SUPER \((a, d)\)-CYCLES-ANTIMAGIC LABELING OF SUBDIVISION OF A FAN GRAPH

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Abstract

We consider a simple, connected and undirected graph \(G(V, E)\) with vertex set \(V(G)\) and edge set \(E(G)\). There is a super \((a, d)\)-\(H\)-antimagic total labeling on the graph \(G(V, E)\) if there exists a bijection \(f : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\}\) such that for all subgraphs isomorphic to \(H\), the total \(H\)-weights \(W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)\) form an arithmetic sequence.

Received: August 8, 2017; Accepted: November 19, 2017
2010 Mathematics Subject Classification: 05C78.
Keywords and phrases: \(H\)-covering, super \((a, d)\)-\(H\)-antimagic total labeling, cycle-antimagic labeling, subdivisions of fan graph.
\[ \{a, a + d, a + 2d, \ldots, a + (m - 1)d\} \text{, where } a > 0 \text{ is the smallest value, } d \text{ is the feasible difference, and } m \text{ is the number of all subgraphs isomorphic to } H. \] In this paper, we investigate the existence of super \((a, d)\)-\(H\)-antimagic total labeling for subdivisions of a fan graph \(S(F_m)\), when subgraphs \(H\) are cycles.

1. Introduction

Given that a graph \(G = (V, E)\) is nontrivial, finite, simple, undirected and connected graph of vertex set \(V\) and edge set \(E\). For more details on graph, see [10, 3, 4]. A covering of \(G\) is a family of subgraphs \(H_1, H_2, \ldots, H_n\) such that all vertices \(V(G)\) and edges \(E(G)\) belong to at least one of the subgraphs \(H_i, i = 1, 2, \ldots, n\) taken into account as a cover. In this case, we say that \(G\) admits \((H_1, H_2, \ldots, H_n)\)-covering if every subgraph \(H_i\) is isomorphic to a given graph \(H\) admits a special property to be an \(H\)-labeling.

A graph \(G\) is said to be an \((a, d)\)-\(H\)-antimagic total graph if there exists a bijective function \(f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}\) such that for all subgraphs isomorphic to \(H\), the total \(H\)-weights

\[
 w(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)
\]

form an arithmetic sequence \(\{a, a + d, a + 2d, \ldots, a + (m - 1)d\}\), where \(a\) and \(d\) are positive integers and \(n\) is the number of all subgraphs isomorphic to \(H\). If such a function exists, then \(f\) is called an \((a, d)\)-\(H\)-antimagic total labeling of \(G\), see [11]. The total \(H\)-weight is the sum of both vertex and edge labels belonging to a subgraph \(H\), under a given labeling \(f\). The \(H\)-weight under a labeling \(f\) is denoted by \(w(H)\). Such a labeling is called super if the smallest possible labels appear on the vertices. If \(G\) admits a super \((a, d)\)-\(H\)-antimagic total labeling, then we say that \(G\) is a super \((a, d)\)-\(H\)-antimagic graph. For \(d = 0\), it is called \(H\)-magic or \(H\)-supermagic.
Some relevant results have been published in many journals, some of them can be found in [1, 2, 8, 9]. Furthermore, Lladó and Moragas [12] proved that wheels, windmills, books and prisms are $C_t^k$-magic for some $t$. Inayah et al. in [11] proved that for any $H$ and any integer $k \geq 2$, $\text{shack}(H, v, k)$ which contains exactly $k$ subgraphs isomorphic to $H$ admits $H$-super antimagic. Dafik et al. in [5, 6] also obtained a cycle-super antimagicness of connected and disconnected tensor product of graphs, and constructed $H$-antimagic graphs by using smaller edge-antimagic graphs. Furthermore, Dafik et al. in [7] also determined the super $H$-antimagicness of an edge comb product of graphs with subgraph as a terminal of its amalgamation.

2. The Results

We study the subdivision of graph $G$. By subdivision of graph, denoted by $S(G)$, we mean a graph obtained from $G$ by replacing each edge $uv$ of $G$ by a new vertex $y$ and the two new edges $uy$ and $vy$. For details on the subdivision of graph $G$, see [4]. The vertex $y$ is called a subdivision vertex on $uv$.

