

**LAPORAN AKHIR PENELITIAN
PENELITIAN PASCA DOKTOR (PPD)
TAHUN ANGGARAN 2020
KONTRAK PENUGASAN Nomor : 3479/UN25.3.1/LT/2020**



JUDUL PENELITIAN:
**MODIFIKASI OPTIMASI KETIDAKPASTIAN
DENGAN METODE TITIK INTERIOR**

PENELITI
Dr. Agustina Pradjaningsih, S.Si., M.Si, 0002087111

**FAKULTAS MATEMATIKA DAN ILMU PENGETAHUAN ALAM
UNIVERSITAS JEMBER
KEMENTERIAN PENDIDIKAN DAN KEBUDAYAAN
DESEMBER, 2020**

Halaman Pengesahan

1. Judul Kegiatan:

Modifikasi Optimasi Ketidakpastian Dengan Metode Titik Interior

2. Penanggung Jawab Kegiatan

- a. Nama Lengkap : Dr. Agustina Pradjaningsih, S.Si., M.Si.
- b. NIDN : 0002087111
- c. Fakultas/Jurusan : FMIPA/Matematika
- d. Alamat : Jl. Kalimantan 37 Jember
- e. Telpon/Faks : +62 331 334293 / +62 331 330225
- f. Alamat Rumah : Tegal Besar Permai I T/25 Jember
- g. Telpon/Faks/E-mail : 087856872999
- h. Bidang keahlian : Riset Operasi/Matematika Terapan

3. Dosen Pembina

- a. Nama Lengkap : Prof. Slamim, M.Comp.Sc., Ph.D
- b. NIDN : 0020046701
- c. Fakultas/Jurusan : FASILKOM
- d. Bidang keahlian : Kombinatorik/Graph

4. Durasi Waktu Kegiatan : Oktober – Desember 2020

5. Pembiayaan

- a. Jumlah biaya dari UNEJ : Rp 22.447.000,-
- b. Jumlah biaya dari sumber pembiayaan lain : Rp -
- Total : Rp 22.447.000,-

Jember, 10 Desember 2020

Mengetahui :

Dekan Fakultas MIPA

Ketua Peneliti

Drs. Achmad Sjaifullah, M.Sc., Ph.D
NIP. 195910091986021001


Dr. Agustina Pradjaningsih, S.Si.M.Si
NIP. 197108022000032009

Ketua LP2M Universitas Jember

Prof. Ir. Achmad Subagio, M.Agr., Ph.D
NIP. 196905171992111001

RINGKASAN

Pada berbagai situasi kehidupan terdapat pengambilan keputusan yang menyangkut permasalahan optimasi yang berkaitan dengan pengoptimuman baik maksimum ataupun minimum dalam penggunaan sumber daya yang terbatas. Dari situasi tersebut dapat dikatakan optimasi merupakan masalah yang berhubungan dengan keputusan dan cara penentuan solusi terbaik atau memuaskan. **Tetapi pada berbagai situasi pula dalam dunia riil ada faktor ketidakpastian yang dikhawatirkan pengoptimalan tidak tercapai.** Semisal dalam perusahaan produksi, terkadang perusahaan menghadapi beberapa kendala seperti: ada dan tidaknya atau banyak dan sedikitnya ketersediaan sumberdaya, naik-turunnya harga, dan juga daya beli konsumen yang berubah-ubah. Hal inilah yang disebut dengan ketidakpastian dalam pemodelan, dimana sering terjadi dalam kehidupan nyata. Permasalahan pemodelan ketidakpastian inilah yang menjadi kesulitan para pemilik perusahaan. Hal ini harus diantisipasi oleh pemilik perusahaan agar tujuan perusahaan tersebut tetap tercapai yaitu mengoptimalkan keuntungan dan mengefisienkan sumber daya yang dipergunakan. **Untuk mengantisipasi permasalahan yang tidak pasti tersebut maka pada nilai-nilai koefisien dan variabel pada pemodelan yang dibuat dalam bentuk interval.**

Tahun 2017, Agustina Pradjaningsih telah menyelesaikan penelitian disertasi doktor dengan judul Konstruksi Metode Titik Interior pada Pemrograman Linier Interval dengan Menggunakan Nilai Batas Bawah Terbesar dan Nilai Batas Bawah Terkecil. Selanjutnya Pradjaningsih, et all (2019) dalam artikel yang ditulisnya, telah menemukan konsep, metode penyelesaian pemrograman linier dengan koefisien interval menggunakan *Affine Scaling*. Tahun 2019, peneliti masuk dalam dua kelompok Riset (KeRis) yang ada di Jurusan Matematika Universitas Jember yaitu *Mathematical Optimization and Computation* dan yaitu KeRis Pemodelan Matematika. Berpijak dari ketiga hal tersebut dan adanya **kesinambungan atau kelinieran penelitian, maka penelitian ini bertujuan untuk melanjutkan penerapan metode titik interior tersebut dalam skala yang lebih luas yaitu modifikasi optimasi ketidakpastian dengan menggunakan metode titik interior yang berkelanjutan.**

Hasil atau Luaran dari penelitian ini merupakan publikasi ilmiah internasional dengan Tingkat Kesiapan Terapan Teknologi (TKT) pada level 3 yaitu menghasilkan formulasi konsep dan atau aplikasi teknologi hingga pembuktian konsep penting secara analitis dan eksperimental. Sesuai dengan rencana pada proposal, **satu artikel telah accepted pada *Far East Journal of Applied Mathematics* (FJAM).** Disamping itu hasil penelitian ini telah dipresentasikan pada seminar internasional IcoMathApp (In the **1st International Conference on Mathematics and Its Applications** “Strengthening Researches on Mathematics for the Challenge of Global Society” yang diselenggarakan oleh Universitas Negeri Malang). **Hasil penelitian lain adalah diperoleh draft satu artikel** yang direncanakan untuk dipublikasikan pada tahun 2022 (sesuai janji luaran pada proposal)

PRAKATA

Puji dan syukur kami panjatkan ke hadirat Allah Subhanahu wa ta'ala yang telah melimpahkan rahmat dan karunia-Nya kepada peneliti sehingga peneliti dapat menyusun laporan akhir penelitian ini. Shalawat dan Salam semoga senantiasa tercurah kepada junjungan kita Nabi Muhammad Sholallahu Alaihi Wassalam. Laporan Akhir ini peneliti susun sebagai salah satu pertanggung jawaban sebagai penerima Hibah Pasaca Doktor Internal Universitas Jember. Terimakasih peneliti haturkan Lembaga Penelitian dan Pengabdian Kepada Masyarakat Universitas Jember selaku penyandang dana dalam penelitian ini.

Jember, Desember 2020

Peneliti



DAFTAR ISI

	Halaman
Halaman Sampul	1
Halaman Pengesahan	2
Ringkasan	3
Prakata	4
Daftar Isi	5
Bab 1 Pendahuluan	6
Bab 2 Tinjauan Pustaka	8
Bab 3 Metode Penelitian	15
Bab 4 Hasil Yang Dicapai	17
Daftar Pustaka	18
Lampiran-Lampiran	
Lampiran 1 Accepted Artikel di Far East Journal of Applied Mathematics	
Lampiran 2 Draft Artikel tahun 2022	
Lampiran 3 Sertifikat dan PPT Ppresentasi Seminar Internasional IcoMathApp 2020	

BAB 1. PENDAHULUAN

Pada berbagai situasi kehidupan terdapat pengambilan keputusan yang menyangkut permasalahan optimasi yang berkaitan dengan pengoptimuman baik maksimum ataupun minimum dalam penggunaan sumber daya yang terbatas. Dari situasi tersebut dapat dikatakan optimasi merupakan masalah yang berhubungan dengan keputusan dan cara penentuan solusi terbaik atau memuaskan. Tetapi pada berbagai situasi pula dalam dunia riil ada faktor ketidakpastian yang dikhawatirkan pengoptimalan tidak tercapai

Semisal dalam suatu perusahaan yang menghasilkan suatu barang, maka akan terdapat suatu aktifitas produksi dan siklus produksi dan. Aktivitas produksi pada umumnya adalah aktivitas mengubah bahan baku menjadi bahan jadi. Hal ini terjadi pada perusahaan industri baik perusahaan hasil industri pertanian, peternakan ataupun perikanan. Siklus produksi memiliki rangkaian aktivitas yang di selenggarakan secara bertahap. Salah satu tahapan itu adalah pemilihan model yang tepat dalam penentuan jumlah barang yang akan diproduksi. Pada tahapan ini, terkadang perusahaan menghadapi beberapa kendala yang yang bervariasi seperti misalnya : ada dan tidaknya atau banyak dan sedikitnya ketersediaan sumberdaya, naik-turunnya harga, dan juga daya beli konsumen yang berubah-ubah. Hal inilah yang disebut dengan ketidakpastian dalam pemodelan, dimana sering terjadi dalam kehidupan nyata. Permasalahan pemodelan ketidakpastian inilah yang menjadi kesulitan para pemilik perusahaan suatu hasil industri. Kesulitan dalam pemodelan inilah yang harus diantisipasi oleh pemilik perusahaan agar tujuan perusahaan tersebut tetap tercapai yaitu mengoptimasikan keuntungan dan mengefisienkan sumber daya yang dipergunakan.

Penyelesaian permasalahan optimisasi disesuaikan dengan karakteristik permasalahan yang akan diselesaikan. Jika karakteristik permasalahan bersifat linear maka disebut dengan pemrograman linear. Pemrograman linear adalah suatu alat matematika yang dikembangkan untuk menangani permasalahan optimasi yang terdiri dari fungsi tujuan dan fungsi kendala berbentuk linear. Fungsi tujuan berbentuk persamaan linear sedangkan fungsi kendala dapat berbentuk persamaan linear ataupun pertidaksamaan linear. Metode penyelesaian permasalahan pemrograman linier salah satunya adalah metode titik interior. Seperti telah diketahui bahwa pola umum masalah yang dapat dikategorikan dalam pemrograman linear adalah : jika memenuhi salah satu asumsi yaitu *certainty*/kepastian Asumsi mengenai *certainty*/kepastian adalah bahwa semua koefisien model (koefisien fungsi tujuan ataupun fungsi kendala) merupakan konstanta-konstanta yang diketahui dengan pasti. Namun dalam situasi sesungguhnya, terkadang terdapat permasalahan yang tidak pasti baik dalam koefisien model ataupun variabel model. Untuk mengantisipasi permasalahan yang tidak pasti tersebut

maka pada nilai-nilai koefisien dan variabel pemrograman linear dibuatlah suatu nilai dalam bentuk interval.

Metode penyelesaian permasalahan pemrograman linier salah satunya adalah metode titik interior. Metode titik interior ini diperkenalkan oleh Narendra Karmakar (1984) dari AT dan T Bell Laboratories, pada papernya yang berjudul *A New Polynomial Time Algorithm for Linear Programming*. Seperti telah diketahui bahwa pola umum masalah yang dapat dikategorikan dalam pemrograman linear adalah : jika memenuhi asumsi-asumsi *proportionality*, *additivity*, *divisibility*, dan *certainty*/kepastian (Winston, 1994). Asumsi mengenai *certainty*/kepastian adalah bahwa semua koefisien model (koefisien fungsi tujuan ataupun fungsi kendala) merupakan konstanta-konstanta yang diketahui dengan pasti. Namun dalam situasi sesungguhnya, terkadang terdapat permasalahan yang tidak pasti. Untuk mengantisipasi permasalahan yang tidak pasti tersebut maka pada nilai-nilai koefisien dan nilai variabel pemrograman linear dibuatlah suatu nilai dalam bentuk interval. Hal ini seiring pula dengan berkembangnya analisis interval oleh Moore (1966).

Tahun 2017, Agustina Pradjaningsih telah menyelesaikan penelitian disertasi doktor dengan judul *Konstruksi Metode Titik Interior pada Pemrograman Linier Interval dengan Menggunakan Nilai Batas Bawah Terbesar dan Nilai Batas Bawah Terkecil*. Selanjutnya Pradjaningsih, et all (2019) dalam artikel yang ditulisnya, telah menemukan konsep, metode dan algoritma metode titik interior untuk menyelesaikan pemrograman linier dengan koefisien interval menggunakan *Affine Scaling*. Tahun 2019, peneliti masuk dalam kelompok Riset yang ada di Jurusan Matematika Universitas Jember yaitu *Mathematical Optimization and Computation* (MOC_o) dan KeRis Pemodelan. KeRis MOC_o yaitu kelompok riset yang bertujuan dapat membaca permasalahan-permasalahan nyata yang terjadi lalu menyelesaikannya menggunakan teori matematika, khususnya optimasi dan komputasi matematika, sedangkan KeRis Pemodelan melakukan kegiatan penelitian dan pembelajaran dibidang pemodelan matematika yang mencakup model proses di beberapa bidang kajian dan pengembangan seperti pengembangan model matematika di bidang pertanian industrial, pengembangan model *traffic flow*, pemetaan potensi pangan dan pengembangan model dinamik ekosistem.. Berpijak dari ketiga hal tersebut dan adanya kesinambungan atau kelinieran penelitian, maka penelitian ini bertujuan untuk melanjutkan penerapan metode titik interior tersebut dalam skala yang lebih luas modifikasi optimasi ketidakpastian dengan menggunakan metode titik interior yang berkelanjutan.

BAB 2. TINJAUAN PUSTAKA

Berikut ini disajikan konsep yang berkaitan dengan algoritma titik interior, analisis interval dan pemrograman linier interval serta hasil-hasil penelitian terdahulu yang berkaitan dengan pemrograman linear interval. Untuk selanjutnya semua bilangan yang dimaksudkan dalam penelitian ini adalah bilangan dalam himpunan bilangan real (R).

2.1. Algoritma Titik Interior

Berikut ini disajikan definisi dan sifat-sifat yang berhubungan dengan metode titik interior yang digunakan dalam penelitian ini. Metode titik interior yang dibahas disini adalah metode titik interior untuk permasalahan pemrograman linear standar. Selanjutnya untuk mencapai tujuan penelitian maka metode ini akan dipakai untuk permasalahan pemrograman linear interval.

Definisi 1. Suatu sistem persamaan dan pertidaksamaan linear sebarang, memenuhi kondisi titik interior jika terdapat penyelesaian fisibel yang tepat memenuhi semua batas kendala pertidaksamaan pada sistem tersebut. (Roos et al, 2005)

Pemrograman linier klasik dengan fungsi tujuan memaksimumkan didefinisikan sebagai berikut : (Hillier & Lieberman, 2010).

Maksimumkan

$$Z = \sum_{j=1}^n c_j x_j, \quad (1)$$

dengan kendala

$$\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m, \quad (2)$$

$$x_j \geq 0, \quad j = 1, 2, \dots, n, \quad (3)$$

dan x_j variabel keputusan, c_j koefisien fungsi tujuan, a_{ij} koefisien fungsi kendala, b_i batas sumber daya, Z fungsi tujuan, serta $x_j \in \mathbb{R}^+$, $c_j, a_{ij}, b_i \in \mathbb{R}$.

Pemrograman linier yang dirumuskan pada Persamaan (1), Persamaan (2) dan Persamaan (3) dapat direpresentasikan dalam bentuk matriks sebagai berikut :

Maksimumkan

$$\mathbf{Z} = \mathbf{c}^T \mathbf{x}, \quad (4)$$

dengan kendala

$$\mathbf{Ax} \leq \mathbf{b}, \quad (5)$$

$$\mathbf{x} \geq \mathbf{0}, \quad (6)$$

dan

$$\mathbf{c}^T = [c_1, c_2, \dots, c_n],$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{0}_{n \times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

dengan $\mathbf{c}, \mathbf{x}, \mathbf{0} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, dan $\mathbf{A} \in \mathbb{R}^{m \times n}$.

Permasalahan pemrograman linear yang dirumuskan pada Persamaan (4), Persamaan (5) dan Persamaan (6) diubah ke permasalahan pemrograman linier bentuk standar sebagai berikut :

Maksimumkan

$$\mathbf{Z} = \mathbf{C}^T \mathbf{X}, \tag{7}$$

dengan kendala

$$\mathbf{A}\mathbf{X} = \mathbf{b}, \tag{8}$$

$$\mathbf{X} \geq \mathbf{0}, \tag{9}$$

dan

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix},$$

dengan $\mathbf{C}, \mathbf{X}, \mathbf{0} \in \mathbb{R}^{n+m}$, $\mathbf{b} \in \mathbb{R}^m$, dan $\mathbf{A} \in \mathbb{R}^{m \times (n+m)}$.

Untuk menyelesaikan permasalahan pemrograman linear dengan metode titik interior, diperlukan beberapa langkah penyelesaian yang tersusun dalam algoritma titik interior. Gagasan dasar dari algoritma titik interior pada pemrograman linier adalah dimulai dengan mengambil titik interior sebarang pada daerah penyelesaian fisibel, sehingga pada proses perhitungan menghasilkan nilai-nilai interior fisibel. Langkah penting dalam algoritma ini adalah titik interior awal yang dapat ditentukan terlebih dahulu kemudian mencari penyelesaian pada daerah penyelesaian fisibel yang dibatasi oleh fungsi-fungsi kendala sehingga tercapai nilai optimum (Hillier & Lieberman, 2010).

Algoritma titik interior dimulai dengan suatu titik interior awal $\tilde{\mathbf{X}}^0$ yang memenuhi kondisi titik interior, sedemikian hingga $\mathbf{A}\tilde{\mathbf{X}}^0 = \mathbf{b}$, $\tilde{\mathbf{X}}_j^0 > 0$ untuk $j = 1, 2, \dots, n + m$.

Kondisi $\tilde{\mathbf{X}}_j^0 > 0$ membuat nilai variabel-variabel $(x_1, x_2, \dots, x_{n+m})$ selalu berada didalam batas daerah penyelesaian fisibel. Algoritma titik interior menghasilkan titik interior fisibel yang berurutan yaitu $\tilde{\mathbf{X}}^1$ (diperoleh dari iterasi ke-1), $\tilde{\mathbf{X}}^2$ (diperoleh dari iterasi ke-2), \dots , $\tilde{\mathbf{X}}^k$ (diperoleh dari iterasi ke-[k-1]), $\tilde{\mathbf{X}}^{k+1}$ (diperoleh dari iterasi ke-k), dan seterusnya. Diagram alir algoritma titik interior dapat dilihat pada Gambar 2.2. Langkah-langkah algoritma titik interior tersusun sebagai berikut : (Hillier & Lieberman, 2010)

Langkah awal : Menentukan titik interior awal $\tilde{\mathbf{X}}^0 = (x_1, x_2, \dots, x_{n+m})$ sebarang yang memenuhi batas kendala permasalahan dan menghitung nilai \mathbf{Z}_0 , dengan

$$\mathbf{Z}_0 = \mathbf{C}^T \tilde{\mathbf{X}}^0. \quad (10)$$

Tahapan Iterasi ke- $i+1$ ($i = 0, 1, 2, \dots$)

Langkah 1 : Menentukan matriks diagonal \mathbf{D}_{i+1} , yaitu

$$\mathbf{D}_{i+1} = \text{diag}(\tilde{\mathbf{X}}^i) = \begin{bmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{n+m} \end{bmatrix}. \quad (11)$$

Langkah 2 : Menentukan

$$\left. \begin{aligned} \mathbf{A}_{i+1} &= \mathbf{A} \mathbf{D}_{i+1} \\ \mathbf{C}_{i+1} &= \mathbf{D}_{i+1} \mathbf{C} \end{aligned} \right\}. \quad (12)$$

Langkah 3 : Menentukan

a. Matriks proyeksi, yaitu

$$\mathbf{P}_{i+1} = \mathbf{I} - \mathbf{A}_{i+1}^T (\mathbf{A}_{i+1} \mathbf{A}_{i+1}^T)^{-1} \mathbf{A}_{i+1}, \quad (13)$$

dengan \mathbf{I} matriks identitas.

b. Tingkat kemiringan/gradien yang diproyeksikan (gradien terhadap garis yang memuat nilai penyelesaian), yaitu

$$\mathbf{C}_{P_{i+1}} = \mathbf{P}_{i+1} \mathbf{C}_{i+1}. \quad (14)$$

Langkah 4 : Menentukan

$$\left. \begin{aligned} \mathbf{V}_{i+1} &= |\min(\mathbf{C}_{P_{i+1}})| \\ \mathbf{M}_{i+1} &= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\alpha}{v_{i+1}} \mathbf{C}_{P_{i+1}} \end{aligned} \right\}, \quad (15)$$

$\mathbf{M}_{i+1} \in \mathbb{R}^{n+m}$ dan $\alpha \in (0, 1)$, α merupakan proporsi yang digunakan untuk mempertahankan agar titik interior selalu dalam batas kendala atau selalu dalam daerah fisibel.

Langkah 5 : Menentukan

$$\tilde{\mathbf{X}}^{i+1} = \mathbf{D}_{i+1}\mathbf{M}_{i+1}, \quad (16)$$

yang merupakan calon titik interior untuk iterasi berikutnya.

