\( P_2 \triangleright H \)-super antimagic total labeling of comb product of graphs

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Abstract

Let \( L \) and \( H \) be two simple, nontrivial and undirected graphs. Let \( o \) be a vertex of \( H \), the comb product between \( L \) and \( H \), denoted by \( L \triangleright H \), is a graph obtained by taking one copy of \( L \) and \( |V(L)| \) copies of \( H \) and grafting the \( i \)th copy of \( H \) at the vertex \( o \) to the \( i \)th vertex of \( L \). By definition of comb product of two graphs, we can say that \( V(L \triangleright H) = \{(a, v)| a \in V(L), v \in V(H)\} \) and \( (a, v)(b, w) \in E(L \triangleright H) \) whenever \( a = b \) and \( vw \in E(H) \), or \( ab \in E(L) \) and \( v = w = o \). Let \( G = L \triangleright H \) and \( P_2 \triangleright H \subseteq G \), the graph \( G \) is said to be an \((a, d)-P_2 \triangleright H\)-antimagic total graph if there exists a bijective function \( f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\} \) such that for all subgraphs isomorphic to \( P_2 \triangleright H \), the total \( P_2 \triangleright H \)-weights \( W(P_2 \triangleright H) = \sum_{v \in V(P_2 \triangleright H)} f(v) + \sum_{e \in E(P_2 \triangleright H)} f(e) \) form an arithmetic sequence \([a, a + d, a + 2d, \ldots, a + (n - 1)d]\), where \( a \) and \( d \) are positive integers and \( n \) is the number of all subgraphs isomorphic to \( P_2 \triangleright H \). An \((a, d)-P_2 \triangleright H\)-antimagic total labeling \( f \) is called super if the smallest labels appear in the vertices. In this paper, we study a super \((a, d)-P_2 \triangleright H\)-antimagic total labeling of \( G = L \triangleright H \) when \( L = C_n \).

Keywords: Super \( H \)-antimagic total labeling; Comb product; Cycle graph

1. Introduction

All graphs in this paper are simple, nontrivial and undirected, see [1,2] for more detail definition of graph. A comb product of \( L \) and \( H \), denoted by \( L \triangleright H \), is a graph obtained by taking one copy of \( L \) and \( |V(L)| \) copies of \( H \) and grafting the \( i \)th copy of \( H \) at the vertex \( o \) to the \( i \)th vertex of \( L \). Thus, we have \( V(L \triangleright H) = \{(a, v)| a \in V(L), v \in V(H)\} \) and \( (a, v)(b, w) \in E(L \triangleright H) \) whenever \( a = b \) and \( vw \in E(H) \), or \( ab \in E(L) \) and \( v = w = o \), see Saputro, et al. in [3]. Susilowati in [4] explains in detail about a generalized comb product of graph.

Let \( G = L \triangleright H \) and let \( P_2 \triangleright H \subseteq G \), the graph \( G \) is said to be an \((a, d)-P_2 \triangleright H\)-antimagic total graph if there exist a bijective function \( f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\} \) such that for all subgraphs isomorphic to

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Let \( P_2 \triangleright H \), the total \( P_2 \triangleright H \)-weights \( W(P_2 \triangleright H) = \sum_{v \in V(P_2 \triangleright H)} f(v) + \sum_{e \in E(P_2 \triangleright H)} f(e) \) form an arithmetic sequence \( \{a, a + d, a + 2d, \ldots, a + (n - 1)d\} \), where \( a \) and \( d \) are positive integers and \( n \) is the number of all subgraphs isomorphic to \( P_2 \triangleright H \). Inayah et al. in [5] proved that, for \( H \) is a non-trivial connected graph and \( k \geq 2 \) is an integer, \( sh(B, k) \) which contains exactly \( k \) subgraphs isomorphic to \( H \) is \( H \)-super antimagic. Some other relevant results can be found in [5-9] and [10-14], but their study only covered a fixed order of the covering \( H \). In this paper, we study a super \((a, d)\)-\( P_2 \triangleright H \)-antimagic total labeling of \( G = L \triangleright H \) when \( L = K_n \), and the covering is the subgraph which is isomorphic to \( P_2 \triangleright H \) where \( H \) is any graph. The resulting graphs of \( \text{comb product} \ G = L \triangleright H \) are not unique, but for the antimagicness of total labeling study, we will give the same set of weight even we consider different resulting graphs. Thus, we do not consider a certain linkage vertex \( o \) of this graph operation.