We deal with the super cycle-antimagic total labelings of subdivision of a fan graph, denoted by $S(F_m)$.

**Observation 1.** Let $S(F_m)$ be a subdivision of a fan graph. The order and size of graph $S(F_m)$ are, respectively, $|V(S(F_m))| = 3m$ and $|E(S(F_m))| = 4m - 2$.

**Proof.** The graph $S(F_m)$ is a connected graph with vertex set $V(S(F_m)) = \{x\} \cup \{x_i; 1 \leq i \leq m\} \cup \{y_i; 1 \leq i \leq m\} \cup \{z_i; 1 \leq i \leq m - 1\}$ and edge set

$$E(S(F_m)) = \{y_iz_i; 1 \leq i \leq m - 1\} \cup \{xix_i; 1 \leq i \leq m\} \cup \{z_iz_{i+1}; 1 \leq i \leq m - 1\} \cup \{y_iz_i; 1 \leq i \leq m\}.$$ 

Thus, the order of the graph $S(F_m)$ is $|V(S(F_m))| = 3m$ and the size of the graph $S(F_m)$ $|E(S(F_m))| = 4m - 2$. 

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For illustration, we give an example of subdivision of a fan graph $S(F_m)$ depicted in Figure 1.

![Diagram of subdivision of a fan graph](image)

**Figure 1.** Example of subdivision of a fan graph $S(F_m)$.

**Observation 2.** Let $C^k_t$ be a cycle of $t$ vertices of subdivision of a fan graph $S(F_m)$, where $t = 6, 8, \ldots, 2m - 2$. The number of cycles of order $t$ which is a cover $H \equiv C^k_t$ of $S(F_m)$ is given by $|H| = m - \frac{t - 4}{2}$.

**Proof.** Let $C^k_t$ be a cycle of $t$ vertices of subdivision of a fan graph $S(F_m)$, where $t = 6, 8, \ldots, 2m - 2$ for $3 \leq m \leq 4$ and $t = 6, 8, \ldots, 2m - 2$ for $m \geq 5$. The $t$th cycle of $C^k_t$ can be formed by the following set of vertices

$$C^k_t = \{x, x_k, y_k, z_k, y_{k+1}, z_{k+1}, y_{k+2}, z_{k+2}, \ldots, z_{k-6}, y_{k-6}, y_{k-4}, x, x_{k-4}, x_{k-6}, x\}.$$

It is easy to see that $k = 1, 2, \ldots, m - \left\lfloor \frac{t - 4}{2} \right\rfloor$. Thus $|C^k_t| = m - \frac{t - 4}{2}$. It concludes the proof. □

Furthermore, we can determine the $C^k_t$-weight of the cycle $C^k_t$, $k = 1, 2, \ldots, m - \left\lfloor \frac{t - 4}{2} \right\rfloor$ under a total labeling $g$: 
\[ w_g(C^k_t) = \sum_{v \in V(C^k_t)} f(v) + \sum_{e \in E(C^k_t)} f(e) \]
\[ = \sum_{s=0}^{t} \left[ g(y_{k+s}) + g(z_{k+s}) + g(y_{k+s}z_{k+s}) + g(z_{k+s}y_{k+s+1}) \right] \]
\[ + g\left( y_k + \left( \frac{t-4}{2} \right) \right) + g(x_k) + g\left( x_k + \left( \frac{t-4}{2} \right) \right) + g(x) \]
\[ + g\left( y_k + \left( \frac{t-4}{2} \right)x + \left( \frac{t-4}{2} \right) \right) + g(y_kx_k) + g(x_kx) + g\left( x_k + \left( \frac{t-4}{2} \right)x \right). \quad (1) \]

From now on, we show our main results. We have found that the graph \( S(F_m) \) admits super \((a, d)\)-\( C^k_t \) antimagic labeling for differences \( d = \{0, 1, 2, 4\} \).

