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On the r-Dynamic Chromatic Number of Corona Product of Star Graph

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Abstract A proper k coloring of graph G such that the neighbors of any vertex $v \in V(G)$ where at least $\min\{r, d(v)\}$ different colors is defined an r-dynamic coloring. The minimum k such that graph G has an r-dynamic k coloring is defined the r-dynamic chromatic number, denoted by $\chi_r(G)$. In this paper, we study the r-dynamic chromatic number of corona product of star graph.

MSC: 05C78
Keywords: r-dynamic chromatic number; star graph; corona product

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1. Introduction

Let $G = (V, E)$ be a simple graph. The degree of v is denoted by $d(v)$ and the neighbor of a vertex $v \in G$, denoted $N(v)$ is all vertices adjacent to v. The maximum degree and minimum degree of graph G are denoted by $\Delta(G)$ and $\delta(G)$. By an r-dynamic coloring of a graph G, we mean a proper k-coloring of graph G such that the neighbors of any vertex v receive at least $\min\{r, d(v)\}$ different colors. The r-dynamic chromatic number of graph G as $\chi_r(G)$ which is the minimum k such that graph G has an r-dynamic k-coloring. This concept was introduced by Montgomery [1, 2].

In recent year, Kang et al [3] also found the r-dynamic chromatic number of grid graph. Jahanbekam et al, [4] found χ_r on grids and toroidal grids. Taherkhani in [5], introduced two upper bounds for $_2(G)(G)$. Kristiana et.al in [6] studied r-dynamic chromatic number of corona product by complete graph. Furthermore, Kristiana et.al [7, 8] studied r-dynamic coloring of corona product of graphs.

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Proposition 1.1 ([1]). *Let $\Delta(G)$ be the maximum degree of graph G. It holds $\chi_r(G) \geq \min\{\Delta(G), r\}+1$.*

Proposition 1.2 ([7]). *Let $S_n \odot H$ be a corona product of star graph and $H \neq K_m, C_m, W_m$, for $n \geq 3, m \geq 4$:*

$$\chi_r(S_n \odot H) \geq \begin{cases} \delta(H)+2, \ 1 \leq r \leq \delta(H)+1 \\ r+1, \delta(H)+1 \leq r \leq n+|V(H)| \\ n+|V(H)|+1, r \geq n+|V(H)|+1 \end{cases}$$

Proposition 1.3 ([7]). *Let $H \odot S_m$ be a corona product of star graph and $H \neq K_n, C_n, W_n$, for $n \geq 4, m \geq 3$:*

$$\chi_r(H \odot S_m) \geq \begin{cases} 3, \ r=1,2 \\ r+1, 3 \leq r \leq m+\Delta(H)+1 \\ \Delta(H)+m+2, r \geq m+\Delta(H)+2 \end{cases}$$

The corona product of G and H is the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the i^{th} vertex of G adjacent to every vertex of the i^{th} copy of H, where $1 \leq i \leq |V(G)$ in [9]. The star graph with $n+1$ vertices is a tree graph, In a star graph one vertex has degree n and the remaining n vertices have vertex degree 1, denoted by $K_{1,n}$. Double star graph is the graph obtained by joining the center of two stars $K_{1,n}$ and $K_{1,m}$ with an edge in [10].

2. Results

In this paper, we determine the r-dynamic chromatic number of corona product of star graph namely $S_n \odot P_m, S_n \odot C_m, P_n \odot S_m$, and $DS_n \odot S_m$ for $m, n \geq 3$.