To show those existence, we will use an integer set partition technique introduced by [15,16]. This technique used in determining the feasible difference \( d \). Let \( n, m \) and \( d \) be positive integers. We consider the partition \( \mathcal{P}^{n}_{m,d}(i, j) \) of the set \( \{1, 2, \ldots, mn\} \) into \( n \) columns, \( n \geq 2 \), \( m \)-rows such that the difference between the sum of the numbers in the \( (j + 1) \)th \( m \)-rows and the sum of the numbers in the \( j \)th \( m \)-rows is always equal to the constant \( d \), where \( j = 1, 2, \ldots, n - 1 \). Thus these sums form an arithmetic sequence with the difference \( d \). By the symbol \( \mathcal{P}^{n}_{m,d}(i, j) \) we denote the \( j \)th \( m \)-rows in the partition with the difference \( d \), where \( j = 1, 2, \ldots, n \). Let \( \sum \mathcal{P}^{n}_{m,d}(i, j) \) be the sum of the numbers in \( \mathcal{P}^{n}_{m,d}(i, j) \), thus \( d = \sum \mathcal{P}^{n}_{m,d}(j + 1) - \sum \mathcal{P}^{n}_{m,d}(j) \).

In this study, we will focus for the connected version of the graph \( G = L \triangleright H \). Let \( L, H \) be two graphs of order \( |V(L)|, |V(H)| \) and size \( |E(L)|, |E(H)| \) respectively. The graph \( G = L \triangleright H \) is a connected graph with \( |V(G)| = |V(L)| |V(H)| \) and \( |E(G)| = |V(L)| |E(H)| + |E(L)| \). When \( L = C_n \), thus \( |V(L)| = |E(L)| = n \). Let \( p_H = |V(H)|, q_H = |E(H)| \), the vertex set and edge set of the graph \( G = C_n \triangleright H \) can be split in the following sets: \( V(G) = \{x_j; 1 \leq j \leq n\} \cup \{x_0; 1 \leq i \leq p_H - 1, 1 \leq j \leq n\} \) and \( E(G) = \{x_jx_{j+1}, x_kx_{k+1}; 1 \leq j \leq n - 1\} \cup \{e_j; 1 \leq l \leq q_H, 1 \leq j \leq n\} \). Thus \( |V(G)| = np_H \) and \( |E(G)| = nq_H + n \).

The upper bound of feasible \( d \) for \( G = C_n \triangleright H \) to be a super \((a, d)\)-\( P_2 \triangleright H \)-antimagic total labeling follows the following lemma, proved by [7].

**Lemma 1 ([7]).** Let \( G \) be a simple graph of order \( n \) and size \( q \). If \( G \) is super \((a, d)\)-\( H \)-antimagic total labeling then \( d \leq \frac{(p_G - p_H)p_H + (q_G - q_H)q_H}{p_H - q_H} \), for \( p_G = |V(G)|, q_G = |E(G)|, p_H = |V(H)|, q_H = |E(H)|, n = |H| \).

If \( G = C_n \triangleright H \), the upper bound of feasible \( d \) follows the following corollary.

**Corollary 1.** Let \( K = P_2 \triangleright H \), for odd integer \( n \geq 3 \), if the graph \( G = C_n \triangleright H \) admits super \((a, d)\)-\( K \)-antimagic total labeling with \( p_K = 2p_H \) and \( q_K = 2q_H + 1 \), then \( d \leq \frac{(p_K^2 + q_K^2)}{2} - \frac{n(n - 1)}{2} \). The following theorem will be useful to show the variation of feasible \( d \) for \( G = C_n \triangleright H \) admits super \((a, d)\)-\( K \)-antimagic total labeling.

**Theorem 1 ([17]).** The number of \( r \)-combinations, with repetition allowed (multisets of size \( r \)), that can be selected from a set of \( n \) elements is \( \binom{r + n - 1}{r} \). This equals with the number of ways of choosing \( r \) objects which can be selected from \( n \) categories of objects with allowed repetition.