**Theorem 1.** Let \( t = 6, 8, \ldots, 2m - 2 \) for \( 3 \leq m \leq 4 \) and \( t = 6, 8, \ldots, 2m - 2 \) for \( m \geq 5 \). Let \( k = 1, 2, \ldots, m - \left( \frac{t-4}{2} \right) \). The subdivision of fan \( S(F_m) \) admits a super \((a, d)\)-\( C^k_t \)-antimagic labeling for \( d = 0 \).

**Proof.** We define the labeling
\[ g_1, g_1 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, \ldots, p_{S(F_m)} + q_{S(F_m)}\} \]
in the following way:
\[ g_1(y_i) = 2i - 1; \ 1 \leq i \leq m \quad g_1(z_i) = 2i; \ 1 \leq i \leq m - 1 \]
\[ g_1(x_i) = 3m + i - 1; \ 1 \leq i \leq m \quad g_1(x) = 2m \]
\[ g_1(y_iz_i) = 5m - 2i; \ 1 \leq i \leq m - 1 \quad g_1(y_iz_i) = 5m + 2i - 3; \ 1 \leq i \leq m \]
\[ g_1(x_iz_i) = 7m - 2i; \ 1 \leq i \leq m \quad g_1(z_iz_{i+1}) = 5m - 2i - 1; \ 1 \leq i \leq m - 1. \]

Evidently, it is easy to see that \( g_1 \) is a bijective function, as it is a map \( g_1 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, \ldots, 3m, \ldots, 5m - 1, 5m, \ldots, 7m - 2\} \). The total weight of \( V(S(F_m)) \cup E(S(F_m)) = \{y_i; \ 1 \leq i \leq m\} \cup \{y_iz_i; \ 1 \leq i \leq m - 1\} \)
under the labeling \( g_1 \), is given by

\[
g_1(y_i) + g_1(z_i) = g_1(z_i) + g_1(z_i y_{i+1}) = [2i - 1] + \lfloor 5m - 2i \rfloor = 5m - 1. \tag{2}
\]

The total edge-weight of

\[
E(S(F_m)) = \{xx_i; 1 \leq i \leq m\} \cup \{y_i x_i; 1 \leq i \leq m\}
\]

is

\[
g_1(xx_i) + g_1(y_i x_i) = [5m + 2i - 3] + [7m - 2i] = 12m - 3. \tag{3}
\]

From equations (1), (2) and (3), we obtain the total \( C_t^k \)-weight as follows:

\[
w_{g_1}(C_t^k) = \sum_{v \in V(C_t^k)} f(v) + \sum_{e \in E(C_t^k)} f(e)
\]

\[
= \left[ (t - 4)(5m - 1) \right] + 2 \times [12m - 3] + \left[ 3m + k + \frac{t - 4}{2} \right]
\]

\[
+ [3m - k + 1] + 2m
\]

\[
= 5m(t - 4) - t + 4 + 24m - 6 + 3m + k + \frac{t - 4}{2} - 2
\]

\[
+ 3m - k + 1 + 2m
\]

\[
= m(5t + 12) - \frac{t}{2} - 3.
\]

It is easy to see that the total \( C_t^k \)-weights of \( S(F_m) \), under the labeling \( g_1 \), when \( t = 6, 8, \ldots, 2m \) for \( 3 \leq m \leq 4 \) and when \( t = 6, 8, \ldots, 2m - 2 \) for \( m \geq 5 \), and for \( k = 1, 2, \ldots, m - \left( \frac{t - 4}{2} \right) \), constitute the following sets:

\[
C_t^k = \left\{ m(5t + 12) - \frac{t}{2} - 3, m(5t + 12) - \frac{t}{2} - 3, \ldots, m(5t + 12) - \frac{t}{2} - 3 \right\}.
\]

It concludes that the subdivision of fan \( S(F_m) \) admits a super \((a, d)\)-\( C_t^k \)-antimagic total labeling with feasible \( d = 0 \). \( \Box \)
Theorem 2. Let \( t = 6, 8, \ldots, 2m - 2 \) for \( 3 \leq m \leq 4 \) and \( t = 6, 8, \ldots \), \( 2m - 2 \) for \( m \geq 5 \). Let \( k = 1, 2, \ldots, m - \left( \frac{t - 4}{2} \right) \). Then subdivision of fan \( S(F_m) \) admits a super \((a, d)\)-\( C^k_t \)-antimagic labeling for \( d = 1 \).