Theorem 2.1. *Let $S_n \odot P_m$ be a corona product of star graph and path graph with $n, m \geq 3$, the r-dynamic chromatic number is*

$$\chi_r(S_n \odot P_m) = \begin{cases} 3, \ r=1,2 \\ r+1, \ 3 \leq r \leq m+n \\ m+n+1, \ r \geq m+n+1 \end{cases}$$

Proof. The vertex set $V(S_n \odot P_m) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \cup \{x_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_j; 1 \leq j \leq m\}$ and $|V(S_n \odot P_m)| = mn+m+n+1$. Thus, the maximum and minimum degrees of $S_n \odot P_m$, respectively are $\Delta(S_n \odot P_m) = m+n$ and $\delta(S_n \odot P_m) = 2$. We divide into three cases for the proof as follows.
Case 1. For $r = 1, 2$
Based on Proposition 2, the lower bound $\chi_r(S_n \odot P_m) \geq \delta(P_m)+2 = 1+2 = 3$. We show that $\chi_r(S_n \odot P_m) \leq 3$, it can be explained that the central vertex of the star graph is colored 1, according to the definition of proper vertex coloring then the other vertices on the star graph are colored 2. Based on the corona product, the path graph is colored 1,3,1,3,...,1,3. Hence, there are 3 color for $r = 1, 2$.
Case 2. For $3 \leq r \leq m+n$
Based on Proposition 2, the lower bound $\chi_r(S_n \odot P_m) \geq r+1$. We show that $\chi_r(S_n \odot P_m) \leq r+1$, it can be explained that Identical to case 1, the color of path graph is moving corresponds to r with $3 \leq r \leq m+n$.
Case 3. For $r \geq m+n+1$
Based on Proposition 2, the lower bound $\chi_r(S_n \odot P_m) \geq m+n+1$. We show that

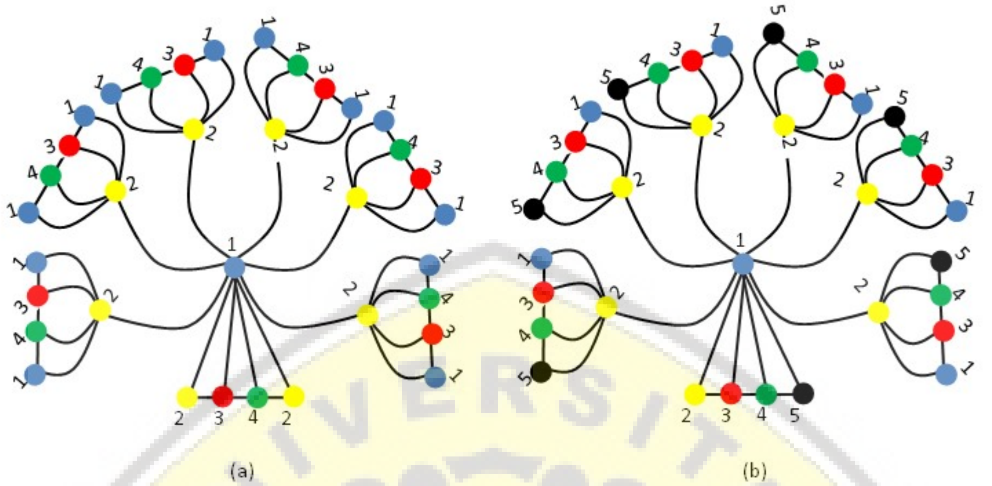

FIGURE 1. The r-dynamic chromatic number, (a) $\chi_3(S_6 \odot P_4) = 4$ (b) $\chi_4(S_6 \odot P_4) = 5$

$\chi_r(S_n \odot P_m) \leq m + n + 1$, define $c : V(S_n \odot P_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows:

$$c(x) = 1$$
$$c(x_i) = 1 + n, 1 \leq i \leq n$$
$$c(x_{ij}) = 1 + n + j, 1 \leq i \leq n,\ 1 \leq j \leq m$$
$$c(y_j) = 1 + n + j, 1 \leq j \leq m$$

It appears that c is a map $c : V(S_n \odot P_m) \to \{1, 2, \ldots, 1 + n + m\}$. Hence, the upper bound of the r-dynamic chromatic number of $S_n \odot P_m$ is $\chi_r(S_n \odot P_m) \leq m + n + 1$. It gives $\chi_r(S_n \odot P_m) = m + n + 1$. The Figure 1 is the illustration of r-dynamic chromatic number of $S_6 \odot P_4$. The proof is completed.

