On Super Edge Local Antimagic Total Labeling by Using an Edge Antimagic Vertex Labeling Technique

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Abstract: In this paper, we consider that all graphs are finite, simple and connected. Let $G(V,E)$ be a graph of vertex set V and edge set E. By a edge local antimagic total labeling, we mean a bijection $f:V(G)\cup E(G)\rightarrow\{1,2,3,\ldots,|V(G)|+|E(G)|\}$ satisfying that for any two adjacent edges e_1 and e_2, $w_t(e_1)\neq w_t(e_2)$, where for $e=uv\in G, w_t(e)=f(u)+f(v)+f(uv)$. Thus, any edge local antimagic total labeling induces a proper edge coloring of G if each edge e is assigned the color $w_t(e)$. It is considered to be a super edge local antimagic total coloring, if the smallest labels appear in the vertices. The chromatic number of super edge local antimagic total, denoted by $\gamma_{leat}(G)$, is the minimum number of colors taken over all colorings induced by super edge local antimagic total labelings of G. In this paper, we investigate the lower bound of super edge local antimagic total coloring of graphs and the existence the chromatic number of super edge local antimagic total labeling of ladder graph L_n, caterpillar graph $C_{n,m}$, and graph coronations $P_n\odot P_2$ and $C_n\odot P_2$..

Index Terms: antimagic total labeling, super edge local antimagic total labeling, chromatic number.

1 INTRODUCTION

We consider that all graphs in this paper are connected, finite, and simple graph, for detail definition of graph can be seen on [3, 4]. The labeling of graph is a bijection mapping a natural number to the vertices of a graph. In this type of labeling, we consider all weights associated with each edge of graph G. The labeling called antimagic if all the edge weights show different values. The concept of antimagic labeling of a graph introduced by Hartsfield and Ringel [5]. There are a lot of results regarding to antimagic labeling, can be found in Dafik *et. al* [7], [8]. They study about super edge-antimagic total labelings and determined the super edge-antimagic total labelings of $mK_{n,n}$ and super edge-antimagicness for disconnected graphs, respectively. In this paper, we study and identify the relation between coloring and antimagic labeling of graph, that is edge local antimagic total labeling. The proper edge coloring of a graph G is a coloring of all edges of graph G assigned by natural number such that every two adjacent edges receive different colors. The definition of edge local antimagic total labeling is a bijection $f:V(G)\cup E(G)\rightarrow\{1,2,3,\cdots,p+q\}$ where $p=|V(G)|$ and $q=|E(G)|$ such that for every two adjacent edges e_1 and e_2 for $e=ab\in G$ and $w_t(e)=f(a)+f(b)+f(ab)$, $w_t(e_1)\neq w_t(e_2)$. Thus, any edge local antimagic total labeling induces a proper edge coloring of G if each edge e is assigned by the color $w_t(e)$. If the smallest labels appear in the vertices, then it is considered to be a super edge local antimagic total labeling. The super edge local antimagic total labeling chromatic number denoted by $\gamma_{leat}(G)$ is the minimum number of colors taken over all colorings induced by super edge local antimagic total labeling of graph G. This paper just initiate to study the super edge local antimagic total labeling, thus we have not found any relevant results yet. But, there are some results related to vertex local antimagic labeling. The concept local antimagic coloring of a graph G firstly introduced by Arumugam $et\ al.$ [6]. They gave a lower bound and an upper bound of vertex local antimagic edge labeling of joint graph and also gave an exact value of vertex local antimagic edge labeling for some graph there are path, cycle, complete graph, friendship, wheel, bipartite and complete bipartite. Ika, $et.al.$ [13] has determined the concept local antimagic coloring of a graph, their study examine the lower bound of the chromatic number of edge local antimagic vertex labeling, denoted by $\gamma_{lea}(G)\geq\Delta(G)$. If $\Delta(G)$ is maximum degrees of G then $\gamma_{lea}(G)\geq\Delta(G)$. Kurniawati, *et. al.* [14] also study the local antimagic of graph, their study edge local antimagic total labeling of graph operation and determine the chromatic number of edge local antimagic total labeling of comb product graph. Their paper determined the lower bound of edge local antimagic total labeling of comb product graph and denoted by $\chi(P\_n\rhd H)\leq\chi(P_n)+\chi(H)$. This paper discusses and determine the existence of super edge local antimagic total labeling of some special graphs and also analyze the lower bound of chromatic number super edge local antimagic total labeling. Prior to show our new results, we recall the definition of edge antimagic vertex labeling and super edge local antimagic total labeling in the following definitions.