Proof. We define the labeling

\[ g_2, \quad g_2 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, \ldots, p_{S(F_m)} + q_{S(F_m)}\} \]

in the following way:

\[ g_2(y_i) = i; \quad 1 \leq i \leq m \quad g_2(x_i) = 2m - i + 1; \quad 1 \leq i \leq m \]

\[ g_2(z_i) = 3m - i; \quad 1 \leq i \leq m - 1 \quad g_2(x) = 3m \]

\[ g_2(y_iz_i) = 3m + i; \quad 1 \leq i \leq m - 1 \quad g_2(y_ix_i) = 5m + 2i - 3; \quad 1 \leq i \leq m \]

\[ g_2(xx_i) = 7m - 2i; \quad 1 \leq i \leq m \quad g_2(z_iz_{i+1}) = 5m - i - 1; \quad 1 \leq i \leq m - 1. \]

Evidently, it is easy to see that \( g_2 \) is a bijection, as it is a map \( g_2 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, \ldots, 3m, \ldots, 5m - 1, 5m, \ldots, 7m - 2\} \). The total weight of

\[ V(S(F_m)) \cup E(S(F_m)) = \{y_i; \quad 1 \leq i \leq m\} \cup \{z_i; \quad 1 \leq i \leq m - 1\} \]

under the labeling \( g_2 \), is

\[ g_2(y_i) + g_2(y_iz_i) + g_2(z_i) + g_2(z_iz_{i+1}) = i + 3m + i + 3m - i + 5m - i - 1 = 11m - 1. \]  \( \text{(4)} \)

The total edge-weight of \( E(S(F_m)) = \{xx_i; \quad 1 \leq i \leq m\} \cup \{y_ix_i; \quad 1 \leq i \leq m\} \) is as follows:

\[ g_2(xx_i) + g_2(y_ix_i) = [5m + 2i - 3] + [7m - 2i] = 12m - 3. \]  \( \text{(5)} \)

From equations (1), (4) and (5), we obtain the total \( C^k_t \)-weight as follows:
\[ w_{g_2}(C^k_t) = \sum_{v \in V(C^k_t)} f(v) + \sum_{e \in E(C^k_t)} f(e) \]

\[ = \left[ \left( \frac{t-4}{2} \right)(11m-1) \right] + 2 \times [12m - 3] + [2m + 1] \]

\[ + [2m - k + 1] + 3m \]

\[ = 11m \left( \frac{t-4}{2} \right) - \frac{t-4}{2} + 24m - 6 + 2m + 1 + 2m - k + 1 + 3m \]

\[ = 11m \frac{t}{2} + 9m - \frac{t}{2} - 3 - k. \]

It is easy to see that the total \( C_i \)-weights of \( S(F_m) \), under the labeling \( g_2, t = 6, 8, ..., 2m \) for \( 3 \leq m \leq 4 \) and \( t = 6, 8, ..., 2m - 2 \) for \( m \geq 5 \) and \( k = 1, 2, ..., m - \left( \frac{t-4}{2} \right) \), constitute the following sets:

\[ C^k_t = \left\{ 11m \frac{t}{2} + 9m - \frac{t}{2} - 3 - k, ..., 11m \frac{t}{2} + 9m - \frac{t}{2} - 3 - 2, \right. \]

\[ \left. 11m \frac{t}{2} + 9m - \frac{t}{2} - 3 - 1 \right\}. \]

It concludes that the subdivision of fan \( S(F_m) \) admits a super \((a, d)\)\(-\)antimagic total labeling with feasible \( d = 1 \). \( \square \)

**Theorem 3.** Let \( t = 6, 8, ..., 2m - 2 \) for \( 3 \leq m \leq 4 \) and \( t = 6, 8, ..., 2m - 2 \) for \( m \geq 5 \). Let \( k = 1, 2, ..., m - \left( \frac{t-4}{2} \right) \). Then subdivision of fan \( S(F_m) \) admits a super \((a, d)\)\(-\)antimagic labeling for \( d = 4 \).