Theorem 2.2. *Let $S_n \odot C_m$ be a corona product of star graph and cycle graph with $n, m \geq 3$, the r-dynamic chromatic number of $S_n \odot C_m$ is*

$$\chi_{r=1,2}(S_n \odot C_m) = \begin{cases} 3, & m, \text{ even} \\ 4, & m, \text{ odd} \end{cases}$$

$$\chi_{r=3}(S_n \odot C_m) = \begin{cases} 4, & m = 3k,\ k \geq 1 \\ 6, & m = 5 \\ 5, & m \text{ else} \end{cases}$$

$$\chi_{r\geq 4}(S_n \odot C_m) = \begin{cases} 6, & r = 4,\ m = 4 \\ r + 1, & 4 \leq r \leq m + n \\ m + n + 1, & r \geq m + n + 1 \end{cases}$$

Proof . The vertex set $V(S_n \odot C_m) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \cup \{x_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_j; 1 \leq j \leq m\}$ and the edge set $E(S_n \odot C_m) = \{xx_i; 1 \leq i \leq n\} \cup \{x_i x_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{xy_i; 1 \leq i \leq n\} \cup \{y_j y_{j+1}; 1 \leq j \leq m - 1\} \cup \{x_{ij} x_{i(j+1)}; 1 \leq i \leq$

$n, 1 \leq j \leq m-1\} \cup \{y_1y_m\} \cup \{x_{i1}x_{im}\}$. The cardinality of vertex set and edge set, respectively are $|V(S_n \odot C_m)| = mn + m + n + 1$ and $|E(S_n \odot C_m)| = 2mn + 2m + 1$. Thus, $\Delta(S_n \odot C_m) = m + n$. We divide into some cases for the proof as follows.

Case 1. For $r = 1, 2$ and m is even

Based on Proposition 1, the lower bound $\chi_r(S_n \odot C_m) \geq \min\{\Delta, r\} + 1 = 2 + 1 = 3$. To obtain the upper bound $\chi_r(S_n \odot C_m)$, define $c_1 : V(S_n \odot C_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows.

$$c_1(x) = 1$$

$$c_1(x_i) = 2, 1 \leq i \leq n$$

$$c_1(x_{ij}) = \begin{cases} 1, & \text{for } j \text{ odd}, \ 1 \leq i \leq n, \ 1 \leq j \leq m \\ 3, & \text{for } j \text{ even}, \ 1 \leq i \leq n, \ 1 \leq j \leq m \end{cases}$$

$$c_1(y_j) = \begin{cases} 2, & \text{for } j \text{ is odd}, \ 1 \leq j \leq m \\ 3, & \text{for } j \text{ is even}, \ 1 \leq j \leq m. \end{cases}$$

It appears that c_1 is a map $c_1 : V(S_n \odot C_m) \to \{1, 2, 3\}$. Hence, the upper bound of the r-dynamic chromatic number of $S_n \odot C_m$ is $\chi_r(S_n \odot C_m) \leq 3$. It gives $\chi_{r=1,2}(S_n \odot C_m) = 3$.