Definition 2.1. A map $f:V(G)\rightarrow\{1,2,3,\ldots,|V(G)|\}$ is called an (a,d)-edge antimagic vertex labeling if the set of edge weights $w(uv)=f(u)+f(v)$, of all the edges in G, form an arithmetic sequence $\{a,a+d,a+2d,\cdots,a+(q-1)d\}$ where $a>0$ and $d\geq 0$ are two fixed integers.

Definition 2.2. Let $G(V,E)$ be a graph of vertex set V and edge set E. A bijection $f:V(G)\cup E(G)\rightarrow\{1,2,3,\cdots,p+q\}$ where $p=|V(G)|$ and $q=|E(G)|$ such that for every two adjacent edges e_1 and e_2 for $e=ab\in G$ and $w_t(e)=f(a)+f(b)+f(ab)$, $w_t(e_1)\neq w_t(e_2)$. It is considered to be a super edge local antimagic total labeling, if the smallest labels appear in the vertices.

We know that, any super edge local antimagic total labeling induces a proper edge coloring of G if each edge e is assigned by the color $w_t(e)$. The lower bound concept of local antimagic graph is shown in the following observation in Arumugam paper, see [6].

Observation 2.1. [6] For any graph G, the vertex local antimagic edge labeling chromatic number $\chi_{la}(G)\geq\chi(G)$, where $\chi(G)$ is a chromatic number of vertex coloring of G.r.

2. MAIN RESULT

We start to present our result by showing the lower bound of super edge local antimagic total labeling chromatic number for any graph in the following lemma. This lower bound will be used to proved the obtained theorems, and it is sharp.

Lemma 3.1. If $\Delta(H)$ is maximum degrees of G, then we have $\gamma_{leat}(H) \geq \Delta(H)$.

Proof. Let f be a super edge local antimagic total labeling of G. For the edge coloring induced by f, the color of each edge ab is assigned by $f(a)+f(b)+f(ab)$. If v is a vertex which is incident with $\Delta(G)$ edges, then there must be at least $\Delta(G)$ edges colors to be a proper edge coloring. Hence, all the edges receive distinct colors, thus $\gamma_{leat}(G) \geq \Delta(G)$.■

We now need to recall an edge antimagic vertex labeling (EAVL for short) lemma. This lemma is important to construct of a super local antimagic total edge coloring. It was introduced by Bača *et al.* in [1]. This lemma also described the connection between edge-antimagic vertex labeling and super edge-antimagic total labeling.

Theorem 3.1. [1] If G has an (a,d)-edge-antimagic vertex labeling then G has super $(a+|V|+1, d+1)$-edge-antimagic total labeling and super $(a+|V|+|E|, d-1)$-edge-antimagic total labeling.

Corollary 3.2. Let G be any simple and connected graph. If G admits an edge antimagic vertex labeling f with different $d=1$, then the edge coloring of G, assigned by color $w(ab) = f(a)+f(ab)+f(b)$, will give the same color.

Proof. Since G admits an edge antimagic vertex labeling f with different $d=1$, by Theorem 3.1, G has super $(a+|V|+|E|, 0)$-edge-antimagic total labeling. It implies that all the edge weights have the same weights. It concludes the proof. ■

We now present an important permutation which is very useful in constructing super edge local antimagic total coloring.

Lemma 3.2. Let α and β be a sequence $\alpha = \{a, a+d, a+2d, \cdots, a+kd\}$ and $\beta = \{b, b+d, b+2d, \cdots, b+kd\}$, where $d \geq 1$ and odd $k \geq 0$ are integer numbers. There exists a permutation $\Pi(\alpha)$ of the elements α such that $\beta + \Pi(\alpha) = \{a+b+(k-1)d, a+b+(k+1)d, \cdots, a+b+(k-1)d, a+b+(k+1)d\}$.

Proof. Let α and β be a sequence $\alpha = \{a+(i-1)d,\ 1 \leq i \leq k+1\}$ and $\beta = \{b+(i-1)d, 1 \leq i \leq k+1\}$, where $d \geq 1$ and odd $k \geq 0$ are integer numbers. Define a permutation $\Pi(\alpha) = \{h(i),\ 1 \leq i \leq k+1\}$ of the elements of α as follows:

$$h(i) = \begin{cases} a+(k-i)d & if\ 1 \leq i \leq k, \quad i \equiv 1(mod\ 2) \\ a+2d+(k-i)d & if\ 2 \leq i \leq k+1, \ i \equiv 0(mod\ 2) \end{cases}$$