Furthermore, a partition theorem has been developed by Dafik et al. in [16]. This theorem is used to have a different permutation of partition technique.

**Lemma 2 ([16]).** Let \( n \) and \( m \) be positive integers. The sum of \( \mathcal{P}^{n}_{m,d_1}(i, j) = \{(i - 1)n + j, 1 \leq i \leq m\} \) and \( \mathcal{P}^{n}_{m,d_2}(i, j) = \{(j - 1)m + i; 1 \leq i \leq m\} \) forms an arithmetic sequence of difference \( d_1 = m, d_2 = m^2 \), respectively.

2. The result

Establishing some lemmas related to the partition \( \mathcal{P}^{n}_{m,d}(i, j) \) is a first important step prior to developing the super \((a, d)\)-\( P_2 \triangleright H \)-antimagic total labeling of \( G = C_n \triangleright H \) when \( K = P_2 \triangleright H \). We have \( p_G = |V(G)| = n \frac{d_K}{2} \) and \( q_G = |E(G)| = n \frac{(d_K - 1)}{2} + 1 \).

Based on **Lemma 2**, we can derive two new lemmas with \( d_1 = m \) and \( d_2 = m^2 \), but it has a different bijective function to **Lemma 2**.
Lemma 3. Let $n, m$ be positive integers. For $1 \leq j \leq n$, the sum of
\[ P_{m,d_1}^n(i, j) = \begin{cases} \frac{j+1}{2} (i-1)n + 1 & \text{if } i \leq m; j \text{ odd} \\ \frac{n}{2} + j(i-1)n + 1 & \text{if } i \leq m; j \text{ even} \end{cases} \]
forms an arithmetic sequence of difference $d_1 = m$.

Proof. By simple calculation, it gives $\sum_{i=1}^{m} P_{m,d_1}^n(i, j) = P_{m,d_1}^n(j)$, where
\[ P_{m,d_1}^n(j) = \begin{cases} \frac{(j+1)m + i}{2} ; 1 \leq i \leq m; j \text{ odd} \\ \frac{m}{2}(j + 1) + \frac{m^2}{2} ; j \leq m; j \text{ even} \end{cases} \]
Since $\left[ \frac{n}{2} \right] = \frac{n+1}{2}$ for $n$ odd, and $\left[ \frac{n}{2} \right] = \frac{n}{2}$ for $n$ even, it is easy to see that $P_{m,d_1}^n(j) = \{ \frac{mn}{2}, \frac{mn}{2} + \frac{m^2n}{2} - mn + m, \frac{mn}{2} + \frac{m^2n}{2} - mn + 2m, \ldots, \frac{m^2n}{2} + \frac{m^2n}{2} \}$ form an arithmetic sequence of difference $d_1 = m$. \hfill \Box

Lemma 4. Let $n, m$ be positive integers. For $1 \leq j \leq n$, the sum of
\[ P_{m,d_2}^n(i, j) = \begin{cases} \frac{(j-1)m + i}{2} ; 1 \leq i \leq m; j \text{ odd} \\ \frac{m}{2}(j + 1) + \frac{m^2}{2} (j-1) + \frac{m}{2} ; j \leq m; j \text{ even} \end{cases} \]
forms an arithmetic sequence of difference $d_2 = m^2$.

Proof. By simple calculation, it gives $\sum_{i=1}^{m} P_{m,d_2}^n(i, j) = P_{m,d_2}^n(j)$, where
\[ P_{m,d_2}^n(j) = \begin{cases} \frac{(j-1)m + i}{2} ; 1 \leq i \leq m; j \text{ odd} \\ \frac{m}{2}(j + 1) + \frac{m^2}{2} (j-1) + \frac{m}{2} ; j \leq m; j \text{ even} \end{cases} \]
Similarly, since $\left[ \frac{n}{2} \right] = \frac{n+1}{2}$ for $n$ odd, and $\left[ \frac{n}{2} \right] = \frac{n}{2}$ for $n$ even, it is easy to see that $P_{m,d_2}^n(j) = \{ \frac{m^2}{2} + \frac{m}{2}, \frac{m^2}{2} + \frac{3m^2}{2}, \ldots, m^2n - \frac{m^2}{2} + \frac{m}{2} \}$ form an arithmetic sequence of difference $d_2 = m^2$. It concludes the proof. \hfill \Box

Now, we are ready to present our main theorem related to the existence of super $(a, d)$-$P_2 \triangleright H$-antimagic total labeling of $G = L \triangleright H$ when $L = C_n$.