**Proof.** We define the labeling

\[ g_3, g_3 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, ..., p_{S(F_m)} + q_{S(F_m)}\} \]

in the following way:
\[ g_3(y_i) = 2i - 1; \quad 1 \leq i \leq m \]
\[ g_3(z_i) = 2i; \quad 1 \leq i \leq m - 1 \]
\[ g_3(x_i) = 2m + i - 1; \quad 1 \leq i \leq m \]
\[ g_3(x_i) = 3m \]
\[ g_3(y_i) = 7m - 2i; \quad 1 \leq i \leq m - 1 \]
\[ g_3(z_i) = 5m - 2i + 1; \quad 1 \leq i \leq m \]
\[ g_3(x_i) = 3m + 2i; \quad 1 \leq i \leq m \]
\[ g_3(z_i) = 7m - 2i - 1; \quad 1 \leq i \leq m - 1. \]

Evidently, it is easy to see that \( g_3 \) is a bijection as it is a map \( g_3 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, \ldots, 3m, \ldots, 5m - 1, 5m, \ldots, 7m - 2\}. \) The total weight of \( V(S(F_m)) \cup E(S(F_m)) = \{y_i; 1 \leq i \leq m\} \cup \{y_i; 1 \leq i \leq m - 1\} \) under the labeling \( g_3 \), is given by

\[ g_3(y_i) + g_3(y_i) = g_3(z_i) + g_3(z_i) = [2i - 1] + [7m - 2i] = 7m - 1. \] (6)

The total edge-weight of \( E(S(F_m)) = \{xx_i; 1 \leq i \leq m\} \cup \{y_i; 1 \leq i \leq m\} \) is as follows:

\[ g_3(x_i) + g_3(x_i) = [5m - 2i + 1] + [3m + 2i] = 8m + 1. \] (7)

From equations (1), (6) and (7), we obtain the total \( C^k_t \)-weight as follows:

\[
\begin{align*}
  w_{g_3}(C^k_t) &= \sum_{v \in V(C^k_t)} f(v) + \sum_{e \in E(C^k_t)} f(e) \\
  &= \left[ (t - 4)(7m - 1) + 2 \times [8m + 1] + \left[ 2(m + 2) + 3 \left( k + \frac{t}{2} - 4 \right) \right] \right] \\
  &\quad + [2m + k - 1] + 3m \\
  &= (t - 4)7m - t + 4 + 16m + 2 + 2m + 4 + 3 \left( k + \frac{t}{2} \right) \\
  &\quad - 12 + 2m + k - 1 + 3m \\
  &= m(7t - 5) - 3 + \frac{t}{2} + 4k.
\end{align*}
\]

It is easy to see that the total \( C_t \)-weights of \( S(F_m) \), under the labeling \( g_3 \), \( t = 6, 8, \ldots, 2m \) for \( 3 \leq m \leq 4 \) and \( t = 6, 8, \ldots, 2m - 2 \) for \( m \geq 5 \) and \( k = 1, 2, \ldots, m = \left( \frac{t - 4}{2} \right) \), constitute the following sets:
It concludes that the subdivision of fan $S(F_m)$ admits a super $(a, d)$-$C_t^k$-antimagic total labeling with feasible $d = 4$. □

**Theorem 4.** Let $t = 6, 8, \ldots, 2m - 2$ for $3 \leq m \leq 4$ and $t = 6, 8, \ldots$, $2m - 2$ for $m \geq 5$. Let $k = 1, 2, \ldots, m - \left\lceil \frac{t - 4}{2} \right\rceil$. Then subdivision of fan $S(F_m)$ admits a super $(a, d)$-$C_t^k$-antimagic labeling for $d = 2$.

