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Several classes of graphs and their $r$-dynamic chromatic numbers

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Abstract. Let $G$ be a simple, connected and undirected graph. Let $r, k$ be natural numbers. By a proper $k$-coloring of a graph $G$, we mean a map $c: V(G) \rightarrow S$, where $|S| = k$, such that any two adjacent vertices receive different colors. An $r$-dynamic $k$-coloring is a proper $k$-coloring of $G$ such that $|c(N(v))| \geq \min\{r, d(v)\}$ for each vertex $v$ in $V(G)$, where $N(v)$ is the neighborhood of $v$ and $c(S) = \{c(v) : v \in S\}$ for a vertex subset $S$. The $r$-dynamic chromatic number, written as $\chi_r(G)$, is the minimum $k$ such that $G$ has an $r$-dynamic $k$-coloring. By simple observation it is easy to see that $\chi_{r+1}(G) - \chi_r(G)$ does not always show a small difference for any $r$. Thus, finding an exact value of $\chi_r(G)$ is significantly useful. In this paper, we will study some of them especially when $G$ are prism graph, three-cyclical ladder graph, joint graph and circulant graph.

Keywords: $r$-dynamic chromatic number, graph coloring, special graphs.

1. Introduction

The $r$-dynamic chromatic number, introduced by Montgomery [8] and written as $\chi_r(G)$, is the least $k$ such that $G$ has an $r$-dynamic $k$-coloring. Note that the 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number of graph has been studied under the name a dynamic chromatic number, denoted by $\chi_d(G)$. In [8], he conjectured $\chi_2(G) \leq \chi(G) + 2$ when $G$ is regular, which remains open. Akbari et.al. [4] proved Montgomery’s conjecture for bipartite regular graphs, as well as Lai, et.al. [9] proved $\chi_2(G) \leq \Delta(G) + 1$ for $\Delta(G) \leq 3$ when no component is the 5-cycle. Some other results can be site in [1, 2, 3, 14].

By a greedy coloring algorithm, Jahanbekama [7] proved that $\chi_r(G) \leq r \Delta(G) + 1$, and equality holds for $\Delta(G) > 2$ if and only if $G$ is $r$-regular with diameter 2 and girth 5. They improved the bound to $\chi_r(G) \leq \Delta(G) + 2r - 2$ when $\delta(G) > 2r \ln n$ and...
\( \chi_r(G) \leq \Delta(G) + r \) when \( \delta(G) > r^2 \ln n \). For further results of \( r \)-dynamic chromatic number can be seen in [6, 10, 11, 12, 5].

The following observation is useful to find the exact values of \( r \)-dynamic chromatic number.

Observation 1. Let \( \delta(G) \) and \( \Delta(G) \) be a minimum and maximum degree of a graph \( G \), respectively. Then the followings hold

- \( \chi_r(G) \geq \min\{\Delta(G), r\} + 1 \),
- \( \chi(G) \leq \chi_2(G) \leq \cdots \leq \chi_{\Delta(G)}(G) \),
- \( \chi_{r+1}(G) \geq \chi_r(G) \) and if \( r \geq \Delta(G) \) then \( \chi_r(G) = \chi_{\Delta(G)}(G) \).

Tahekhani in [13], proved the following theorem

Theorem 1. [13] Let \( G \) be a \( d \)-regular graph and \( r \) be a positive integer with \( 2 \leq r \leq \frac{\delta}{\log(2e\delta(\Delta^2 + 1))} \). Then the \( r \)-dynamic chromatic number of \( G \) is \( \chi_r(G) \leq \chi(G) + (r - 1)\left[ \frac{\delta}{e} \log(2e\delta(\Delta^2 + 1)) \right] \), where \( e \) euler’s number.

2. The Results

We are ready to show our main theorems. There are four theorems found in this study. These deals with prism graph, three-cyclical ladder graph, joint graph and circulant graph.