Theorem 2. Let $K = P_2 \triangleright H$, and let $p_H = m_1 + m_2$ and $q_H = r_1 + r_2$ be the number of vertices and edges of graph $H$, respectively. For odd integer $n \geq 3$, if we assign the linear combination of $P_{m,m}^n$ and $P_{m,m^2}^n$ as a label of all elements in $G$, then $G = C_n \triangleright H$ admits a super $(a, d)$-$P_2 \triangleright H$ antimagic total labeling with $d = m_1 + m_2^2 + r_1 + r_2 + 1$.

Proof. The graph $G = C_n \triangleright H$ is a connected graph with vertex set and edge set of the graph $G = C_n \triangleright H$ can be split in the following sets: $V(G) = \{ x_j ; 1 \leq j \leq n \} \cup \{ x_{ij} ; 1 \leq i \leq p_H - 1, 1 \leq j \leq n \}$ and $E(G) = \{ x_jx_{j+1}, x_{1n}; 1 \leq j \leq n - 1 \} \cup \{ x_{ij}; 1 \leq i \leq l \leq q_H, 1 \leq j \leq n \}$. Then $p_G = |V(G)| = np_H$ and $q_G = |E(G)| = q_H + n$. Since the cover is $K = P_2 \triangleright H$, and let $p_H = m_1 + m_2$ and $q_H = r_1 + r_2$, we can define the vertex labeling $f_1 : V(G) \cup E(G) \rightarrow \{ 1, 2, \ldots, p_G + q_G \}$ by using the linear combination of $P_{m,m}^n$ and $P_{m,m^2}^n$. By Lemmas 3 and 4, we use $m_1$ and $r_1$ for the partition $P_{m,m}^n(i, j)$ and we use $m_2$ and $r_2$ for the partition $P_{m,m^2}^n(i, j)$. For $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n$, the total labels can be expressed as follows
\[
 f_1(x_j) = \{ P_{m,m}^n \} \cup \{ P_{m,m^2}^n \} \\
 f_1(x_{ij}) = \{ mn + 1 \} \\
 f_1(x_{ij+1}) = \{ mn + 1 + j ; 1 \leq j \leq n - 1 \} \\
 f_1(x_{ij}) = \{ P_{r_1,r_1}^n \} \cup \{ P_{r_2,r_2}^n + [mn + n] \} \cup \{ P_{r_1,r_1}^n \} \cup \{ P_{r_2,r_2}^n + [n(r_1) + mn + n] \}.
\]
The vertex labeling $f_1$ is a bijective function $f_1 : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, pg + qG\}$. The total edge-weights of $G = C_n \bowtie H$ under the labeling $f_1$, for $1 \leq j \leq n - 1$, constitute the following sets:

$$w^1_{f_1} = \left[ \sum_{i=1}^{m} P^n_{m_1,m_1}(i, j) + m_1n + \sum_{i=1}^{m} P^n_{m_1,m_1}(i, j + 1) + m_1n \right] + \left[ \sum_{i=1}^{m} P^n_{m_2,m_2}(i, j) + m_2n(m_1 + 1) + \sum_{i=1}^{m} P^n_{m_2,m_2}(i, j + 1) + m_2n(m_1 + 1) \right]$$

$$= [P^n_{m_1,m_1}(j) + nm_1 + P^n_{m_1,m_1}(j + 1) + nm_1] + [P^n_{m_2,m_2}(j) + nm_2(m_1 + 1)$$