**Proof.** We define the labeling $g_3$,

$$g_3 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, \ldots, P_{S(F_m)} + q_{S(F_m)}\}$$

in the following way:

$$g_4(y_i) = 2i - 1; 1 \leq i \leq m$$

$$g_4(x_i) = 3m - i + 1; 1 \leq i \leq m$$

$$g_4(z_i) = 2i; 1 \leq i \leq m - 1$$

$$g_4(x) = 2m$$

$$g_4(y_i z_i) = 5m - i - 1; 1 \leq i \leq m - 1$$

$$g_4(y_i x_i) = 5m + 2i - 3; 1 \leq i \leq m$$

$$g_4(xx_i) = 7m - 2i; 1 \leq i \leq m$$

$$g_4(z_i y_{i+1}) = 4m - i; 1 \leq i \leq m - 1.$$

Evidently, it is easy to see that $g_4$ is a bijection as it is a map $g_3 : V(S(F_m)) \cup E(S(F_m)) \rightarrow \{1, 2, \ldots, 3m, \ldots, 5m - 1, 5m, \ldots, 7m - 2\}$. The total weight of

$$V(S(F_m)) \cup E(S(F_m)) = \{z_i; 1 \leq i \leq m - 1\} \cup \{y_i z_i; 1 \leq i \leq m - 1\}$$

$$\cup \{x_i y_{i+1}; 1 \leq i \leq m - 1\}$$

under the labeling $g_3$, is as follows:

$$g_4(z_i) + g_4(y_i z_i) + g_4(z_i y_{i+1}) = (5m - i - 1) + 2i + 4m - i = 9m - 1. \ (8)$$

The total edge-weights of $E(S(F_m)) = \{xx_i; 1 \leq i \leq m\} \cup \{y_i x_i; 1 \leq i \leq m\}$ are as follows:

$$g_4(xx_i) + g_4(y_i x_i) = [5m + 2i - 3] + [7m - 2i] = 12m - 3. \ (9)$$
From equations (1), (8) and (9), we obtain the total $C_t^k$-weight in the following way:

$$w_{g1}(C_t^k) = \sum_{v \in V(C_t^k)} f(v) + \sum_{e \in E(C_t^k)} f(e)$$

$$= \left[\left(\frac{t-4}{2}\right)(9m-1)\right] + 2 \times [12m - 3] + [2m] + [2k - 1 + 2k + t - 5] + [3m - k + 1] + 3m - \frac{t}{2} + 2 - k$$

$$= 9m\left(\frac{t-4}{2}\right) + 32m + 2k - 7.$$  

It is easy to see that the total $C_t$-weights of $S(F_m)$, under the labeling $g_4, t = 6, 8, ..., 2m$ for $3 \leq m \leq 4$ and $t = 6, 8, ..., 2m - 2$ for $m \geq 5$ and $k = 1, 2, ..., m - \left(\frac{t-4}{2}\right)$, constitute the following sets:

$$C_t^k = \{9m\left(\frac{t-4}{2}\right) + 32m + 2 - 7, 9m\left(\frac{t-4}{2}\right) + 32m + 4 - 7, ..., 9m\left(\frac{t-4}{2}\right) + 32m + 2\left(m - \left(\frac{t-4}{2}\right)\right) - 7\}.$$  

It concludes that the subdivision of fan $S(F_m)$ admits a super $(a, d)$-$C_t^k$-antimagic total labeling with feasible $d = 2$. \hspace{1cm} \square

3. Concluding Remarks

We have shown the existence of super $(a, d)$-$H$-antimagicness of subdivision of fan graphs $S(F_m)$, when $H$ is a cycle. We can prove that $d = \{0, 1, 2, 4\}$. As we have not found the result for another difference, we propose the following:

**Problem.** Find a super $(a, d)$-$H$-antimagic labeling of the subdivision of a fan graph for $d \neq \{0, 1, 2, 4\}$. 
Acknowledgement

We gratefully acknowledge the support HIKOM DP2M Ristekdikti - Indonesia and CGANT - University of Jember of year 2017.

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