By direct computation, we obtain that $\beta + \Pi(\alpha) = \{b+(i-1)d + h(i)\ |1 \leq i \leq k+1\} = \{a+b+(k-1)d|\ \{i \equiv 1(mod\ 2), 1 \leq i \leq k\} \cup \{a+b+(k+1)d\ |\ \{i \equiv 0(mod\ 2), 2 \leq i \leq k+1\} = \{a+b+(k-1)d, a+b+(k+1)d, \cdots, a+b+(k-1)d, a+b+(k+1)d\}$. We arrive at the desired result.■

Furthermore, we also present this useful lemma in constructing super edge local antimagic total labeling. We consider the partition $\mathcal{P}^n_{3,d}(i)$ of the set $\{1,2,\cdots,3n\}$ into n columns, $n \geq 2$, 3-rows such that the difference between the sum of the numbers in the $(i+1)$th 3-rows and the sum of the numbers in the ith 3-rows is always equal to the constant d, where $i = 1,2,\cdots,n-1$. Thus $d = \sum \mathcal{P}^n_{3,d}(i+1) - \sum \mathcal{P}^n_{3,d}(i)$.

Lemma 3.3. Let n be an odd positive integer. For $1 \leq i \leq n$, the sum of

$$\mathcal{P}^n_{3,d}(i) = \{g_1(i), g_2(i), g_3(i)\}$$

with

$$g_1(i) = \begin{cases} \dfrac{n+1+i}{2}; i \equiv 0(mod\ 2) \\ \dfrac{1+i}{2}; i \equiv 1(mod\ 2) \end{cases}$$

$$g_2(i) = \begin{cases} \dfrac{i}{2}; i \equiv 0(mod\ 2) \\ \dfrac{n+i}{2}; i \equiv 1(mod\ 2) \end{cases}$$

$$g_3(i) = n+1-i$$

form an aritmatic sequence of difference $d = 0$.

Proof. By simple calculation. It gives $\mathcal{P}^n_{3,d}(i) = g_1(i), +g_2(i) + g_3(i)\}$, thus

$$\sum_{i=1}^{n} \mathcal{P}^n_{3,d}(i) = \begin{cases} \dfrac{n+1+i}{2} + \dfrac{i}{2} + n - i + 1; i \equiv 0(mod\ 2) \\ \dfrac{i+1}{2} + \dfrac{n+i}{2} + n - i + 1; i \equiv 1(mod\ 2) \end{cases}$$

It is easy to see that $\sum_{i=1}^{n} \mathcal{P}^n_{3,d}(i) = \frac{3}{2}(n+1)$ form an aritmatic sequence of difference $d = \sum \mathcal{P}^n_{3,d}(i+1) - \sum \mathcal{P}^n_{3,d}(i) = 0$.■

From now on, by those lemmas in hand, we are ready to prove the following results.

Theorem 3.3. Let n be an odd positive integer. Given that L_n is a ladder graph of order n. The chromatic number of super edge local antimagic total labeling of L_n is $\gamma_{leat}(L_n) = 3$.

Proof. The graph L_n is a connected graph with vertex set $V(L_n) = \{x_i, y_i : 1 \leq i \leq n$ and edge set $E(L_n) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\}$. Hence $|V(L_n)| = 2n,\ |E(L_n)| = 3n-2$ and $\Delta(L_n) = 3$. Based on Lemma 3.1, the lower bound is $\gamma_{leat}(L_n) \geq \Delta(L_n) = 3$.

Now we will prove that the upper bound is \$\gamma_{leat}(L_n)\le 3\$. By using Lemma 3.3, we define the vertex labeling f_1 of ladder by the following.

$$f_1(x_i) = g_1(i)$$

$$f_1(y_i) = g_2(i) \oplus n$$

The vertex labeling f_1 is a bijective function from $f: V(L_n) \to \{1,2,3,\cdots,|V(L_n)|\}$. The edge-weights $w(uv) = f(u)+f(v)$, where $u,v \in L_n$ and under the labeling f_1, is $w = \left\{\frac{1}{2}(n+3) + k; 1 \leq k \leq 3n-2\right\}$, which from a a consecutive sequence of $d=1$. Hence L_n admits an $\left(\frac{1}{2}(n+3)+1,1\right)$ –edge - antimagic vertex labeling.

Together with Theorem 3.1 and Corollary 3.2, there is a set of edge labeling $\alpha = \{5n-1-k; 1 \le k \le 3n-2\}$ such that it will give all the edge weights of L_n have the same edge weights. Then the edge coloring of L_n, assigned by color $w(uv) = f(u) + f(uv) + f(v)$, will give the same colors.