Theorem 2. Let \( P_{n,2} \) be a prism graph, the \( r \)-dynamic chromatic number is:

\[
\chi(P_{n,2}) = \begin{cases} 
 2, & n \text{ even} \\
 3, & n \text{ odd}
\end{cases} \quad \chi_d(P_{n,2}) = \begin{cases} 
 3, & n = 3k, k \in N \\
 4, & \text{otherwise}
\end{cases}
\]

For \( r \geq 3 \), we have

\[
\chi_r(P_{n,2}) = \begin{cases} 
 4, & n = 4k, k \in N \\
 6, & n = 3, 7, 11 \\
 5, & \text{otherwise}
\end{cases}
\]

Proof. A prism graph, denoted by \( P_{n,2} \), is a connected graph with vertex set \( V(P_{n,2}) = \{x_i, y_i | 1 \leq i \leq n\} \), and edge set \( E(P_{n,2}) = \{x_i x_i+1, y_i y_i+1 | 1 \leq i \leq n - 1\} \cup \{x_n x_1 \} \cup \{y_n y_1 \} \cup \{x_i y_i | 1 \leq i \leq n\} \). The order and size of \( P_{n,2} \), \( n \geq 3 \) are \( |V(P_{n,2})| = 2n \) and \( |E(P_{n,2})| = 3n \). A prism graph is regular graph of degree 3, thus \( P_{n,2} \), \( \delta(P_{n,2}) = \Delta(P_{n,2}) = 3 \). By Observation 1, \( \chi_r(P_{n,2}) \geq \min\{\Delta(P_{n,2}), r\} + 1 = \min\{3, r\} + 1 \). To find the exact value of \( r \)-dynamic chromatic number of \( P_{n,2} \), we define three cases, namely \( \chi(P_{n,2}), \chi_2(P_{n,2}) \) and \( \chi_{r\geq3}(P_{n,2}) \). For \( \chi(P_{n,2}) \), the lower bound \( \chi(P_{n,2}) \geq \min\{3, 1\} + 1 = 2 \). We will prove that \( \chi(P_{n,2}) \leq 2 \) by defining a map \( c_1 : V(P_{n,2}) \rightarrow \{1, 2, \ldots, k\} \) for \( n \geq 3 \), by the following:

\[
c_1(x_1, x_2, \ldots, x_n) = \begin{cases} 
 21 \ldots 21, & n \text{ even} \\
 12 \ldots 12, & n \text{ odd}
\end{cases}
\]

\[c_1(y_1, y_2, \ldots, y_n) = \begin{cases} 
 12 \ldots 12, & n \text{ even} \\
 312\ldots 12, & n \text{ odd}
\end{cases} \]
It is easy to see that $c_1$ gives $\chi(P_{n,2}) \leq 2$ for $n$ even, but for $n$ odd, we could not avoid to have $\chi(P_{n,2}) \leq 3$, otherwise there are at least two adjacent vertices assigned the same colors. Thus $\chi(P_{n,2}) = 2$ for $n$ even and $\chi(P_{n,2}) = 3$, for $n$ odd.

For $\chi_2(P_{n,2})$, the lower bound $\chi_2(P_{n,2}) \geq \min\{3, 2\} + 1 = 3$. We will prove that $\chi_2(P_{n,2}) \leq 3$ by defining a map $c_2 : V(P_{n,2}) \rightarrow \{1, 2, \ldots, k\}$ where $n \geq 3$, by the following

$$c_2(x_1, x_2, \ldots, x_{n-1}) = \begin{cases} 
12123, & n = 5 \\
123 \ldots 123, & n \equiv 0 \pmod{3}, \\
123 \ldots 123 4, & n \equiv 1 \pmod{3}, \\
123 \ldots 123 41234, & n \equiv 2 \pmod{3}.
\end{cases}$$

$$c_2(y_1, y_2, \ldots, y_{n-1}) = \begin{cases} 
23434, & n = 5 \\
312 \ldots 312, & n \equiv 0 \pmod{3}, \\
4123 \ldots 123, & n \equiv 1 \pmod{3}, \\
4123 \ldots 123 4123, & n \equiv 2 \pmod{3}.
\end{cases}$$

It is easy to see that $c_2$ gives $\chi_2(P_{n,2}) \leq 3$, for $n = 3k, k \in N$, but apart $n = 3k$ we could not avoid to have $\chi_2(P_{n,2}) \leq 4$ otherwise there are at least two adjacent vertices assigned the same colors. Thus $\chi_2(P_{n,2}) = 3$ for $n = 3k$ and $\chi_2(P_{n,2}) = 4$ for otherwise.