Case 2. For $r = 1, 2$, and m is odd or $r = 3$, $m \neq 4$ and m is even

Based on Proposition 1, the lower bound $\chi_r(S_n \odot C_m) \geq \min\{\Delta, r\} + 1 = 2 + 1 = 3$. It appears $\chi_r(S_n \odot C_m) \leq 4$. We obtain the upper bound $\chi_r(S_n \odot C_m)$, define $c_2 : V(S_n \odot C_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows.

$$c_2(x) = 1$$

$$c_2(x_i) = 2, 1 \leq i \leq n$$

$$c_2(x_{ij}) = \begin{cases} 1, & \text{for } j \text{ odd}, \ 1 \leq i \leq n, \ 1 \leq j \leq m-1 \\ 3, & \text{for } j \text{ even}, \ 1 \leq i \leq n, \ 1 \leq j \leq m \\ 4, & \text{for } j = m \end{cases}$$

$$c_2(y_j) = \begin{cases} 2, & \text{for } j \text{ is odd}, \ 1 \leq j \leq m-1 \\ 3, & \text{for } j \text{ is even}, \ 1 \leq j \leq m \\ 4, & \text{for } j = m \end{cases}$$

It appears that c_2 is a map $c_2 : V(S_n \odot C_m) \to \{1, 2, 3, 4\}$. Hence, the upper bound of the r-dynamic chromatic number of $S_n \odot C_m$ is $\chi_r(S_n \odot C_m) \leq 4$. It gives $\chi_{r=1,2}(S_n \odot C_m) = 4$.

Case 3. For $r = 3$ and $m = 3k$, $k \geq 1$

Based on Proposition 1, the lower bound $\chi_r(S_n \odot C_m) \geq \min\{\Delta, r\} + 1 = 3 + 1 = 4$. It appears $\chi_r(S_n \odot C_m) \leq 4$. We obtain the upper bound $\chi_r(S_n \odot C_m)$, define $c_3 : V(S_n \odot C_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows.

$$c_3(x) = 1$$

$$c_3(x_i) = 2, 1 \leq i \leq n$$

$$c_3(x_{ij}) = \begin{cases} 1, & \text{for } j \equiv 1 \mod (3), \ 1 \leq j \leq m, \ 1 \leq i \leq n \\ 3, & \text{for } j \equiv 2 \mod (3), \ 1 \leq j \leq m, \ 1 \leq i \leq n \\ 4, & \text{for } j \equiv 0 \mod (3), \ 1 \leq j \leq m, \ 1 \leq i \leq n \end{cases}$$

$$c_3(y_j) = \begin{cases} 2, & \text{for } j \equiv 1 \mod (3), \ 1 \leq j \leq m \\ 3, & \text{for } j \equiv 2 \mod (3), \ 1 \leq j \leq m \\ 4, & \text{for } j \equiv 0 \mod (3), \ 1 \leq j \leq m \end{cases}$$

It appears that c_3 is a map $c_3 : V(S_n \odot C_m) \to \{1, 2, 3, 4\}$. Hence, the upper bound of the r-dynamic chromatic number of $S_n \odot C_m$ is $\chi_r(S_n \odot C_m) \leq 4$. It gives $\chi_r(S_n \odot C_m) = 4$.

Case 4. For $r = 3$ and $m = 5$
Based on Proposition 1, the lower bound $\chi_r(S_n \odot C_m) \geq \min\{\Delta, r\} + 1 = 3 + 1 = 4$. It appears $\chi_r(S_n \odot C_m) \leq 6$. We obtain the upper bound $\chi_r(S_n \odot C_m)$, define $c_4 : V(S_n \odot C_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows.

$$c_4(x) = 1$$

$$c_4(x_i) = 2, 1 \leq i \leq n$$

$$c_4(x_{ij}) = \begin{cases} 1, & \text{for } j = 1 \\ j+1, & \text{for } 2 \leq j \leq 5,\ 1 \leq i \leq n \end{cases}$$

$$c_4(y_j) = \begin{cases} 2, & \text{for } j = 1 \\ j+1, & \text{for } 2 \leq j \leq 5 \end{cases}$$