However, definition of a proper coloring, the adjacent edges can not be assigned by the same colors. Therefore, we need to re-assign a color to the adjacent edges. By using Lemma 3.2, let $W = \{(w_i, \alpha_i);\ w_i \in w, \alpha_i \in \alpha\}$ be the ordered pair of set which gives the total edge weight of L_n of $d = 0$. There are subset $w_j, \alpha_j \subset W$ which all of them are the adjacent edge weights of L_n. Based on Lemma 3.2, there are a permutation $\Pi(w_{i,1})$ and $\Pi(w_{i,2})$ such that $W_1' = W_2' = a_{i,1} + \Pi(w_{i,1}) = a_{i,2} + \Pi(w_{i,2})$ gives two different colors. Those all colors are distinct with the other non adjacent edge weights of L_n.

Therefore, we can define the following edge labeling

$$f_1(x_i x_{i+1}) = h(i) \oplus 4n - 1$$
$$f_1(y_i y_{i+1}) = h(i) \oplus 2n$$
$$f_1(x_i y_i) = g_3(i) \oplus 3n - 1$$

where $h(i)$ is the permutation set $\Pi(\alpha)$ mentioned in Lemma 3.2, with $a = 1, d = 1, k = n-2$. The edge labeling f is a bijective function from $f: V(L_n) \cup E(L_n) \to \{1,2,3,\cdots,|V(L_n)| + |E(L_n)|\}$.

Hence, from the super local antimagic total edge labelings above, it easy to see that $W = \left\{\frac{1}{2}(11n-1), \frac{1}{1}(11n+1), \frac{1}{1}(11n+3)\right\}$ contains only three element which induces a proper edge coloring of L_n. Thus, it gives $\gamma_{\{leat\}}(L_n) \le 3$. It concludes that $\gamma_{\{leat\}}(L_n) = 3$.■

For illustration, we give the following example.

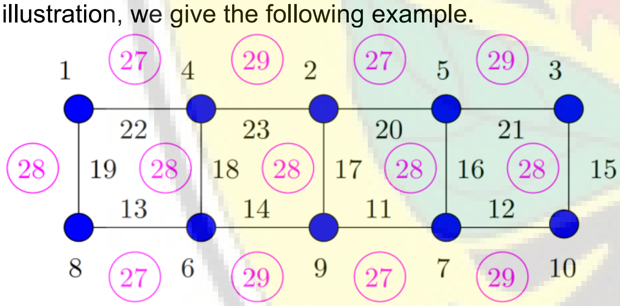

Figure 1. Super edge local antimagic total labeling of L_5

Theorem 3.4. For n odd and m be positive integers. Let $C_{n,m}$ be caterpillar graph. The chromatic number of super edge local antimagic total labeling of $C_{n,m}$ is $\gamma_{leat}(C_n, m) = m + 2$.

Proof. The caterpillar graph $C_{n,m}$ is a connected graph with vertex set $V(C_{n,m}) = \{x_i, y_{i,j}: 1 \le i \le n, 1 \le j \le m$ and edge set $E(C_{n,m}) = \{x_i y_{i,j}: 1 \le i \le n\} \cup \{x_i x_{i+1}: 1 \le i \le n-1\}$. Hence $|V(C_{n,m})| = n(m+1)$ and $|E(C_{n,m})| = n(m+1) - 1$. Based on Lemma 3.1, the lower bound is $\gamma_{leat}(C_{n,m}) \ge \Delta(C_{n,m}) = m + 2$.

Now we will prove that the upper bound is $\gamma_{leat}(C_n, m) \le 3$. By using Lemma 3.3, we define the vertex labeling f_2 of caterpillar by the following.

$$f_2(x_i) = g_1(i)$$

$$f_2(y_{i,j}) = g_2(i) \oplus jn$$

The vertex labeling f_1 is a bijective function from $f: V(C_{n,m}) \to \{1,2,3,\ldots,|V(C_{n,m})|\}$. The edge-weights $w(uv) = f(u) + f(v)$, where $u, v \in C_{n,m}$ and under the labeling f_1, is $w = \left\{\frac{1}{2}(n+3) + i; 1 \le i \le n-1\right\} \cup \left\{\frac{1}{2}(n+1) + i + jn; 1 \le i \le n;\ 1 \le i \le m\right\}$, which from a consecutive sequence of $d = 1$. Hence $C_{n,m}$ admits an $\left(\frac{1}{2}(n+3) + 1, 1\right)$-edge-antimagic vertex labeling.

Together with Theorem 3.1 and Corollary 3.2, there is a set of edge labeling $\alpha = \{a_k; 1 \le k \le nm + n - 1\}$ such that it will give all the edge weights of $C_{n,m}$ have the same edge weights. Then the edge coloring of $C_{n,m}$, assigned by color $w(uv) = f(u) + f(uv) + f(v)$, will give the same colors.