For $\chi_r(P_{n,2})$ and $r \geq 3$, the lower bound $\chi_2(P_{n,2}) \geq \min\{3, 3\} + 1 = 4$. We will prove that $\chi_3(P_{n,2}) \leq 4$ by defining a map $c_3 : V(P_{n,2}) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$, by the following.

$$c_3(x_1, x_2, \ldots, x_n) = \begin{cases} 
123, & n = 3, \\
1234 \ldots 1234, & n \equiv 0 \pmod{4}, \\
1234 \ldots 1234 5, & n \equiv 1 \pmod{4}, \\
123456, & n = 6, \\
1234563, & n = 7, \\
1234 \ldots 1234 512345, & n \equiv 2 \pmod{4}, n \geq 10, \\
12345123456, & n = 11, \\
1234 \ldots 1234 51234512345, & n \equiv 3 \pmod{4}, n \geq 15.
\end{cases}$$

$$c_3(y_1, y_2, \ldots, y_{n-1}) = \begin{cases} 
456, & n = 3, \\
341234 \ldots 1234 12, & n \equiv 0 \pmod{4}, \\
451234 \ldots 1234 123, & n \equiv 1 \pmod{4}, \\
561234, & n = 6, \\
6412345, & n = 7, \\
45123451234 \ldots 1234 123, & n \equiv 0 \pmod{4}, n \geq 10, \\
561234512345, & n = 11, \\
451234 \ldots 1234 51234512345, & n \equiv 0 \pmod{4}, n \geq 15.
\end{cases}$$

It is easy to see that $c_3$ gives $\chi_3(P_{n,2}) \leq 4$, for $n = 4k, k \in N$, but for $n = 3, 6, 7, 11$ we are forced to have $\chi_3(P_{n,2}) \leq 6$ as well as $\chi_3(P_{n,2}) \leq 5$ for $n$ otherwise. Thus $\chi_3(P_{n,2}) = 4$, for $n = 4k$, $\chi_3(P_{n,2}) = 6$ for $n = 3, 6, 7, 11$ and $\chi_3(P_{n,2}) = 5$ for $n$ otherwise. By Observation 1, since $r \geq \Delta(P_{n,2}) = 3$, it immediately gives $\chi_3(P_{n,2}) = \chi_r(P_{n,2})$ for $n \geq 3$. \[\Box\]
Theorem 3. Let $G$ be three-cyclical ladder graph ($TCL_n$) for $n \geq 2$, $r$-dynamic chromatic number of $TCL_n$ is

$$\chi(TCL_n) = \chi_d(TCL_n) = 3, \chi_3(TCL_n) = 4, \chi_4(TCL_n) = 5, \chi_r(TCL_n) = 6, r \geq 5$$

Proof. The graph three-cyclical ladder graph, denoted by $TCL_n$, is connected graph with vertex set $V(TCL_n) = \{x_i, y_j, z_j ; 1 \leq i \leq n; 1 \leq j \leq n + 1\}$ and edge set $E(TCL_n) = \{y_jz_j ; 1 \leq j \leq n + 1\} \cup \{x_iy_{j+1} ; 1 \leq i \leq n\} \cup \{x_iy_i; x_{i+1}z_i; x_{i+1}z_{i+1}; 1 \leq i \leq n\}$. Thus, $p = |V(TCL_n)| = 3n + 2, q = |E(TCL_n)| = 6n + 1, \Delta(TCL_n) = 5$.

By Observation 1, $\chi_r(TCL_n) \geq \min\{\Delta(TCL_n), r\} + 1 = \min\{5, r\} + 1$. To find the exact value of $r$-dynamic chromatic number of $TCL_n$, we define three cases, namely for $\chi(TCL_n), \chi_d(TCL_n), \chi_3(TCL_n)$ and $\chi_4(TCL_n)$.