It appears that c_4 is a map $c_4 : V(S_n \odot C_m) \to \{1, 2, \ldots, 6\}$. Hence, the upper bound of the r-dynamic chromatic number of $S_n \odot C_m$ is $\chi_r(S_n \odot C_m) \leq 6$. It gives $\chi_r(S_n \odot C_m) = 6$.
Case 5. For $r \geq m + n + 1$
Based on Proposition 1, the lower bound $\chi_r(S_n \odot C_m) \geq \min\{\Delta, r\} + 1 = m + n + 1$. It appears $\chi_r(S_n \odot C_m) \leq m + n + 1$. We obtain the upper bound $\chi_r(S_n \odot C_m)$, define $c_5 : V(S_n \odot C_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows.

$$c_5(x) = 1$$

$$c_5(x_i) = i + 1, 1 \leq i \leq n$$

$$c_5(x_{ij} = 1 + n + j, 1 \leq j \leq m$$

$$c_5(y_j) = 1 + n + j, 1 \leq j \leq m$$

It appears that c_5 is a map $c_5 : V(S_n \odot C_m) \to \{1, 2, \ldots, 1 + n + m\}$. Hence, the upper bound of the r-dynamic chromatic number of $S_n \odot C_m$ is $\chi_r(S_n \odot C_m) \leq m + n + 1$. It gives $\chi_r(S_n \odot C_m) = m + n + 1$. It concludes the proof.

Theorem 2.3. *Let $P_n \odot S_m$ be a corona product of path graph and star graph with $n, m \geq 3$, the r-dynamic chromatic number of $P_n \odot S_m$ is*

$$\chi_r(P_n \odot S_m) = \begin{cases} 3,\ r = 1, 2 \\ r + 1,\ 3 \leq r \leq m + 3 \\ m + 4,\ r \geq m + 4 \end{cases}$$

Proof. The graph $P_n \odot S_m$ is a connected graph with vertex set, $V(P_n \odot S_m) = \{x_i, y_i, y_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\}$ and the order of graph is $|V(P_n \odot S_m)| = mn + 2n$ Thus, $\Delta(P_n \odot S_m) = m + 3$ and $\delta(P_n \odot S_m) = 2$. We divide into three cases for the proof as follows.
Case 1. For $r = 1, 2$ Based on Proposition 3, the lower bound $\chi_r(P_n \odot S_m) \geq 3$. We obtain the upper bound $\chi_r(P_n \odot S_m)$, define $c_6 : V(P_n \odot S_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows

$$c_6(x_i) = \begin{cases} 1, & \text{for } i \text{ odd},\ 1 \leq i \leq n \\ 2, & \text{for } i \text{ even},\ 1 \leq i \leq n \end{cases}$$

$$c_6(y_i) = \begin{cases} 2, & \text{for } i \text{ odd},\ 1 \leq i \leq n \\ 1, & \text{for } i \text{ even},\ 1 \leq i \leq n \end{cases}$$