$$+ P^n_{m_2,m_2}(j + 1) + nm_2(m_1 + 1)]$$

$$= (m_1\left(\frac{j+1}{2}\right) + \left(\frac{m_1+m_1^2}{2}\right)n - m_1n) + [m_1\left(\frac{n}{2}\right) + \frac{j+1}{2} + \left(\frac{m_1^2-m_1}{2}\right)n] + [m_2\left(\frac{2j+1}{2}\right) + \frac{m_2^2j}{2} + \frac{m_2}{2} + nm_1m_2]$$

$$= \sum_{i=1}^{m} P^n_{m_1,m_1}(i, j + 1) + m_1j + m_1 + m_1^2n - m_1n + [m_2\left(\frac{n}{2}\right) + m_2^2j + m_2 + 2nm_2m_1]$$

$$w^2_{f_1} = \sum_{i=1}^{r} P^n_{r_1,r_1}(i, j) + r_1(mn + 2n) + \sum_{i=1}^{r} P^n_{r_1,r_1}(i, j + 1) + r_1(mn + 2n)]$$

$$+ \left[ \sum_{i=1}^{r} P^n_{r_2,r_2}(i, j) + r_2(nr_1 + mn + 2n) + \sum_{i=1}^{r} P^n_{r_2,r_2}(i, j + 1) + r_2(nr_1 + mn + 2n) \right]$$

$$= [P^n_{r_1,r_1}(j) + r_1(mn + 2n) + P^n_{r_1,r_1}(j + 1) + r_1(mn + 2n)] + [P^n_{r_2,r_2}(j)$$

$$+ r_2(nr_1 + mn + 2n) + P^n_{r_2,r_2}(j + 1) + r_2(nr_1 + mn + 2n)]]$$

$$= \left[ r_1\left(\frac{j+1}{2}\right) + \left(\frac{r_1+r_1^2}{2}\right)n - r_1n + r_1(mn+n) \right] + \left[ r_1\left(\frac{n}{2}\right) + \frac{j+1}{2} + \left(\frac{r_1^2-r_1}{2}\right)n + r_1(mn+n) \right] + [r_2\left(\frac{n}{2}\right) + r_2^2j + r_2 + 2r_2(nr_1 + mn + n)]$$

$$W^1_j = w^1_{f_1} + f_1(x_jx_{j+1}) + w^2_{f_1} = w^1_{f_1} + mn + j + 1 + w^2_{f_1} = C_1 + j[m_1 + m_2^2 + r_1 + r_2^2 + 1]$$

where $C_1 = \{m_1\left[\frac{n}{2}\right] + m_1 + m_1^2n - m_1n\} + \{m_2^2\left[\frac{n}{2}\right] + m_2 + 2nm_2m_1\} + mn + \{r_1\left[\frac{n}{2}\right] + r_1 + r_1^2n - r_1n + 2r_1(mn+n)\} + \{r_2^2\left[\frac{n}{2}\right] + r_2 + 2r_2(nr_1 + mn + n)\} + 1$. While the total $K$-weight for $j = 1$, $n$ is as follows:

$$w^1_{f_1} = [\sum_{i=1}^{m} P^n_{m_1,m_1}(i, 1) + m_1n + \sum_{i=1}^{m} P^n_{m_1,m_1}(i, n) + m_1n] + [\sum_{i=1}^{m} P^n_{m_2,m_2}(i, 1) + m_2n(m_1 + 1) + \sum_{i=1}^{m} P^n_{m_2,m_2}(i, n) + m_2n(m_1 + 1)$$

$$= [P^n_{m_1,m_1}(1) + nm_1 + P^n_{m_1,m_1}(n) + nm_1] + [P^n_{m_2,m_2}(1) + nm_2(m_1 + 1)$$

$$+ P^n_{m_2,m_2}(n) + nm_2(m_1 + 1)]$$

$$= \left[ m_1\left(\frac{j+1}{2}\right) + \left(\frac{m_1+m_1^2}{2}\right)n - m_1n \right] + \left[ m_1\left(\frac{n}{2}\right) + \frac{m_1^2}{2}n - m_1n \right] + [\frac{m_2^2}{2}(m_2 + 1) + nm_1m_2] + [\frac{m_2}{2}(m_2n + 1) + nm_1m_2] + 1.$$
Let $K$.