Langkah 6 : Menentukan nilai optimum

$$\mathbf{Z}_{i+1} = \mathbf{C}^T\tilde{\mathbf{X}}^{i+1}, \quad (17)$$

- a. Jika $\mathbf{Z}_{i+1} > \mathbf{Z}_i$, maka dilanjutkan ke iterasi berikutnya dengan langkah yang sama seperti pada Langkah 1-5 pada iterasi sebelumnya. $\tilde{\mathbf{X}}^{i+1}$ dipilih menjadi titik interior untuk iterasi berikutnya tersebut.
- b. Jika $\mathbf{Z}_{i+1} \leq \mathbf{Z}_i$, maka iterasi berhenti dan diperoleh penyelesaian optimum yaitu titik optimum dan nilai optimum (\mathbf{Z}_i dan $\tilde{\mathbf{X}}^i$).

2. 2. Analisis Interval

Analisis interval ini digunakan sebagai alat kontrol otomatis dari kesalahan perhitungan yang mungkin muncul dari kesalahan input, kesalahan pembulatan selama perhitungan dan kesalahan pemotongan dalam penggunaan pendekatan numerik untuk matematika. (Caprani dkk, 2002). Definisi dan konsep analisis interval ini diambil dari Moore (1966) dan Alefeld & Herzberg (1983).

Definisi 2. Interval tertutup dinotasikan $\underline{x} = [x_l, x_s] \in I(\mathbb{R})$ didefinisikan sebagai

$$\underline{x} = [x_l, x_s] = \{x_l, x_s \in \mathbb{R} | x_l \leq x \leq x_s\}$$

dengan x_l adalah *infimum* \underline{x} dan x_s adalah *supremum* \underline{x} .

Definisi 3. Suatu interval bilangan real \underline{x} disebut interval tak terbatas jika *infimum* x_l atau *supremum* x_s adalah tak berhingga.

Definisi 4. Suatu interval bilangan real $\underline{x} = [x_l, x_s]$ disebut interval *degenerate* jika $x_l = x_s$.

Definisi 5. Dua interval $\underline{x} = [x_l, x_s]$ dan $\underline{y} = [y_l, y_s]$ dikatakan sama jika memenuhi $x_l = y_l$ dan $x_s = y_s$.

Definisi 6. Misalkan diberikan interval-interval $\underline{x}, \underline{y} \in I(\mathbb{R})$ dengan $\underline{x} = [x_l, x_s]$ dan $\underline{y} = [y_l, y_s]$. Operasi atas interval \underline{x} dan \underline{y} didefinisikan sebagai berikut :

1. Penjumlahan : $\underline{x} + \underline{y} = [x_l + y_l, x_s + y_s]$
2. Pengurangan : $\underline{x} - \underline{y} = [x_l - y_s, x_s - y_l]$
3. Perkalian : $\underline{x}\underline{y} = [\min\{x_ly_l, x_ly_s, x_sy_l, x_sy_s\}, \max\{x_ly_l, x_ly_s, x_sy_l, x_sy_s\}]$

4. Pembagian : $\frac{\underline{x}}{\underline{y}} = [x_I, x_S] \left[\frac{1}{y_S}, \frac{1}{y_I} \right]$ dengan $0 \notin \underline{y}$
5. Titik tengah interval : $m(\underline{x}) = \frac{1}{2}(x_I + x_S)$
6. lebar interval : $w(\underline{x}) = \frac{1}{2}(x_S - x_I)$
7. Nilai mutlak interval : $|\underline{x}| = \max\{|x_I|, |x_S|\}$

Definisi 7. Suatu matriks interval $\underline{\mathbf{A}} \in I(\mathbb{R}^{m \times n})$, adalah matriks $\underline{\mathbf{A}} = (\underline{a}_{ij})$ dengan $\underline{a}_{ij} = [a_{ijI}, a_{ijS}] \in I(\mathbb{R})$, dengan a_{ijI} adalah *infimum* \underline{a}_{ij} dan a_{ijS} adalah *supremum* \underline{a}_{ij} , untuk setiap $i = 1, 2, \dots, m, j = 1, 2, \dots, n$.

Definisi 8. Titik tengah suatu matriks interval $\underline{\mathbf{A}}$ adalah matriks $m(\underline{\mathbf{A}}) = \left(m(\underline{a}_{ij}) \right)$ dengan $m(\underline{a}_{ij}) = \frac{1}{2}(a_{ijI} + a_{ijS})$.

Definisi 9. Lebar suatu matriks interval $\underline{\mathbf{A}}$ adalah matriks $w(\underline{\mathbf{A}}) = \left(w(\underline{a}_{ij}) \right)$ dengan $w(\underline{a}_{ij}) = \frac{1}{2}(a_{ijS} - a_{ijI})$.

Definisi 10. Nilai mutlak suatu matriks interval $\underline{\mathbf{A}}$ adalah matriks $|\underline{\mathbf{A}}| = \left(|\underline{a}_{ij}| \right)$.

2.3. Pemrograman Linear dengan Koefisien dan Variabel Interval

Model pemrograman linier dengan koefisien dan variabel interval telah dirumuskan sebagai berikut : (Suprajitno & Mohd, 2008)

Maksimumkan

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}] [x_{jI}, x_{jS}], \quad (18)$$

dengan kendala

$$\sum_{j=1}^n [a_{ijI}, a_{ijS}] [x_{jI}, x_{jS}] \leq [b_{iI}, b_{iS}], \quad i = 1, 2, \dots, m, \quad (19)$$

$$[x_{jI}, x_{jS}] \geq 0, \quad j = 1, 2, \dots, n, \quad (20)$$

dan $[x_{jI}, x_{jS}] \in I(\mathbb{R}^+)$ serta $[c_{jI}, c_{jS}], [a_{ijI}, a_{ijS}], [b_{iI}, b_{iS}] \in I(\mathbb{R})$.

Berikut ini disajikan teorema dan definisi yang merujuk pada Suprajitno & Mohd (2008).

Teorema 1. Jika ada pertidaksamaan interval pada fungsi kendala

$$\sum_{j=1}^n [a_{ijI}, a_{ijS}] [x_{jI}, x_{jS}] \leq [b_{iI}, b_{iS}], \quad (21)$$

dengan $[x_{jI}, x_{jS}] \geq 0$, untuk setiap j dengan $j = 1, 2, \dots, n$ maka

(a) Daerah penyelesaian fisibel terbesar adalah daerah penyelesaian yang memenuhi

$$\sum_{j=1}^n \min\{a_{ijI}x_{jI}, a_{ijS}x_{jS}\} \leq b_{iS}, \quad (22)$$

(b) Daerah penyelesaian fisibel terkecil adalah daerah penyelesaian yang memenuhi

$$\sum_{j=1}^n \max\{a_{ijs}x_{jI}, a_{ijs}x_{jS}\} \leq b_{iI}. \quad (23)$$

Teorema 2. Jika diberikan fungsi tujuan

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}][x_{jI}, x_{jS}], \quad (24)$$

dengan $[x_{jI}, x_{jS}] \geq 0$ untuk setiap j dengan $j = 1, 2, \dots, n$ maka untuk sebarang vektor interval $\underline{x} = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n)^T$ pada daerah penyelesaian fisibel berlaku

$$\sum_{j=1}^n \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\} \leq \sum_{j=1}^n \max\{c_{jS}x_{jI}, c_{jS}x_{jS}\}. \quad (25)$$

Definisi 11. Jika diberikan fungsi tujuan

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}][x_{jI}, x_{jS}], \quad (26)$$

dengan $[x_{jI}, x_{jS}] \geq 0$ ($j = 1, 2, \dots, n$) maka

1) Batas atas fungsi tujuan diberikan oleh

$$z_S = \sum_{j=1}^n \max\{c_{jS}x_{jI}, c_{jS}x_{jS}\}, \quad (27)$$

2) Batas bawah fungsi tujuan diberikan oleh

$$z_I = \sum_{j=1}^n \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\}. \quad (28)$$

Definisi 12. Jika diberikan fungsi tujuan seperti pada Persamaan (26) dan jika ada pertidaksamaan interval pada fungsi kendala seperti pada Persamaan (21), maka

1) Masalah optimum terbaik diberikan oleh

Maksimumkan

$$z_S = \sum_{j=1}^n \text{maks}\{c_{jS}x_{jI}, c_{jS}x_{jS}\}, \quad (29)$$

dengan kendala

$$\sum_{j=1}^n \min\{a_{ijI}x_{jI}, a_{ijS}x_{jS}\} \leq b_{iS}, \quad (30)$$

dan $x_{jI}, x_{jS} \geq 0, (j = 1, 2, \dots, n)$.

2) Masalah optimum terburuk diberikan oleh

Maksimumkan

$$z_I = \sum_{j=1}^n \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\}, \quad (31)$$

dengan kendala

$$\sum_{j=1}^n \text{maks}\{a_{ijs}x_{jI}, a_{ijs}x_{jS}\} \leq b_{iI}, \quad (32)$$

dan $x_{jI}, x_{jS} \geq 0, (j = 1, 2, \dots, n)$.

BAB 3. METODE PENELITIAN

Metode penelitian terdiri dari tiga tahap untuk mencapai tujuan akhir yang akan diteliti. **Pada tahap I** akan dimulai dengan studi literatur dan analisis ketidakpastian. **Tahap II** membuat model penyelesaian pemrograman linear dengan koefisien interval dan variabel interval dengan menggunakan hitung batas interval. Selanjutnya **Tahap III** akan disimulasikan model pada Tahap ke II untuk optimasinya. Terakhir **Tahap IV** Pembuatan/Pengiriman Artikel dan Seminar

Tahap I : studi literatur dan analisis ketidakpastian

Tahap II : Pemrograman linier dengan koefisien interval dan variabel interval berdasarkan hitung batas interval pada metode titik interior

Penyelesaian pemrograman linier dengan koefisien interval dan variabel interval pada Tahap II ini dengan langkah-langkah sebagai berikut :

Langkah 1 : Permasalahan

Maksimumkan

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}] [x_{jI}, x_{jS}], \quad (33)$$

dengan kendala

$$\sum_{j=1}^n [a_{ijI}, a_{ijS}] [x_{jI}, x_{jS}] \leq [b_{iI}, b_{iS}], \quad i = 1, 2, \dots, m, \quad (34)$$

$$[x_{jI}, x_{jS}] \geq 0, \quad j = 1, 2, \dots, n, \quad (35)$$

dan $[x_{jI}, x_{jS}] \in I(\mathbb{R}^+)$, $[c_{jI}, c_{jS}]$, $[a_{ijI}, a_{ijS}]$, $[b_{iI}, b_{iS}] \in I(\mathbb{R})$.

Langkah 2 : Menentukan daerah penyelesaian fisibel yang ditentukan oleh fungsi kendala pada persamaan (34).

- a) Daerah penyelesaian fisibel terbesar
- b) Daerah penyelesaian fisibel terkecil

Langkah 3 : Menentukan nilai-nilai batas atas dan batas bawah pada fungsi tujuan dari persamaan (33).

- a) Batas atas fungsi tujuan
- b) Batas bawah fungsi tujuan

Langkah 4 : Membentuk dua sub-model pemrograman linier berdasarkan fungsi tujuan pada Langkah 3 dan daerah penyelesaian fisibel pada Langkah 2.

- a) Masalah optimum terbaik
- b) Masalah optimum terburuk

Langkah 5 : Menyelesaikan permasalahan optimum terbaik pada Langkah 4 untuk memperoleh nilai optimum terbaik z_S

- a) Membentuk permasalahan optimum terbaik ke dalam bentuk pemrograman linier standar seperti pada Persamaan (7), Persamaan (8) dan Persamaan (9).
- b) Menggunakan algoritma titik interior untuk menyelesaikan permasalahan optimum terbaik.

Langkah 6 : Menyelesaikan permasalahan optimum terburuk pada Langkah 4 untuk memperoleh nilai optimum terburuk z_I

- a) Membentuk permasalahan optimum terburuk ke dalam bentuk pemrograman linier standar seperti pada Persamaan (7), Persamaan (8) dan Persamaan (9).
- b) Menggunakan algoritma titik interior untuk menyelesaikan permasalahan optimum terburuk.

Langkah 7 : Mengidentifikasi/memeriksa penyelesaian pada Langkah 5 dan Langkah 6.

Langkah 8 : Membentuk penyelesaian interval.

Langkah 9 : Dari Langkah 5-8 diperoleh penyelesaian optimum yaitu titik optimum dan nilai optimum berbentuk interval $\underline{x}_i = [x_{iI}, x_{iS}]$ dan $\underline{z} = [z_I, z_S]$.

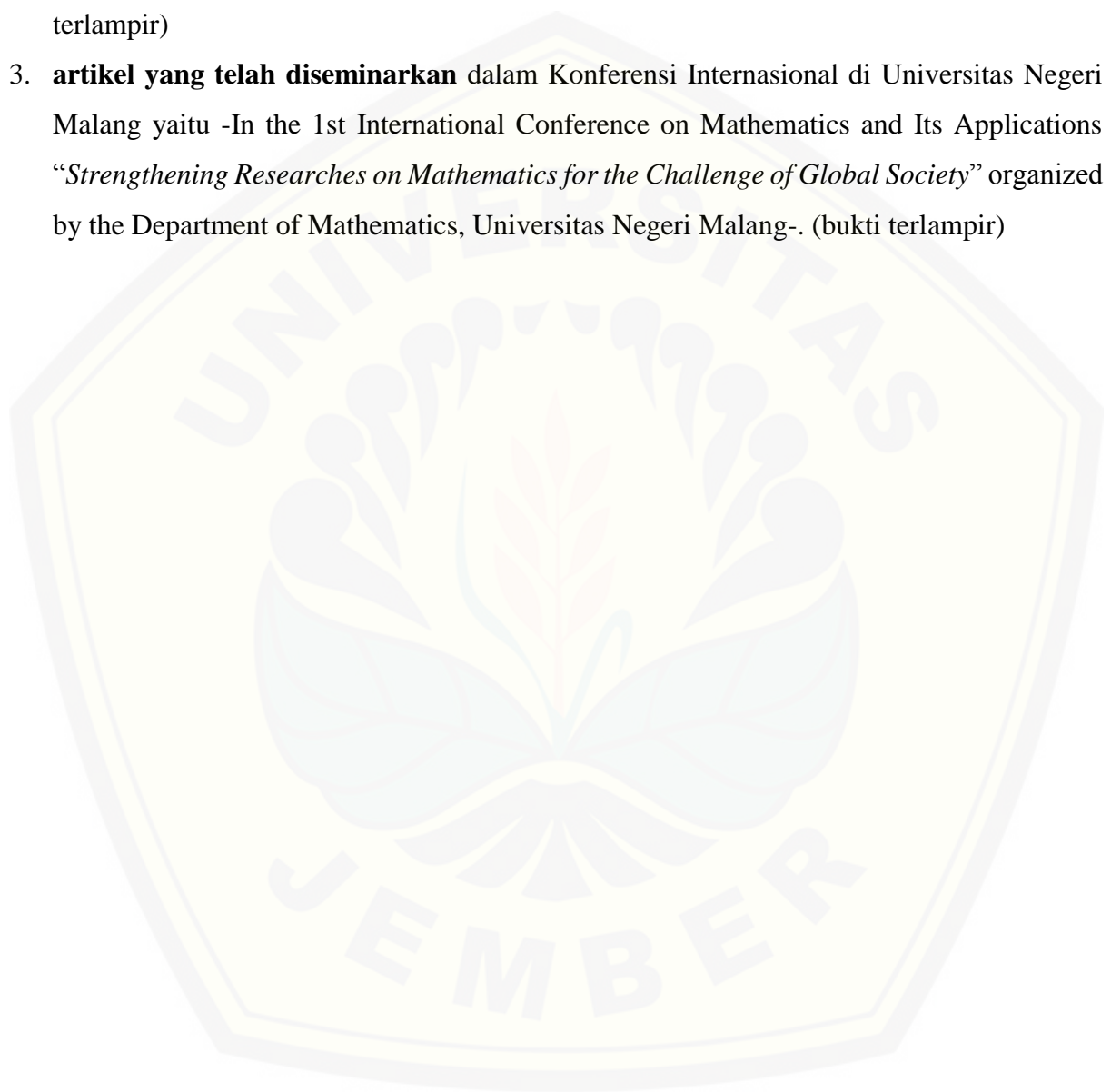
Tahap III Aplikasikan/simulasikan model pada Tahap ke II untuk optimasinya

Tahap IV Pembuatan/Pengiriman Artikel dan Seminar

BAB 4. HASIL YANG DICAPAI

Sampai dengan Laporan Akhir ini dibuat dan sesuai dengan janji luaran pada proposal, hasil yang dicapai dari penelitian ini adalah :

1. satu artikel telah **accepted** pada *Far East Journal of Applied Mathematics* - FJAM. (bukti terlampir)
2. **draft satu artikel** yang direncanakan untuk dipublikasikan pada tahun 2022 (bukti terlampir)
3. **artikel yang telah diseminarkan** dalam Konferensi Internasional di Universitas Negeri Malang yaitu -In the 1st International Conference on Mathematics and Its Applications “*Strengthening Researches on Mathematics for the Challenge of Global Society*” organized by the Department of Mathematics, Universitas Negeri Malang-. (bukti terlampir)



DAFTAR PUSTAKA

- Alefeld, G. & Herzberger J., Translate by Rokne, J., 1983, *Introduction To Interval Computation*, Academic Press, 111 Fifth Avenue New York 10003
- Caprani, O., Madsen, K. & Nielsen, H.B., 2002, *Introduction to Interval Analysis*, IMM-DTU.
- Hillier, F.S. & Lieberman, G.J., 1990, *Introduction to Operation Research*, (Fifth Edition) McGraw-Hill, Inc.
- Karmakar, N., 1984, *A new Polynomial Time Algorithm for Linear Programming*, *Combinatorica* 4, 373-395.
- Moore, R.E., 1966, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ.
- Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2015. Construction Interior Point Method to Linear Programming with Interval Coeficients, Internasonal Seminar ICOWOBAS-5, 16-17 September 2015, Surabaya
- Pradjaningsih, A., 2016. Konstruksi Metode Titik Interior pada Pemrograman Linear Interval dengan Menggunakan Nilai Batas Bawah Terbesar dan Nilai Batas Atas Terkecil, *Penelitian Disertasi Doktor*.
- Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2016. Construction Interior Point Method to Linear Programming with Interval Variable, Internasional Seminar AMC, 25-29 Juli 2016, Bali
- Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2017. Konstruksi Metode Titik Interior untuk Penyelesaian Pemrograman Linear dengan Koefisien Interval, Seminar Nasional SNMA, 21 Oktober 2017, Surabaya.
- Pradjaningsih, A., Suprajitno, H., & Fatmawati, 2017. Konstruksi Metode Titik Interior untuk Penyelesaian Pemrograman Linear dengan Variabel Interval, Seminar Nasional Matematika dan Pembelajarannya, 25 November 2017, Malang.
- Pradjaningsih, A., Fatmawati & Suprajitno, H., 2019, *Interior Point Method for Solving Linear Programming with Interval Coefficients using Affine Scaling*, *Far East Journal of Mathematical Sciences* Vol 111 (2), 195-211.
- Roos, C., Terlaky, T. & Vial, J.-Ph., 2005, *Interior Point Methods for Linear Optimization*, Springer-Verlag US.
- Suprajitno, H. & bin Mohd., I. 2008. Interval Linear Programming, *Proceedings of the 3 International Conference on Mathematics and Statistics (ICOMS-3)*, Institut Pertanian Bogor, Indonesia. 5-6 Agustus 2008.
- Winston, W.L, 1994, *Operation Research : Applications and Algorithms*, Wadsworth, Inc.



Far East Journal of Applied Mathematics (ISSN: 0972-960)

Our reference no.: PPH-2012053-AM

To

Professor Agustina Pradjaningsih
Department of Mathematics
Faculty of Mathematics and Natural Science
Jember University
68121, Jember, Indonesia

January 11, 2021

Dear Professor Pradjaningsih,

I am happy to inform you that your paper entitled "*Solution of linear programming with interval variables using interior point method based on interval boundary calculations*" jointly written with Fatmawati and Herry Suprajitno has been accepted for publication in the Far East Journal of Applied Mathematics.

Thank you for submitting your work to Far East Journal of Applied Mathematics.

Yours sincerely



(ARUN AZAD)

Paper No.: PPH-2012053-AM

Title: "Solution of linear programming with interval variables using interior point method based on interval boundary calculations".

Authors: Agustina Pradjaningsih, Fatmawati and Herry Suprajitno

Recommendation: Accept (✓) Revision () Reject ()

In continuation to their earlier work, the authors provide an alternative way to solve the linear interval programming problem using the interior point method. The interior point method, which can solve linear programming with interval variables by converting the problem into two classical linear programming models has been presented in detail. The optimal solution in the form of intervals is obtained from constructing the two models. The paper is systematically developed, the algorithm presented is well described and the findings are elaborately discussed. As I see merit and also scope for future investigation, I recommend that this paper be accepted for publication in its present form in an earlier issue of the Far East Journal of Applied Mathematics.