$$c_6(y_{ij}) = 3, 1 \leq i \leq n,\ 1 \leq j \leq m$$

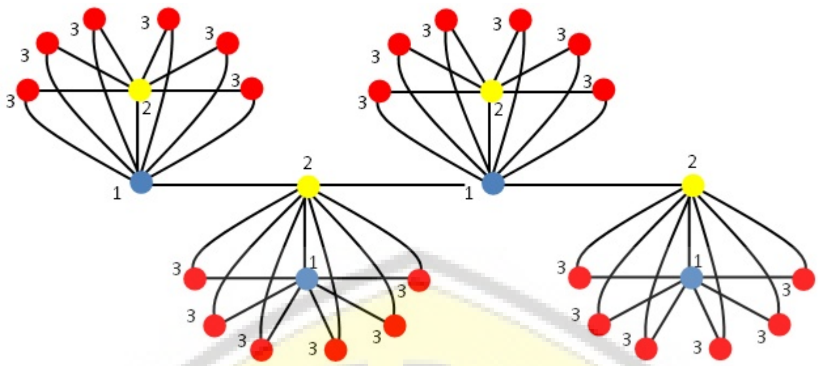

FIGURE 2. The 2-dynamic chromatic number of $P_4 \odot S_6$

It appears that c_6 is a map $c_6 : V(P_n \odot S_m) \to \{1, 2, 3\}$. Hence, the upper bound of the r-dynamic chromatic number of $P_n \odot S_m$ is $\chi_r(P_n \odot S_m) \leq 3$. It gives $\chi_r(P_n \odot S_m) = 3$.
Case 2. For $3 \leq r \leq m + 3$
Based on Proposition 3 we known $\Delta(P_n) = 2$ so $\chi_{3 \leq r \leq m+3} \geq r + 1$. We obtain the upper bound, we have construction of the pattern as follows.

$$n = 4, m = 3, r = 4,\ \chi_{\mathbf{4}}(P_4 \odot S_3) = \mathbf{5}$$
$$n = 4, m = 3, r = 5,\ \chi_{\mathbf{5}}(P_4 \odot S_3) = \mathbf{6}$$
$$n = 4, m = 3, r = 6,\ \chi_{\mathbf{6}}(P_4 \odot S_3) = \mathbf{7}$$
$$n = 4, m = 5, r = 3,\ \chi_{\mathbf{3}}(P_4 \odot S_5) = \mathbf{4}$$
$$n = 4, m = 5, r = 6,\ \chi_{\mathbf{6}}(P_4 \odot S_5) = \mathbf{7}$$
$$n = 4, m = 5, r = 7,\ \chi_{\mathbf{7}}(P_4 \odot S_5) = \mathbf{8}$$

It concludes that $\chi_{3 \leq r \leq m+3} \leq r + 1$.
Case 3. For $r \geq m + 4$
Based on Proposition 3, the lower bound $\chi_r(P_n \odot S_m) \geq m + 4$. We obtain the upper bound $\chi_r(P_n \odot S_m)$, define $c_7 : V(P_n \odot S_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows.

$$c_7(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 (mod\ 3),\ 1 \leq i \leq n \\ 2, & \text{for } i \equiv 2 (mod\ 3),\ 1 \leq i \leq n \\ 3, & \text{for } i \equiv 0 (mod\ 3),\ 1 \leq i \leq n \end{cases}$$

$$c_7(y_i) = 4, 1 \leq i \leq n$$

$$c_7(y_{ij}) = 4 + j, 1 \leq i \leq n,\ 1 \leq j \leq m$$

It appears that c_7 is a map $c_7 : V(P_n \odot S_m) \to \{1, 2, \ldots, m + 4\}$. Hence, the upper bound of the r-dynamic chromatic number of $P_n \odot S_m$ is $\chi_r(P_n \odot S_m) \leq m + 3$. It gives $\chi_r(P_n \odot S_m) = m + 4$. The Figure 2 is the illustration of 2-dynamic chromatic number of $P_4 \odot S_6$. The proof is completed.

Theorem 2.4. *Let $DS_n \odot S_m$ be a corona product of double star graph and star graph with $n, m \geq 3$, the r-dynamic chromatic number of $DS_n \odot S_m$ is*

$$\chi_r(DS_n \odot S_m) = \begin{cases} 3, & r = 1, 2 \\ r + 1, & 3 \leq r \leq m + n + 2 \\ m + n + 3, & r \geq m + n + 3 \end{cases}$$