However, definition of a proper coloring, the adjacent edges can not be assigned by the same colors. Therefore, we need to re-assign a color to the adjacent edges. By using Lemma 3.2, let $W = \{(w_i, a_i);\ w_i \in w, a_i \in a\}$ be the ordered pair of set which gives the total edge weight of $C_{n,m}$ of $d = 0$. There are subset $w_j, a_j \subset W$ which all of them are the adjacent edge weights of $C_{n,m}$. Based on Lemma 3.2, there are a permutation $\Pi(w_{i,1})$ and $\Pi(w_{i,2})$ such that $W_1' = W_2' = a_{i,1} + \Pi(w_{i,1}) = a_{i,2} + \Pi(w_{i,2})$ gives two different colors. Those all colors are distinct with the other non adjacent edge weights of $C_{n,m}$.

Therefore, we can define the following edge labeling

$$f_2(x_i y_{i,j}) = g_3(i) \oplus (m+j)n$$

$$f_2(x_i x_{i+1}) = h(i) \oplus n(2m+1)$$

where $h(i)$ is the permutation set $\Pi(\alpha)$ mentioned in Lemma 3.2, with $a = 1, d = 1, k = n-2$. The edge labeling f is a bijective function from $f: V(C_{n,m}) \cup E(C_{n,m}) \to \{1,2,3,\cdots,|V(C_{n,m})| + |E(C_{n,m})|\}$.

Hence, from the super local antimagic total edge labelings above, it easy to see that $W = \left\{\left(m + \frac{3}{2} + 2\right)n + \frac{3}{2}, \left(m + \frac{3}{2} + 4\right)n + \frac{3}{2}, \ldots, \left(m + \frac{3}{2} + 2m\right)n + \frac{3}{2}, \left(2m + \frac{5}{2}\right)n + \frac{1}{2}, \left(2m + \frac{5}{2}\right)n + \frac{5}{2}\right\}$ contains $m + 2$ element which induces a proper edge coloring of $C_{n,m}$. Thus, it gives $\gamma_{leat}(C_n, m) \le m + 2$. It concludes that $\gamma_{leat}(C_n, m) = m + 2$.■

Theorem 3.5. Let G be any graph of order $n \ge 3$. Let P_2 be path graph, then we have $\gamma_{leat}(G \odot P_2) \ge \Delta(G \odot P_2) + 1$.

Proof. Let G be a graph of order $n \ge 3$ and vertex set $V(H) = \{x_i: 1 \le i \le n\}$. Let $G \odot P_2$ be connected graph with vertex set $V(G \odot P_2) = V(G) \cup \{y_{1,i}, y_{2,i}: 1 \le i \le n\}$ and the edge set $E(G \odot P_2) = V(G) \cup \{x_i y_{1,i}, x_i y_{2,i}: 1 \le i \le n\}$. Hence $|V(G \odot P_2)| = 3n$ and $|E(G \odot P_2)| = |E(G)| + 3n$. The maximum degree of $G \odot P_2$ is $\Delta(G \odot P_2) = \Delta(G) + 2$. The graph $G \odot P_2$ have $|V(G)|$ subgraph $K_1 + P_2$. In process, we can be construction some condition for total edge weight in $e \in G \odot P_2$ as follows

(1) We assume that $e_1 \in E(G)$, $(e_2)_i \in E(H_i)$ with $H_i \cong P_2$ and $(e_3)_i = x_i v$ where $x_i \in V(G)$, $v \in V(H_i)$.
(2) Suppose G admits an edge local antimagic total labeling with $\gamma_{leat}(G)$ and based on a proper edge coloring which there must be at least $\Delta(G)$ edges colors.
(3) Based on definition coronation that the edges e_1 which incident to $u_i \in V(G)$ vertices are adjacent to the edges

$(e_3)_i$. Thus, we obtain that the edge weight of G different with the edge weight of $(e_3)_i$.

(4) Since the edges $(e_2)_i \in E(H_i)$ are adjacent to the edges $(e_3)_i$ such that the edges e_2 have distinct color to the edges $(e_3)_i$. Thus, we have 1 colors for the edges $(e_2)_i$ and we can claim that the edges $(e_2)_i$ in i-th subgraph $H_i \cong P_2$ have same color.