For $\chi(TCL_n), \chi_d(TCL_n)$, the lower bound $\chi_4(TCL_n) \geq \min\{5, 2\} + 1 = 3$. We will show that $\chi_1(TCL_n) \leq 3$, by defining a map $c_4 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 2$ by the following

$$c_4(x_i) = 3, 1 \leq i \leq n$$

$$c_4(y_j) = \begin{cases} 1, & j \equiv 1 \mod 2, 1 \leq j \leq n + 1, \\ 2, & j \equiv 0 \mod 2, 1 \leq j \leq n + 1. \end{cases}$$

$$c_4(z_j) = \begin{cases} 1, & j \equiv 0 \mod 2, 1 \leq j \leq n + 1, \\ 2, & j \equiv 1 \mod 2, 1 \leq j \leq n + 1. \end{cases}$$

It easy to see that $c_4$ gives $\chi(TCL_n) \leq 3$ and $\chi_d(TCL_n) \leq 3$. Thus $\chi(TCL_n) = 3$ and $\chi_d(TCL_n) = 3$.

For $r = 3$, the lower bound $\chi_3(TCL_n) \geq \min\{5, 3\} + 1 = 4$. We will show that $\chi_3(TCL_n) \leq 4$, by defining a map $c_5 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 2$ by the following

$$c_5(x_i) = \begin{cases} 1, & i \equiv 2 \mod 3, 1 \leq i \leq n, \\ 2, & i \equiv 0 \mod 3, 1 \leq i \leq n, \\ 3, & i \equiv 1 \mod 3, 1 \leq i \leq n. \end{cases}$$

$$c_5(y_j) = \begin{cases} 1, & j \equiv 1 \mod 3, 1 \leq j \leq n + 1, \\ 2, & j \equiv 2 \mod 3, 1 \leq j \leq n + 1, \\ 3, & j \equiv 0 \mod 3, 1 \leq j \leq n + 1. \end{cases}$$

$$c_5(z_j) = 4, \text{for } 1 \leq i \leq n + 1$$

It is easy to understand that $c_5$ gives $\chi_3(TCL_n) \leq 4$. Thus $\chi_3(TCL_n) = 4$.

For $r = 4$, the lower bound $\chi_4(TCL_n) \geq \min\{5, 4\} + 1 = 5$. We will show that $\chi_4(TCL_n) \leq 5$, by defining a map $c_6 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 2$ by the following

$$c_6(x_i) = \begin{cases} 3, & i \equiv 1 \mod 3, 1 \leq i \leq n, \\ 4, & i \equiv 2 \mod 3, 1 \leq i \leq n, \\ 5, & i \equiv 0 \mod 3, 1 \leq i \leq n. \end{cases}$$

$$c_6(y_j) = \begin{cases} 1, & j \equiv 1 \mod 2, 1 \leq j \leq n + 1, \\ 2, & j \equiv 0 \mod 2, 1 \leq j \leq n + 1. \end{cases}$$
The graph $c_6(z_j) = \begin{cases} 
3, & j \equiv 0 \mod{3}, 1 \leq j \leq n + 1, \\
4, & j \equiv 1 \mod{3}, 1 \leq j \leq n + 1, \\
5, & j \equiv 2 \mod{3}, 1 \leq j \leq n + 1. 
\end{cases}
$

It is easy to see that $c_6$ gives $\chi_4(TCL_n) \leq 5$. Thus $\chi_4(TCL_n) = 5$.

For $r = 5$, the lower bound $\chi_5(TCL_n) \geq \min\{5, 5\} + 1 = 6$. We will show that $\chi_5(TCL_n) \leq 6$, by defining a map $c_7 : V(TCL_n) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 2$ by the following

$c_7(x_i) = \begin{cases} 
4, & i \equiv 1 \mod{3}, 1 \leq i \leq n, \\
5, & i \equiv 2 \mod{3}, 1 \leq i \leq n, \\
6, & i \equiv 0 \mod{3}, 1 \leq i \leq n. 
\end{cases}
$