SOLUTION OF LINEAR PROGRAMMING WITH INTERVAL VARIABLES USING INTERIOR POINT METHOD BASED ON INTERVAL BOUNDARY CALCULATIONS

Agustina Pradjaningsih^{1,*}, Fatmawati² and Herry Suprajitno²

¹Department of Mathematics
Faculty of Mathematics and Natural Science
Jember University Kalimantan 37
Jember, 68121, Indonesia
e-mail: tinapradja.math@gmail.com

²Department of Mathematics
Faculty of Science and Technology
Universitas Airlangga
Kampus C, Mulyorejo, Surabaya
60115, Indonesia

Abstract

Linear programming with interval variables is the development of linear programming with interval coefficients. In this paper, linear programming with interval variables is solved using the interior point method based on interval boundary calculations. This solution's initial procedure is to change the linear programming model with interval variables into a pair of classical linear programming models. This pair or two classical linear programming models correspond to an interval

Received: December 28, 2020; Accepted: January 11, 2021

2020 Mathematics Subject Classification: 90-10, 90C05.

Keywords and phrases: interval linear programming, interval variables, interior point method.

*Corresponding author

2 Agustina Pradjaningsih, Fatmawati and Herry Suprajitno

boundary called the least upper bound and the greatest lower bound. The least upper bound is determined using the largest feasible region and the greatest lower bound is determined by using the smallest feasible region. The least upper bound and the greatest lower bound are solved using the interior point method. Finally, the optimal solution in the form of intervals is obtained from constructing the two models. This paper provides an alternative solution to solve the linear interval programming problem using the interior point method.

1. Introduction

An optimization problem could be categorized as a linear programming model if it satisfies assumptions of proportionality, additivity, divisibility, and certainty [8]. The certainty assumption is that all coefficients and variables in the model are known. However, in real situations, sometimes the values of coefficients and variables are not certainty known. The uncertainty problem could be overcome using the interval approach to the coefficients and variables. This approach supported the concept and theory of interval analysis developed by [13]. Linear programming with interval coefficients is linear programming where the coefficients on the objective function and the constraint function are intervals. Suppose the coefficient and variable in the objective function and the constraint function are both in the form of intervals; in that case, it is called linear programming with interval variables.

Researches on linear programming with interval coefficients have been discussed by [2, 3, 12, 14]. The research on linear programming with interval variables was inspired by [15-19]. The researchers [15-19] used the simplex method to solve it. The interior point method has been used to solve linear programming [4, 6, 7, 8, 10, 11]. Meanwhile, several researchers have discussed using the interior point method to solve the linear programming of fuzzy numbers [20]. The advantages of the interior point method are used in solving linear programming problems with many variables. For several reasons that have been stated, this paper will discuss the interior point method as an alternative solution to linear programming with interval variables.

This paper presents a solution to linear programming with interval variables using an interior point method. The steps to obtain the optimal solution in the form of intervals are: first, to transform the problem into linear programming with interval variables model, meaning that the coefficients and variables in the objective function and the constraints function are intervals.

The second step is to change linear programming with interval variables into a pair of the classical linear programming model. This pair or two classical linear programming models correspond to an interval boundary called the least upper bound and the greatest lower bound. For the least upper bound is determined using the largest feasible region and for the greatest lower bound is determined by using the smallest feasible region. A final step, the least upper bound, and the greatest lower bound are solved using the interior point method. A more detailed explanation of the last step is in subsection 2.3.

2. Preliminaries

In this section, we review some of the concepts needed, such as interval arithmetic and formula of linear programming with interval variables. For more details, we refer to [1, 8, 13, 18].

2.1. Interval arithmetic

The basic concepts, definition, and properties of interval number, interval arithmetic, and comparison of two intervals can be found in [1, 13, 18]. Let R denote the set of all real numbers.

Definition 2.1. A closed real interval $\underline{x} = [x_I, x_S]$ is a real interval number which can be defined by

$$\underline{x} = [x_I, x_S] = \{x \in R \mid x_I \leq x \leq x_S, x_I, x_S \in R\}, \quad (2.1)$$

where x_I and x_S are called infimum and supremum of \underline{x} , respectively.

Definition 2.2. An interval of real numbers \underline{x} is called an *unbounded interval* if the infimum x_I or supremum x_S is infinite.

4 Agustina Pradjaningsih, Fatmawati and Herry Suprajitno

Definition 2.3. A real interval number $\underline{x} = [x_I, x_S]$ is called a *degenerate*, if $x_I = x_S$.

Definition 2.4. The width of an interval \underline{x} is the real number $w(\underline{x}) = \frac{1}{2}(x_S - x_I)$.

Definition 2.5. The midpoint of an interval \underline{x} is a real number $m(\underline{x}) = \frac{1}{2}(x_I + x_S)$.

Definition 2.6. Let $I(R)$ is the set of all interval on R . A real interval vector $\underline{x} \in I(R^n)$, is a vector in the form $\underline{x} = (\underline{x}_i)_{n \times 1}$, where $\underline{x}_i = [x_{iI}, x_{iS}] \in I(R)$, $i = 1, 2, \dots, n$.

Definition 2.7. Let $\underline{x}, \underline{y} \in I(R)$, where $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$, then

(a) $\underline{x} + \underline{y} = [x_I + y_I, x_S + y_S]$ (addition),

(b) $\underline{x} - \underline{y} = [x_I - y_S, x_S - y_I]$ (subtraction),

(c) $\underline{x} \cdot \underline{y} = [\min\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}, \max\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}]$

(multiplication),

(d) $\underline{x}/\underline{y} = [x_I, x_S][1/y_S, 1/y_I]$, $0 \notin \underline{y}$ (division).

Definition 2.8. Let $\underline{x}, \underline{y} \in I(R)$, $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$. $\underline{x} \leq \underline{y}$ if only if $ux_I + vx_S \leq uy_I + vy_S$, where $u, v \in (0, 1]$ and $u \leq v$.

2.2. Linear programming with interval variables

The general form of linear programming with interval variables is defined as follows:

Maximize (objective function)

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}] [x_{jI}, x_{jS}], \quad (2.2)$$

subject to

$$\sum_{j=1}^n [a_{ijI}, a_{ijS}] [x_{jI}, x_{jS}] \leq [b_I, b_S], \quad (2.3)$$

$$[x_{jI}, x_{jS}] \geq 0, \quad (2.4)$$

where $i = 1, 2, \dots, m, j = 1, 2, \dots, n$, $[x_{jI}, x_{jS}]$ is a decision variable, $[c_{jI}, c_{jS}]$ is a coefficient of the objective function, $[a_{ijI}, a_{ijS}]$ is a coefficient of the constraint function, $[b_I, b_S]$ is a coefficient of resource constraint, \underline{Z} is the objective function, and

$$[x_{jI}, x_{jS}], [c_{jI}, c_{jS}], [a_{ijI}, a_{ijS}], [b_I, b_S] \in I(\mathbb{R}).$$

Theorem 2.9 and Theorem 2.10 have been presented in [18]. Theorem 2.9 determines the largest feasible region and the smallest feasible region of linear programming with interval variables. Theorem 2.10 described some property of the objective function of linear programming with interval variables model.

Theorem 2.9. *Suppose the interval inequality of linear programming with interval variables:*

$$\sum_{j=1}^n [a_{jI}, a_{jS}] [x_{jI}, x_{jS}] \leq [b_I, b_S], \quad (2.5)$$

where $[x_{jI}, x_{jS}] \geq 0$, for every $j = 1, 2, \dots, n$ is given. Then

(a) *The largest feasible region is a region satisfies the following inequality,*

$$\sum_{j=1}^n \min\{a_{jI}x_{jI}, a_{jS}x_{jS}\} \leq b_S, \quad (2.6)$$

(b) *The smallest feasible region is a region satisfies the following inequality*

$$\sum_{j=1}^n \max\{a_{jS}x_{jI}, a_{jS}x_{jS}\} \leq b_I. \quad (2.7)$$

Theorem 2.10. *Suppose an objective function of linear programming with interval variables*

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}][x_{jI}, x_{jS}], \quad (2.8)$$

where $[x_{jI}, x_{jS}] \geq 0$ ($j = 1, 2, \dots, n$) is given. Then for every $\underline{x} = (x_1, x_2, \dots, x_n)^T$ in the feasible region,

$$\sum_{j=1}^n \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\} \leq \sum_{j=1}^n \max\{c_{jS}x_{jI}, c_{jI}x_{jS}\}. \quad (2.9)$$

2.3. Interior point algorithm

The interior point algorithm starts with an initial interior point $\tilde{\mathbf{X}}^0$ which satisfies the interior point conditions, such that $\mathbf{A}\tilde{\mathbf{X}}^0 = \mathbf{b}$, $\tilde{\mathbf{X}}_j^0 > 0$ for $j = 1, 2, \dots, n + m$. The condition $\tilde{\mathbf{X}}_j^0 > 0$ makes the variable values $(x_1, x_2, \dots, x_{n+m})$ always within the boundary of the feasible area. The interior point algorithm produces a feasible interior point sequence, namely $\tilde{\mathbf{X}}^1$ (obtained from the 1st iteration), $\tilde{\mathbf{X}}^2$ (obtained from the 2nd iteration), ..., $\tilde{\mathbf{X}}^k$ (obtained from the k -iteration), $\tilde{\mathbf{X}}^{k+1}$ (obtained from the $k + 1$ -iteration), and so on, with iterations following $i + 1$, ($i = 0, 1, 2, \dots$). The steps for the interior point algorithm are as follows: [8].

Initial step: Determine the initial interior point $\tilde{X}^0 = (x_1, x_2, \dots, x_{n+m})$ which satisfies the boundary constraints of the problem and calculate the value of Z_0 with

$$Z_0 = C^T \tilde{X}^0. \quad (2.10)$$

Step 1. Determine the diagonal matrix \mathbf{D}_{i+1} , i.e.,

$$\mathbf{D}_{i+1} = \text{diag}(\tilde{\mathbf{X}}^i) = \begin{bmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{n+m} \end{bmatrix}. \quad (2.11)$$

Step 2. Determine the matrix \mathbf{A}_{i+1} and \mathbf{C}_{i+1} , i.e.,

$$\left. \begin{aligned} \mathbf{A}_{i+1} &= \mathbf{A} \mathbf{D}_{i+1} \\ \mathbf{C}_{i+1} &= \mathbf{D}_{i+1} \mathbf{C} \end{aligned} \right\}. \quad (2.12)$$

Step 3. Determine the projection matrix \mathbf{P}_{i+1} and projected gradient $\mathbf{C}_{P_{i+1}}$,

(1) The projection matrix, i.e.,

$$\mathbf{P}_{i+1} = \mathbf{I} - \mathbf{A}_{i+1}^T (\mathbf{A}_{i+1} \mathbf{A}_{i+1}^T)^{-1} \mathbf{A}_{i+1}, \quad (2.13)$$

where \mathbf{I} is an identity matrix.

(2) The projected gradient, i.e.,

$$\mathbf{C}_{P_{i+1}} = \mathbf{P}_{i+1} \mathbf{C}_{i+1}. \quad (2.14)$$

Step 4. Determine the scalar \mathbf{V}_{i+1} and matrix \mathbf{M}_{i+1} ,

8 Agustina Pradjaningsih, Fatmawati and Herry Suprajitno

$$\mathbf{M}_{i+1} = \left. \begin{array}{l} \mathbf{V}_{i+1} = | \min(\mathbf{C}_{P_{i+1}}) | \\ \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \right] + \frac{\alpha}{\mathbf{V}_{i+1}} \mathbf{C}_{P_{i+1}} \end{array} \right\}, \quad (2.15)$$

where \mathbf{M}_{i+1} is $(n + m) \times 1$ matrix and $\alpha \in (0, 1)$.

Step 5. Determine the candidate interior point for the next iteration, i.e.,

$$\tilde{\mathbf{X}}^{i+1} = \mathbf{D}_{i+1} \mathbf{M}_{i+1}, \quad (2.16)$$

Step 6. Determine the optimum value, i.e.,

$$\mathbf{Z}_{i+1} = \mathbf{C}^T \tilde{\mathbf{X}}^{i+1}. \quad (2.17)$$

(1) If $Z_{i+1} > Z_i$, then proceed to the next iteration with the same steps as in Steps 1-5 in the previous iteration. $\tilde{\mathbf{X}}^{i+1}$ is chosen to be the next iteration's interior point.

(2) If $Z_{i+1} \leq Z_i$, then the iteration stops, and the optimum solution is obtained, namely the optimum point and the optimum value (Z and $\tilde{\mathbf{X}}^i$).

3. A Solution of Linear Programming with Interval Variables Based on Interval Boundary Calculations

The solution of linear programming with interval variables using an interior point method based on interval boundary calculations will be constructed in this section, given linear programming with interval variables in equations (2.2)-(2.4). The steps of the solution are as follows.

Step 1. Determine the largest feasibility region and the smallest feasibility region from a given linear programming with interval variables. Equation (2.6) showed the largest feasibility region, and equation (2.7) showed the smallest feasibility region.

Step 2. Determine the least upper bound and the greatest lower bound of the objective function.

(a) The least upper bound satisfies the following inequality,

$$z_S = \sum_{j=1}^n \max\{c_{jS}x_{jI}, c_{jS}x_{jS}\}, \quad (3.1)$$

(b) The greatest lower bound satisfies the following inequality,

$$z_I = \sum_{j=1}^n \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\}. \quad (3.2)$$

Step 3. Forming two linear programming sub-models based on the objective function in Step 2 and the feasible solution region in Step 1. These two submodels are called the best optimum problem and the worst optimum problem.

(a) The best optimum problem

Maximize (objective function)

$$z_S = \sum_{j=1}^n \max\{c_{jS}x_{jI}, c_{jS}x_{jS}\}, \quad (3.3)$$

subject to

$$\sum_{j=1}^n \min\{a_{ijI}x_{jI}, a_{ijI}x_{jS}\} \leq b_{iS}, \quad i = 1, 2, \dots, m, \quad (3.4)$$

$$x_{jI}, x_{jS} \geq 0, \quad j = 1, 2, \dots, n. \quad (3.5)$$

(b) The worst optimum problem,

Maximize (objective function)

$$z_I = \sum_{j=1}^n \min\{c_{jI}x_{jI}, c_{jI}x_{jS}\}, \quad (3.6)$$

10 Agustina Pradjaningsih, Fatmawati and Herry Suprajitno

subject to

$$\sum_{j=1}^n \max\{a_{ijS}x_{jI}, a_{ijS}x_{jS}\} \leq b_{iI}, \quad i = 1, 2, \dots, m, \quad (3.7)$$

$$x_{jI}, x_{jS} \geq 0, \quad j = 1, 2, \dots, n. \quad (3.8)$$

Step 4. Using an interior point algorithm (subsection 2.3), to solve the best optimum problem in Step 3 to obtain the best optimum value z_S .

Step 5. Using an interior point algorithm (subsection 2.3), to solve the worst optimum problem in Step 3 to obtain the worst optimum value z_I .

Step 6. Check completion in Steps 4-5.

(a) If the best optimum solution is feasible with a z_S value and the worst optimum solution is feasible with a z_I value, then go directly to Step 7.

(b) If the linear programming with interval variables does not have the largest feasible region and the smallest feasible region, the linear programming with interval variables does not have any solution.

(c) If the best optimum solution for the best problem or the worst optimum solution for the worst problem is unbounded, select the optimum problem that is unbounded. Replace the constraint function with a new constraint function obtained from the combination of the best optimum problem constraint function and the worst optimum problem so that a new model is formed, then return to Step 4 or Step 5.

Step 7. Form the completion interval.

(a) From the best optimum problem solving, the supremum value corresponding to the least upper bound is taken.

(b) From the worst optimum problem solving, the infimum value that corresponds to the greatest lower bound is taken.

(c) If the solution formed (from Step 7a and Step 7b) is an interval, go to Step 8.

(d) If the solution formed (from Step 7a and Step 7b) is not in the form of an interval, then the solution to the worst optimum problem is taken.

(1) If the worst-case optimal solution for each variable has the form of an interval, then proceed to Step 8.

(2) If the worst optimum problem solving there is a part of the solution that is not in the form of an interval, the portion is given infimum value and will form the degenerate intervals.

Step 8. From Steps 4-7, the optimum solution is obtained, namely the optimum point and the optimum value in the form of intervals

$$\underline{x}_i = [x_{iI}, x_{iS}] \quad i = 1, 2, \dots, m \quad \text{and} \quad \underline{Z} = [z_I, z_S]. \quad (3.9)$$

4. Numerical Example

In this section, one example of interval linear programming has been solved in [5, 9, 19, 21], and we will compare the results.

Maximize (objective function)

$$Z = [26, 30][x_{1I}, x_{1S}] - [5.5, 6][x_{2I}, x_{2S}] \quad (4.1)$$

subject to

$$\left. \begin{aligned} [8, 10][x_{1I}, x_{1S}] - [12, 14][x_{2I}, x_{2S}] &\leq [3.8, 4.2] \\ [1, 1.1][x_{1I}, x_{1S}] + [0.19, 0.2][x_{2I}, x_{2S}] &\leq [6.5, 7] \\ [x_{1I}, x_{1S}], [x_{2I}, x_{2S}] &\geq 0 \end{aligned} \right\} \quad (4.2)$$

Step 1. Determine the largest feasibility region and the smallest feasibility region is given in Table 1.

Table 1. The largest and the smallest feasible region

(1) The Largest Feasible Region	(2) The Smallest Feasible Region
$8x_{1I} - 14x_{2S} \leq 4.2$	$10x_{1S} - 12x_{2I} \leq 3.8$
$x_{1I} + 0.19x_{2I} \leq 7$	$1.1x_{1S} + 0.2x_{2S} \leq 6.5$
$x_{1I}, x_{1S}, x_{2I}, x_{2S} \geq 0$	$x_{1I}, x_{1S}, x_{2I}, x_{2S} \geq 0$

Step 2. Determine the least upper bound and the greatest lower bound of the objective function is given in Table 2.

Table 2. The least upper bound and the greatest lower bound

(1) The Least Upper Bound	(2) The Greatest Lower Bound
Maximize $z_S = 30x_{1S} - 5.5x_{2I}$	Maximize $z_I = 26x_{1I} - 6x_{2S}$

Step 3. Forming two linear programming sub-models are called the best optimum problem and the worst optimum problem given in Table 3.

Table 3. The best optimum problem and the worst optimum problem

(1) The Best Optimum Problem	(2) The Worst Optimum Problem
Maximize $z_S = 30x_{1S} - 5.5x_{2I}$	Maximize $z_I = 26x_{1I} - 6x_{2S}$
subject to	subject to
$\left. \begin{array}{l} 8x_{1I} - 14x_{2S} \leq 4.2 \\ x_{1I} + 0.19x_{2I} \leq 7 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \geq 0 \end{array} \right\}$	$\left. \begin{array}{l} 10x_{1S} - 12x_{2I} \leq 3.8 \\ 1.1x_{1S} + 0.2x_{2S} \leq 6.5 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} \geq 0 \end{array} \right\}$

Step 4. Solve the best optimum problem in Step 3 to obtain the best optimum Z_S value. From Step 3, an unbounded solution is obtained.

Step 5. Solve the worst optimum problem in Step 3 to get the worst optimum Z_I value. From Step 3, an unbounded solution is obtained.

Step 6. Checking completion in Step 4 and Step 5.

Because the best optimum solution and the worst optimum solution are unbounded, each constraint function is added to the other model constraint functions. The best optimum problem-new and the worst optimum problem-new are formed. The next two new models were completed in Step 4 and Step 5. The best optimum problem-new and the worst optimum problem-new can be seen in Table 4.

Table 4. The best optimum problem-new and the worst optimum problem-new

(1) The Best Optimum Problem-New	(2) The Worst Optimum Problem-New
Maximize $z_S = 30x_{1S} - 5.5x_{2I}$	Maximize $z_I = 26x_{1I} - 6x_{2S}$
subject to	subject to
$\left. \begin{aligned} 8x_{1I} - 14x_{2S} &\leq 4.2 \\ x_{1I} + 0.19x_{2I} &\leq 7 \\ 10x_{1S} - 12x_{2I} &\leq 3.8 \\ 1.1x_{1S} + 0.2x_{2S} &\leq 6.5 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} &\geq 0 \end{aligned} \right\}$	$\left. \begin{aligned} 10x_{1S} - 12x_{2I} &\leq 3.8 \\ 1.1x_{1S} + 0.2x_{2S} &\leq 6.5 \\ 8x_{1I} - 14x_{2S} &\leq 4.2 \\ x_{1I} + 0.19x_{2I} &\leq 7 \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} &\geq 0 \end{aligned} \right\}$

Step 7. By using the interior point algorithm (subsection 2.3) and $\alpha = 0.95$ [8], to solve the best optimum problem-new in Step 6, the best optimum value is obtained as follows:

$$x_{1I} \cong 0.2485568309, x_{1S} \cong 5.9090910433,$$

$$x_{2I} \cong 4.6075757564, x_{2S} \cong 4.207017516410^{-8} \cong 0.$$

Step 8. By using the interior point algorithm (subsection 2.3) and $\alpha = 0.95$ [8], to solve the worst optimum problem-new in Step 6, the worst optimum value is obtained as follows:

$$x_{1I} \cong 6.9999999424, x_{1S} \cong 0.1626671934,$$

$$x_{2I} \cong 4.842505022710^{-7} \cong 0, x_{2S} \cong 3.7000001840.$$

Step 9. Form a completion interval.