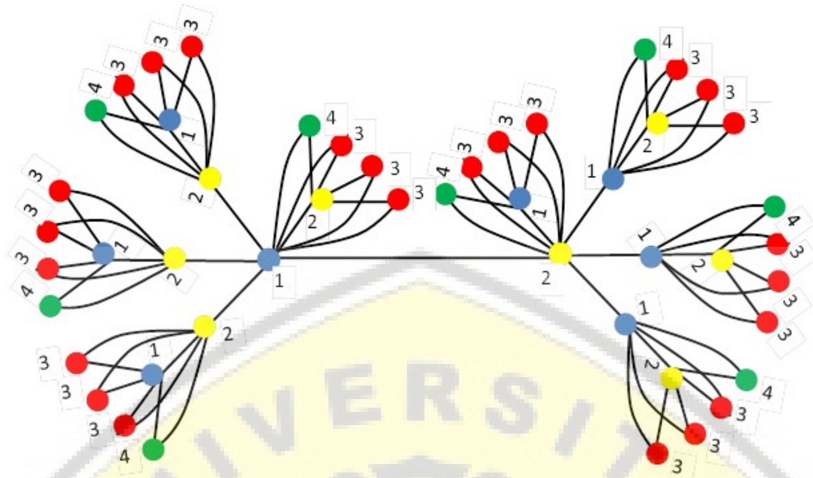

FIGURE 3. The 3-dynamic chromatic number of $DS_3 \odot S_4$

Proof. The graph $DS_n \odot S_m$ is a connected graph with vertex set, $V(DS_n \odot S_m) = \{x^s, x_i^s, p^s, p_j^s, y_i^s, y_{ij}^s; s = 1, 2;\ 1 \le i \le n;\ 1 \le j \le m\}$ and the order of graph is $|V(DS_n \odot S_m)| = 2nm + 4n + 2m + 4$. Thus, $\Delta(DS_n \odot S_m) = m + n + 2$ and $\delta(DS_n \odot S_m) = 2$. We divide into three cases for the proof as follows.

Case 1. For $r = 1, 2$

Based on Proposition 3, the lower bound $\chi_r(DS_n \odot S_m) \ge 3$. We obtain the upper bound $\chi_r(DS_n \odot S_m)$, define $c_8 : V(DS_n \odot S_m) \to \{1, 2, \cdots, k\}$ where $n, m \ge 3$, as follows.

$$c_8(x_i^s) = \begin{cases} 1, & \text{for } s = 2,\ 1 \le i \le n \\ 2, & \text{for } s = 1,\ 1 \le i \le n \end{cases}$$

$$c_8(x^s) = s,\ \ s = 1, 2$$

$$c_8(p^s) = 3 - s,\ \ s = 1, 2$$

$$c_8(p_j^s) = 3,\ \ s = 1, 2,\ \ 1 \le j \le m$$

$$c_8(y_i^s) = s,\ \ s = 1, 2,\ \ 1 \le i \le n$$

$$c_8(y_{ij}^s) = 3,\ \ s = 1, 2, 1 \le i \le n,\ \ 1 \le j \le m$$

It appears that c_8 is a map $c_8 : V(DS_n \odot S_m) \to \{1, 2, 3\}$. Hence, the upper bound of the r-dynamic chromatic number of $DS_n \odot S_m$ is $\chi_r(DS_n \odot S_m) \le 3$. It gives $\chi_r(DS_n \odot S_m) = 3$.

Case 2. For $3 \le r \le m + n + 2$

Based on Proposition 3 we known $\Delta(DS_n) = 2$ so $\chi_{3 \le r \le m+n+2} \ge r + 1$. We obtain the upper bound, we have construction of the pattern as follows.

$$n = 4, m = 3, r = 4,\ \chi_{\mathbf{4}}(DS_4 \odot S_3) = \mathbf{5}$$
$$n = 4, m = 3, r = 5,\ \chi_{\mathbf{5}}(DS_4 \odot S_3) = \mathbf{6}$$
$$n = 4, m = 3, r = 8,\ \chi_{\mathbf{8}}(DS_4 \odot S_3) = \mathbf{9}$$
$$n = 4, m = 5, r = 3,\ \chi_{\mathbf{3}}(DS_4 \odot S_5) = \mathbf{4}$$
$$n = 4, m = 5, r = 9,\ \chi_{\mathbf{9}}(DS_4 \odot S_5) = \mathbf{10}$$
$$n = 4, m = 5, r = 10,\ \chi_{\mathbf{10}}(P_4 \odot S_5) = \mathbf{11}$$

It concludes that $\chi_{3\leq r\leq m+3} \leq r+1$. The Figure 3 is the illustration of r-dynamic chromatic number of $DS_3 \odot S_4$.