**Proof.** The graph $G = C_n \cup W_s$ can be split in the following sets: $V(G) = \{x_j; 1 \leq j \leq n\} \cup \{x_{ij}; 1 \leq i \leq (s + 1) - 1, 1 \leq j \leq n\}$ and $E(G) = \{x_jx_{j+1}, x_jx_{n+1}, 1 \leq j \leq n - 1\} \cup \{x_jx_{j-1}, x_jx_{n+1}, 1 \leq j \leq n; 1 \leq i \leq s - 2\} \cup \{x_jx_i; 1 \leq j \leq n; 1 \leq i \leq s - 1\}$. Thus $p_G = |V(G)| = n(s + 1)$ and $q_G = |E(G)| = 2ns$. Since the cover is $K = P_2 \cup W_s$, and let $p_{w_j} = m_1 + m_2$ and $q_{w_i} = r_1 + r_2$, we can define the vertex labeling $f_1 : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p_G + q_G\}$ by using the linear combination of $\mathcal{P}_{m,m}^n$ and $\mathcal{P}_{m,m}^{n+2}$: By Lemma and 4, we use $m_1$ and $r_1$ for the partition $\mathcal{P}_{m,m}^n(i,j)$ and we use $m_2$ and $r_2$ for the partition $\mathcal{P}_{m,m}^n(i,j)$. For $i = 1, 2, \ldots, m, l = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n$, the total labels can be expressed as follows

$$f_2(x_j \cup x_{i,j}) = \{\mathcal{P}_{m_1,m_1}^n \cup \{\mathcal{P}_{m_2,m_2}^n \oplus nm_1\}\}$$

$$f_2(x_{i,n}) = \{mn + 1\}$$

$$f_2(x_{j,x_{j+1}}) = \{mn + 1 + j; 1 \leq j \leq n - 1\}$$

$$f_2(x_{j,x_{j+1}}) = \{mn + 1 + j; 1 \leq j \leq n - 1\}$$

$$f_2(x_{j,x_{j+1}}) = \{mn + 1 + j; 1 \leq j \leq n - 1\}$$

From the two $K$-weights, we have the following

$$W_j = \{C_1, C_1 + [m_1 + m_2 + r_1 + r_2 + 1], C_1 + 2[m_1 + m_2 + r_1 + r_2 + 1], \ldots, C_1 + (n - 1)[m_1 + m_2 + r_1 + r_2 + 1]\}.$$
The vertex labeling $f_2$ is a bijective function $f_2 : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p_G + q_G\}$. The total edge-weights of $G = C_n \triangleright W_5$ under the labeling $f_2$, for $1 \leq j \leq n - 1$, constitute the following sets:

\[w^1_{f_2} = \left\{ \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, j) + m_1n + \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, j + 1) + m_1n \right\} + \left\{ \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2}(i, j) + m_2n(m_1 + 1) + \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2}(i, j + 1) + m_2n(m_1 + 1) \right\} = \left[ \mathcal{P}^n_{m_1,m_1}(j) + nm_1 + \mathcal{P}^n_{m_1,m_1}(j + 1) + nm_1 \right] + \left[ \mathcal{P}^n_{m_2,m_2}(j) + nm_2(m_1 + 1) \right]
\]

\[= \left\{ (m_1(j + 1)/2) + (m_1^2 + 2m_1n - m_1n) \right\} + \left\{ (m_2^2 + 2m_1n + m_1) \right\} = \left\{ (m_1(j + 1)/2) + (m_1^2 + 2m_1n - m_1n) \right\} + \left\{ (m_2^2 + 2m_1n + m_1) \right\}
\]

\[w^2_{f_2} = \left\{ \sum_{i=1}^{r} \mathcal{P}^n_{r_1,r_1}(l, j) + r_1(mn + 2n) + \sum_{i=1}^{r} \mathcal{P}^n_{r_2,r_2}(l, j + 1) + r_1(mn + 2n) \right\} + \left\{ (r_2^2 - r_1^2) + r_2^2 + r_1 + r_2 \right\} + \left\{ (r_2^2 + r_1 + r_2 + r_3 + r_2r_3) + r_1 + r_2 + r_3 + r_2r_3 + r_1 + r_2 + r_3 \right\}
\]