By $(2), (3)$ and (4), we can construction of the lower bound of the local antimagic total edge coloring of $G \odot P_2$ as follows.

$$\begin{aligned}\gamma_{leat}(G \odot P_2) \geq\ & |\{w((e_1)), \in V(G)\}| + |\{w((e_3)_i), (e_3)_i \in V((K_1 + P_n)_i)\}| + \\ & |\{w((e_2)_i, (e_2)_i \in V(H_i)\}| \\ \geq\ & \Delta(G) + 2 + 1 \\ =\ & \Delta(G \odot P_2) + 1\end{aligned}$$

Hence, we get that the lower bound of the local antimagic total edge coloring of $G \odot P_2$ is $\gamma_{leat}(G \odot P_2) \geq \Delta(G \odot P_2) + 1$.■

Theorem 3.6. For n be odd positive integers with $n \geq 2$, we have $\gamma_{leat}(P_n \odot P_2) = 5$.

Proof. The graph $P_n \odot P_2$ is a connected graph with vertex set $V(P_n \odot P_2) = \{x_i, y_{1,i}, y_{2,i} : 1 \leq i \leq n\}$ and edge set $E(P_n \odot P_2) = \{x_i y_{1,i}, x_i y_{2,i}, y_{1,i} y_{2,i} : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\}$. Hence $|V(P_n \odot P_2)| = 3n$, $|E(P_n \odot P_2)| = 4n - 1$ and $\Delta(P_n \odot P_2) = 4$. Based on Theorem 3.5, the lower bound is $\gamma_{leat}(P_n \odot P_2) \geq \Delta(P_n \odot P_2) = 4 + 1$.

Now we will prove that the upper bound is $\gamma_{leat}(P_n \odot P_2) \leq \Delta(P_n \odot P_2) = 5$. By using Lemma 3.3, we define the vertex labeling f_3 of $P_n \odot P_2$ by the following.

$$f_3(x_i) = g_1(i)$$
$$f_3(y_{1,i}) = g_3(i) \oplus n$$
$$f_3(y_{2,i}) = g_2(i) \oplus 2n$$

The vertex labeling f_1 is a bijective function from $f: V(P_n \odot P_2) \to \{1,2,3,\ldots,|V(P_n \odot P_2)|\}$. The edge-weights $w(uv) = f(u) + f(v)$, where $u, v \in P_n \odot P_2$, under the labeling f_1, is $w = \left\{\frac{1}{2}(n+3) + k; 1 \leq k \leq 4n - 1\right\}$, which from a consecutive sequence of $d = 1$. Hence $P_n \odot P_2$ admits an $\left(\frac{1}{2}(n+3) + 1, 1\right)$-edge-antimagic vertex labeling.

Together with Theorem 3.1 and Corollary 3.2, there is a set of edge labeling $a = \{a_k; 1 \leq k \leq 4n - 1\}$ such that it will give all the edge weights of $P_n \odot P_2$ have the same edge weights. Then the edge coloring of $P_n \odot P_2$, assigned by color $w(uv) = f(u) + f(uv) + f(v)$, will give the same colors.

However, definition of a proper coloring, the adjacent edges can not be assigned by the same colors. Therefore, we need to re-assign a color to the adjacent edges. By using Lemma 3.2, let $W = \{(w_i, a_i);\ w_i \in w, a_i \in a\}$ be the ordered pair of set which gives the total edge weight of $P_n \odot P_2$ of $d = 0$. There are subset $w_j, a_j \subset W$ which all of them are the adjacent edge weights of $P_n \odot P_2$. Based on Lemma 3.2, there are a permutation $\Pi(w_{i,1})$ and $\Pi(w_{i,2})$ such that $W_1' = W_2' = \{a_{i,1}\} + \Pi(w_{i,1}) = \{a_{i,2}\} + \Pi(w_{i,2})$ gives two different colors. Those all colors are distinct with the other non adjacent edge weights of $P_n \odot P_2$.

Therefore, we can define the following edge labeling

$$f_3(y_{1,i} y_{2,i}) = g_1(i) \oplus 3n$$
$$f_3(x_i y_{1,i}) = g_2(i) \oplus 4n$$
$$f_3(x_i y_{2,i}) = g_3(i) \oplus 5n$$
$$f_3(x_i x_{i+1}) = h(i) \oplus 6n$$

where $h(i)$ is the permutation set $\Pi(a)$ mentioned in Lemma 3.2, with $a = 1, d = 1, k = n - 2$. The edge labeling f is a bijective function from $f: V(P_n \odot P_2) \cup E(P_n \odot P_2) \to \{1,2,3,\cdots,|V(P_n \odot P_2)| + |E(P_n \odot P_2)|\}$.