$c_7(y_j) = \begin{cases} 
1, & j \equiv 1 \mod{3}, 1 \leq j \leq n + 1, \\
2, & j \equiv 2 \mod{3}, 1 \leq j \leq n + 1, \\
3, & j \equiv 0 \mod{3}, 1 \leq j \leq n + 1. 
\end{cases}
$

$c_7(z_j) = \begin{cases} 
4, & j \equiv 0 \mod{3}, 1 \leq j \leq n + 1, \\
5, & j \equiv 1 \mod{3}, 1 \leq j \leq n + 1, \\
6, & j \equiv 2 \mod{3}, 1 \leq j \leq n + 1. 
\end{cases}
$

It clearly shows that $c_7$ gives $\chi_5(TCL_n) \leq 6$. Thus $\chi_5(TCL_n) = 6$. Since for $r \geq 5$, we have $r \geq \Delta(TCL_n)$. By Observation 1, $\chi_r(TCL_n) = \chi_5(TCL_n) = 6$. It concludes the proof.

**Theorem 4.** Let $P_n + C_m$ be a joint graph of $P_n$ and $C_m$, the $r$-dynamic chromatic number is

$\chi_{1 \leq r \leq 4}(P_n + C_m) = \begin{cases} 
5, & m = 3k, k \in N, \\
6, & m \text{ otherwise}. 
\end{cases}
$

$\chi_5(P_n + C_m) = \begin{cases} 
5, & m = 5, \\
6, & m = 3, \\
7, & m \text{ otherwise}. 
\end{cases}
$

For $r \geq 6$, we have

$\chi_r(P_n + C_m) = \begin{cases} 
r + m - 2, & 3 \leq m \leq r - 2, m \geq r - 1, n \geq m - 1, \\
2r - 3, & m \text{ lainnya}, n \geq r - 1. 
\end{cases}
$

**Proof.** The graph $P_n + C_m$ is a connected graph with vertex set $V(P_n + C_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq m\}$ and edge set $E(P_n + C_m) = \{x_i; x_{i+1}; 1 \leq i \leq n - 1\} \cup \{y_j; y_{j+1}; 1 \leq j \leq m\}$. The order and size of this graph are $p = |V(P_n + C_m)| = m + n$, $q = |E(P_n + C_m)| = mn + m - 1$. Since all vertices in $P_n$ joint with all vertices in $C_m$, it gives $\Delta(P_n + C_m) = m + 2$

By Observation 1, $\chi_r(P_n + C_m) \geq \min\{\Delta(P_n + C_m), r\} + 1 = \min\{m + 2, r\} + 1$. To find the exact value of $r$-dynamic chromatic number of $P_n + C_m$, we define three cases, namely for $\chi_{1 \leq r \leq 4}(P_n + C_m), \chi_5(P_n + C_m)$ and $\chi_{r \geq 6}(P_n + C_m)$.
For $\chi_{1 \leq r \leq 4}(P_n + C_m)$, define a map $c_8 : V(P_n + C_m) \to \{1, 2, \ldots, k\}$ where $n \geq 3$, by the following:

$$c_8(x_0, x_1, x_2, \ldots, x_{n-1}) = \begin{cases} 
123 \ldots 123, n \equiv 0 \pmod{3}, m \equiv 2 \pmod{3}, \\
123 \ldots 123 1, n \equiv 1 \pmod{3}, m \equiv 2 \pmod{3}, \\
123 \ldots 123 12, n \equiv 2 \pmod{3}, m \equiv 2 \pmod{3}, \\
12 \ldots 12, n \text{ even}, m \text{ otherwise}, \\
12 \ldots 12 1, n \text{ odd}, m \text{ otherwise}.
\end{cases}$$

It is easy to see that $c_8$ gives $\chi_{1 \leq r \leq 4}(P_n + C_m) = 5$, for $m = 3k, k \in N$ and $\chi_{1 \leq r \leq 4}(P_n + C_m) = 6$ for $m$ otherwise.