The solution to the optimum point formed from Step 7 and Step 8 is not in the form of an interval, so a solution to the worst optimum problem is taken, namely the solution in Step 8 only. From Step 8, there is an optimum point that is not in the form of an interval, so at that optimum point, only the infimum value is given. So that the optimum point completion is obtained, namely

14 Agustina Pradjaningsih, Fatmawati and Herry Suprajitno

$$\underline{x}_1 \cong [6.9999999424, 6.9999999424], \underline{x}_2 \cong [0.3.7000001840].$$

Step 10. Obtained the optimum point solution and the optimum value in the form of intervals, namely

$$\underline{x}_1 \cong [6.9999999424, 6.9999999424], \underline{x}_2 \cong [0.3.7000001840],$$

$$\underline{Z} \cong [159.7999973984, 209.9999982720].$$

The optimum point and optimum value from previous researchers' results, and this paper are presented in Table 5.

Table 5. The results of previous researchers and this paper

No	Researcher	\underline{x}_j	\underline{Z}
1	Huang et al. [9]	$\underline{x}_1 = [5.21, 6.34],$ $\underline{x}_2 = [3.32, 4.03]$	$\underline{Z} = [111.4, 171.8]$
2	Zhou et al. [21]	$\underline{x}_1 = [4.574, 6.336],$ $\underline{x}_2 = [3.320, 3.495]$	$\underline{Z} = [97.954, 171.82]$
3	Suprajitno et al. [19]	$\underline{x}_1 = [5.1816, 5.1816],$ $\underline{x}_2 = [4.0013, 4.0013]$	$\underline{Z} = [110.71, 133.440]$
4	Fan et al. [5]	$\underline{x}_1 = [5.21, 6.23],$ $\underline{x}_2 = [3.26, 4.03]$	$\underline{Z} = [111.38, 169.1]$
5	This paper	$\underline{x}_1 \cong [6.99, 6.99],$ $\underline{x}_2 \cong [0, 3.70]$	$\underline{Z} \cong [159.799, 209.99]$

According to [21], two criteria can be used to determine the best solution for a linear program with interval variables. These criteria are the value of the width of the interval ($w(\underline{Z})$ the smallest) and the value of the degree of uncertainty ($w(\underline{Z})/m(\underline{Z})$ the smallest). Table 6 presents a width of an interval and the degree of uncertainty from previous researchers' results, and this paper.

Table 6. The results of a width of an interval and a degree of uncertainty

No	Researcher	$w(\underline{Z})$	$m(\underline{Z})$	Degree of Uncertainty
1	Huang et al. [9]	30.2	141.6	21.3276%
2	Zhou et al. [21]	36.933	134.887	27.3807%
3	Suprajitno et al. [19]	11.36352	122.077	9.3085%
4	Fan et al. [5]	28.86	140.24	20.579%
5	This paper	25.1	184.899	13.5749

It can be seen from Table 4.6 that the solution of linear programming with interval variables using interior point method based on interval boundary calculations has a smaller value for a width of interval and a degree of uncertainty compared to [5, 9, 21]. So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with interval variables based on interval boundary calculations.

5. Conclusions

This paper presents the interior point method, which can solve linear programming with interval variables. Linear programming with interval variables can be solved by converting the problem into two classical linear programming models. Furthermore, both models were solved using the interior point method. If using the criteria of interval's width and degree of uncertainty, solutions using the linear programming with interval variables get better results than previous researchers and satisfies the criteria for constraint values. So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with interval variables based on interval boundary calculations.

Acknowledgments

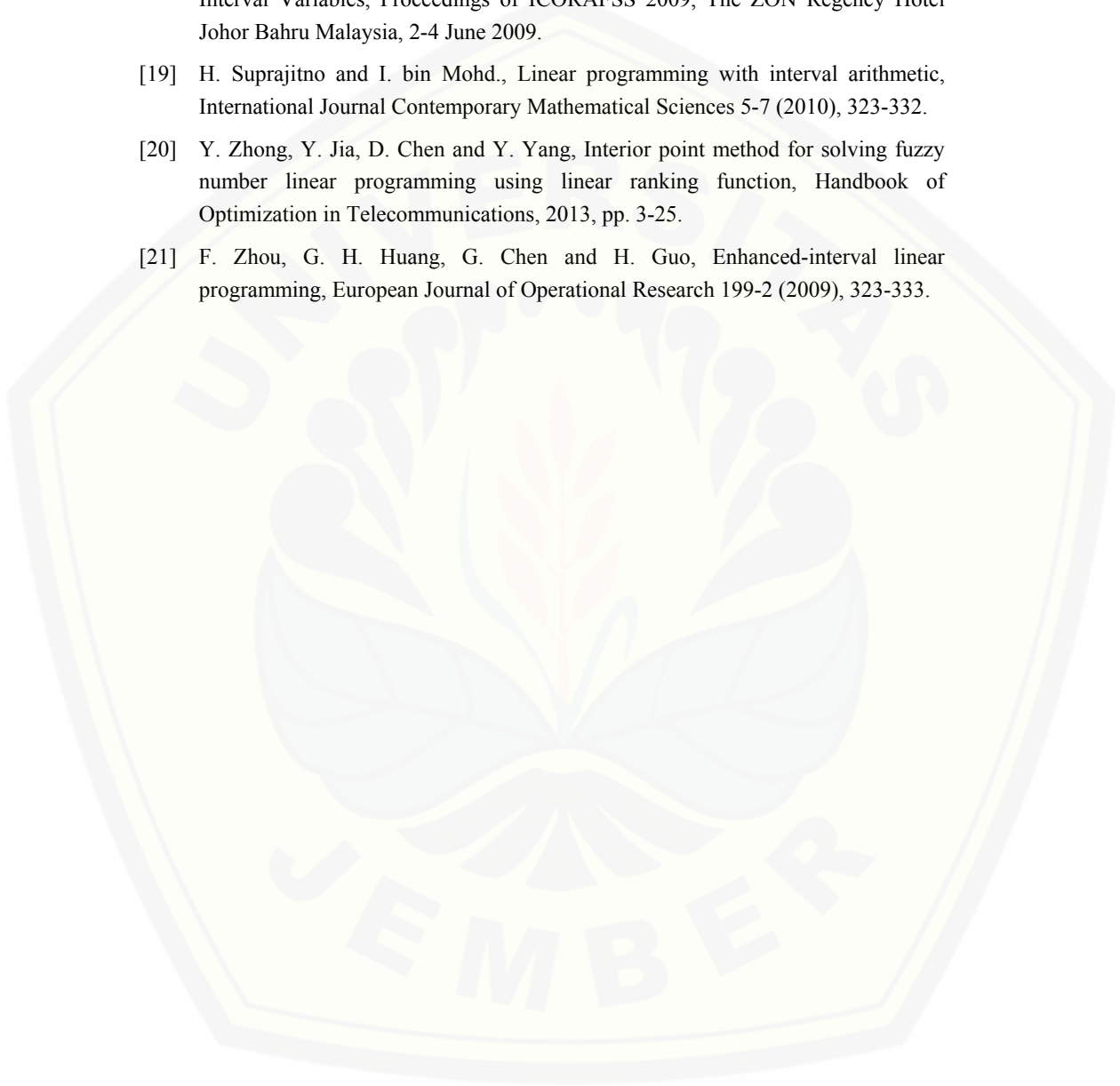
This project is supported by a Post-Doctoral Research Grant from the Jember University with Contract No 3479/UN25.3.1/LT/2020.

References

- [1] G. Alefeld and J. Herzberger, Introduction to Interval Computation, Published: 2nd December 2012, Academic Press, 1983.

- [2] I. Alolyan, Algorithm for Interval Linear Programming Involving Interval Constraints Proceedings of the World Congress on Engineering and Computer Science I, 2013.
- [3] J. W. Chinnek and K. Ramadhan, Linear Programming with Interval Coefficients, Journal of the Operational Research Society 51-2 (2000), 209-220.
- [4] E. K. P. Chong and H. Z. Stanislaw, An Introduction to Optimization Ed.2th, John Wiley & Sons, Inc., New York, 2001.
- [5] Y. R. Fan and G. H. Huang, A robust two-step method for solving interval linear programming problems within an environmental management context, Journal of Environmental Informatics 19(1) (2012), 1-9.
- [6] J. Gondzio, Interior point methods 25 years later, European Journal of Operational Research 218(3) (2012), 587-601.
- [7] D. D. Hertog, Interior point approach to linear, Quadratic and Convex Programming Springer Science + Business Media, BV, 1994.
- [8] F. S. Hillier and G. J. Lieberman, Introduction to Operation Research Ed. 10th McGraw-Hill Higher Education, 2014.
- [9] G. H. Huang, B. W. Baetz and G. G. Patry, A grey linear programming approach for municipal solid waste management planning under uncertainty, Civil Engineering Systems 9-4 (1992), 319-335.
- [10] N. Karmakar, A new polynomial time algorithm for linear programming, Combinatorica 4-4 (1984), 373-395.
- [11] J. E. Mitchell, K. Farwell and D. Ramsden, Interior point methods for large-scale linear programming in resende M.G.C., Pardalos P.M. (eds.), Handbook of Optimization in Telecommunications (2006), 3-25.
- [12] A. A. Molai and E. Khorram, Linear programming problem with interval coefficients and interpretation for its constraint, Iranian Journal of Science & Technology, Transaction A 31-A4 (2007), 369-390.
- [13] R. E. Moore, Interval Analysis, Prentice-Hall, 1966.
- [14] A. Pradjaningsih, Fatmawati and H. Suprajitno, Interior point method for solving linear programming with interval coefficients using affine scaling, Far East Journal of Mathematical Sciences (FJMS) 111(2) (2019), 195-211.
- [15] G. Ramesh and K. Ganesan, Interval linear programming with generalized interval arithmetic, International Journal of Scientific & Engineering Research 2-11 (2011), 1-8.

- [16] G. Ramesh and K. Ganesan, Duality theory for interval linear programming, *IOSR Journal of Mathematics (IOSRJM)* 4-4 (2012), 39-47.
- [17] H. Suprajitno and I. bin Mohd., Interval Linear Programming, *Proceedings of the 3 International Conference on Mathematics and Statistics (ICOMS-3)* Institut Pertanian Bogor, Indonesia, 5-6 Agustus 2008.
- [18] H. Suprajitno, I. bin Mohd., M. Mamat and F. Ismail, Linear Programming with Interval Variables, *Proceedings of ICORAFSS 2009*, The ZON Regency Hotel Johor Bahru Malaysia, 2-4 June 2009.
- [19] H. Suprajitno and I. bin Mohd., Linear programming with interval arithmetic, *International Journal Contemporary Mathematical Sciences* 5-7 (2010), 323-332.
- [20] Y. Zhong, Y. Jia, D. Chen and Y. Yang, Interior point method for solving fuzzy number linear programming using linear ranking function, *Handbook of Optimization in Telecommunications*, 2013, pp. 3-25.
- [21] F. Zhou, G. H. Huang, G. Chen and H. Guo, Enhanced-interval linear programming, *European Journal of Operational Research* 199-2 (2009), 323-333.



Paper No. PPH-2012053-AM

Kindly return the proof after correction to:


*The Publication Manager
Pushpa Publishing House
Vijaya Niwas
198, Mumfordganj
Prayagraj-211002 (India)*

along with the print charges*.

***Invoice attached**

Proof read by: Agustina Pradjaningsih.....

Copyright transferred to the Pushpa Publishing House

Signature:

Date:18 January 2021.....

Tel: +62 812 345 5367.....

e-mail: tinapradja.math@gmail.com.....

Number of additional reprints required

.....

Cost of a set of 25 copies of additional reprints @ U.S. Dollars 15.00 per page.

(25 copies of reprints are provided to the corresponding author ex-gratis)

1. INTRODUCTION

There are decision-making regarding optimization problems related to optimizing both the maximum and minimum in limited resources in various life situations. From this situation, it can be said that optimization is a problem related to decisions and how to determine the best or satisfactory solution, for example, minimizing financing, the number of employees, raw materials, or maximizing profits. The notion of optimization is a process to determine the best conditions that provide the maximum or minimum value of the objective function that fulfills the given constraint function (Rao, 2009).

Linear programming is mathematical programming developed to handle optimization problems that involve linear equations in the objective function and the constraint function. Several linear programming applications include resource allocation, transportation problems, operation scheduling, and management information systems. One method of solving linear programming is the simplex method introduced by George B. Dantzig in 1947 (Bazaraa et al., 2010).

One of the basic assumptions that exist in linear programming problems is the assumption of certainty. The assumption of certainty shows that all coefficients of decision variables in the model are constants that are known with certainty (Hillier & Liebermann, 2010). However, in real situations or problems, there may be coefficients or uncertain decision variables. Based on the concepts and theory of interval analysis developed by Moore (1966), this uncertainty problem is anticipated by making the approximate value in the form of intervals to develop linear interval programming. The development of linear interval programming starts from linear programming with interval coefficients, both in the objective function's coefficient and the coefficient of the constraint function. Furthermore, it developed into linear programming with coefficients and decision variables in the form of intervals.

Shaocheng (1994), Ramadan (1997), Chinnek & Ramadan (2000), and Kuchta (2008) have developed linear programming with interval coefficients. The optimum value of the objective function is obtained by combining the optimum value of the best optimum problem and the worst optimum problem. It is in the form of an interval, while the optimum point is not in the form of an interval. Suprajitno & Mohd (2008) and Suprajitno (2010) describe linear programming problems with interval-shaped coefficients and decision variables to get the optimum solution in the form of intervals, both the optimum point and the optimum value. Suprajitno et al. (2009), Suprajitno (2010), Suprajitno & Mohd (2010), Ramesh & Ganesan (2011), and Ramesh & Ganesan (2012) also develop linear programming solutions with interval coefficients and decision variables using the interval arithmetic programming approach with the simplex method. Interval arithmetic programming aims to obtain direct linear programming solutions with interval coefficients and variables.

Until now, the simplex method is the method most often used to solve linear programming problems. In 1984, Karmakar studied an algorithm for solving large (complex) linear programming problems in terms of a large number of variables and the large number of constraint functions that must be solved. According to den Hertog (1994), the interior point method has been used in linear programming and quadratic programming problems, and therefore as an alternative method of solving linear programming problems, this interior point method needs to get a more comprehensive appreciation, the same as Gondzio's (2011) simplex method.). So far, the interior point method is used to solve classical linear programming problems that meet the certainty assumptions, with coefficients and variables in the form of constants (Hillier & Lieberman, 2010). From the search, the interior point method has not been applied to solve interval linear programming until now. This has led to the desire to discuss the interior point method in solving linear interval programming.

Based on the background and literature review stated, it is still open to research the use of the interior point method in interval linear programming. This study aims to construct an interior point method used to solve interval linear programming. Furthermore, this construction will be applied to several examples taken from Hillier & Liebermann (2010), Ashayerinasab et al. (2018), Huang (1992), Zhou et al. (2009) in Suprajitno (2010), and Fan & Huang (2012).

2. LITERATURE REVIEW

2.1. Linear Programming

This section discusses general forms of classical linear programming and standard linear programming. The discussion in this sub-chapter comes from Taha (2007), Bazaraa et al. (2010) and Hillier & Lieberman (2010).

Classical linear programming with the objective maximize function is defined as follows:

Maximize

$$Z = \sum_{j=1}^n c_j x_j, \quad (2.1)$$

Subject to

$$\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m, \quad (2.2)$$

$$x_j \geq 0, \quad j = 1, 2, \dots, n, \quad (2.3)$$

and x_j decision variables, c_j objective function coefficient, a_{ij} constraint function coefficient, b_i resource limit, Z objective function, and $x_j \in \mathbb{R}^+$, $c_j, a_{ij}, b_i \in \mathbb{R}$.

(Hillier & Lieberman, 2010)

The linear programming problems formulated in Equation (2.1), Equation (2.2) and Equation (2.3) must fulfill the following basic assumptions:

- 1) Proportionality assumption. This assumption means that the increase and decrease in the value of the destination function and use of the available sources or facilities will change proportionally with changes in the level of activity.
- 2) Assumption of additivity. This assumption ensures that the contribution of a decision variable to the objective function and constraint function is independent of other variables. This assumption is fulfilled if the objective function is a direct addition of each of the decision variables. As for the constraint function, this assumption is met if the value of the resource is the total use of each decision variable.
- 3) Assumption of divisibility. This assumption states that the output generated by each activity can be a non-integer number. Likewise with the resulting goal value.
- 4) Assumption of certainty. This assumption ensures that all coefficients both the objective function coefficient and the constraint function coefficient of the linear programming problem are known constants with certainty.

(Bazaraa et al., 2010; Hillier & Lieberman, 2010)

The linear programming formulated in Equation (2.1), Equation (2.2) and Equation (2.3) can be represented in the form of a matrix as follows:

Maximize

$$\mathbf{Z} = \mathbf{c}^T \mathbf{x}, \quad (2.4)$$

Subject to

$$\mathbf{Ax} \leq \mathbf{b}, \quad (2.5)$$

$$\mathbf{x} \geq \mathbf{0}, \quad (2.6)$$

and

$$\mathbf{c}^T = [c_1, c_2, \dots, c_n],$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0}_{n \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

with $\mathbf{c}, \mathbf{x}, \mathbf{0} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{A} \in \mathbb{R}^{m \times n}$.

(Hillier & Lieberman, 2010)

Furthermore, it discusses the definition of the set of feasible solutions, the area of feasible solutions, the optimal solution, the convex set and the corner theorem.

Definitions 2.1. The set of all ordered pairs

(x_1, x_2, \dots, x_n) that satisfies all the constraints of Equation (2.2) and Equation (2.3) is called a feasible solution. If it does not meet the constraints it is called an infisible solution.

(Winston, 2004)

Definitions 2.2. The set of all visible solutions is called the visible region.

(Winston, 2004)

Definition 2.3. A feasible solution that optimizes the objective function in Equation (2.1) is called the optimal solution, and Z is the optimal value.

(Winston, 2004)

2.2. Interior Point Algorithm

Linear programming problems are generally solved by the simplex method, but there is another method that can be used to solve linear programming problems, namely the interior point method. The interior point method is composed of several steps called the interior point algorithm (Hillier & Lieberman, 2010). In the completion of linear

programming using the simplex method, the concern is that the solutions of the angular point are feasible at the boundary of the feasible settlement area. Meanwhile, the interior point method takes an initial value in the form of an interior point in the visible settlement area. A system of linear equations and inequalities, satisfies the interior point conditions if there is a feasible feasible solution that satisfies all the limits of the inequality constraints in the system (Roos et al., 2005)

Solving linear programming problems using an interior point algorithm, requires a standard form or canonical form. The standard form is a form of a system of linear equations that contains the basis variables (variables that have a coefficient of +1). Standard linear programming is a form of linear programming that has a formulation with the following properties:

- 1) All constraints must be in the form of an equation (marked =) with a non-negative right side / resource boundary.
- 2) All decision variables are non-negative variables.
- 3) The objective function can be in the form of maximizing or minimizing.

(Hillier & Lieberman, 2010)

The transformation from a non-standard linear programming problem into a standard linear programming problem is as follows:

- 1) Changing the objective function can be done from the problem of minimizing to the problem of maximizing or vice versa by multiplying the objective function by -1.
- 2) To form the constraint function into canonical form, it is necessary to add new base variables. The new base variables are:
 - a) If the i th constraint is in the form of inequality (marked \leq), then to change it to the standard form, add the slack variable $s_i \geq 0$, with $i = 1, 2, \dots, m$. The slack variable $s_i \geq 0$ is the new base variable.
 - b) If the i th constraint is in the form of inequality (marked \geq), then to change it to standard form, it is reduced by the surplus variable $t_i \geq 0$ with $i = 1, 2, \dots, m$. The surplus variable $t_i \geq 0$ is not the basis variable (when on the left side the coefficient is not +1).
 - c) Artificial variables, namely variables added to the constraint function that have not included the base variable in point b.
 - d) If the i -th constraint is in the form of an equation (signed =), then to change it to standard form by adding the artificial variable $q_i \geq 0$ dengan $i = 1, 2, \dots, m$, in order to form a base.
 - e) The right side / resource boundary of an equation can be converted into a non-negative number by multiplying both the left and right sides by -1.
 - f) The direction of the inequality can change if both the left and right sides are multiplied by -1.
- 3) An infinite y_i variable in sign, can be expressed as two non-negative variables by using the substitute $y_i = y_i' - y_i''$ with $y_i' - y_i'' \geq 0$. This substitution is performed on all constraint functions and objective functions.

(Hillier & Lieberman, 2010)

The linear programming problem formulated in Equation (2.4), Equation (2.5) and Equation (2.6) is converted to a standard linear programming problem as follows:

Maximize

$$\mathbf{Z} = \mathbf{C}^T \mathbf{X}, \quad (2.7)$$

Subject to

$$\mathbf{A} \mathbf{X} = \mathbf{b}, \quad (2.8)$$

$$\mathbf{X} \geq \mathbf{0}, \quad (2.9)$$

and

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix},$$

with $\mathbf{C}, \mathbf{X}, \mathbf{0} \in \mathbb{R}^{n+m}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{A} \in \mathbb{R}^{m \times (n+m)}$.