Hence, from the super local antimagic total edge labelings above, it easy to see that $W = \left\{\frac{1}{2}(13n+3), \frac{1}{2}(15n+3), \frac{1}{2}(17n+3), \frac{1}{2}(15n+1), \frac{1}{2}(15n+5)\right\}$ contains 5 element which induces a proper edge coloring of $P_n \odot P_2$. Thus, it gives $\gamma_{leat}(P_n \odot P_2) \leq 5$. It concludes that $\gamma_{leat}(P_n \odot P_2) = 5$. ■

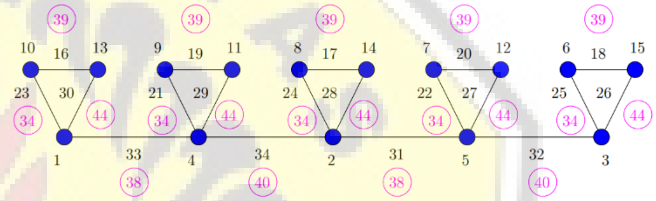

Figure 1. Example of Edge Local Antimagic Total Labeling of $P_5 \odot P_2$

Theorem 3.7. For n be odd positive integers with $n \geq 3$, we have $\gamma_{leat}(C_n \odot P_2) = 5$.

Proof. The graph $C_n \odot P_2$ is a connected graph with vertex set $V(C_n \odot P_2) = \{x_i, y_{1,i}, y_{2,i} : 1 \leq i \leq n\}$ and edge set $E(C_n \odot P_2) = \backslash\{x_i y_{1,i}, x_i y_{2,i}, y_{1,i} y_{2,i} : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_1 x_n\}$. Hence $|V(C_n \odot P_2)| = 3n$, $|E(C_n \odot P_2)| = 4n$ and $\Delta(C_n \odot P_2) = 4$. Based on Theorem 3.5, the lower bound is $\gamma_{leat}(C_n \odot P_2) \geq \Delta(C_n \odot P_2) = 4 + 1$.

Now we will prove that the upper bound is $\gamma_{leat}(C_n \odot P_2) \geq \Delta(C_n \odot P_2) = 5$. By using Lemma 3.3, we define the vertex labeling f_4 of $C_n \odot P_2$ by the following.

$$f_4(x_i) = g_1(i)$$
$$f_4(y_{1,i}) = g_3(i) \oplus n$$
$$f_4(y_{2,i}) = g_2(i) \oplus 2n$$

The vertex labeling f_1 is a bijective function from $f: V(C_n \odot P_2) \to \{1,2,3,\ldots,|V(C_n \odot P_2)|\}$. The edge-weights $w(uv) = f(u) + f(v)$, where $u, v \in C_n \odot P_2$, under the labeling f_1, is $w = \left\{\frac{1}{2}(n+3) + k; 1 \leq k \leq 4n\right\}$, which from a consecutive sequence of $d = 1$. Hence $C_n \odot P_2$ admits an $\left(\frac{1}{2}(n+3) + 1, 1\right)$ –edge-antimagic vertex labeling.

Together with Theorem 3.1 and Corollary 3.2, there is a set of edge labeling $\alpha = \{a_k; 1 \leq k \leq 4n - 1\}$ such that it will give all the edge weights of $C_n \odot P_2$ have the same edge weights. Then the edge coloring of $C_n \odot P_2$, assigned by color $w(uv) = f(u) + f(uv) + f(v)$, will give the same colors.

However, definition of a proper coloring, the adjacent edges can not be assigned by the same colors. Therefore, we need to re-assign a color to the adjacent edges. By using Lemma 3.2, let $W = \{(w_i, a_i);\ w_i \in w, a_i \in a\}$ be the ordered pair of set which gives the total edge weight of $C_n \odot P_2$ of $d = 0$. There are subset $w_j, a_j \subset W$ which all of them are the adjacent edge weights of $C_n \odot P_2$. Based on Lemma 3.2, there are a permutation $\Pi(w_{i,1})$ and $\Pi(w_{i,2})$ such that $W_1' = W_2' = \{a_{i,1}\} + \Pi(w_{i,1}) = \{a_{i,2}\} + \Pi(w_{i,2})$ gives two different colors. Those all colors are distinct with the other non adjacent edge weights of $C_n \odot P_2$.

Therefore, we can define the following edge labeling

$$f_4(y_{1,i}y_{2,i})) = g_1(i) \oplus 3n$$
$$f_4(x_i y_{1,i}) = g_2(i) \oplus 4n$$
$$f_4(x_i y_{2,i}) = g_3(i) \oplus 5n$$
$$f_4(x_1 x_n) = 7n$$
$$f_4(x_i x_{\{i+1\}}) = h(i) \oplus 6n$$

where $h(i)$ is the permutation set $\Pi(\alpha)$ mentioned in Lemma 3.2, with $a = 1, d = 1, k = n - 2$. The edge labeling f is a bijective function from $f: V(C_n \odot P_2) \cup E(C_n \odot P_2) \rightarrow \{1,2,3,\cdots, |V(C_n \odot P_2)| + |E(C_n \odot P_2)|\}$.