For $\chi_5(P_n + C_m)$, define a map $c_9 : V(P_n + C_m) \to \{1, 2, \ldots, k\}$ where $n \geq 3$, by the following:

$$c_9(x_0, x_1, x_2, \ldots, x_{n-1}) = \begin{cases} 
123 \ldots 123, n \equiv 0 \pmod{3}, m \equiv 2 \pmod{3}, \\
123 \ldots 123 1, n \equiv 1 \pmod{3}, m \equiv 2 \pmod{3}, \\
123 \ldots 123 12, n \equiv 2 \pmod{3}.
\end{cases}$$

It is easy to see that $c_9$ gives $\chi_5(P_n + C_m) = 6$, for $m = 3$, $\chi_5(P_n + C_m) = 8$, for $m = 5$, and $\chi_5(P_n + C_m) = 7$ for $m$ otherwise.

The last for $\chi_6(P_n + C_m)$, define a map $c_{10} : V(P_n + C_m) \to \{1, 2, \ldots, k\}$ where $m \geq 3, n \geq r - 2$, by the following

$$c_{10}(x_i) = \begin{cases} 
1, i \equiv 1 \pmod{r - 2}, \\
2, i \equiv 2 \pmod{r - 2}, \\
3, i \equiv 3 \pmod{r - 2}, \\
\vdots \\
r - 3, i = n - 1, \\
r - 2, i = n.
\end{cases}$$
Let $\chi_r(P_n+C_m) = r+m-2$ for $3 \leq m \leq r-2, m \geq r-1, n \geq m-1$ and $\chi_6(P_n+C_m) = 2r-3$ for $n \geq r-1, m$ otherwise. By Observation 1, since $r \geq \Delta(P_n+C_m) = m+2$, it immediately gives $\chi_6(P_n+C_m) = \chi_r(P_n+C_m)$ for $n \geq 4$. □

**Theorem 5.** Let $C_n(1, \frac{n}{2})$ be a circulant graph of order 3, the $r$-dynamic chromatic number is

$$
\chi(C_n(1, \frac{n}{2})) = \begin{cases} 
4, & n = 4, \\
2, & n = 4k + 2, k \in \mathbb{N}, \\
3, & n = 4k + 4, k \in \mathbb{N}.
\end{cases}
$$

For $r \geq 3$, we have

$$
\chi_r(C_n(1, \frac{n}{2})) = \begin{cases} 
n, & n = 4, 6, 8, \\
4, & n = 8k + 4, k \in \mathbb{N}, \\
5, & n = 8k + 6, k \in \mathbb{N}, \\
6, & n \text{ otherwise.}
\end{cases}
$$

**Proof.** The graph $C_n(1, \frac{n}{2})$ is a connected graph with vertex set $V(C_n(1, \frac{n}{2})) = \{x_i, 0 \leq i \leq n - 1\}$ and edge set $E(C_n(1, \frac{n}{2})) = \{x_i x_{i+1}(\text{mod } n), 0 \leq i \leq n - 1\} \cup \{x_i x_{i+\frac{n}{2}}(\text{mod } n), 0 \leq i \leq \frac{n}{2}\}$. The order and size of the graph $C_n(1, \frac{n}{2})$ are $p = |V(C_n(1, \frac{n}{2}))| = n, q = |E(C_n(1, \frac{n}{2}))| = \frac{3n}{2}$. Since $C_n(1, \frac{n}{2})$ is a regular graph of degree 3, thus $\delta(C_n(1, \frac{n}{2})) = \Delta(C_n(1, \frac{n}{2})) = 3$.

By Observation 1, $\chi_r(C_n(1, \frac{n}{2})) = \min\{\Delta(C_n(1, \frac{n}{2})), r\} + 1 = \min\{3, r\} + 1$. In the same way, to find the exact value of $r$-dynamic chromatic number of $C_n(1, \frac{n}{2})$, we define three cases, namely for $\chi(C_n(1, \frac{n}{2})), \chi_2(C_n(1, \frac{n}{2}))$ and $\chi_{r \geq 3}(C_n(1, \frac{n}{2}))$.