(Hillier & Lieberman, 2010)

The basic idea of the interior point algorithm in linear programming is to start by taking any interior point in the feasible solution area, so that the calculation process produces visible interior values. An important step in this algorithm is an initial interior point that can be determined first then looking for a solution in a feasible solution area that is limited by constraint functions so that an optimum value is achieved (Hillier & Lieberman, 2010).

The interior point algorithm starts with an initial interior point $\tilde{\mathbf{X}}^0$ which satisfies the interior point conditions, such that $\mathbf{A} \tilde{\mathbf{X}}^0 = \mathbf{b}$, $\tilde{\mathbf{X}}_j^0 > 0$ for $j = 1, 2, \dots, n + m$. The condition $\tilde{\mathbf{X}}_j^0 > 0$ makes the values of the variables $(x_1, x_2, \dots, x_{n+m})$ always within the boundary of the feasible settlement area. The interior point algorithm produces a fissile interior point in a sequence, namely $\tilde{\mathbf{X}}^1$ (obtained from 1st iteration), $\tilde{\mathbf{X}}^2$ (obtained from 2nd iteration), $\dots, \tilde{\mathbf{X}}^k$

(obtained from the (k-1) iteration), $\tilde{\mathbf{X}}^{k+1}$ (obtained from the k- iteration), dan seterusnya. and so on. The steps for the interior point algorithm are as follows:

Initial step: Determine the initial interior point $\tilde{\mathbf{X}}^0 = (x_1, x_2, \dots, x_{n+m})$ which satisfies the constraints of the problem and calculate the value of \mathbf{Z}_0 , with

$$\mathbf{Z}_0 = \mathbf{C}^T \tilde{\mathbf{X}}^0. \quad (2.10)$$

Stage iteration- $i+1$ ($i = 0,1,2, \dots$)

Step 1: Determine the diagonal matrix \mathbf{D}_{i+1} , i.e.

$$\mathbf{D}_{i+1} = \text{diag}(\tilde{\mathbf{X}}^i) = \begin{bmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{n+m} \end{bmatrix}. \quad (2.11)$$

Step 2: Determine

$$\left. \begin{aligned} \mathbf{A}_{i+1} &= \mathbf{A} \mathbf{D}_{i+1} \\ \mathbf{C}_{i+1} &= \mathbf{D}_{i+1} \mathbf{C} \end{aligned} \right\} \quad (2.12)$$

Step 3: Determine

a) Projection matrix, i.e.

$$\mathbf{P}_{i+1} = \mathbf{I} - \mathbf{A}_{i+1}^T (\mathbf{A}_{i+1} \mathbf{A}_{i+1}^T)^{-1} \mathbf{A}_{i+1}, \quad (2.13)$$

with \mathbf{I} identity matrix.

b) The level of projected slope / gradient (gradient against the line containing the completion value), i.e

$$\mathbf{C}_{P_{i+1}} = \mathbf{P}_{i+1} \mathbf{C}_{i+1}. \quad (2.14)$$

Step 4: Determine

$$\left. \begin{aligned} \mathbf{V}_{i+1} &= |\min(\mathbf{C}_{P_{i+1}})| \\ \mathbf{M}_{i+1} &= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\alpha}{v_{i+1}} \mathbf{C}_{P_{i+1}} \end{aligned} \right\} \quad (2.15)$$

$\mathbf{M}_{i+1} \in \mathbb{R}^{n+m}$ and $\alpha \in (0, 1)$, α is the proportion used to maintain that the interior point is always within the constraint boundary or always in the visible area.

Step 5: Determine

$$\tilde{\mathbf{X}}^{i+1} = \mathbf{D}_{i+1} \mathbf{M}_{i+1}, \quad (2.16)$$

which is a candidate interior point for the next iteration.

Step 6: Determine the optimum value

$$\mathbf{Z}_{i+1} = \mathbf{C}^T \tilde{\mathbf{X}}^{i+1}, \quad (2.17)$$

a) If $\mathbf{Z}_{i+1} > \mathbf{Z}_i$, then proceed to the next iteration with the same steps as in Steps 1-5 in the previous iteration. $\tilde{\mathbf{X}}^{i+1}$ is chosen to be the interior point for the next iteration.

b) If $\mathbf{Z}_{i+1} \leq \mathbf{Z}_i$, then the iteration stops and the optimum solution is obtained, namely the optimum point and the optimum value (\mathbf{Z}_i dan $\tilde{\mathbf{X}}^i$).

(Hillier & Lieberman, 2010)

2.3. Analisis Interval

Interval analysis can be used as an automatic control tool for calculation errors that may arise in the use of a numerical approach to solving mathematical problems. This calculation error may arise from rounding errors during calculations, truncation errors, and input errors (Caprani et al., 2002).

The definition and concept of interval analysis are taken from Moore (1966) and Alefeld & Herzberg (1983).

Definition 2.4. The closed interval is denoted $\underline{x} = [x_l, x_s] \in I(\mathbb{R})$ is defined as

$$\underline{x} = [x_l, x_s] = \{x_l, x_s \in \mathbb{R} | x_l \leq x \leq x_s\}$$

where x_l is *infimum* \underline{x} and x_s is *supremum* \underline{x} .

Definitions 2.5. An interval of real numbers \underline{x} is called an infinite interval if the infimum x_l or *supremum* x_s is infinite

Definition 2.6. An interval of real numbers $\underline{x} = [x_l, x_s]$ is called the degenerate interval if $x_l = x_s$.

Definitions 2.7. The two intervals $\underline{x} = [x_l, x_s]$ and $\underline{y} = [y_l, y_s]$ are said to be the same if they satisfy $x_l = y_l$ and $x_s = y_s$.

Definition 2.8. The width of an interval \underline{x} is the real number $w(\underline{x}) = \frac{1}{2}(x_s - x_l)$.

Definition 2.9. The center of an interval x is a real number \underline{x} is $m(\underline{x}) = \frac{1}{2}(x_l + x_s)$.

Definitions 2.10. The absolute value of an interval x is a real number $\underline{|x|}$ is $|\underline{x}| = \max\{|x_l|, |x_s|\}$.

Definition 2.11. Suppose two interval $\underline{x}, \underline{y} \in I(\mathbb{R})$ where $\underline{x} = [x_l, x_s]$ and $\underline{y} = [y_l, y_s]$, then the operations of addition, subtraction, multiplication and division of the intervals x and y are defined as the following:

1) Sum

$$\underline{x} + \underline{y} = [x_l + y_l, x_s + y_s].$$

2) Subtraction

$$\underline{x} - \underline{y} = [x_l, x_s] - [y_l, y_s] = [x_l, x_s] + [-y_s, -y_l] = [x_l - y_s, x_s - y_l].$$

3) Multiplication

$$\underline{xy} = [\min\{x_l y_l, x_l y_s, x_s y_l, x_s y_s\}, \max\{x_l y_l, x_l y_s, x_s y_l, x_s y_s\}].$$

4) Division

$$\frac{\underline{x}}{\underline{y}} = \underline{\frac{x}{y}} = [x_l, x_s] \left[\frac{1}{y_s}, \frac{1}{y_l} \right], 0 \notin \underline{y}$$

Definition 2.12. An interval vector $\underline{V} \in I(\mathbb{R}^n)$, is a set of the form $\underline{V} = \left(\underline{V}_i \right)_{n \times 1}$, where $\underline{V}_i = [x_{il}, x_{is}] \in I(\mathbb{R})$, $x_{il}, x_{is} \in \mathbb{R}$, and $i = 1, 2, \dots, n$.

Some ideas for comparison of two intervals are taken from Alefeld & Herzberger, (1983), Chanas & Kuchta (1996), Klatte et al. (1999), Maleki et al. (2000), Sengupta et al. (2001) and Kuchta (2008) which have been summarized in Suprajitno & Mohd (2010).

Definisi 2.13. suppose \underline{x} and \underline{y} intervals.

- 1) $\underline{x} < \underline{y}$ if and only if $x_s \leq y_l$
- 2) $\underline{x} < \underline{y}$ if and only if $x_l \geq y_l$ and $x_s \leq y_s$
- 3) $\underline{x} < \underline{y}$ if and only if $x_l \leq y_l$ and $x_s \leq y_s$
- 4) (a). $\underline{x} < \underline{y}$ if and only if $x_l \leq y_l$ and $m(\underline{x}) \leq m(\underline{y})$
 (b). $\underline{x} < \underline{y}$ if and only if $x_s > y_l$ and $m(\underline{x}) < m(\underline{y})$
 (c). $\underline{x} < \underline{y}$ if and only if $m(\underline{x}) \leq m(\underline{y})$ and $w(\underline{x}) \geq w(\underline{y})$.
- 5) (a). $\underline{x} < \underline{y}$ if and only if $x_l - \varepsilon \leq y_s$ with ε is a real number
 (b). $\underline{x} < \underline{y}$ if and only if $x_s - \varepsilon \leq y_l$ with ε is a real number
- 6) $\underline{x} < \underline{y}$ if and only if $x_l + x_s \leq y_l + y_s$
- 7) $\underline{x} < \underline{y}$ if and only if $u x_l + v x_s \leq u y_l + v y_s$ with $u, v \in (0,1]$ and $u \leq v$

The following shows the results in the form of definitions, theorems, propositions and examples of interval matrices referring to Nirmala et al. (2011).

Definition 2.14. An interval matrix $\underline{\mathbf{A}} \in I(\mathbb{R}^{m \times n})$, is a matrix $\underline{\mathbf{A}} = (\underline{a}_{ij})$ where $\underline{a}_{ij} = [a_{ijl}, a_{ijs}] \in I(\mathbb{R})$, with a_{ijl} is *infimum* \underline{a}_{ij} and a_{ijs} is *supremum* \underline{a}_{ij} , for each $i = 1, 2, \dots, m, j = 1, 2, \dots, n$.

Definitions 2.15. The center point of an interval matrix $\underline{\mathbf{A}}$ is the matrix $m(\underline{\mathbf{A}}) = \left(m(\underline{a}_{ij}) \right)$ where $m(\underline{a}_{ij}) = \frac{1}{2}(a_{ijl} + a_{ijs})$.

Definition 2.16. The width of an interval matrix $\underline{\mathbf{A}}$ is the matrix $w(\underline{\mathbf{A}}) = \left(w(\underline{a}_{ij}) \right)$ where $w(\underline{a}_{ij}) = \frac{1}{2}(a_{ijs} - a_{ijl})$.

Definition 2.17. The absolute value of an interval matrix $\underline{\mathbf{A}}$ is the matrix $|\underline{\mathbf{A}}| = \left(|\underline{a}_{ij}| \right)$.

2.4. Linear Programming with Interval Coefficients

The linear programming problem with interval coefficients is defined as follows:

Maximize

$$\underline{Z} = \sum_{j=1}^n [c_{jl}, c_{js}] x_j, \tag{2.18}$$

Subject to

$$\sum_{j=1}^n [a_{ijl}, a_{ijs}]x_j \leq [b_{il}, b_{is}], i = 1, 2, \dots, m, \quad (2.19)$$

$$x_j \geq 0, \quad j = 1, 2, \dots, n, \quad (2.20)$$

and $x_j \in \mathbb{R}^+, [c_{jl}, c_{js}], [a_{ijl}, a_{ijs}], [b_{il}, b_{is}] \in I(\mathbb{R})$.

(Ramadan, 1997; Chinnek & Ramadan, 2000)

According to Chinneck & Ramadan (2000), if each inequality of $-i, i = 1, 2, \dots, m$ in Equation (2.19) has an p interval coefficient, it can be transformed into 2^{p+1} different extreme inequalities. The inequality is obtained by combining the upper limit value and the lower limit value of the coefficient. The optimum solution is obtained by selecting a version of the extreme inequality of 2^{p+1} the extreme inequality that optimizes the objective function. The special inequality formula obtained from setting each lower limit or upper limit of the interval coefficient is called the characteristic formula.

Some terms that need to be defined are related to the $-i, i = 1, 2, \dots, m$ inequality constraint in Equation (2.19), namely S_k is the set of k -th extreme inequality solutions between 2^{p+1} different extreme inequalities, S_S is the combination of all the set of extreme inequalities solutions and S_I is the slice of all the set of extreme inequalities solutions, with

$$S_S = \bigcup_{k=1}^{2^{p+1}} S_k \quad \text{dan} \quad S_I = \bigcap_{k=1}^{2^{p+1}} S_k$$

In the following, some definitions and theorems related to linear programming problems with interval coefficients are presented

Definition 2.18. For each inequality constraint in Equation (2.19), if there is a characteristic formula such that the set of solutions is equal to S_S or S_I , then the characteristic formulas are called the largest or least feasible area of solution..

(Ramadan, 1997)

Theorem 2.1. If there is an interval inequality in the constraint function

$$\sum_{j=1}^n [a_{ijl}, a_{ijs}]x_j \leq [b_{il}, b_{is}], i = 1, 2, \dots, m, \quad (2.21)$$

with $x_j \geq 0 (j = 1, 2, \dots, n)$ then

- 1) The largest feasible settlement area is the satisfactory settlement area

$$\sum_{j=1}^n a_{ijl}x_j \leq b_{is}, \quad (2.22)$$

- 2) The smallest feasible settlement area is the satisfactory settlement area

$$\sum_{j=1}^n a_{ijs}x_j \leq b_{il}. \quad (2.23)$$

(Chinnek & Ramadan, 2000)

Theorem 2.2. If given the objective function

$$Z = \sum_{j=1}^n [c_{jl}, c_{js}]x_j, \quad (2.24)$$

with $x_j \geq 0 (j = 1, 2, \dots, n)$ then for any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ in the feasible solution area apply

$$\sum_{j=1}^n c_{js}x_j \geq \sum_{j=1}^n c_{jl}x_j. \quad (2.25)$$

(Chinnek & Ramadan, 2000)

Definition 2.19. If given the objective function

$$Z = \sum_{j=1}^n [c_{jl}, c_{js}]x_j, \quad (2.26)$$

with $x_j \geq 0 (j = 1, 2, \dots, n)$ thena

- 1) The best objective function is given by

$$z_S = \sum_{j=1}^n c_{js}x_j, \quad (2.27)$$

- 2) The worst objective function is given by

$$z_I = \sum_{j=1}^n c_{jl}x_j. \quad (2.28)$$

(Chinnek & Ramadan, 2000)

Definition 2.20. If given the objective function as in Equation (2.26) and if there is an interval inequality in the constraint function as in Equation (2.21), then

- 1) The best optimum problem is given by
Maximize

$$z_S = \sum_{j=1}^n c_{jS} x_j, \quad (2.29)$$

Subject to

$$\sum_{j=1}^n a_{ijI} x_j \leq b_{iS}, \quad (2.30)$$

and $x_j \geq 0$ ($j = 1, 2, \dots, n$).

- 2) The worst optimum problem is given by
Maximize

$$z_I = \sum_{j=1}^n c_{jI} x_j, \quad (2.31)$$

Subject to

$$\sum_{j=1}^n a_{ijS} x_j \leq b_{iI}, \quad (2.32)$$

And $x_j \geq 0$ ($j = 1, 2, \dots, n$).

(Chinnek & Ramadan, 2000)

Ramadan (1997) explains that there is one major difference in the coefficients of linear programming problems, namely between coefficients in the form of constants and coefficients in the form of intervals. The difference is that the constant can express the relationship of the equation into two inequalities. For example if $a = b$ then $a \geq b$ and $a \leq b$. However, this property does not apply to coefficients in the form of intervals. This is described by Ramadan (1997) as in the following Theorem 2.4 which states about the equation constraints involving interval coefficients. Henceforth, Theorem 2.4 can be used to take the initial interior point at the interior point in the form of intervals.

Theorem 2.3. If $[a, b] = [c, d]$, then $b \geq c$ and $a \leq d$.

Proof:

From Definition 2.13, states that:

$$[a, b] = [c, d], \text{ if } a = c \text{ and } b = d. \quad (2.33)$$

Meanwhile, based on Definition 2.10,

$$\text{for } [a, b], \text{ the condition } a \leq b \text{ must apply} \quad (2.34)$$

$$\text{and for } [c, d], \text{ the condition } c \leq d \text{ must apply} \quad (2.35)$$

From Equations (2.33) and (2.34) it is obtained $a \leq b = d$ so that $a \leq d$.

From Equations (2.33) and (2.35), it is obtained $c \leq d = b$ so that $c \leq b$ or $b \geq c$.

So it is proven that if $[a, b] = [c, d]$, then we get $b \geq c$ and $a \leq d$.

(Ramadan, 1997)

2.5. Linear Programming with Coefficients and Interval Variables

The linear programming model with coefficients and interval variables has been formulated as follows:

Maximize

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}] [x_{jI}, x_{jS}], \quad (2.36)$$

Subject to

$$\sum_{j=1}^n [a_{ijI}, a_{ijS}] [x_{jI}, x_{jS}] \leq [b_{iI}, b_{iS}], \quad i = 1, 2, \dots, m, \quad (2.37)$$

$$[x_{jI}, x_{jS}] \geq 0, \quad j = 1, 2, \dots, n, \quad (2.38)$$

and $[x_{jI}, x_{jS}] \in I(\mathbb{R}^+)$, $[c_{jI}, c_{jS}]$, $[a_{ijI}, a_{ijS}]$, $[b_{iI}, b_{iS}] \in I(\mathbb{R})$.

(Suprajitno & Mohd, 2008)

The following shows the theorems and definitions that refer to Suprajitno & Mohd (2008).

Theorem 2.4. If there is an interval inequality in the constraint function

$$\sum_{j=1}^n [a_{ijI}, a_{ijS}] [x_{jI}, x_{jS}] \leq [b_{iI}, b_{iS}], \quad (2.39)$$

with $[x_{jI}, x_{jS}] \geq 0$, for every j where $j = 1, 2, \dots, n$ then

- 1) The largest feasible settlement area is the satisfactory settlement area

$$\sum_{j=1}^n \min\{a_{ijl}x_{jl}, a_{ijs}x_{js}\} \leq b_{is}, \quad (2.40)$$

2) The smallest feasible settlement area is the satisfactory settlement area

$$\sum_{j=1}^n \max\{a_{ijs}x_{jl}, a_{ijs}x_{js}\} \leq b_{il}. \quad (2.41)$$

(Suprajitno & Mohd, 2008)

Theorem 2.5. If given the objective function

$$\underline{Z} = \sum_{j=1}^n [c_{jl}, c_{js}][x_{jl}, x_{js}], \quad (2.42)$$

with $[x_{jl}, x_{js}] \geq 0$ for every j where $j = 1, 2, \dots, n$ then for any interval vector $\underline{x} = (x_1, x_2, \dots, x_n)^T$ in the feasible solution area applies

$$\sum_{j=1}^n \min\{c_{jl}x_{jl}, c_{jl}x_{js}\} \leq \sum_{j=1}^n \max\{c_{js}x_{jl}, c_{js}x_{js}\}. \quad (2.43)$$

(Suprajitno & Mohd, 2008)

Definition 2.21. If given the objective function

$$\underline{Z} = \sum_{j=1}^n [c_{jl}, c_{js}][x_{jl}, x_{js}], \quad (2.44)$$

with $[x_{jl}, x_{js}] \geq 0$ ($j = 1, 2, \dots, n$) then

1) The upper limit of the objective function is given by

$$z_s = \sum_{j=1}^n \max\{c_{js}x_{jl}, c_{js}x_{js}\}, \quad (2.45)$$

2) The lower bound of the objective function is given by

$$z_l = \sum_{j=1}^n \min\{c_{jl}x_{jl}, c_{jl}x_{js}\}. \quad (2.46)$$

(Suprajitno & Mohd, 2008)

Definition 2.22. If given the objective function as in Equation (2.44) and if there is an interval inequality in the constraint function as in Equation (2.39), then

1) The best optimum problem is given by

Maximize

$$z_s = \sum_{j=1}^n \max\{c_{js}x_{jl}, c_{js}x_{js}\}, \quad (2.47)$$

Subject to

$$\sum_{j=1}^n \min\{a_{ijl}x_{jl}, a_{ijs}x_{js}\} \leq b_{is}, \quad (2.48)$$

and $x_{jl}, x_{js} \geq 0, (j = 1, 2, \dots, n)$.

2) The worst optimum problem is given by

Maximize

$$z_l = \sum_{j=1}^n \min\{c_{jl}x_{jl}, c_{jl}x_{js}\}, \quad (2.49)$$

Subject to

$$\sum_{j=1}^n \max\{a_{ijs}x_{jl}, a_{ijs}x_{js}\} \leq b_{il}, \quad (2.50)$$

and $x_{jl}, x_{js} \geq 0, (j = 1, 2, \dots, n)$.

(Suprajitno & Mohd, 2008)

3. HYPOTHESIS AND SCIENTIFIC CONCEPTS

3.1. Hypothesis

The interior point method can be used to solve linear programming with interval coefficients and variables directly using interval operations involving all the components of the interval.