\[= \left\{ (r_1(j + 1)/2) + (r_1^2 + 2r_1n - r_1n) \right\} + \left\{ (r_2^2 - r_1^2) + r_2^2 + r_1 + r_2 \right\} + \left\{ (r_2^2 + r_1 + r_2 + r_3 + r_2r_3) + r_1 + r_2 + r_3 + r_2r_3 + r_1 + r_2 + r_3 \right\}
\]

\[W^2_{f_1} = w^1_{f_1} + f_1(x_1,x_{i+1}) = w^2_{f_1} = w^1_{f_1} + mn + j + 1 + w^2_{f_1} = C_j + j(m_1 + m_2^2 + r_1 + r_2^2 + 1)
\]

where $C_j = \{ m_1(n^2/2) + m_1 + m_1^2n - m_1n \} + \{ m_2^2(n^2/2) + m_2 + 2mnm_1 \} + mn + \{ r_1(n^2/2) + r_1 + r_2^n - r_1n + 2r_1(mn + 1) \} + \{ r_1^2(n^2/2) + r_2 + 2r_2r_3(nr_1 + mn) \} + 1$. While the total $K$-weight for $j = 1, n$ is as follows:

\[w^1_{f_2} = \left\{ \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, 1) + m_1n + \sum_{i=1}^{m} \mathcal{P}^n_{m_1,m_1}(i, 1) + m_1n \right\} + \left\{ \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2}(i, 1) + m_2n(m_1 + 1) + \sum_{i=1}^{m} \mathcal{P}^n_{m_2,m_2}(i, 1) + m_2n(m_2 + 1) \right\} = \left[ \mathcal{P}^n_{m_1,m_1}(1) + nm_1 + \mathcal{P}^n_{m_1,m_1}(n) + nm_1 \right] + \left[ \mathcal{P}^n_{m_2,m_2}(1) + nm_2(m_1 + 1) \right]
\]

\[= \left\{ m_1(j + 1)/2) + (m_1^2 + 2m_1n - m_1n) \right\} + \left\{ m_2^2 + 2m_1n + m_1) \right\} = \left\{ m_1(j + 1)/2) + (m_1^2 + 2m_1n - m_1n) \right\} + \left\{ m_2^2 + 2m_1n + m_1) \right\}
\]
\[
\begin{align*}
\sum_{i=1}^{r} P_{r,1}^n(l,1) + r_1(mn + 2n) + \sum_{i=1}^{r} P_{r,1}^n(l,1) + r_1(mn + 2n) + \sum_{i=1}^{r} P_{r,2}^n(l,1) + r_2(nr_1 + mn + 2n) + \sum_{i=1}^{r} P_{r,2}^n(l,1) + r_2(nr_1 + mn + 2n) + mn + 2n)
= [P_{r,1}^n(1) + r_1(mn + 2n) + P_{r,1}^n(1) + r_1(mn + 2n) + P_{r,2}^n(1) + r_2(nr_1 + mn + 2n)]
= \{r_1(\frac{j+1}{2} + \frac{r_1 + r_2^2}{2})n - r_1n + r_1(mn + n)\} + \{r_1(\frac{n+1}{2} + \frac{r_1 + r_2}{2})n - r_1n + r_1(mn + n)\}
= \{(r_1^2 + r_1)n - \frac{3}{2}r_1n + \frac{3}{2}r_1 + 2r_1(mn + n)\} + \{(r_2^2 + r_2)n - \frac{3}{2}r_2n + \frac{3}{2}r_2 + 2r_2nr_1 + mn + n)\}
= W_2^2 = w_1^2 + f_2(x_1, x_2) + w_2^2 = w_1^2 + mn + 1 + w_2^2 = C_2.
\end{align*}
\]

From the two \(K\)-weights, we have the following
\[
\bigcup_{t=1}^{2} W_{f_2}^2 = \{C_2, C_2 + [m_1 + m_2^2 + r_1 + r_2^2 + 1], C_2 + 2[m_1 + m_2^2 + r_1 + r_2^2 + 1], \ldots, C_2 + (n - 1)(m_1 + m_2^2 + r_1 + r_2^2 + 1)\}.
\]

