (\text{mod } 4), 0\leq i\leq \frac{n}{2}-7,\\ 2, & i\equiv 1 (\text{mod } 4), 1\leq i\leq \frac{n}{2}-6,\\ 3, & i\equiv 2 (\text{mod } 4), 2\leq i\leq \frac{n}{2}-5,\\ 4, & i\equiv 3 (\text{mod } 4), 3\leq i\leq \frac{n}{2}-4,\\ 5, & i=\frac{n}{2}-12 \text{ atau } i=n-1,\end{cases}$$

$$c_{13}(x_i) = \begin{cases} i, & i\equiv 0 (\text{mod } \frac{n}{2}-2), \frac{n}{2}-2\leq i\leq n-5,\\ i-1, & i\equiv 1 (\text{mod } \frac{n}{2}-2), \frac{n}{2}-1\leq i\leq n-4,\\ i-2, & i\equiv 2 (\text{mod } \frac{n}{2}-2), \frac{n}{2}\leq i\leq n-3,\\ i-3, & i\equiv 3 (\text{mod } \frac{n}{2}-2), \frac{n}{2}+1\leq i\leq n-2.\end{cases}$$

- For $n = 8k + 8, k \in N$

$$c_{13}(x_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 4), 0 \le i \le \frac{n}{2} - 8, \text{ or} \\ & i \equiv 0(\text{mod } \frac{n}{2} - 2), \frac{n}{2} - 2 \le i \le n - 6, \\ 2, & i \equiv 1(\text{mod } 4), 1 \le i \le \frac{n}{2} - 7, \text{ or} \\ & i \equiv 1(\text{mod } \frac{n}{2} - 2), \frac{n}{2} - 1 \le i \le n - 5, \\ 3, & i \equiv 2(\text{mod } 4), 2 \le i \le \frac{n}{2} - 6, \text{ or} \\ & i \equiv 2(\text{mod } \frac{n}{2} - 2), \frac{n}{2} \le i \le n - 4, \\ 4, & i \equiv 3(\text{mod } 4), 3 \le i \le \frac{n}{2} - 5, \text{ or} \\ & i \equiv 3(\text{mod } \frac{n}{2} - 2), \frac{n}{2} + 1 \le i \le n - 3, \\ 5, & i = \frac{n-8}{2} \text{or } i = n - 2, \\ 6, & i = \frac{n-6}{2} \text{or } i = n - 1. \end{cases}$$

- For $n = 8k + 10, k \in N$

$$c_{13}(x_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 4), 0 \le i \le \frac{n}{2} - 6, \text{ or} \\ & i \equiv 0(\text{mod } \frac{n}{2} - 2), \frac{n}{2} - 2 \le i \le n - 7, \\ 2, & i \equiv 1(\text{mod } 4), 1 \le i \le \frac{n}{2} - 5, \text{ or} \\ & i \equiv 1(\text{mod } \frac{n}{2} - 2), \frac{n}{2} - 1 \le i \le n - 6, \\ 3, & i \equiv 2(\text{mod } 4), 2 \le i \le \frac{n}{2} - 4, \text{ or} \\ & i \equiv 2(\text{mod } \frac{n}{2} - 2), \frac{n}{2} \le i \le n - 5, \\ 4, & i \equiv 3(\text{mod } 4), 3 \le i \le \frac{n}{2} - 3, \text{ or} \\ & i \equiv 3(\text{mod } \frac{n}{2} - 2), \frac{n}{2} + 1 \le i \le n - 4, \text{or } i = n - 1, \\ 5, & \text{For } i = n - 3, \\ 6, & \text{For } i = n - 2, \end{cases}$$

It easy to see that c_{13} gives $\chi_3(C_n(1, \frac{n}{2})) = 4, 6, 8$ for $n = 4, 6, 8$, $\chi_3(C_n(1, \frac{n}{2})) = 4$ for $n = 8k + 4$, $\chi_3(C_n(1, \frac{n}{2})) = 5$ for $n = 8k + 6$, and $\chi_3(C_n(1, \frac{n}{2})) = 6$ for n otherwise. By Observation 1, since $r \ge \Delta(C_n(1, \frac{n}{2})) = 4$, it immediately gives $\chi_3(C_n(1, \frac{n}{2}))=\chi_r(C_n(1, \frac{n}{2}))$ for $n \ge 4$. □

Concluding Remarks

We have found some $r-$dynamic chromatic number of several graphs, namely prism graph, three-cyclical ladder graph, joint graph and circulant graph. All numbers attain a best lower bound. For the characterization of the lower bound of for any connected graphs G, we have not found any result yet, thus we propose the following open problem.

Open Problem

Given that any connected graphs G, determine the sharp lower bound of $\chi_r(G)$

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