3.1. Scientific Concept

The scientific conceptual framework of research can be seen in Figure 3.1.

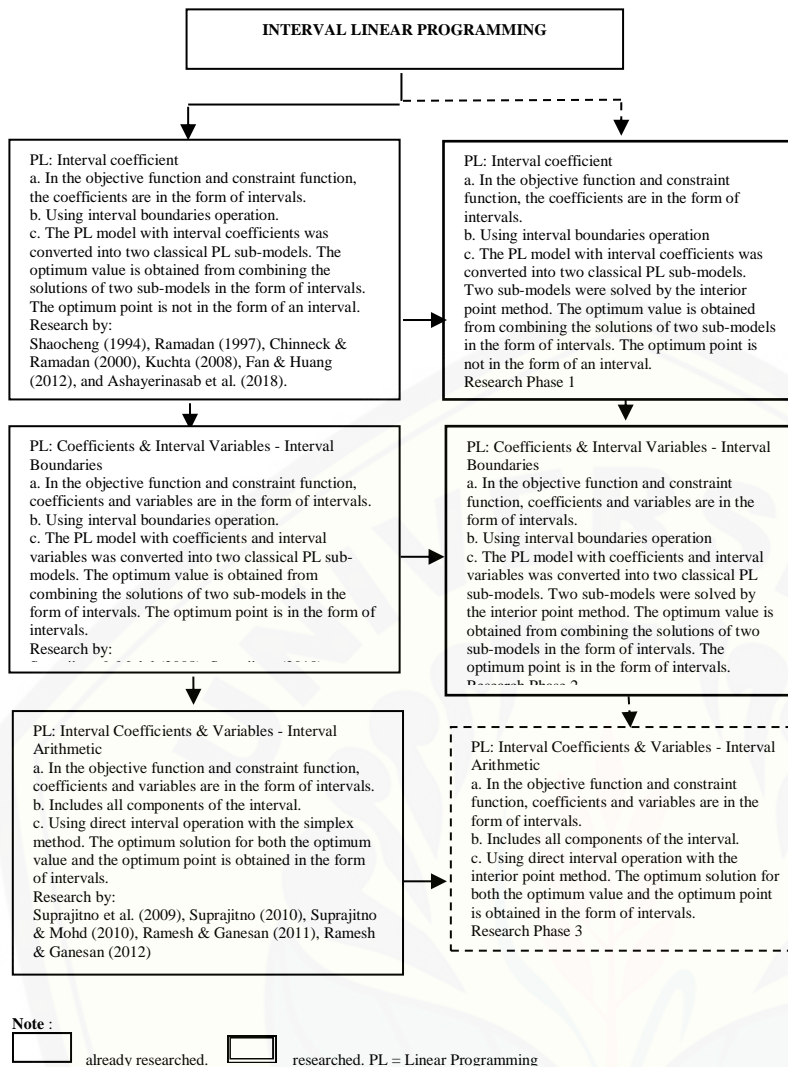


Figure 3.1. Research conceptual framework.

4. RESEARCH METHOD

In this study, a solution to the linear interval programming problem will be constructed by involving all interval components directly, namely linear programming with coefficients and interval variables based on interval arithmetic calculations.

Step 1: The problem

Maximize

$$\underline{Z} = \sum_{j=1}^n \underline{c}_j \underline{x}_j, \tag{4.7}$$

Subject to

$$\sum_{j=1}^n \underline{a}_{ij} \underline{x}_j \leq \underline{b}_i, \quad i = 1, 2, \dots, m, \tag{4.8}$$

$$\underline{x}_j \geq 0, \quad j = 1, 2, \dots, n, \tag{4.9}$$

and $\underline{x}_j \in I(\mathbb{R}^+)$, $\underline{c}_j, \underline{a}_{ij}, \underline{b}_i \in I(\mathbb{R})$.

Step 2: Forming the problem in Step 1 into a standard linear programming form as in Equation (2.7- (2.9), with elements in the form of intervals.

Maximize

$$\underline{Z} = \underline{C}^T \underline{X}, \tag{4.10}$$

Subject to

$$\underline{A} \underline{X} = \underline{b}, \tag{4.11}$$

$$\underline{X} \geq \underline{0}, \tag{4.12}$$

and

$$\underline{C} = \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \vdots \\ \underline{c}_n \\ \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \end{bmatrix}, \underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \\ \underline{x}_{n+1} \\ \underline{x}_{n+2} \\ \vdots \\ \underline{x}_{n+m} \end{bmatrix}, \underline{b} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_m \end{bmatrix}, \underline{0} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \end{bmatrix},$$

$$\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \dots & \underline{a}_{1n} & \underline{1} & \underline{0} & \dots & \underline{0} \\ \underline{a}_{21} & \underline{a}_{22} & \dots & \underline{a}_{2n} & \underline{0} & \underline{1} & \dots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{a}_{m1} & \underline{a}_{m2} & \dots & \underline{a}_{mn} & \underline{0} & \underline{0} & \dots & \underline{1} \end{bmatrix},$$

with $\underline{C}, \underline{X}, \underline{0} \in I(\mathbb{R}^{n+m})$, $\underline{b} \in I(\mathbb{R}^m)$, and $\underline{A} \in I(\mathbb{R}^{m \times (n+m)})$.

Step 3: Determine the initial interior point $\underline{\tilde{X}}^0 = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n+m})$ which satisfies the boundary constraint of the problem with $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n+m} \in I(\mathbb{R})$, $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n+m} > 0$ and calculates the value of $\underline{Z}_0 = \underline{C}^T \underline{\tilde{X}}^0$.

Step 4: Determine the diagonal matrix, namely $\underline{D}_{i+1} = \text{diag}(\underline{\tilde{X}}^i)$.

Step 5 : Determine $\underline{A}_{i+1} = \underline{A} \underline{D}_{i+1}$ and $\underline{C}_{i+1} = \underline{D}_{i+1} \underline{C}$.

Step 6 : Determine

a) Projection matrix $\underline{P}_{i+1} = \underline{I} - \underline{A}_{i+1}^T (\underline{A}_{i+1} \underline{A}_{i+1}^T)^{-1} \underline{A}_{i+1}$.

b) Projected gradient rate $\underline{C}_{P_{i+1}} = \underline{P}_{i+1} \underline{C}_{i+1}$.

Step 7 : Determine

$$\left. \begin{aligned} \underline{V}_{i+1} &= \left| \min(\underline{C}_{P_{i+1}}) \right| \\ \underline{M}_{i+1} &= \begin{bmatrix} [1, 1] \\ [1, 1] \\ \vdots \\ [1, 1] \end{bmatrix} + \frac{\alpha}{\underline{v}_{i+1}} \underline{C}_{P_{i+1}} \end{aligned} \right\}$$

Step 8: Determine the candidate interior point for the next iteration, namely $\underline{\tilde{X}}^{i+1} = \underline{D}_{i+1} \underline{M}_{i+1}$. Steps to determine candidate interior points are presented in chapter V Results and Discussion..

Step 9 : Determine the optimum value $\underline{Z}_{i+1} = \underline{C}^T \underline{\tilde{X}}^{i+1}$. The steps for determining the optimum value are presented in chapter V Results and Discussion.

Step 10: Obtained the optimum solution, namely the optimum point and the optimum value in the form of intervals $\underline{x}_i = [x_{iU}, x_{iS}]$ dan $\underline{Z} = [z_U, z_S]$.

5. RESULTS AND DISCUSSION

Chapter V presents the results obtained by researchers. In line with the research hypothesis and method, the results and discussion will begin with linear programming with interval coefficients, followed by linear programming with interval coefficients and variables whose operations only involve interval boundaries. The last part of this chapter will present the results and discussion of linear programming problems with interval coefficients and variables involving all interval components. All stages and problems of linear interval programming will be solved by using the interior point method. To clarify the problem solving, at the end of each algorithm, examples of algorithm usage are given. Starting with the example of ordinary linear programming which is transformed into linear programming with interval coefficients or transformed into linear programming with coefficients and interval variables. The examples in this chapter are taken from some existing literature, so that these results can be analyzed and compared with the results of solving problems using the interior point method.

Example 5.1. Determine the solution to the linear programming problem taken from (Hillier & Lieberman, 2010), assuming the coefficient is an interval.

Maximize
$$Z = x_1 + 2x_2 \tag{5.11}$$

Subject to
$$\left. \begin{aligned} x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned} \right\} \tag{5.12}$$

Based on Hillier & Lieberman (2010), the optimum solution to this problem is $Z = 16$ with the optimum point $(x_1, x_2) = (0, 8)$.

According to Zhou et al. (2009), there are three criteria that can be used to determine the best solution of interval linear programming problems. These criteria are:

- 1) Meets a constraint function
- 2) The width of the interval from the optimum value (narrowest)
- 3) The degree of uncertainty from the ratio of the width of the optimum value interval to the midpoint of the optimum value interval (smallest).

Table 5.4 provides a comparison between the completion of Example 5.1 and the results obtained by Hillier & Lieberman (2010)

Table 5.4. Comparison of completion of Example 5.1

Kriteria	Penyelesaian	
	Hillier & Lieberman	Contoh 5.1
Nilai optimum	[16, 16]	[15.999999888, 15.999999888]
Lebar interval	0	0
Titik tengah interval	16	15.999999888
Derajat ketidakpastian	0%	0%

From Table 5.4 it can be seen that the optimum value, the interval width, the interval midpoint and the degree of uncertainty, correspond (almost to the same) as that obtained by Hillier & Lieberman (2010). This suitability indicates that the interior point method can be used as an alternative to solve linear programming problems with interval coefficients.

Example 5.2. Determine the solution to the linear programming problem with the following interval coefficients (Ashayerinasab et al., 2018).

Minimize
$$\underline{Z} = [1, 5]\underline{x}_1 + [3, 6]\underline{x}_2 \tag{5.19}$$

Subject to
$$\left. \begin{aligned} [2, 5]\underline{x}_1 + \underline{x}_2 &\geq [3, 7] \\ \underline{x}_1, \underline{x}_2 &\geq 0 \end{aligned} \right\} \tag{5.20}$$

Based on Ashayerinasab et al. (2018) obtained the optimum solution of the problem is $\underline{Z} = [0.6, 17.5]$ with the optimum interval point $\underline{x}_1 = [0.6, 3.5]$ and $\underline{x}_2 = [0, 0]$.

Table 5.6 provides a comparison between the completion of Example 5.2 and the results obtained by Ashayerinasab et al. (2018).

Table 5.6. Comparison of completion of Example 5.2

Kriteria	Penyelesaian	
	Ashayerinasab et al.	Contoh 5.2
Nilai optimum	[0.6, 17.5]	[0.5999992587, 17.4999999754]
Lebar interval	8.45	8.450000
Titik tengah interval	9.05	9.049999

Kriteria	Penyelesaian	
	Ashayerinasab <i>et al.</i>	Contoh 5.2
Derajat ketidakpastian	93.370%	93.370%

From Table 5.6 it can be seen that the optimum value, the width of the interval, the midpoint of the interval and the degree of uncertainty, correspond (almost the same) to that obtained by Ashayerinasab *et al.* (2018). This suitability indicates that the interior point method can be used as an alternative to solve linear programming problems with interval coefficients.

Example 5.3. Determine the solution of the linear programming problem with interval coefficients taken from Huang (1992), Zhou *et al.* (2009) in Suprajitno (2010) and Fan & Huang (2012) as follows:

Maximize

$$\underline{Z} = [26, 30]\underline{x}_1 - [5.5, 6]\underline{x}_2$$

Subject to

$$\left. \begin{aligned} [8, 10]\underline{x}_1 - [12, 14]\underline{x}_2 &\leq [3.8, 4.2] \\ [1, 1.1]\underline{x}_1 + [0.19, 0.2]\underline{x}_2 &\leq [6.5, 7] \\ \underline{x}_1, \underline{x}_2 &\geq 0 \end{aligned} \right\}$$

Based on Huang (1992), Zhou *et al.* (2009) in Suprajitno (2010) and Fan & Huang (2012) obtained the optimum solution to these problems which are written in Table 5.7.

Table 5.7. Previous research results from Example 5.3.

No	Peneliti	\underline{x}_j	\underline{Z}
1	Huang (1992)	$\underline{x}_1 = [5.21, 6.34]$ $\underline{x}_2 = [3.32, 4.03]$	$\underline{Z} = [111.4, 171.8]$
2	Zhou <i>et al.</i> (2009)	$\underline{x}_1 = [4.574, 6.336]$ $\underline{x}_2 = [3.320, 3.495]$	$\underline{Z} = [97.954, 171.82]$
3	Fan & Huang (2012)	$\underline{x}_1 = [5.21, 6.23]$ $\underline{x}_2 = [3.26, 4.03]$	$\underline{Z} = [111.38, 169.1]$

Step 1: The initial form of the problem is in the form of linear programming with interval coefficients with the objective function of maximizing, so the problem is rewritten according to its initial form, namely

Maximize

$$Z = [26, 30]\underline{x}_1 - [5.5, 6]\underline{x}_2 \tag{5.27}$$

Subject to

$$\left. \begin{aligned} [8, 10]\underline{x}_1 - [12, 14]\underline{x}_2 &\leq [3.8, 4.2] \\ [1, 1.1]\underline{x}_1 + [0.19, 0.2]\underline{x}_2 &\leq [6.5, 7] \\ \underline{x}_1, \underline{x}_2 &\geq 0 \end{aligned} \right\} \tag{5.28}$$

Table 5.9 provides a comparison between the completion of Example 5.3 and the results obtained by previous researchers.

Table 5.9. Comparison of completion of Example 5.3.

Penyelesaian	Kriteria		
	Lebar interval	Titik tengah interval	Derajat ketidakpastian
Huang	30.2	141.6	21.3276836%
Zhou et al	36.933	134.887	27.3806964%
Fan & Huang	28.86	140.24	20.5790074%
Contoh 5.3	30.95272	141.665876	21.8491008%

If using two criteria from Zhou *et al.*, (2009), namely the width of the interval and the degree of uncertainty and the comparison of solutions in Table 5.9, it can be seen that the solution of Example 5.3 is between the best solution of Fan & Huang (2012) and the worst solution of Zhou *et al.* . (2009). This shows that the interior point method can be used as an alternative to solve linear programming problems with interval coefficients.

The following shows Example 5.4, Example 5.5 and Example 5.6 to clarify the use of Algorithm 5.2. These examples will be transformed into linear programming problems with coefficients and interval variables. Furthermore, it will be solved using Algorithm 5.2.

Example 5.4. Determine the solution of the linear programming problem which can be seen in Equation (5.11) and Equation (5.12), assuming the coefficients and variables are intervals. Based on Hillier & Lieberman (2010), the optimum solution to this problem is $Z = 16$ with the optimum point $(x_1, x_2) = (0, 8)$, which is equivalent to $\underline{Z} = [16, 16]$, $\underline{x}_1 = [0, 0]$ and $\underline{x}_2 = [8, 8]$.

Step 1: Forming the initial problem from Equation (5.11) and Equation (5.12) becomes

Maximize

$$Z = [1, 1][x_{1I}, x_{1S}] + [2, 2][x_{2I}, x_{2S}] = [x_{1I} + 2x_{2I}, x_{1S} + 2x_{2S}] \quad (5.47)$$

Subject to

$$\left. \begin{aligned} [1, 1][x_{1I}, x_{1S}] + [1, 1][x_{2I}, x_{2S}] &\leq [8, 8] \\ [x_{1I}, x_{1S}], [x_{2I}, x_{2S}] &\geq 0 \end{aligned} \right\}$$

or

$$\left. \begin{aligned} [x_{1I} + x_{2I}, x_{1S} + x_{2S}] &\leq [8, 8] \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} &\geq 0 \end{aligned} \right\} \quad (5.48)$$

Table 5.14 provides a comparison between the completion of Example 5.4 and the results obtained by Hillier & Lieberman (2010).

Table 5.14. Comparison of completion of Example 5.4.

Kriteria	Penyelesaian	
	Hillier & Lieberman	Contoh 5.4
Titik optimum	$\underline{x}_1 = [0, 0]$ $\underline{x}_2 = [8, 8]$	$\underline{x}_1 = [0, 0]$ $\underline{x}_2 = [7.9999999815, 7.9999999815]$
Nilai optimum	[16, 16]	[15.9999999888, 15.9999999888]
Nilai kendala	[8, 8] ≤ [8, 8]	[7.9999999815, 7.9999999815] ≤ [8, 8]
Lebar interval	0	0
Titik tengah interval	16	15.9999999888
Derajat ketidakpastian	0%	0%

From Table 5.14 it can be seen that all the criteria were achieved by the completion of Example 5.4. or said to be in conformity (almost the same) as that obtained by Hillier & Lieberman (2010). For the constraint value fulfills 8 characteristics of the 10 characteristics in Definition 2.21, namely properties 1, 3, 4a, 4c, 5a, 5b, 6, and 7. The fulfillment of these characteristics can be seen in Appendix 4. The more number of fulfillment properties according to Definition 2.21 shows the better a solution. So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with coefficients and interval variables based on the interval boundary calculation.

Example 5.5. Determine the solution of the linear programming problem with the interval coefficient which can be seen in Equation (5.19) and Equation (5.20). Based on Ashayerinasab et al. (2018) obtained that the optimum solution to this problem is $\underline{Z} = [0.6, 17.5]$ with the optimum interval $\underline{x}_1 = [0.6, 3.5]$ and $\underline{x}_2 = [0, 0]$.

Step 1: Forming the initial problem from Equation (5.19) and Equation (5.20) becomes

Maximize

$$\begin{aligned} -\underline{Z} &= [-5, -1][x_{1I}, x_{1S}] + [-6, -3][x_{2I}, x_{2S}] \\ &= [-5x_{1S} - 6x_{2S}, -x_{1I} - 3x_{2I}] \end{aligned} \quad (5.53)$$

Subject to

$$\left. \begin{aligned} [-5, -2][x_{1I}, x_{1S}] - [1, 1][x_{2I}, x_{2S}] &\leq [-7, -3] \\ [x_{1I}, x_{1S}], [x_{2I}, x_{2S}] &\geq 0 \end{aligned} \right\}$$

or

$$\left. \begin{aligned} [-5x_{1S} - x_{2S}, -2x_{1I} - x_{2I}] &\leq [-7, -3] \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} &\geq 0 \end{aligned} \right\} \quad (5.54)$$

Table 5.19. presents a comparison between the solutions to Example 5.5 and the results obtained by Ashayerinasab et al. (2018).

Table 5.19. Comparison of completion of Example 5.5.

Kriteria	Penyelesaian	
	Ashayerinasab et al.	Contoh 5.5
Titik optimum	$\underline{x}_1 = [0.6, 3.5]$ $\underline{x}_2 = [0, 0]$	$\underline{x}_1 = [4, 4]$ $\underline{x}_2 = [1, 1]$
Nilai optimum	[0.6, 17.5]	[7, 26]
Nilai kendala	[3, 7] ≤ [1.2, 17.5]	[3, 7] ≤ [9, 21]
Lebar interval	8.450	9.5
Titik tengah interval	9.050	16.5
Derajat ketidakpastian	93.370%	57.575%

If using criteria from Zhou et al. (2009), from Table 5.19 it can be seen that the completion of Example 5.5 obtained a better solution than Ashayerinasab et al. (2018). For the constraint value fulfills the 7 properties of the 10 properties in Definition 2.21, namely properties 1, 3, 4a, 5a, 5b, 6, and 7. The fulfillment of these properties can be seen in Appendix 5. So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with coefficients and interval variables based on the interval limit calculation.

Example 5.6. Determine the solution of the linear programming problem with the interval coefficient which can be seen in Equation (5.27) and Equation (5.28). Based on Huang (1992), Zhou et al. (2009) in Suprajitno (2010), Fan & Huang (2012) obtained the optimum solution to these problems which are written in Table 5.20.

Table 5.20. Previous research results from Example 5.6.

No	Peneliti	\underline{x}_j	\underline{Z}
1	Huang (1992)	$\underline{x}_1 = [5.21, 6.34]$ $\underline{x}_2 = [3.32, 4.03]$	$\underline{Z} = [111.4, 171.8]$
2	Zhou et al. (2009)	$\underline{x}_1 = [4.574, 6.336]$ $\underline{x}_2 = [3.320, 3.495]$	$\underline{Z} = [97.954, 171.82]$
3	Suprajitno (2010)	$\underline{x}_1 = [5.1816, 5.1816]$ $\underline{x}_2 = [4.0013, 4.0013]$	$\underline{Z} = [110.7138, 133.44085]$
4	Fan & Huang (2012)	$\underline{x}_1 = [5.21, 6.23]$ $\underline{x}_2 = [3.26, 4.03]$	$\underline{Z} = [111.38, 169.1]$

Step 1: Forming the initial problem from Equation (5.27) and Equation (5.28) becomes

Maximize

$$Z = [26, 30][x_{1I}, x_{1S}] - [5.5, 6][x_{2I}, x_{2S}] = [26x_{1I} - 6x_{2S}, 30x_{1S} - 5.5x_{2I}] \quad (5.59)$$

Subject to

$$\left. \begin{aligned} [8, 10][x_{1I}, x_{1S}] - [12, 14][x_{2I}, x_{2S}] &\leq [3.8, 4.2] \\ [1, 1.1][x_{1I}, x_{1S}] + [0.19, 0.2][x_{2I}, x_{2S}] &\leq [6.5, 7] \\ [x_{1I}, x_{1S}], [x_{2I}, x_{2S}] &\geq 0 \end{aligned} \right\}$$

or

$$\left. \begin{aligned} [8x_{1I} - 14x_{2S}, 10x_{1S} - 12x_{2I}] &\leq [3.8, 4.2] \\ [x_{1I} + 0.19x_{2I}, 1.1x_{1S} + 0.2x_{2S}] &\leq [6.5, 7] \\ x_{1I}, x_{1S}, x_{2I}, x_{2S} &\geq 0 \end{aligned} \right\} \quad (5.60)$$

The results of previous researchers can be seen in Table 5.20 for the optimum point and the optimum value. Table 5.25. presents a comparison between the completion of Example 5.6 and the results obtained by previous researchers.