Hence, from the super edge local antimagic total labeling above, it easy to see that $W = \left\{\frac{1}{2}(13n+3), \frac{1}{2}(15n+3), \frac{1}{2}(17n+3), \frac{1}{2}(15n+1), \frac{1}{2}(15n+3), \frac{1}{2}(15n+5)\right\}$ contains 5 element which induces a proper edge coloring of $C_n \odot P_2$. Thus, it gives $\gamma_{leat}(C_n \odot P_2) \leq 5$. It concludes that $\gamma_{leat}(C_n \odot P_2) = 5$. ■

Theorem 3.8. Let H be any graph of order $n \geq 3$. Let K_1 be complete graph, then we have $\gamma_{leat}(H \odot rK_1) \geq \gamma_{leat}(H) + r$.

Proof. Let H be any graph with order $n \geq 3$. The vertex set of H is $V(H) = \{a_i : 1 \leq i \leq n\}$. Let $H \odot rK_1$ be connected graph with corona. The vertex set and the edge set of $H \odot rK_1$ are $V(H \odot rK_1) = V(G) \cup \{x_i^j : 1 \leq i \leq n\}$ and $E(H \odot rK_1) = V(G) \cup \{a_i x_i^j : 1 \leq i \leq n\}$.

Suppose H admits a edge local antimagic total labeling with $\gamma_{leat}(H) = k$. We define $f: V(G) \cup E(G) \rightarrow \{1,2,3,\cdots,p+q\}$ where $p = |V(G)|$ and $q = |E(G)|$ as the edge local antimagic total labeling bijection of k colors. Since every vertex of rK_1 connects to every vertex in base graph H, the edge weights of pendant edge must be different with the edge weights of base graph H. It implies that $\gamma_{leat}(H \odot rK_1) \geq k + r$. To show the exact value, firstly we prove that $\gamma_{leat}(H \odot rK_1) \leq k + r$. Define a bijection $g: V(H \odot rK_1) \rightarrow \{1,2,3,\ldots,|V(H)| + rn\}$ by the following way:

$$g(v) = \begin{cases} f(a_i) & if\ v = a_i, 1 \leq i \leq \\ (j+1)n - f(a_i) + 1 & if\ v = x_i^j, 1 \leq i \leq n, 1 \leq j \leq r \end{cases}$$

Based on above function, it can be seen that g is a edge local antimagic total labeling of $H \odot rK_1$ and we have the edge weights as follows:

$$w_g(e) = \begin{cases} w_f(ab) & if\ e = ab, a, b \in V(G) \\ (j+1)n + 1 & if\ e = a_i x_i^j, 1 \leq i \leq n, 1 \leq j \leq r \end{cases}$$

It is easy to see that $w_g(ab) < w_g(a_i x_i^j)$ for every $1 \leq i \leq n, 1 \leq j \leq r$ and each label f is at most n. Thus, $f(a) < f(b)$ or $f(a) < f(b)$ for $a, b \in V(H)$. The edge weight

$$w_g(ab) = g(a) + g(b)$$
$$= f(a) + f(b)$$
$$= 2n - 1$$

and

$$w_g(a_i x_i^j) = g(a_i) + g(x_i^j)$$
$$= f(a_i) + (j+1)n - f(a_i) + 1$$
$$\leq (j+1)n + 1$$

Clearly, that for $n \geq 3$ we have $2n - 1 < (j+1)n + 1$. Thus for every $1 \leq i \leq n, 1 \leq j \leq r$ is

$$w_g(ab) < w_g(a_i x_i^j)$$

Based on the labeling, we know that the edge weight of pendants are larger than the edge weight of the base graph H. Therefore, it is easy to see that g is a edge local antimagic total labeling of $H \odot rK_1$.

$$\gamma_{leat}(H \odot rK_1) \leq |w_g(e)|$$
$$= |w_f(ab)| + |w_g(a_i x_i^j)|$$
$$= \gamma_{leat}(H) + r$$
$$= k + r$$

Hence, from the above edge weight it is easy to see that the upper bound of the local antimagic total edge chromatic number of $H \odot rK_1$ is $\gamma_{leat}(H \odot rK_1) \leq k + r$. It concludes that $\gamma_{leat}(H \odot rK_1) = k + r = \gamma_{leat}(H) + r$.

3. Conclusion

We have found that most of the local antimagic total edge chromatic numbers attain the best lower bound and in this paper we study and determine the chromatic number of the edge local antimagic total labeling of special graph and its operations. However, we need to characterize more general result for any graphs G, especially the connection with the edge local antimagic total labeling of graph..

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