Case 3. For $r \geq m+n+3$
Based on Proposition 3, the lower bound $\chi_r(DS_n \odot S_m) \geq m+n+3$. We obtain the upper bound $\chi_r(DS_n \odot S_m)$, define $c_9 : V(DS_n \odot S_m) \to \{1, 2, \cdots, k\}$ where $n, m \geq 3$, as follows.

$$c_9(x^s) = s, \; s = 1, 2$$
$$c_9(x_i^s) = 2 + i, \; s = 1, 2, \; 1 \leq i \leq n$$
$$c_9(p^s) = n + m + 2, \; s = 1, 2$$
$$c_9(p_j^s) = 2 + n + j, \; s = 1, 2, \; 1 \leq j \leq m$$
$$c_9(y_i^s) = n + m + 2, \; s = 1, 2, \; 1 \leq i \leq n$$
$$c_9(y_{ij}^s) = 2 + n + j, \; s = 1, 2, 1 \leq i \leq n, \; 1 \leq j \leq m$$

It appears that c_9 is a map $c_9 : V(DS_n \odot S_m) \to \{1, 2, \ldots, m+n+3\}$. Hence, the upper bound of the r-dynamic chromatic number of $DS_n \odot S_m$ is $\chi_r(DS_n \odot S_m) \leq m+n+3$. It gives $\chi_r(DS_n \odot S_m) = m+n+3$. The proof is completed.
The Figure 3 is the illustration of 3-dynamic chromatic number of $DS_3 \odot S_4$.

3. Conclusion

We have obtained the exact value of r-dynamic chromatic number of corona product by star graph. Based on result, we have characterized that $\chi_r(Pn \odot S_m) = \chi_r(DS_n \odot S_m) = \chi_r(S_m)$.

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References

[1] B. Montgomery, Dynamic coloring of graphs (Ph.D Dissertation), West Virginia University, 2001.
[2] S. Akbari, M. Ghanbari, S. Jahanbekam, On the dynamic chromatic number of graphs, *Comb and Grapgs, in: Contemp Math - American Mathematical Society* 531 (2010) 11–18.
[3] R. Kang, T. Muller, Douglas B. West, On r-dynamic coloring of grids, Discrete Appl. Math. 186 (2015) 286–290
[4] S. Jahanbekam, J. Kimb, Suil O, Douglas B. West, On r-dynamic Coloring of Graphs, Discrete Applied Mathematics 206 (2016) 65–72
[5] A. Taherkhani, On r-dynamic chromatic number of graphs, Discrete Appl. Math. 201 (2016) 222–227.
[6] A. I. Kristiana, M.I. Utoyo, Dafik, On the r-dynamic chromatic number of the corronation by complete graph, Journal of Physics: Conference Series, 1008 (2018) 012–033.
[7] A. I. Kristiana, M.I., Utoyo, R. Alfarisi, Dafik, On r-dynamic coloring of the corona product of graphs, Discrete Mathematics, Algorithms and Applications, 12 (02) (2020) 2050019

[8] A. I. Kristiana, M.I. Utoyo, Dafik, The lower bound of the r-dynamic chromatic number of corona product product by wheel graphs, AIP Conference Proceedings, 2014 (2017) 020054

[9] S. Mohan, J. Geetha, K. Somasundaram, Total coloring of the corona product of two graphs, Australasian Journal of Combinatorics, 68(1) (2017) 15–22.

[10] S. P. Jeyakokila and P. Sumathi, A note on soEnergy of Stars, Bistars and Double stars graphs. Bulletin of the international Mathematical Virtual Institute 6 (2016) 105–113.