Table 5.25. Comparison of completion of Example 5.6.

Penyelesaian	Kriteria				
	Nilai kendala 1	Nilai kendala 2	Lebar interval	Titik tengah interval	Derajat ketidakpastian
Huang	$[-14.74, 23.56]$ $\leq [3.8, 4.2]$	$[5.8408, 7.78]$ $\leq [6.5, 7]$	30.2	141.6	21.3276%
Zhou et al	$[-12.338, 23.52]$ $\leq [3.8, 4.2]$	$[5.2048, 7.6686]$ $\leq [6.5, 7]$	36.933	134.887	27.3807%
Suprajitno	$[-14.5654, 3.8004]$ $\leq [3.8, 4.2]$	$[5.94184, 6.5000]$ $\leq [6.5, 7]$	11.36352	122.077	9.3085%
Fan & Huang	$[-14.74, 23.18]$ $\leq [3.8, 4.2]$	$[5.8294, 7.659]$ $\leq [6.5, 7]$	28.86	140.24	20.579%
Contoh 5.6	$[4.19999, 69.9999]$ $\leq [3.8, 4.2]$	$[6.9999, 8.4399] \leq$ $[6.5, 7]$	25.1	184.899	13.5749%

If using criteria from Zhou et al. (2009), from Table 5.25 it can be seen that the completion of Example 5.6. is ranked second after the best finish from Suprajitno (2010). To fulfill the constraint value in Definition 2.21 can be seen in Appendix 6. So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with interval coefficients and variables based on interval boundary calculations.

Completion of Linear Programming with Interval Coefficients and Variables Based on Interval Arithmetic Calculations

In this sub-chapter, a linear programming solution will be constructed with interval coefficients and variables involving the interval component directly. This linear interval programming completion algorithm is based on an interior point algorithm which is further modified. This is an important contribution from this paper. Modifications are intended to conform to the definitions and theorems that apply to the interior point method, interval operations as well as limitations on the problem to be solved. The steps for solving linear programming problems with coefficients and interval variables based on interval arithmetic are compiled in Algorithm 5.3.

Algorithm 5.3.

Step 1: the problem

Maximize

$$\underline{Z} = \sum_{j=1}^n c_j \underline{x}_j, \quad (5.65)$$

Subject to

$$\sum_{j=1}^n a_{ij}x_j \leq \underline{b}_i, \quad i = 1, 2, \dots, m, \quad (5.66)$$

$$\underline{x}_j \geq 0, \quad j = 1, 2, \dots, n., \quad (5.67)$$

and $\underline{x}_j \in I(\mathbb{R}^+)$, $\underline{c}_j, \underline{a}_{ij}, \underline{b}_i \in I(\mathbb{R})$.

Step 2: Shaping the problem in Step 1 into standard interval linear programming form

Maximize

$$\underline{\mathbf{Z}} = \underline{\mathbf{C}}^T \underline{\mathbf{X}}, \quad (5.68)$$

Subject to

$$\underline{\mathbf{A}} \underline{\mathbf{X}} = \underline{\mathbf{b}}, \quad (5.69)$$

$$\underline{\mathbf{X}} \geq \underline{\mathbf{0}}, \quad (5.70)$$

and

$$\underline{\mathbf{C}} = \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \vdots \\ \underline{c}_n \\ \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \end{bmatrix}, \quad \underline{\mathbf{X}} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \\ \underline{x}_{n+1} \\ \underline{x}_{n+2} \\ \vdots \\ \underline{x}_{n+m} \end{bmatrix}, \quad \underline{\mathbf{b}} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_m \end{bmatrix}, \quad \underline{\mathbf{0}} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \end{bmatrix},$$

$$\underline{\mathbf{A}} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \dots & \underline{a}_{1n} & \underline{1} & \underline{0} & \dots & \underline{0} \\ \underline{a}_{21} & \underline{a}_{22} & \dots & \underline{a}_{2n} & \underline{0} & \underline{1} & \dots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{a}_{m1} & \underline{a}_{m2} & \dots & \underline{a}_{mn} & \underline{0} & \underline{0} & \dots & \underline{1} \end{bmatrix},$$

with $\underline{\mathbf{C}}, \underline{\mathbf{X}}, \underline{\mathbf{0}} \in I(\mathbb{R}^{n+m})$, $\underline{\mathbf{b}} \in I(\mathbb{R}^m)$, and $\underline{\mathbf{A}} \in I(\mathbb{R}^{m \times (n+m)})$.

Step 3: Determine the initial interior point $\underline{\tilde{\mathbf{X}}}^0 = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n+m})$ and calculate the value of (\underline{Z}_0) , with

$\underline{\mathbf{Z}}_0$, with

$$\underline{\mathbf{Z}}_0 = \underline{\mathbf{C}}^T \underline{\tilde{\mathbf{X}}}^0, \quad (5.71)$$

and $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n+m} \in I(\mathbb{R})$, $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n+m} > 0$.

Step 4: Determine the diagonal matrix, namely $\underline{\mathbf{D}}_{i+1} = \text{diag}(\underline{\tilde{\mathbf{X}}}^i)$, i.e.

$$\underline{\mathbf{D}}_{i+1} = \text{diag}(\underline{\tilde{\mathbf{X}}}^i) = \begin{bmatrix} \underline{x}_1 & 0 & 0 & \dots & 0 \\ 0 & \underline{x}_2 & 0 & \dots & 0 \\ 0 & 0 & \underline{x}_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \underline{x}_{n+m} \end{bmatrix} \quad (5.72)$$

Step 5: Determine

$$\left. \begin{aligned} \underline{\mathbf{A}}_{i+1} &= \underline{\mathbf{A}} \underline{\mathbf{D}}_{i+1} \\ \underline{\mathbf{C}}_{i+1} &= \underline{\mathbf{D}}_{i+1} \underline{\mathbf{C}} \end{aligned} \right\} \quad (5.73)$$

Step 6: Determine

a) Projection matrix

$$\underline{\mathbf{P}}_{i+1} = \underline{\mathbf{I}} - \underline{\mathbf{A}}_{i+1}^T (\underline{\mathbf{A}}_{i+1} \underline{\mathbf{A}}_{i+1}^T)^{-1} \underline{\mathbf{A}}_{i+1}, \quad (5.74)$$

with $\underline{\mathbf{I}}$ the identity matrix

b) The level of the gradient projected

$$\underline{\mathbf{C}}_{P_{i+1}} = \underline{\mathbf{P}}_{i+1} \underline{\mathbf{C}}_{i+1}. \quad (5.75)$$

Step 7: Determine

$$\left. \begin{aligned} \underline{\mathbf{V}}_{i+1} &= \left| \min(\underline{\mathbf{C}}_{P_{i+1}}) \right| \\ \underline{\mathbf{M}}_{i+1} &= \begin{bmatrix} [1, 1] \\ [1, 1] \\ \vdots \\ [1, 1] \end{bmatrix} + \frac{\alpha}{\underline{\mathbf{V}}_{i+1}} \underline{\mathbf{C}}_{P_{i+1}} \end{aligned} \right\} \quad (5.76)$$

$\underline{\mathbf{M}}_{i+1} \in I(\mathbb{R}^{n+m})$ and $\alpha \in (0, 1)$.

Step 8: Determine the interior point candidate for the next iteration, ie

$$\underline{\tilde{\mathbf{X}}}^{i+1} = \underline{\mathbf{D}}_{i+1} \underline{\mathbf{M}}_{i+1} \quad (5.77)$$

Inspection of candidate interior points (non-negative test and boundary constraint test)

a) If the candidate interior points meet the constraint boundary, proceed to Step 9

- b) If the candidate interior point does not meet the boundary constraint it stops. Return to the previous iteration and take the interior point of the last interval that meets the constraint boundary, continue to Step 9.

Step 9: Determine the optimum value $\underline{Z}_{i+1} = \underline{C}^T \underline{\tilde{X}}^{i+1}$ (5.78)

Check the optimum value (optimality test)

- a) If $\underline{Z}_{i+1} > \underline{Z}_i$, then proceed to the next iteration with the same steps as in Step 1-Step 8 in the previous iteration. $\underline{\tilde{X}}^{i+1}$ is chosen to be the interior point for the iteration.
 b) If $\underline{Z}_{i+1} \leq \underline{Z}_i$, then stop, proceed to Step 10..

Langkah 10 : The optimum solution is obtained, namely the optimum point and the optimum value in the form of intervals

$$\underline{x}_i = [x_{iL}, x_{iS}], i = 1, 2, \dots, m, \text{ dan } \underline{Z} = [z_L, z_S]. \quad (5.79)$$

The following shows Example 5.7, Example 5.8 and Example 5.9 to clarify the use of Algorithm 5.3. The calculation of interior interval algorithm iteration steps in this section uses the help of Octave16 Software.

Example 5.7. Determine the solution of the linear programming problem which can be seen in Equation (5.11) and Equation (5.12), assuming the coefficients and variables are intervals. Based on Hillier & Lieberman (2010), the optimum solution to this problem is $Z = 16$ with the optimum point $(x_1, x_2) = (0, 8)$, which is equivalent $\underline{Z} = [16, 16]$, $\underline{x}_1 = [0, 0]$ and $\underline{x}_2 = [8, 8]$.

Step 1: Forming the initial problem from Equation (5.11) and Equation (5.12) becomes

Maximize

$$\underline{Z} = [1, 1]\underline{x}_1 + [2, 2]\underline{x}_2 \quad (5.80)$$

Subject to

$$\left. \begin{aligned} [1, 1]\underline{x}_1 + [1, 1]\underline{x}_2 &\leq [8, 8] \\ \underline{x}_1, \underline{x}_2 &\geq 0 \end{aligned} \right\} \quad (5.81)$$

Step 2: Shaping the problem in Step 1 into standard interval linear programming form

Maximize

$$Z = [1, 1]\underline{x}_1 + [2, 2]\underline{x}_2 + 0\underline{S} \quad (5.82)$$

Subject to

$$\left. \begin{aligned} [1, 1]\underline{x}_1 + [1, 1]\underline{x}_2 + 0\underline{S} &= [8, 8] \\ \underline{x}_1, \underline{x}_2, \underline{S} &\geq 0 \end{aligned} \right\} \quad (5.83)$$

Step 3: Take any initial interior interval, which satisfies $\underline{X}^0 = ([2, 2], [5, 5], [1, 1])$ which satisfies the constraint boundary in equation (5.83). Obtained the value $\underline{Z}_0 = [12, 12]$.

Calculation Steps 4–9 in this dissertation uses the Octave16 software. The iteration results are presented in Table 5.26.

Table 5.26. Iteration Example 5.7.

Iterasi	X_i	Uji		Z_i	UO	Kep
		$X_i \geq 0$	$AX_i \leq B$			
0	[2,2], [5.5], [1,1]	$X_0 \geq 0$	$AX_0 \leq B$	[12, 12]	Z_0	
1	[0.3112, 0.31121] [7.6837, 7.6388] [0.050052, 0.05003]	$X_1 \geq 0$	$AX_1 \leq B$	[15.588, 15.589]	$Z_1 > Z_0$	L
2	[0.015565, 0.015627] [7.9401, 7. ⁹⁵⁹²] [0.034744, 0.034747]	$X_2 \geq 0$	$AX_2 \leq B$	[15.896, 15.934]	$Z_2 > Z_1$	L
3	[0.012211, 0.012362] [-0.2203, 16.173] [0.0017373, 0.0020528]	$X_3 \leq 0$	-	-	-	S

UO = Uji optimalitas ; Kep : Keputusan ; L : lanjut ; S : Berhenti

Step 10: Optimum finish

$$\underline{x}_1 \cong [0.015565, 0.015627], \underline{x}_2 \cong [7.9401, 7.9592], \underline{Z} \cong [15.896, 15.934]$$

From the use of Algorithm 5.3, the optimum point is $\underline{x}_1 \cong [0.015565, 0.015627]$, $\underline{x}_2 \cong [7.9401, 7.9592]$ and the optimum value $\underline{Z} \cong [15.896, 15.934]$. Table 5.27 provides a comparison between the completion of Example 5.7 and the results obtained by Hillier & Lieberman (2010).

Table 5.27. Comparison of completion of Example 5.7.

Kriteria	Penyelesaian	
	Hillier & Lieberman	Contoh 5.7
Titik optimum	$\underline{x}_1 = [0, 0]$ $\underline{x}_2 = [8, 8]$	$\underline{x}_1 = [0.015565, 0.015627]$ $\underline{x}_2 = [7.9401, 7.9592]$
Nilai optimum	[16, 16]	[15.896, 15.934]

Kriteria	Penyelesaian	
	Hillier & Lieberman	Contoh 5.7
Nilai kendala	$[8, 8] \leq [8, 8]$	$[7.9401, 7.9592] \leq [8, 8]$
Lebar interval	0	0.019
Titik tengah interval	16	15.915
Derajat ketidakpastian	0%	0.119%

If using a constraint value, the completion of Example 5.7 fulfills 8 characteristics of the 10 characteristics in Definition 2.21, namely properties 1, 3, 4a, 4c, 5a, 5b, 6, and 7. The fulfillment of these properties can be seen in Appendix 4. Whereas if referring to criteria from Zhou et al. (2009), from Table 5.27 it can be seen that Hillier & Lieberman's (2010) solution is better than that of Example 5.7. This can be identified as follows:

- 1) Difficulty taking early interior intervals (Ramadan, 1997)
- 2) The inverse interval calculation is still using the approach (Nirmala et al., 2011).

So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with interval coefficients and variables based on interval arithmetic.

Example 5.8. Determine the solution of the linear programming problem with the interval coefficient which can be seen in Equation (5.19) and Equation (5.20). Based on Ashayerinasab et al. (2018) obtained the optimum solution to the problem is $\underline{Z} = [0.6, 17.5]$ with the optimum interval $\underline{x}_1 = [0.6, 3.5]$ and $\underline{x}_2 = [0, 0]$.

Step 1: Forming the initial problem from Equation (5.19) and Equation (5.20) becomes

$$\text{Maximize} \quad -\underline{Z} = [-5, -1]\underline{x}_1 + [-6, -3]\underline{x}_2 \quad (5.84)$$

$$\text{Subject to} \quad \left. \begin{aligned} [-5, -2]\underline{x}_1 - [1, 1]\underline{x}_2 &\leq [-7, -3] \\ \underline{x}_1, \underline{x}_2 &\geq 0 \end{aligned} \right\} \quad (5.85)$$

Step 2: Shaping the problem in Step 1 into standard interval linear programming form

$$\text{Maximize} \quad -\underline{Z} = [-5, -1]\underline{x}_1 + [-6, -3]\underline{x}_2 + 0\underline{s} \quad (5.86)$$

$$\text{Subject to} \quad \left. \begin{aligned} [-5, -2]\underline{x}_1 - [1, 1]\underline{x}_2 + \underline{s} &= [-7, -3] \\ \underline{x}_1, \underline{x}_2, \underline{s} &\geq 0 \end{aligned} \right\} \quad (5.87)$$

Langkah 3 : Step 3: Take any initial interior interval, which satisfies the constraint boundary in Equation (5.87) and uses Theorem 2.4, $\underline{\tilde{X}}^0 = ([2, 2], [1, 1], [3, 3])$, obtained the value $\underline{Z}_0 = [-16, -5]$.

Calculation of Steps 4–9 in this dissertation using Octave16 software, obtained the optimum solution for $\underline{x}_1 \cong [2, 2]$, $\underline{x}_2 \cong [1, 1]$ and Maximum $-\underline{Z} \cong [-16, -5]$.

Step 10: Optimum finish

$$\underline{x}_1 = [2, 2], \underline{x}_2 = [1, 1]$$

$$\text{Maximum } -\underline{Z} = [-16, -5] \text{ is equivalent to Minimum } \underline{Z} = [5, 16]$$

From the use of Algorithm 5.3, it is obtained the optimum point $\underline{x}_1 \cong [2, 2]$, $\underline{x}_2 \cong [1, 1]$ and the optimum value $\underline{Z} \cong [5, 16]$. Table 5.28. presents a comparison between the solutions to Example 5.8 and the results obtained by Ashayerinasab et al. (2018).

Table 5.28. Comparison of settlement of Example 5.8

Kriteria	Penyelesaian	
	Ashayerinasab et al.	Contoh 5.5
Titik optimum	$\underline{x}_1 = [0.6, 3.5]$	$\underline{x}_1 = [2, 2]$
	$\underline{x}_2 = [0, 0]$	$\underline{x}_2 = [1, 1]$
Nilai optimum	$[0.6, 17.5]$	$[5, 16]$
Nilai kendala	$[3, 7] \leq [1.2, 17.5]$	$[3, 7] \leq [5, 16]$
Lebar interval	8.450	5.5
Titik tengah interval	9.050	10.5
Derajat ketidakpastian	93.370%	52.380%

If using criteria from Zhou et al. (2009), from Table 5.27 it can be seen that the completion of Example 5.8 obtained a better solution than Ashayerinasab et al. (2018). The value of the constraint meets the 7 properties of the 10 properties in Definition 2.21, namely properties 3, 4a, 4b, 5a, 5b, 6, and 7. The fulfillment of these properties can be seen in Appendix 5. So it can be concluded that the interior point method can be used as alternative to solve linear programming problems with coefficients and interval variables based on interval arithmetic calculations.

Example 5.9. Determine the solution of the linear programming problem with the interval coefficient which can be seen in Equation (5.27) and Equation (5.28). Based on Huang (1992), Zhou et al. (2009) in Suprajitno (2010), Fan & Huang (2012) obtained the optimum solution to these problems which are written in Table 5.29.

Step 1: The initial form of the problem is already in the form of linear programming with coefficients and interval variables with the objective function to maximize, so the problem is only rewritten according to the original form, namely

Maximize

$$\underline{Z} = [26, 30]\underline{x}_1 - [5.5, 6]\underline{x}_2 \quad (5.88)$$

Subject to

$$\left. \begin{aligned} [8, 10]\underline{x}_1 - [12, 14]\underline{x}_2 &\leq [3.8, 4.2] \\ [1, 1.1]\underline{x}_1 + [0.19, 0.2]\underline{x}_2 &\leq [6.5, 7] \\ \underline{x}_1, \underline{x}_2 &\geq 0 \end{aligned} \right\} \quad (5.89)$$

Table 5.29. Previous research results from Example 5.9.

No	Peneliti	\underline{x}_j	\underline{Z}
1	Huang (1992)	$\underline{x}_1 = [5.21, 6.34]$ $\underline{x}_2 = [3.32, 4.03]$	$\underline{Z} = [111.4, 171.8]$
2	Zhou et al. (2009)	$\underline{x}_1 = [4.574, 6.336]$ $\underline{x}_2 = [3.320, 3.495]$	$\underline{Z} = [97.954, 171.82]$
3	Suprajitno (2010)-MS	$\underline{x}_1 = [3.7, 4.1]$ $\underline{x}_2 = [2.7, 3.1]$	$\underline{Z} = [77.6, 108.15]$
4	Fan & Huang (2012)	$\underline{x}_1 = [5.21, 6.23]$ $\underline{x}_2 = [3.26, 4.03]$	$\underline{Z} = [111.38, 169.1]$

Step 2: Shaping the problem in Step 1 into standard interval linear programming form

$$Z = [26, 30]\underline{x}_1 - [5.5, 6]\underline{x}_2 + 0\underline{S}_1 + 0\underline{S}_2 \quad (5.90)$$

Subject to

$$\left. \begin{aligned} [8, 10]\underline{x}_1 - [12, 14]\underline{x}_2 + \underline{S}_1 &= [3.8, 4.2] \\ [1, 1.1]\underline{x}_1 + [0.19, 0.2]\underline{x}_2 + \underline{S}_2 &= [6.5, 7] \\ \underline{x}_1, \underline{x}_2, \underline{S}_1, \underline{S}_2 &\geq 0 \end{aligned} \right\} \quad (5.91)$$

Step 3: Take any initial interior interval, which satisfies the constraint boundary in equation (5.91) and uses Theorem 2.4, $\underline{X}^0 = ([2, 2], [5, 5], [51, 51], [3.5, 3.5])$, obtained the value $\underline{Z}_0 = [22, 32.5]$.