For $\chi(C_n(1, \frac{n}{2}))$, define a map $c_{11} : V(C_n(1, \frac{n}{2})) \to \{1, 2, \ldots, k\}$ where $n \geq 3$, by the following:

$$
c_{11}(x_0, x_1, \ldots, x_{n-1}) = \begin{cases} 
1234, & n = 4, \\
12 \ldots 12, & n = 4k + 2, k \in \mathbb{N}.
\end{cases}
$$

$$
c_{11}(x_0, x_1, \ldots, x_{\frac{n}{2}}) = 12 \ldots 12 13, n = 4k + 4, k \in \mathbb{N}.
$$

$$
c_{11}(x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_{n-1}) = 21 \ldots 21, 32, n = 4k + 4, k \in \mathbb{N}.
$$
It easy to see that for $n = 4k + 2, k \in N$, and $\chi(C_n(1, \frac{n}{2})) = 3$, for $n = 4k + 4, k \in N$. 

For $\chi_2(C_n(1, \frac{n}{2}))$, define a map $c_{12} : V(C_n(1, \frac{n}{2})) \rightarrow \{1, 2, \ldots, k\}$ where $n \geq 3$, by the following:

\[
c_{12}(x_0, x_1, \ldots, x_{n-1}) = 1234, \quad \text{for } n = 4
\]
\[
c_{12}(x_0, x_1, \ldots, x_{n-2}) = 12 \ldots 12, \quad \text{for } n = 4k + 2, k \in N
\]
\[
c_{12}(x_{n-2}^2+1, x_{n-2}^2+2, \ldots, x_{n-1}) = 34 \ldots 34, \quad \text{for } n = 4k + 2, k \in N
\]

It easy to see that $c_{12}$ gives $\chi_2(C_n(1, \frac{n}{2})) = 4$ for any $n$.

For $\chi_2(C_n(1, \frac{n}{2}))$, and $r \geq 3$, define a map $c_{13} : V(C_n(1, \frac{n}{2})) \rightarrow \{1, 2, \ldots, k\}$ where $n \geq 3$, by the followings

- For $n = 4$, $c_{13}(x_i) = i + 1, \quad 0 \leq i \leq n - 1$
- For $n = 10$

\[
c_{13}(x_i) = \begin{cases} 
1, & i \equiv 0 \pmod{4}, 0 \leq i \leq n - 4, \\
2, & i \equiv 1 \pmod{4}, 1 \leq i \leq n - 3, \\
3, & i \equiv 2 \pmod{4}, 2 \leq i \leq n - 2, \\
4, & i \equiv 3 \pmod{4}, 3 \leq i \leq n - 1.
\end{cases}
\]

- For $n = 8k + 4, k \in N$

\[
c_{13}(x_i) = \begin{cases} 
1, & i \equiv 0 \pmod{4}, 0 \leq i \leq \frac{n}{4} - 7, \\
2, & i \equiv 1 \pmod{4}, 1 \leq i \leq \frac{n}{4} - 6, \\
3, & i \equiv 2 \pmod{4}, 2 \leq i \leq \frac{n}{4} - 5, \\
4, & i \equiv 3 \pmod{4}, 3 \leq i \leq \frac{n}{4} - 4, \\
5, & i = \frac{n}{2} - 12 \text{ atau } i = n - 1,
\end{cases}
\]

- For $n = 8k + 6, k \in N$

\[
c_{13}(x_i) = \begin{cases} 
i, & i \equiv 0 \pmod{\frac{n}{2} - 2}, \frac{n}{2} - 2 \leq i \leq n - 5, \\
i - 1, & i \equiv 1 \pmod{\frac{n}{2} - 2}, \frac{n}{2} - 1 \leq i \leq n - 4, \\
i - 2, & i \equiv 2 \pmod{\frac{n}{2} - 2}, \frac{n}{2} - 2 \leq i \leq n - 3, \\
i - 3, & i \equiv 3 \pmod{\frac{n}{2} - 2}, \frac{n}{2} + 1 \leq i \leq n - 2.
\end{cases}
\]
Acknowledgement

Given that any connected graphs $G$, we have not found any result yet, thus we propose the following open problem.

Open Problem

Given that any connected graphs $G$, determine the sharp lower bound of $\chi_r(G)$

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References