It is easy to see that all total \(K\)-weight elements form an arithmetic sequence with the smallest value \(C_2\) and the difference \(d = m_1 + m_2^2 + r_1 + r_2^2 + 1\). It concludes the proof. \(\square\)

**Fig. 1** shows an example of super \((a, d)\)-antimagic total covering of graph \(G = C_5 \uplus W_5\) using a linear combination of \(P_{m,m}^n(i,j)\) and \(P_{m,m}^n(i,j)\). We use linear combination \(P_{4,4}^5(i,j)\) and \(P_{2,2}^5(i,j)\) for vertex labeling and linear combination \(P_{5,5}^5(i,j)\) and \(P_{5,5}^5(i,j)\) for edge labeling. Thus the value of \(d = 4 + 2^2 + 5 + 5^2 + 1 = 39\) and the smallest value is \(a = 1351\).

We have shown the theorem above, the question now, how many feasible values of \(d = m_1 + m_2^2 + r_1 + r_2^2\) can we have? The following theorem will describe its number of possibility feasible \(d\).

**Theorem 3.** Let \(m\) and \(r\) be positive integer of \(m = m_1 + m_2 + r_1 + r_2\). If \(d = m_1 + m_2^2 + r_1 + r_2^2\) then the number of possible different \(d\) is at least \(m\) for \(m > r\), at least \(r\) for \(r > m\), and at most \(mr\).

**Proof.** Let \(d_1 = m_1 + m_2^2\) and \(d_2 = r_1 + r_2^2\). Based on **Theorem 1**, the equation \(m_1 + m_2 = m\) has \(\binom{m+2}{m} - 1\) number of solutions. When we substitute all the possible solutions it will possibly gives the same \(d_1\). Take \(m_2 = 1, m_1 = m - 1\) and \(m_1 = m, m_2 = 0,\) and substitute into \(d_1\) yields the following:

\[
\begin{align*}
d_1 &= m_1 + m_2^2 = m - 1 + (1)^2 = m, \quad \text{or} \\
d_1 &= m_1 + m_2^2 = m + (0)^2 = m.
\end{align*}
\]

Thus, the number of possible solution is less than one. It implies that the number of possible solution \(m_1 + m_2 = m\) satisfying for different \(d_1 = m_1 + m_2^2\) is the following

\[
\begin{align*}
\binom{m+2}{m} - 1 &= \binom{m+1}{m} - 1 \\
&= \frac{(m+1)!}{m!1!} - 1 \\
&= \frac{(m+1)(m)!}{m!1!} - 1 \\
&= m.
\end{align*}
\]
By the same manner, we will get the number of solution such that the feasible $d_2$ has different $r$. Since $d = d_1 + d_2$ and we consider an optimal parameter $d_1$ or $d_2$, with number of possible $d_1$ and $d_2$ are respectively $m$ and $r$, the number of different solution of $d$, for $m > r$ and for $r > m$ are $m$ and $r$ respectively. Furthermore, since $d_1$ and $d_2$ has respectively at most $m$ and $r$ solutions, $d = d_1 + d_2$ has at most $mr$ solutions. □

3. Concluding remarks

We have shown the existence of super antimagicness of comb product of any graphs $G = L \triangleright H$ when $L = C_n$ and $K = P_2 \triangleright H$. By using a partition technique we can prove that, for odd $n \geq 3$, $G = C_n \triangleright H$ admits a super($a$, $d$)-$P_2 \triangleright H$-antimagic total labeling with difference $d = m_1^2 + m_2 + r_1^2 + r_2 + 1$. For more illustration of our general theorem, we have taken a special $H = W_s$. For odd $n \geq 3$, $G = C_n \triangleright W_s$ admits a super($a$, $d$)-$P_2 \triangleright W_s$-antimagic total labeling. However, for $n$ is even we have not found any result yet. Thus, we propose the following open problem.

**Open Problem 1.** For even $n \geq 3$, do the graphs $G = C_n \triangleright H$ admit a super ($a$, $d$)-$P_2 \triangleright H$-antimagic total labeling with all feasible $d$?

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**References**