Calculation of Steps 4–9 in this dissertation using Octave16 software, obtained the optimum solution for $\underline{x}_1 \cong [3.8109, 4.9579]$, $\underline{x}_2 \cong [2.3034, 5.7705]$ and $\underline{Z} \cong [64.462, 136.07]$

Step 10: Optimum finish

$$\underline{x}_1 \cong [3.8109, 4.9579], \underline{x}_2 \cong [2.3034, 5.7705]$$

$$\underline{Z} \cong [64.462, 136.07]$$

From the use of Algorithm 5.3. The optimum point is $\underline{x}_1 \cong [3.8109, 4.9579]$, $\underline{x}_2 \cong [2.3034, 5.7705]$ and the optimum value $\underline{Z} \cong [64.462, 136.07]$. The results of previous researchers can be seen in Table 5.29 for the optimum point and the optimum value. Table 5.30. presents a comparison between the completion of Example 5.9 and the results obtained by the previous researcher.

Table 5.30. Comparison of completion of Example 5.9.

Penyelesaian	Kriteria				
	Nilai kendala 1	Nilai kendala 2	Lebar interval	Titik tengah interval	Derajat ketidakpastian
Huang	$[-14.74, 23.56] \leq [3.8, 4.2]$	$[5.8408, 7.78] \leq [6.5, 7]$	30.2	141.6	21.3276%
Zhou et al	$[-12.338, 23.52] \leq [3.8, 4.2]$	$[5.2048, 7.6686] \leq [6.5, 7]$	36.933	134.887	27.3807%
Suprajitno – MS	$[-13.8, 8.6] \leq [3.8, 4.2]$	$[4.213, 5.13] \leq [6.5, 7]$	15.275	92.875	16.4468%
Fan & Huang	$[-14.74, 23.18] \leq [3.8, 4.2]$	$[5.8294, 7.659] \leq [6.5, 7]$	28.86	140.24	20.579%
Contoh 5.9	$[-50.2998, 21.9382] \leq [3.8, 4.2]$	$[6.612091, 8.621515] \leq [6.5, 7]$	35.804	100.266	35.709%

If using the constraint value, the solution to Example 5.9 fulfills the 9 characteristics of Definition 2.21 which can be seen in Appendix 6. This solution is better than Huang (1992), Zhou et al. (2009) and Fan & Huang (2012). Meanwhile, referring to the criteria of Zhou et al. (2009), from Table 5.30 it can be seen that the completion of Example 5.9. not the best solution. The best solution is obtained from Suprajitno (2010) with the MS method. This can be identified as follows:

- 1) Difficulty taking early interior intervals (Ramadan, 1997)
- 2) The inverse interval calculation is still using the approach (Nirmala et al., 2011).

So it can be concluded that the interior point method can be used as an alternative to solve linear programming problems with interval coefficients and variables based on interval arithmetic with improvements in taking the initial interior intervals and determining the appropriate inverse matrix.

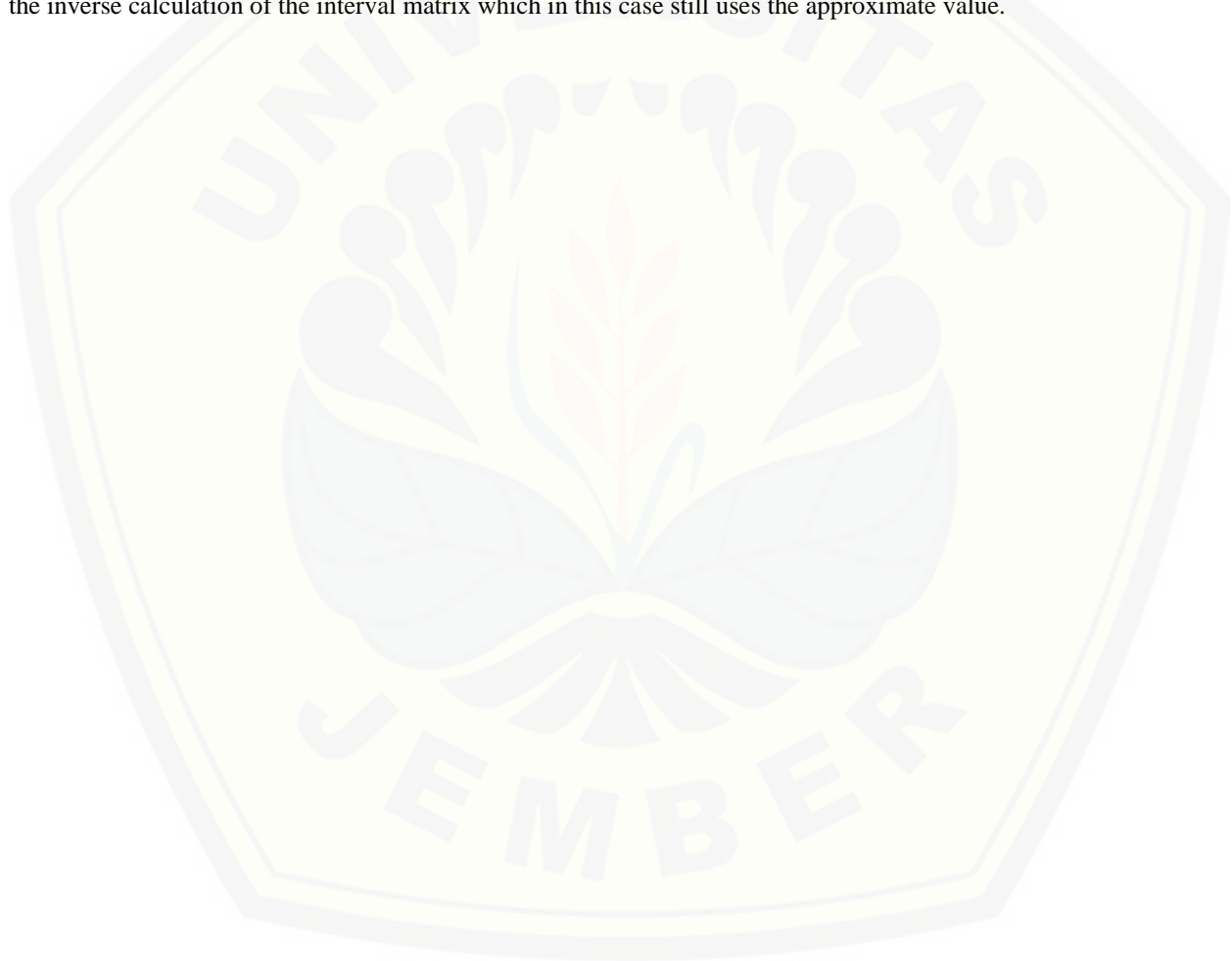
6. CONCLUSIONS AND SUGGESTIONS

6.1. Conclusion

Solving linear programming with interval coefficients and variables can also involve all interval components directly, so it is called linear programming with coefficients and interval variables based on interval arithmetic calculations. The first step in solving linear programming problems with coefficients and variables in the form of intervals, forms the problem into the standard interval linear programming form. Interval standard linear programming is classical standard linear programming which is modified by substituting its point elements in the form of intervals. Next, take any initial interior intervals that meet the problem constraints. The next step is to solve the problem using the interior point method which has been modified by replacing the point elements in the form of intervals, so that the optimum solution is obtained, namely the optimum point and the optimum value in the form of an interval. The difficulty in solving linear programming with interval coefficients and variables involving all interval components directly is in determining the initial interior interval and the determination of the interval matrix inverse. Sometimes this difficulty results in the solution obtained is not the best solution when compared to the method used by previous researchers.

6.2. Suggestion

In general, the study of interval linear programming using the interior point method can be continued with the application of interval linear programming problems with a large number of variables and a large / complex number of constraints, in this case what is meant is the number of variables and a large number of constraints. In particular, in solving the linear interval programming using the interior point method which involves direct interval components, there are difficulties in determining the initial interior interval and the determination of the interval matrix inverse. Therefore, this research can be continued with the study of the formulation for determining the initial interior interval which will facilitate the calculation of completion and the study of the formulation of settlement stability to anticipate the inverse calculation of the interval matrix which in this case still uses the approximate value.



REFERENSI

- Alefeld, G. & Herzberger, J. 1983. *Introduction to Interval Computation*, Academic Press, Inc., New York.
- Ashayerinasab, H. A., Mishmast Nehi, H., & Allahdadi, M. 2018. Solving The Interval Linear Programming Problem : A New Algorithm for A General Case. *Expert Systems With Applications*, 93: 39–49.
- Bartle, R.G. 1976. *The Elements of Real Analysis*, Ed. 2nd. John Wiley & Sons, Inc., New York.
- Bazaraa, M.S., Jarvis, J.J., & Sherali, H.D. 2010. *Linear Programming and Network Flows*, Ed. 4th. John Wiley & Sons, New York.
- Calvert, J.E. & Voxman, W.L. 1989. *Linear Programming*. Academic Press, New York.
- Caprani, O., Madsen, K., & Nielsen, H. . 2002. *Introduction to Interval Analysis*, Annual Report Year. DTU Orbit – The Research Information System. IMM-DTU. Technical University of Denmark.
- Chinnek, J.W. & Ramadan, K. 2000. Linear Programming with Interval Coefficients. *Journal of The Operational Research Society*, 51(2): 209-220.
- den Hertog, D. 1994. *Interior Point Approach to Linear, Quadratic and Convex Programming*. Springer Science+Business Media, BV.
- Fan, Y.R. & Huang, G.H. 2012. A Robust Two-Step Method for Solving Interval Linear Programming Problems within An Environmental Management Context. *Journal of Environmental Informatics*, 19(1): 1–9 .
- Gondzio, J. 2012. Interior Point Methods 25 Years Later. *European Journal of Operational Research*, 218(3): 587-601.
- Hillier, F.S. & Lieberman, G.J. 2010. *Introduction to Operation Research*, Ed. 10th. McGraw-Hill, Inc., New York.
- Huang, G.H., Baetz, B.W., & Patry, G.G. 1992. A Grey Linear Programming Approach For Municipal Solid Waste Management Planning Under Uncertainty. *Civil Engineering Systems*. 9(4):319-335
- Karmakar, N. 1984. A New Polynomial Time Algorithm for Linear Programming. *Combinatorica*, 4(4): 373-395.
- Kuchta, D. 2008. A Modification of A Solution Concept of The Linear Programming with Interval Coefficients in The Constraints. *Central European Journal of Operations Research*, 16(3): 307-316.
- Moore, R.E. 1966. *Interval Analysis*. Prentice-Hall, Englewood Cliffs, NJ.
- Nirmala, T., Datta, D., Kushwaha, H.S., & Ganesan, K. 2011. Inverse Interval Matrix: A New Approach. *Applied Mathematical Sciences*, 5(13): 607 – 624.
- Pradjaningsih, A., Fatmawati & Suprajitno, H. 2019. Interior Point Method for Solving Linear Programming with Interval Coefficients Using Affine Scalling. *Far East Journal of Mathematical Sciences*, 111(2).
- Ramadan, K. 1997. Linear Programming with Interval Coefficients. *M.Sc. Thesis*. Carleton University, Canada.
- Rao, S.S. 2009. *Engineering Optimization : Theory and Practice*, Ed. 4th. John Wiley & Sons, Inc., Canada.
- Ramesh, G. & Ganesan, K. 2011. Interval Linear Programming with Generalized Interval Arithmetic. *International Journal of Scientific & Engineering Research*, 2(11): 1-8.
- Ramesh, G. & Ganesan, K. 2012. Duality Theory for Interval Linear Programming. *IOSR Journal of Mathematics (IOSRJM)*, 4(4): 39-47.
- Roos, C., Terlaky, T., & Vial, J.-Ph. 2005. *Interior Point Methods for Linear Optimization*, Springer-Verlag US
- Shaocheng, T. 1994. Interval Number And Fuzzy Number Linear Programmings. *Fuzzy Sets and Systems*, 66(3): 301-306.
- Suprajitno, H. 2010. Penyelesaian Masalah Pengaturan Linear Multimatlamat Menggunakan Aritmetika Selang. *Doktor Falsafah*. Universiti Malaysia Terengganu.
- Suprajitno, H. & bin Mohd., I. 2008. Interval Linear Programming, *Proceedings of the 3 International Conference on Mathematics and Statistics (ICOMS-3)*, Institut Pertanian Bogor, Indonesia. 5-6 Agustus 2008.
- Suprajitno, H. & bin Mohd., I. 2010. Linear Programming with Interval Arithmetic. *International Journal Contemporary Mathematical Sciences*. 5(7): 323-332.
- Suprajitno, H., bin Mohd., I., Mamat, M., & Ismail, F. 2009. Linear Programming with Interval Variables. *Proceedings of ICORAFSS 2009*, The ZON Regency Hotel, Johor Bahru, Malaysia. 2-4 June 2009.
- Taha, H.A. 1997. *Operation Research : An Introduction*, Ed. 8th. New Jersey, Pearson Prentice Hall.
- Winston, W.L. 2004. *Operation Research : Applications and Algorithms*, Ed. 4th. Thomson Brooks/Cole, Canada.
- Zhou, F., Huang, G.H., Chen, G., & Guo, H. 2009. Enhanced-Interval Linear Programming. *European Journal of Operational Research*, 199(2): 323-333.



um
The Learning
University

Digital Repository Universitas Jember

Ministry of Education and Culture

UNIVERSITAS NEGERI MALANG

Faculty of Mathematics and Natural Sciences

Jalan Semarang No 5, Malang, 65145

ICoMathApp
2020

CERTIFICATE

No. 30.9.8/UN32.3/TU/2020

This is to certify

AGUSTINA PRADJANINGSIH

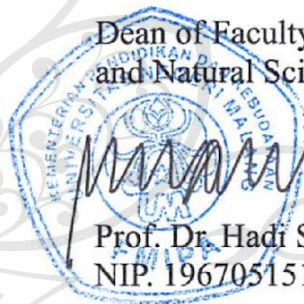
As the

Presenter

of a paper entitled *A Solution of Linear Programming with Interval Variables Using Interior Point Method Based on Interval Boundary Calculations*

In the **1st International Conference on Mathematics and Its Applications**
“Strengthening Researches on Mathematics for the Challenge of Global Society”
organized by the Department of Mathematics, Universitas Negeri Malang
September 30th, 2020, Malang, Indonesia

Dean of Faculty of Mathematics
and Natural Sciences,



Prof. Dr. Hadi Suwono, M.Si
NIP. 196705151991031007

Malang, September 30th, 2020
Chairman,

ICoMathApp
2020

Mochammad Hafiizh, S.Pd, M.Si, Ph.D
NIP. 198804182014041001

A Solution Of Linear Programming With Interval Variables Using Interior Point Method Based On Interval Boundary Calculations

Agustina Pradjaningsih¹, Fatmawati², Herry Suprajitno²
¹Jember University, ²Universitas Airlangga

Presented at
1st International Conference on Mathematics and its Applications
Department of Mathematics - Universitas Negeri Malang
September 30th, 2020

OVERVIEW

an optimization assumptions

- An optimization problem could be categorized as a linear programming model if it satisfies assumptions of proportionality, additivity, divisibility, and certainty [1].
- The certainty assumption is that all coefficients and variables in the model are known.

real situations

- However, in real situations, sometimes the values of coefficients and variables are not certainty known.
- The uncertainty problem could be overcome using the interval approach to the coefficients and variables.
- This approach supported the concept and theory of interval analysis.

OVERVIEW

coefficient and variable in the form of intervals

- Linear programming with interval coefficients is linear programming where the coefficients on the objective function and the constraint function are intervals.
- Suppose the coefficient and variable in the objective function and the constraint function are both in the form of intervals.
- In that case, it is called linear programming with interval variables or interval linear programming.

□ interval linear programming

□ interval variables

□ interior point method

□ the gradient projection algorithm

MODEL 1

The general form of interval linear programming

Maximize (objective function)

$$\underline{Z} = \sum_{j=1}^n [c_{jI}, c_{jS}] [x_{jI}, x_{jS}],$$

subject to

$$\sum_{j=1}^n [a_{ijI}, a_{ijS}] [x_{jI}, x_{jS}] \leq [b_I, b_S],$$

$$[x_{jI}, x_{jS}] \geq 0,$$

where

$$i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.,$$

$[x_{jI}, x_{jS}]$ decision variable,

$[c_{jI}, c_{jS}]$ coefficient of the objective function,

$[a_{ijI}, a_{ijS}]$ coefficient of the constraint function,

$[b_I, b_S]$ is a coefficient of resource constraint,

\underline{Z} is the objective function, and

$[x_{jI}, x_{jS}], [c_{jI}, c_{jS}], [a_{ijI}, a_{ijS}], [b_I, b_S] \in I(R)$.

Gradient Projection Algorithm

- ❑ The gradient projection algorithm is a variant of the interior point method. Due to its popularity, the gradient projection algorithm is called the interior point algorithm.
- ❑ The gradient projection algorithm starts with an initial interior point $\tilde{\mathbf{X}}^0$ which satisfies the interior point conditions, such that $\mathbf{A}\tilde{\mathbf{X}}^0 = \mathbf{b}$, $\tilde{\mathbf{X}}_j^0 > 0$ for $j = 1, 2, \dots, n + m$. The condition $\tilde{\mathbf{X}}_j^0 > 0$ makes the variable values $(x_1, x_2, \dots, x_{n+m})$ always within the boundary of the feasible area.
- ❑ The gradient projection algorithm produces a feasible interior point sequence, namely $\tilde{\mathbf{X}}^1$ (obtained from the 1st iteration), $\tilde{\mathbf{X}}^2$ (obtained from the 2nd iteration), \dots , $\tilde{\mathbf{X}}^k$ (obtained from the iteration- $k-1$), $\tilde{\mathbf{X}}^{k+1}$ (obtained from the k -iteration), and so on

The steps for gradient projection algorithm are as follows:

Initial step: Determine the initial interior point $\tilde{X}^0 = (x_1, x_2, \dots, x_{n+m})$ which satisfies the boundary constraints of the problem and calculate the value of Z_0 with

$$Z_0 = c^T \tilde{X}^0.$$

Step Iteration- $i+1$ ($i = 0, 1, 2, \dots$)

Step 1: Determine the diagonal matrix D_{i+1} , i.e.

$$D_{i+1} = \text{diag}(\tilde{X}^i) = \begin{bmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{n+m} \end{bmatrix}.$$

Step 2: Determine

$$\left. \begin{aligned} A_{i+1} &= A D_{i+1} \\ C_{i+1} &= D_{i+1} C \end{aligned} \right\}$$

The steps for gradient projection algorithm cont.

Step 3: Determine

3. The Projection matrix, i.e.

$$\mathbf{P}_{i+1} = \mathbf{I} - \mathbf{A}_{i+1}^T (\mathbf{A}_{i+1} \mathbf{A}_{i+1}^T)^{-1} \mathbf{A}_{i+1},$$

where \mathbf{I} is an identity matrix

4. The projected gradient, i.e.

$$\mathbf{C}_{P_{i+1}} = \mathbf{P}_{i+1} \mathbf{C}_{i+1}.$$

Step 4: Determine

$$\left. \begin{aligned} \mathbf{V}_{i+1} &= \left| \min(\mathbf{C}_{P_{i+1}}) \right| \\ &\quad \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} \\ \mathbf{M}_{i+1} &= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\alpha}{\mathbf{V}_{i+1}} \mathbf{C}_{P_{i+1}} \end{aligned} \right\}$$

where \mathbf{M}_{i+1} is $(n+m) \times 1$ matrix and $\alpha \in (0,1)$.

The steps for gradient projection algorithm cont.

Step 5: Determine the candidate interior point for the next iteration, i.e.

$$\tilde{\mathbf{X}}^{i+1} = \mathbf{D}_{i+1} \mathbf{M}_{i+1},$$

Step 6: Determine the optimum value, i.e.

$$\mathbf{Z}_{i+1} = \mathbf{C}^T \tilde{\mathbf{X}}^{i+1},$$

1. If $\mathbf{Z}_{i+1} > \mathbf{Z}_i$, then proceed to the next iteration with the same steps as in Steps 1-5 in the previous iteration. $\tilde{\mathbf{X}}^{i+1}$ is chosen to be the next iteration's interior point.
2. If $\mathbf{Z}_{i+1} \leq \mathbf{Z}_i$, then the iteration stops, and the optimum solution is obtained, namely the optimum point and the optimum value (\mathbf{Z} and $\tilde{\mathbf{X}}^i$).

FRAMEWORK

- ❑ linear programming with interval variables is solved using the interior point method based on interval boundary calculations.
- ❑ This solution's initial procedure is to change the linear programming model with interval variables into a pair of classical linear programming models.
- ❑ This pair or two classical linear programming models correspond to an interval boundary called the least upper bound and the greatest lower bound.
- ❑ The least upper bound and the greatest lower bound are solved using the interior point method in this case, the gradient projection algorithm.
- ❑ Finally, the optimal solution in the form of intervals is obtained from constructing the two models.

Agustina Pradjaningsih

agustina.fmipa@unej.ac.id

+62 857 4659 4067

Thank You